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by

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MASTER

Résumé:

En s'inspirant des récents travaux de Chyzhykov-Semochko et Belaïdi sur le φ-ordre, on étudie dans ce mémoire la croissance des solutions de l'équation différentielle linéaire complexe suivante $f^{(k)}+A_{k-1}(z)f^{(k-1)}+\dots+A_0(z)f=0,$ où $A_0(z),\dots,A_{k-1}(z)$ sont des fonctions analytiques dans $\overline{\mathbf{C}}\setminus\{z_0\},\ z_0\in\mathbf{C}$. Sous certaines conditions sur les coefficients, on établit des estimations sur le φ-ordre des solutions de ces équations.

Abstract :

Inspired by the recent works of Chyzhykov-Semochko and Belaïdi on the φ-order, we study in this thesis the growth of the solutions of the following complex linear differential equation $f^{(k)}+A_{k-1}(z)f^{(k-1)}+\dots+A_0(z)f=0,$ where $A_0(z),\dots,A_{k-1}(z)$ are analytic functions in $\overline{\mathbf{C}}\setminus\{z_0\},\ z_0\in\mathbf{C}$. Under certain conditions on the coefficients, we establish some estimates on the φ-order of the solutions of these equations.

الملخص :

مستوحاة من الأعمال الأخيرة لـ شيزيكوف-سيميشكو و بلعيدي على الترتيب ، ندرس في هذه الأطروحة نمو حلول المعادلة التفاضلية الخطية المعقدة $f^{(k)}+A_{k-1}(z)f^{(k-1)}+\dots+A_0(z)f=0,$ حيث $A_0(z),\dots,A_{k-1}(z)$ دوال تحليلية في $\overline{\mathbf{C}}\setminus\{z_0\},\ z_0\in\mathbf{C}$. في ظل شروط معينة على المعاملات ، نقوم بعمل تقديرات حول φ-ترتيب حلول هذه المعادلات.

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Contents

INTRODUCTION

"The shortest path between two truths is the real domain passes through the complex domain "
Jacques Hadamand

Complex analysis is all about the unite number i satisfying the quadratic $x^2 + 1 = 0$ which was the biggest concept ever in the mathematical analysis history. It is helpful in many areas of mathematics, including specially the analytic branches, for instance, algebraic geometry, analytic number theory and analytic combinatorics.

As years passed, a lot of new concepts and theories has been built in order to develop the field of complex analysis. In 1925, a new theory was devised by **Rolf Nevanlinna** [21] and **Hermann Weyl** [24]. It's the so called **Nevanlinna Theory** which deals with meromorphic functions as the main object in an analytical way, it describes the asymptotic behavior of such functions around some point z_0 on the extended complex plane, i. e., the analysis of the given equation $f(z) = a$, where $a \in \bar{\mathbb{C}}$ and f is any meromorphic function, for more one can checks ([17] and [25]).

Nowadays many contexts of science can be expressed by the language of applied mathematics as equations, models or some other problems and specially in the form of a differential system.

Throughout this work we will study the equation

$$f^{(k)} + A_{k-1}(z)\ f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{0.0.1}$$

where the A_j $(0 \leqslant j \leqslant k-1)$ are all analytic in a complex domain and f is a meromorphic function in the punctured plane $\bar{\mathbb{C}} - z_0$ (here z_0 represent a singular point). As in [20] Long and Zeng gave a generalization for the work of Hamouda and Fettouch, see [8] by introducing the $[p, q]$-order of growth of the solutions for the complex linear differential equation near a singular point.

Over the days many efforts has been done by a numerous searchers and collaborators to study the solutions of equation (0.0.1) under some additional conditions, such works have been done using the out breaking discovery of Nevanlinna by introducing the fundamental properties and theorems.

In order to study the behavior of the solutions for the equation (0.0.1) we'll represent new concept for the usual order of growth and type of growth namely the φ-order and the φ-type of a holomorphic function, here the φ is function of a special class, the idea was found on [6, 23]. A well known result can be stated as the following theorem: The equation

$$g^{(k)} + A_{k-1}(z)\ g^{(k-1)} + \cdots + A_0(z)g = 0,\ \ k \geqslant 2 \tag{0.0.2}$$

with analytic coefficients $(A_j)_{0 \leqslant j \leqslant k-1}$ and a leading coefficients A_0 with infinite order has all its nontrivial solutions of infinite order.

Proof outlines comes by using the two main Nevanlinna theorems and their reflections. For the rest denote by an A_j any holomorphic function on $\overline{\mathbb{C}} - \{z_0\} - z_0$, z_0 is a singular point.

By first Chapter 2 there will be listed some definitions. Concerning φ- order and φ - type, related background with a tricky lemmas and main results obtained. Sketches of the proofs will be found on the second chapter.

Chapter 1

Nevanlinna Theory Near an Essential Singular Point

"I tell you, with complex numbres
you can do anything "

John Derbyshire

1.1 Motivation

Taking a look at a given meromorphic function as a mathematical object inspires minds to ask how it actually behaves at each point? The interesting topic to study here is the growth of such objects : some of which may fastly grow up while the other ones act lazy. The creation of the tools for the success of the study comes intuitively and step by step. The first move were done by Rolf Nevanlinna (see [17] and [25]), then numerous researches have been done, some of such arrived to a fascinating, outbreaking results. A first motivational observation is the obvious question : how can we compare two holomorphic functions, which one leads in the sense of the growth ? Yes, they came up to compare each function with a model of the form $z^{a^{z^{b^{z^{\cdot^{\cdot^{\cdot}}}}}}}$, where constants a and b refers to the order and the type of the growth respectively. As in [4] Belaïdi came to find a comparison is another sense (see the definition the φ- order and φ- type) , while the magnitudes $|a|$ and $|b|$ on the tower $z^{a^{z^{b^{z^{\cdot^{\cdot^{\cdot}}}}}}}$ can be found by applying successive logarithms of the form $\underbrace{\log\log\ldots\log}_{p-times}$. Instead of taking logarithms some authors replaced it by special class of function having common properties with $\log_p$.

We aim to study the growth of solution of (0.0.1) providing comparison for the coefficients A_j $(0 \leqslant j \leqslant k-1)$, by each others, this comparison may show which coefficient leads the differential equation and how this will change thing about the behavior of f. By the following, we will emphasize the results obtained by Long and Zeng on [20] through the φ-order, φ-type concepts.

1.2 Concepts near a singular point

Let f be a meromorphic function defined in the punctured plane $\bar{\mathbb{C}} - \{z_0\}$ for some complex singular points.
The Nevanlinna's fundamentals are the most important tool for our study. For that reason, we dedicate the entire chapter to fully developing the theory for a function with singular point z_0.

In order to develop our study, we firstly recall some related notations. Let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the whole extended complex plane, $z_0 \in \mathbb{C}$ is some essential singularity.
Define the counting function of f near z_0 by the following formula

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t} dt - n(\infty,f)\log r,$$

where $n\,(t,f)$ denote the number of poles of f in the region $\{z \in \mathbb{C} : t \leqslant |z - z_0|\} \cup \{\infty\}$ counting its multiplicities, we also define the proximity function near z_0 by

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi.$$

Summing up together, the characteristic function of f near z_0 will be

$$T_{z_0}(r,,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

For $k \geqslant 2$ a positive integer, consider the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)\ f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.2.1}$$

where the coefficients are analytic in some complex domain. Since it's hard to find some general forms for the solutions of (1.2.1), many searchers are interested on the study of the behavior of such solutions and specially the notion of the growth. The strongest tool they used for establishing their results is the Nevanlinna theory which can be found in [1] , [9] , [11] , [17] and [25].

In [13, 14], Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p,q]$-order and obtained some results about their growth. In [18], in order to maintain accordance with general definitions of the entire function f of iterated p-order [16], Liu-Tu-Shi gave a minor modification of the original definition of the $[p,q]$-order given in [13, 14]. With this new concept of $[p,q]$-order, Liu, Tu and Shi [19] have considered equation (1.2.1) with entire coefficients and obtained different results concerning the growth of their solutions. After that, several authors used this new concept to investigate the growth of solutions in the complex plane and in the unit disc [2, 3, 18].

In [6], Chyzhykov and Semochko showed that both definitions of iterated order and of $[p,q]$-order have the disadvantage that they do not cover arbitrary growth, i.e., there exist entire or meromorphic functions of infinite $[p,q]$-order and $p-$th iterated order for arbitrary $p \in \mathbb{N}$, i.e., of infinite degree, see Example 1.4 in [6]. They used more general scale, called the φ-order (see [6, 23]). In recent times, the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [4, 6, 22]).

As in [20] Long and Zeng gave a generalisation for the work of Hamouda and Fettouch (see [8]) by introducing the $[p,q]$- order near a essential singular point. So, we find it very interesting to generalize the work done on [20] by introducing the concept of the φ- order near an essential singular point, that is the coefficients of (1.2.1) are all analytic in $\overline{\mathbb{C}} - \{z_0\}$.
Recently, Chyzhykov and Semochko [6] have given general definition of growth for an entire function in the complex plane by introducing a new class of functions. So, as in [6], let ϕ be the class of positive and bounded increasing functions φ on $[1,\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e ,

$$\forall c > 0 : \lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1.$$

Here some useful properties of a function $\varphi \in \phi$.

Proposition 1.2.1 ([6]) *If $\varphi \in \phi$, then the following holds*

$$i)\quad \forall\delta > 0 : \lim_{x\to+\infty} \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} = +\infty,$$

$$ii)\quad \forall m > 0, k \geqslant 0 : \lim_{x\to+\infty} \frac{\varphi^{-1}(\log x^m)}{x^k} = +\infty.$$

$$iii) \ \ \forall c>0,\ \varphi(ct)\leqslant\varphi(t^c)\leqslant(1+o(1))\varphi(t),\quad t\longrightarrow+\infty.$$

Definition 1.2.1 ([6]) *Let φ be an increasing unbounded function on $[1,+\infty)$. The φ-orders of a meromorphic function f are defined by*

$$\rho_\varphi^0(f):=\limsup_{r\to+\infty}\frac{\varphi\left(e^{T(r,f)}\right)}{\log r},\qquad \rho_\varphi^1(f):=\limsup_{r\to+\infty}\frac{\varphi(T(r,f))}{\log r}.$$

If f is an entire function, then the φ-orders are defined by

$$\tilde\rho_\varphi^0(f):=\limsup_{r\to+\infty}\frac{\varphi(M(r,f))}{\log r},\qquad \tilde\rho_\varphi^1(f):=\limsup_{r\to+\infty}\frac{\varphi(\log M(r,f))}{\log r}.$$

Proposition 1.2.2 ([6]) *Let $\varphi\in\Phi$ and f be an entire function. Then*

$$\rho_\varphi^j(f)=\tilde\rho_\varphi^j(f),\ j=0,1.$$

We now turn our attention to the basic definitions which may be are new concepts. Below, we will define the growth for a meromorphic function of f near a singular point.

Definition 1.2.2 ([15]) *Let φ be an increasing and unbounded function on $[1,+\infty)$. Then, the orders of the growth of a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ is given by*

$$\rho_\varphi^0(f,z_0)=\limsup_{r\to0}\frac{\varphi\left(e^{T_{z_0}(r,f)}\right)}{\log\frac{1}{r}},\ \rho_\varphi^1(f,z_0)=\limsup_{r\to0}\frac{\varphi\left(T_{z_0}(r,f)\right)}{\log\frac{1}{r}}.$$

If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then the φ-orders are defined by

$$\widetilde\rho_\varphi^0(f,z_0)=\limsup_{r\to0}\frac{\varphi(M_{z_0}(r,f))}{\log\frac{1}{r}},\ \widetilde\rho_\varphi^1(f,z_0)=\limsup_{r\to0}\frac{\varphi(\log M_{z_0}(r,f))}{\log\frac{1}{r}},$$

here $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$.

Remark 1.2.1 *A motivational observation for the creation of the above definition is that $\varphi(r)=\log\log r\in\phi$ it's also obviously that $\widetilde\rho_\varphi^0(f,z_0)=\rho_\varphi^0(f,z_0)$ due to the double inequality between M and T in* [11, *p.*41].

Definition 1.2.3 ([15]) *Let φ be an increasing and unbounded function on $[1,\infty)$. Then, the types of the growth of an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\widetilde\rho_\varphi^0(f,z_0)\in(0,+\infty)$ and $\widetilde\rho_\varphi^1(f,z_0)\in(0,+\infty)$ are defined by*

$$\widetilde\tau_\varphi^0(f,z_0)=\limsup_{r\to0}\frac{\exp\{\varphi(M_{z_0}(r,f))\}}{\frac{1}{r^{\widetilde\rho_\varphi^0(f,z_0)}}},$$

$$\widetilde\tau_\varphi^1(f,z_0)=\limsup_{r\to0}\frac{\exp\{\varphi(\log M_{z_0}(r,f))\}}{\frac{1}{r^{\widetilde\rho_\varphi^1(f,z_0)}}}.$$

Recently, Long and Zeng have investigated the $[p,q]$-order of growth of solutions of equation (1.2.1) and obtained some estimations of $[p,q]$-order of growth of solutions of such equation which is a generalization of previous results from Fettouch-Hamouda [8]. Before stating the results of Long and Zeng, we give here the definitions of the $[p,q]$-order and the $[p,q]$-type of a meromorphic function near a singular point.

Definition 1.2.4 ([20]) *Let f a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, for p,q two integers $p \geq q \geq 1$, the $[p,q]-$order of growth is defined by*

$$\rho_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q(\frac{1}{r})}$$

If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then the $[p,q]-$order of growth is defined by

$$\rho_{M,[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q(\frac{1}{r})},$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z-z_0| = r\}$.

Definition 1.2.5 ([20]) *Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho = \rho_{[p,q]}(f,z_0) \in (0,\infty)$. Then the $[p,q]-$type of f is defined by*

$$\tau_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^{\rho}}$$

If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho = \rho_{[p,q]}(f,z_0) \in (0,\infty)$, then the $[p,q]-$type of f is defined by

$$\tau_{M,[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^{\rho}}.$$

Theorem 1.2.1 ([20]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying $\max\left\{\rho_{[p,q]}(A_j,z_0) : j \neq 0\right\} < \rho_{[p,q]}(A_0,z_0) < \infty$. Then, every nontrivial solution of* (1.2.1) *that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0)$.*

Theorem 1.2.2 ([20]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying the following conditions*

$$i)\ \max\left\{\rho_{[p,q]}(A_j,z_0) : j \neq 0\right\} \leqslant \rho_{[p,q]}(A_0,z_0) < \infty,$$

$$ii)\ \max\left\{\tau_{[p,q]}(A_j,z_0):\rho_{[p,q]}(A_j,z_0)=\rho_{[p,q]}(A_0,z_0)\right\}<\tau_{[p,q]}(A_0,z_0).$$

Then, every nontrivial solution of (1.2.1) *that is analytic in* $\overline{\mathbb{C}}-\{z_0\}$*, satisfies*

$$\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0).$$

Theorem 1.2.3 ([20]) *Let* $A_0(z),A_1(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}-\{z_0\}$ *satisfying*

$$\max\left\{\rho_{[p,q]}(A_j,z_0):j\neq s\right\}<\rho_{[p,q]}\left(A_s,z_0\right)<\infty.$$

Then, every nontrivial solution of (1.2.1) *that is analytic in* $\overline{\mathbb{C}}-\{z_0\}$*, satisfies*

$$\rho_{[p+1,q]}(f,z_0)\leqslant\rho_{[p,q]}\left(A_s,z_0\right)\leqslant\rho_{[p,q]}(f,z_0).$$

Here is the full generalization for the work of Long and Zeng given on [20] by using the concept of the φ-order. The following theorem seems like to be a classical version that describes the impact of A_0.

Theorem 1.2.4 ([15]) *Let* $A_0(z),A_1(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}-\{z_0\}$*, all together satisfying* $\max\left\{\widetilde{\rho}_{\varphi}^{0}\left(A_j,z_0\right):j\neq0\right\}<\widetilde{\rho}_{\varphi}^{0}\left(A_0,z_0\right)<\infty$*. Then, every nontrivial solution of* (1.2.1) *is analytic in* $\overline{\mathbb{C}}-\{z_0\}$*, satisfies* $\widetilde{\rho}_{\varphi}^{1}(f,z_0)=\widetilde{\rho}_{\varphi}^{0}(A_0,z_0)$*.*

The following theorem discusses the case of the quality in the condition, namely A_0 still a dominant coefficient but not the only one.

Theorem 1.2.5 ([15]) *Let* $A_0(z),A_1(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}-\{z_0\}$ *all together satisfying the following conditions*

$$i)\ \max\left\{\widetilde{\rho}_{\varphi}^{0}(A_j,z_0):j\neq0\right\}\leqslant\widetilde{\rho}_{\varphi}^{0}(A_0,z_0)<\infty,$$

$$ii)\ \max\left\{\widetilde{\tau}_{\varphi}^{0}(A_j,z_0):\widetilde{\rho}_{\varphi}^{0}(A_j,z_0)=\widetilde{\rho}_{\varphi}^{0}(A_0,z_0)\right\}<\widetilde{\tau}_{\varphi}^{0}(A_0,z_0).$$

Then, every nontrivial solution of (1.2.1) *that is analytic in* $\overline{\mathbb{C}}-\{z_0\}$*, satisfies*

$$\widetilde{\rho}_{\varphi}^{1}(f,z_0)=\widetilde{\rho}_{\varphi}^{0}(A_0,z_0).$$

For the last Theorem we suppose that the dominant coefficient runs over the set $\{0,1,2,...,k-1\}$.

Theorem 1.2.6 ([15]) *Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} - \{z_0\}$ *all together satisfying the following condition*

$$\max\left\{\widetilde{\rho}_{\varphi}^{0}(A_j, z_0) : j \neq s\right\} < \widetilde{\rho}_{\varphi}^{0}(A_s, z_0) < \infty.$$

Then, every nontrivial solution of (1.2.1) *that is analytic in* $\overline{\mathbb{C}} - \{z_0\}$*, satisfies*

$$\widetilde{\rho}_{1}^{1}(f, z_0) \leqslant \widetilde{\rho}_{\varphi}^{0}(A_s, z_0) \leqslant \widetilde{\rho}_{\varphi}^{0}(f, z_0).$$

Remark 1.2.2 ([7]) *The condition that* f *is analytic in* $\overline{\mathbb{C}} - \{z_0\}$ *is necessary. The following example shows that there exists a solution* $f(z)$ *of* (1.2.1) *such that* $f(z)$ *is not analytic in* $\overline{\mathbb{C}} - \{z_0\}$ *provided that all coefficients* $A_j(z)$ $(j = 0, ..., k-1)$ *of* (1.2.1) *are analytic in* $\overline{\mathbb{C}} - \{z_0\}$*. For instance, we consider the equation*

$$f'' + \left(\exp_2\left\{\frac{1}{z_0 - z}\right\} + \frac{1}{z_0 - z}\right) f' + \frac{2}{z_0 - z}\exp_2\left\{\frac{1}{z_0 - z}\right\} f = 0. \tag{1.2.2}$$

The function $f(z) = (z_0 - z)^2$ *solves* (1.2.2)*, and* $f(z)$ *is not analytic in* $\overline{\mathbb{C}} - \{z_0\}$*. So, in our results, we suppose always that* $f(z)$ *is analytic in* $\overline{\mathbb{C}} - \{z_0\}$*.*

1.3 Preliminary results

We now concentrated on the main preliminaries needed for establishing the proofs of our results. We firstly clarify some notations. Denote, the logarithmic measure of a set $E \subset (0, 1)$ by

$$m_l(E) = \int_E \frac{dt}{t}.$$

We also denote by $\nu(r, g)$ the central index of an entire function $g(z)$ in $\mathbb{C}$, for more properties, see [12, p.33 − 35]. Finally, denote the central index of an analytic function f in $\overline{\mathbb{C}} - \{z_0\}$ by $\nu_{z_0}(r, f)$ (reader may check [10, p.996]).

Lemma 1.3.1 *(As in* [20]*, Lemma 2.5). Let* $g : (0, 1) \to \mathbb{R}$*,* $h : (0, 1) \to \mathbb{R}$ *be a monotone decreasing function such that* $g(r) \geqslant h(r)$ *possibly outside an exceptional set* $E \subset (0, 1)$ *that has finite logarithmic measure. Then, for any given* $\beta > 1$*, there exists a constant* $0 < r_0 < 1$ *such that for all* $r \in (0, r_0)$*, we have* $g(r^{\beta}) \geqslant h(r)$*.*

Proof. Set $\alpha = \int_E \frac{dt}{t} < \infty$, and choose $r_0 = \exp\left(\frac{\alpha}{1-\beta}\right) \in (0,1)$. So that for all $0 < r < r_0$, the interval $I_r = [r^\beta, r]$ meets E^c. Since,

$$\int_{I_r} \frac{dt}{t} = \int_{r^\beta}^{r} \frac{dt}{t} = (1-\beta)\log r > (1-\beta)\log r_0 = \alpha.$$

Therefore, by the monotonicity of g and h, there exists $t \in I_r$ for which

$$g(r^\beta) \geqslant g(t) \geqslant h(t) \geqslant h(r).$$

Lemma 1.3.2 ([12]) *Let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ be an entire function. Let $\mu(r)$ and $\nu_f(r)$ denoting respectively the maximum term and the central index of f, i.e., $\mu(r) = \max\{|a_n|r^n; n = 0, 1, \dots\}$ and $\nu_f(r) = \max\{n : \mu(r) = |a_n|r^n\}$. Then, we have*

$$\log \mu(r) = \log|a_0| + \int_0^r \frac{\nu_f(t)}{t}\, dt \quad (|a_0| \neq 0), \tag{1.3.1}$$

$$M(r,f) < \mu(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\} \quad (R > r). \tag{1.3.2}$$

Lemma 1.3.3 ([15]) *Let $\varphi \in \Phi$ and f be an entire function. Then, we have*

$$\rho_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi(\exp \nu_f(r))}{\log r},$$

where $\nu_f(r)$ is the central index of f.

Proof. Denote $\rho := \limsup\limits_{r\to+\infty} \frac{\varphi(\exp\nu_f(r))}{\log r}$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$\nu_f(r) \leqslant \log \varphi^{-1}(\log r^{\rho+\varepsilon}). \tag{1.3.3}$$

By setting $R = 2r$ in (1.3.2), we get

$$M(r,f) < \mu(r)\left(\nu_f(2r) + 2\right) = |a_{\nu_f(r)}| r^{\nu_f(r)} \left(\nu_f(2r) + 2\right). \tag{1.3.4}$$

Since $\{|a_n|\}_{n\geqslant 0}$ is a bounded sequence, then by using (1.3.3) and (1.3.4), we obtain

$$\begin{aligned} M(r,f) &< cr^{\nu_f(r)}\left(\nu_f(2r) + 2\right) \\ &< cr^{\log\varphi^{-1}(\log r^{\rho+\varepsilon})}\left(\log\varphi^{-1}(\log(2r)^{\rho+\varepsilon}) + 2\right) \\ &= ce^{\log\varphi^{-1}(\log r^{\rho+\varepsilon})\log r}\left(\log\varphi^{-1}(\log(2r)^{\rho+\varepsilon}) + 2\right) \\ &\leqslant e^{\log\varphi^{-1}(\log r^{\rho+3\varepsilon})} = \varphi^{-1}(\log r^{\rho+3\varepsilon}), \end{aligned} \tag{1.3.5}$$

where $c > 0$ is a real constant. From (1.3.5), by the monotonicity of φ, we get

$$\frac{\varphi(M(r,f))}{\log r} \leqslant \rho + 3\varepsilon.$$

By arbitrariness of $\varepsilon > 0$ and Proposition 1.2.2, we obtain

$$\rho_\varphi^0(f) \leqslant \rho := \limsup_{r\to+\infty} \frac{\varphi(\exp \nu_f(r))}{\log r}. \tag{1.3.6}$$

Now, we prove the reverse inequality. Without loss of generality, we may assume $|a_0| \neq 0$. It follows from (1.3.1) that

$$\log \mu(2r) = \log |a_0| + \int_0^{2r} \frac{\nu_f(t)}{t}\, dt \geqslant \log |a_0| + \nu_f(r) \int_r^{2r} \frac{dt}{t}$$

$$= \log |a_0| + \nu_f(r) \log 2.$$

By Cauchy's inequality we have $\mu(2r) \leqslant M(2r,f)$ and then

$$\nu_f(r) \leqslant \frac{\log M(2r,f)}{\log 2} - \frac{\log |a_0|}{\log 2} \leqslant c_1 \log M(2r,f), \tag{1.3.7}$$

where $c_1 > 2$ is a real constant. It follows from (1.3.7) and Proposition 1.2.1, especially case (iii), that

$$\frac{\varphi(\exp \nu_f(r))}{\log r} \leqslant \frac{\varphi((M(2r,f))^{c_1})}{\log 2r} \cdot \frac{\log 2r}{\log r} \leqslant \frac{(1+o(1))\, \varphi(M(2r,f))}{\log 2r} \cdot \frac{\log 2r}{\log r}.$$

Hence

$$\limsup_{r\to+\infty} \frac{\varphi(\exp \nu_f(r))}{\log r} \leqslant \limsup_{r\to+\infty} \frac{\varphi(M(2r,f))}{\log 2r} = \rho_\varphi^0(f). \tag{1.3.8}$$

We deduce from (1.3.6) and (1.3.8) that

$$\rho_\varphi^0(f) = \limsup_{r\to+\infty} \frac{\varphi(\exp \nu_f(r))}{\log r}.$$

Lemma 1.3.4 ([15]) *Let f be a non constant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. For a function $\varphi \in \phi$ one has*

$$\limsup_{r\to 0} \frac{\varphi(e^{\nu_{z_0}(r,f)})}{\log \frac{1}{r}} = \rho_\varphi^0(f, z_0).$$

*Proof***.** Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. As the function g is entire [10, Remark 7], it turns out that

$$\nu_{z_0}(r,f) = \nu(R,g),\ R = \frac{1}{r}.$$

By Lemma 1.3.3, we have

$$\rho_{\varphi}^{0}(g)=\limsup_{R\to+\infty}\frac{\varphi(e^{\nu(R,g)})}{\log R}.$$

By Lemma 2.2 in [8], we have

$$T(R,g)=T_{z_0}(r,f).$$

That gives

$$\rho_{\varphi}^{0}(f,z_0)=\rho_{\varphi}^{0}(g)=\limsup_{r\to 0}\frac{\varphi(\exp(\nu_{z_0}(r,g))}{\log\frac{1}{r}}.$$

Lemma 1.3.5 ([15]) *Let f be a non constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho_{\varphi}^{0}(f,z_0)=\rho$. Then, there exists a set $E\subset(0,1)$ with $m_l(E)=+\infty$ such that for all $|z-z_0|=r\in E$,*

$$\lim_{r\to 0}\frac{\varphi(M_{z_0}(r,f))}{\log\frac{1}{r}}=\rho.$$

*Proof***.** By Definition 1.2.1, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1}<\frac{n}{n+1}r_n$ and

$$\lim_{n\to+\infty}\frac{\varphi(M_{z_0}(r_n,f))}{\log\frac{1}{r_n}}=\rho$$

Therefore, there exists $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$ and for every $r\in[\frac{n}{n+1}r_n,r_n]$, we get

$$\frac{\varphi(M_{z_0}(r_n,f))}{\log\frac{1}{\frac{n}{n+1}r_n}}\leqslant\frac{\varphi(M_{z_0}(r,f))}{\log\frac{1}{r}}\leqslant\frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n,f))}{\log\frac{1}{r_n}}.$$

Therefore, since

$$\lim_{n\to+\infty}\frac{\varphi(M_{z_0}(r_n,f))}{\log\frac{1}{\frac{n}{n+1}r_n}}=\lim_{n\to+\infty}\frac{\varphi(M_{z_0}(\frac{n}{n+1}r_n,f))}{\log\frac{1}{r_n}}=\rho,$$

then yielding

$$\lim_{r\to 0}\frac{\varphi(M_{z_0}(r,f))}{\log\frac{1}{r}}=\rho$$

for all $r\in\left[\frac{n}{n+1}r_n,r_n\right]$. By setting $E=\bigcup\limits_{n=n_0}^{\infty}\left[\frac{n}{n+1}r_n,r_n\right]$, the conclusion follows since E fulfills $m_l(E)=+\infty$.

By analogous logic, we establish the same lemma with the limit.

Lemma 1.3.6 ([15]) *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho_{\varphi}^{0}(f,z_0)=\rho$. Then, there exists a set $E\subset(0,1)$ with $m_l(E)=+\infty$ such that for all $|z-z_0|=r\in E$,*

$$\lim_{r\to 0}\frac{\varphi(e^{T_{z_0}(r,f)})}{\log\frac{1}{r}}=\rho.$$

Lemma 1.3.7 ([15]) *Let f be a non constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\widetilde{\rho}_{\varphi}^{0}(f,z_0)=\rho\in(0,\infty)$ and $\widetilde{\tau}_{\varphi}^{0}(f,z_0)=\tau\in(0,\infty)$, for any given $\beta\in(0,\infty)$, there exists a set $E\subset(0,1)$ of infinite logarithmic measure such that for $|z-z_0|=r\in E$,*

$$\varphi\left(M_{z_0}\left(r,f\right)\right)\geqslant\log\left(\frac{\beta}{r^{\rho}}\right).$$

Proof. For the proof, by the same reasoning as the previous Lemma 1.3.5, we obtain the desired conclusion. Here we omit the details.

Lemma 1.3.8 ([15]) *Let $A_j(z)$ be analytic in $\overline{\mathbb{C}}-\{z_0\}$ satisfying all together, the inequality*

$$\widetilde{\rho}_{\varphi}^{0}(A_j,z_0)\leqslant\rho<\infty,\ \ j=0,1,...,k-1.$$

Then every solution of (1.2.1) *that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ satisfies $\widetilde{\rho}_{\varphi}^{1}(f,z_0)\leqslant\rho$.*

Proof. The equation (1.2.1) implies

$$\left|\frac{f^{(k)}}{f}\right|\leqslant|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|+|A_0(z)|. \tag{1.3.9}$$

By the definition of $\widetilde{\rho}_{\varphi}^{0}(f,z_0)$ and since one has the bound $\widetilde{\rho}_{\varphi}^{0}(A_j,z_0)\leqslant\rho$ ($j=0,1,...,k-1$), then for any given $\varepsilon>0$, there exists $r_0\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_0)$, we get

$$|A_j(z)|\leqslant\varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{r}\right),\ \ (\ j=0,1,...,k-1\,). \tag{1.3.10}$$

By throughing reader back to ([10], Theorem 8), there exists a set $E\subset(0,1)$ that has infinite logarithmic measure, such that for all $j\in\{0,1,...,k\}$ and $r\notin E$, we have

$$\left|\frac{f^{(j)}\left(z\right)}{f\left(z\right)}\right|=|1+o(1)|\left(\frac{\nu_{z_0}(r,f)}{r}\right)^{j},\ \ r\to0, \tag{1.3.11}$$

for z in the cercle $|z-z_0|=r$ and $|f(z)|=\max_{|z-z_0|=r}|f(z)|$. Together, combining the three estimations (1.3.9), (1.3.10) and (1.3.11), we get for all $|z-z_0|=r\in(0,r_0)\diagdown E$ and $|f(z)|=M_{z_0}(r,f)$

$$\nu_{z_0(r,f)}\leqslant kr\varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{r}\right)|1+o(1)|. \tag{1.3.12}$$

Finally by Lemma 1.3.1, the last claim, and (1.3.12), the desired conclusion follows.

Lemma 1.3.9 ([5]) *(Logarithmic Derivative Lemma) Let f be a non constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ and $k \geqslant 1$ be an integer. Then, we have*

$$m_{z_0}\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right)$$

for all $r \in (0,1)\diagdown E$, where $m_l(E) = \int_E \frac{dr}{r} < \infty$. If $\rho(f,z_0) < \infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\frac{1}{r}\right).$$

Then the following lemma helps to complete the proof of the third theorem.

Lemma 1.3.10 ([20]) *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$. Then f enjoys the following two properties*

$$i)\ T_{z_0}\left(r, \frac{1}{f}\right) = T_{z_0}(r,f) + O(1),$$

$$ii)\ T_{z_0}(r,f') < O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right),\ \ r \in (0,r_0]\diagdown E,\ \ \text{where } E \subset (0,r_0]\ \ \text{with } m_l(E) < \infty.$$

Proof. By Lemma 2.2 in [8], it is easy to see that T_{z_0} shares some same familiar properties as T on the Nevanlinna theory. So a suitable substitution may prove the lemma. Set

$$g(w) = f\left(z_0 - \frac{1}{w}\right).$$

Then Lemma 2.2 of [8] shows that

$$T\left(R, \frac{1}{g}\right) = T_{z_0}\left(\frac{1}{R}, \frac{1}{f}\right).$$

By the first main Nevanlinna theory, we get

$$T\left(R, \frac{1}{g}\right) = T(R,g) + O(1).$$

Thus

$$T_{z_0}\left(r, \frac{1}{f}\right) = T_{z_0}(r,f) + O(1).$$

So, the conclusion (i) holds. By definition, one has

$$T_{z_0}(r,f') = m_{z_0}(r,f') + N_{z_0}(r,f') \leqslant 2T_{z_0}(r,f) + m_{z_0}\left(r, \frac{f'}{f}\right).$$

From the last inequality and Lemma 1.3.9 it follows that there exists a set $E \subset (0, r_0]$ that has finite logarithmic measure an for all $|z - z_0| = r \in (0, r_0] \diagdown E$,

$$T_{z_0}(r, f') \leqslant O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right)$$

Hence, (ii) is established.

Lemma 1.3.11 ([15]) *Let f_1, f_2 be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\rho^0_\varphi(f_1, z_0) = \rho_1 > 0$, $\rho^0_\varphi(f_2, z_0) = \rho_2 < \infty$ and $\rho_2 < \rho_1$. Then, there exists a set $E \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E$ one has*

$$\lim_{r \to 0} \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} = 0.$$

*Proof***.** By Definition 1.2.1, for any $\varepsilon > 0$ with $\varepsilon < \frac{\rho_1 - \rho_2}{2}$, there exists $r_0 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_0)$ the following holds

$$T_{z_0}(r, f_2) \leqslant \log \varphi^{-1}\left((\rho_2 + \varepsilon) \log \frac{1}{r}\right). \tag{1.3.13}$$

Concerning the Lemma 1.3.6, we have deduce the existence of some sets $E \subset (0, r_0)$ of infinite logarithmic measure such that for all $|z - z_0| = r \in E$

$$T_{z_0}(r, f_1) \geqslant \log \varphi^{-1}\left((\rho_1 - \varepsilon) \log \frac{1}{r}\right). \tag{1.3.14}$$

Combining (1.3.13) and (1.3.14), it follows that for all $|z - z_0| = r \in E \cap (0, r_0)$

$$0 \leqslant \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} \leqslant \frac{\log \varphi^{-1}\left((\rho_2 + \varepsilon) \log \frac{1}{r}\right)}{\log \varphi^{-1}\left((\rho_1 - \varepsilon) \log \frac{1}{r}\right)}$$

as $\rho_2 + \varepsilon < \rho_1 - \varepsilon$, by setting $(\rho_2 + \varepsilon) \log \frac{1}{r} = x$ and $\frac{\rho_1 - \varepsilon}{\rho_2 + \varepsilon} = 1 + \delta$ $(\delta > 0)$ and making use of Proposition 1.2.1 (i)

$$\lim_{r \to 0} \frac{\log \varphi^{-1}\left((\rho_2 + \varepsilon) \log \frac{1}{r}\right)}{\log \varphi^{-1}\left((\rho_1 - \varepsilon) \log \frac{1}{r}\right)} = \lim_{r \to 0} \frac{\log \varphi^{-1}\left((\rho_2 + \varepsilon) \log \frac{1}{r}\right)}{\log \varphi^{-1}\left(\frac{\rho_1 - \varepsilon}{\rho_2 + \varepsilon} (\rho_2 + \varepsilon) \log \frac{1}{r}\right)}$$

$$= \lim_{x \to +\infty} \frac{\log \varphi^{-1}(x)}{\log \varphi^{-1}((1 + \delta) x)} = 0.$$

Therefore yielding

$$\lim_{r \to 0} \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} = 0.$$

Lemma 1.3.12 *For a sequence $(x_j)_{j\in\mathbb{N}_n}$ of no negative reals the following hold.*

- $\log^+(x_i, x_j) \leqslant \log^+ x_i + \log^+ x_j$
- $\log^+(\sum_{j\leqslant n} x_j) \leqslant \sum_{j\leqslant n} \log^+ x_j + \log n$

Proof **.** 1) For the first property we have to treat two cases.

Case 1 : If $x_i x_j \leqslant 1$, then $\log^+ x_i x_j = 0 \leqslant \log^+ x_i + \log^+ x_j$ because the $\log^+$ is positive valued function

Case 2 : If $x_i x_j \geqslant 1$, then $\log^+ x_i x_j = \log x_i x_j = \log x_i + \log x_j$. Since $\log^+ x \geqslant \log x$, we obtain

$$\log^+ x_i x_j \leqslant \log^+ x_i + \log^+ x_j$$

2) Without loss of generality suppose that

$$x_1 \leqslant x_2 \leqslant ... \leqslant x_n$$

Then

$$\log n + \sum_{j=1}^{n} \log^+ x_j \geqslant \log^+ n + \log^+ x_n \geqslant \log^+(n x_n) \leqslant \log^+\left(\sum_{j=1}^{n} x_j\right).$$

The next lemma finishes the preliminaries.

Lemma 1.3.13 ([8]) *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, let $\gamma > 1$, $\varepsilon > 0$ be given real constants and $k \in \mathbb{N}$. Then there exist a set $E \subset (0, r_0]$, $(r_0 \in (0,1))$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on γ and k such that for all $|z - z_0| = r \in (0, r_0] \setminus E$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leqslant \lambda \left[\frac{1}{r^2} T_{z_0}\left(\frac{1}{\gamma} r, f\right) \log T_{z_0}(r, f)\right]^k.$$

Chapter 2

Growth of Solutions of Complex Linear Differential Equations Near an Essential Singular Point

"Mathematical Analysis is as
extensive as nature herself"
Joseph Fourier

In this chapter, we perform initial and essential manipulation toward establishing the three theorems mentioned earlier. So, we will find it convenient to use the lemmas listed on the second chapter in order to emphasize the theorems with no other technologies. All such proofs were explored in a straight direction. These arguments are generalizations of the ones performed by Long and Zeng in [20] but a good remark can be observed.

Remark 2.0.1 *In* [20] *Long and Zeng used some positive integers $q \geqslant 1$ while we took $q = 1$ on the definition of the $\varphi-$order, that is I really can, let q be random on $\mathbb{Z}_{>0}$ while the results remain almost possibly similar based all on the estimation obtained by the definition in a combination with the fundamental results obtained by the other searchers.*

2.1 Proof of Theorems

Proof of Theorem 1.2.4 Set $\widetilde{\rho}_{\varphi}^{0}(A_0, z_0) = \rho$. Choose α and β such that

$$\max\left\{\widetilde{\rho}_{\varphi}^{0}(A_j, z_0) : j \neq 0\right\} < \beta < \alpha < \rho.$$

By Definition 1.2.1, for any $\varepsilon \in \left(0, \min(\frac{\alpha-\beta}{2}, \frac{\rho-\alpha}{2})\right)$, there exists r_1 such that for all $|z - z_0| = r < r_1$,

$$|A_j(z)| \leqslant \varphi^{-1}\left((\beta+\varepsilon)\log\frac{1}{r}\right),\ j = 1, 2, ..., k-1. \tag{2.1.1}$$

By Lemma 1.3.5 for all ε given above, we conclude the existence of some r_2 and a set $E_1 \subset (0,1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in (0, r_2) \cap E_1$ and $|A_0(z)| = M_{z_0}(r, A_0)$

$$|A_0(z)| \geqslant \varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right). \tag{2.1.2}$$

Now, let $r_0 = \min(r_1, r_2)$ and $\gamma > 1$. By Lemma 1.3.13, there exists a set $E_2 \subset (0, r_0]$ that have finite logarithmic measure and a constant λ that depends on γ such that for all $|z - z_0| = r \in (0, r_0] \setminus E_2$ the following occurs

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \lambda\left(\frac{1}{r^2}T_{z_0}\left(\frac{r}{\gamma}, f\right)\log T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^j,\ j = 0, 1, ..., k. \tag{2.1.3}$$

By $(1.2.1)$, we get

$$|A_0(z)| \leqslant \left|\frac{f^{(k)}}{f}\right| + \cdots + |A_j(z)|\left|\frac{f^{(j)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{2.1.4}$$

As the last step, let $E_0 = (0, r_0] \cap E_1 \setminus E_2$, obviously E_0 has infinite logarithmic measure. Consequently, the combination between $(2.1.1)$, $(2.1.2)$, $(2.1.3)$ and $(2.1.4)$ gives for all $|z - z_0| = r \in E_0$,

$$\varphi^{-1}\left((\rho_1-\varepsilon)\log\frac{1}{r}\right) \leqslant \lambda k\left(\frac{1}{r}T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^{2k}\varphi^{-1}\left((\beta+\varepsilon)\log\frac{1}{r}\right). \tag{2.1.5}$$

Without loss of generality assume that ε is fixed small as we want. By contradiction suppose that

$$\rho_{\varphi}^{1}(f, z_0) < \rho.$$

Then we obtain for all such z as above

$$T_{z_0}(r, f) \leqslant \varphi^{-1}\left((\rho_1-\varepsilon)\log\frac{1}{r}\right) \tag{2.1.6}$$

for some $\rho_1 < \rho$. By (2.1.5) and (2.1.6) , we obtain

$$\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right) \leqslant \frac{\lambda k}{r^{2k}}\left(\varphi^{-1}\left((\rho_1-\varepsilon)\log\frac{\gamma}{r}\right)\right)^{2k}\varphi^{-1}\left((\beta+\varepsilon)\log\frac{1}{r}\right).$$

Since $\rho-\varepsilon > \max(\rho_1-\varepsilon, \beta+\varepsilon)$, we get

$$\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right) \leqslant \frac{\lambda k}{r^{2k}}\left(\varphi^{-1}\left(\max\left(\rho_1-\varepsilon,\beta+\varepsilon\right)\log\frac{\gamma}{r}\right)\right)^{2k+1}. \tag{2.1.7}$$

Applying the logarithm on both sides, we find

$$\frac{2k\log r}{(2k+1)\log\varphi^{-1}\left(\max\left(\rho_1-\varepsilon,\beta+\varepsilon\right)\log\frac{\gamma}{r}\right)} + \frac{\log\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right)}{(2k+1)\log\varphi^{-1}\left(\max\left(\rho_1-\varepsilon,\beta+\varepsilon\right)\log\frac{\gamma}{r}\right)}$$

$$< \frac{\log\lambda k}{(2k+1)\log\varphi^{-1}\left(\max\left(\rho_1-\varepsilon,\beta+\varepsilon\right)\log\frac{\gamma}{r}\right)} + 1. \tag{2.1.8}$$

Notice that φ^{-1} is increasing, so by applying the Proposition 1.2.1 (ii), we get

$$\lim_{r\to 0}\frac{\log r}{\log\varphi^{-1}\left(\max(\rho_1-\varepsilon,\beta+\varepsilon)\log\frac{\gamma}{r}\right)} = 0 \tag{2.1.9}$$

As we did earlier in Lemma 1.3.11, we have

$$\lim_{r\to 0}\frac{\log\varphi^{-1}\left((\rho-\varepsilon)\log\frac{1}{r}\right)}{\log\varphi^{-1}\left(\max(\rho_1-\varepsilon,\beta+\varepsilon)\log\frac{\gamma}{r}\right)} = +\infty. \tag{2.1.10}$$

The right hand side on (2.1.8) is finite while the left hand side is infinite, thus contradiction holds i,e.

$$\rho_\varphi^1(f, z_0) \geqslant \rho.$$

Thus by Remark 1.2.1 and as $\varphi \in \phi$ we obtain

$$\widetilde{\rho}_\varphi^1(f, z_0) \geqslant \rho.$$

By Lemma 1.3.8, we get

$$\widetilde{\rho}_\varphi^1(f, z_0) = \widetilde{\rho}_\varphi^0(A_0, z_0).$$

Proof of Theorem 1.2.5 By an analogous progress set $\widetilde{\rho}_\varphi^0(A_0, z_0) = \rho$, $\widetilde{\tau}_\varphi^0(A_0, z_0) = \tau$. If $\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, ..., k-1\} < \tilde{\rho}_\varphi^0(A_0, z_0) = \rho$, then by Theorem 1.2.4, we obtain $\tilde{\rho}_\varphi^1(f, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)$. Suppose that $\max\{\tilde{\rho}_\varphi^0(A_j, z_0) : j = 1, 2, ..., k-1\} = \tilde{\rho}_\varphi^0(A_0, z_0) = \rho$ $(0 < \rho < +\infty)$ and $\max\{\tilde{\tau}_\varphi^0(A_j, z_0) : \tilde{\rho}_\varphi^0(A_j, z_0) = \tilde{\rho}_\varphi^0(A_0, z_0)\} < \tilde{\tau}_\varphi^0(A_0, z_0) = \tau$ $(0 < \tau < +\infty)$. Then, there exists a set $I \subseteq \{1, 2, ..., k-1\}$ such that $\tilde{\rho}^0_{\varphi}(A_j, z_0) = \tilde{\rho}^0_{\varphi}(A_0, z_0) = \rho$ $(j \in I)$ and $\tilde{\tau}^0_{\varphi}(A_j, z_0) < \tilde{\tau}^0_{\varphi}(A_0, z_0)$ $(j \in I)$. Thus, we choose β_1, β_2 satisfying

$$\max\{\tilde{\tau}^0_{\varphi}(A_j, z_0) : (j \in I)\} < \beta_1 < \beta_2 < \tilde{\tau}^0_{\varphi}(A_0, z_0) = \tau.$$

From Definition 1.2.3, there exists $r_0 \in (0,1)$ such that for all $|z - z_0| = r \in (0, r_0)$

$$|A_j(z)| \leqslant \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right) \quad (j \in I) \tag{2.1.11}$$

and

$$|A_j(z)| \leqslant \varphi^{-1}\left(\log \frac{1}{r^{\rho_1}}\right) \leqslant \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right) \quad (j \in \{1, ..., k-1\} \setminus I), \tag{2.1.12}$$

where $0 < \rho_1 < \rho$. We now turns to Lemma 1.3.13, it claims the existence of a set $E_1 \subset (0, r_0]$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on some given $\gamma > 1$ such that for all $|z - z_0| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \lambda \left(\frac{1}{r^2} T_{z_0}\left(\frac{r}{\gamma}, f\right) \log T_{z_0}\left(\frac{r}{\gamma}, f\right)\right)^j \tag{2.1.13}$$

By Lemma 1.3.7, there exists a set $E_2 \subset (0,1)$ of infinite logarithmic measure for which

$$\varphi\left(M_{z_0}(r, A_0)\right) \geqslant \log \frac{\beta_2}{r^{\rho}}$$

equivalently

$$M_{z_0}(r, A_0) \geqslant \varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right). \tag{2.1.14}$$

Set $E_0 = E_2 \setminus E_1$, for sure E_0 has infinite logarithmic measure. Combining (2.1.11), (2.1.12), (2.1.13) and (2.1.14) with (2.1.4) we get for all $|z - z_0| = r \in E_0$,

$$\varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right) \leqslant \lambda k \left(\frac{1}{r} T_{z_0}(\frac{r}{\gamma}, f)\right)^{2k} \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right). \tag{2.1.15}$$

The last inequality implies $\rho^1_{\varphi}(f) \geqslant \rho$. To see why assume there $\rho^1_{\varphi}(f) < \rho$. Then there exists $\rho_2 < \rho$ such that

$$T_{z_0}(r, f) \leqslant \varphi^{-1}\left(\rho_2 \log \frac{1}{r}\right). \tag{2.1.16}$$

Consequently, by (2.1.15) and (2.1.16)

$$\varphi^{-1}\left(\log \frac{\beta_2}{r^{\rho}}\right) \leqslant \lambda k \frac{1}{r^{2k}} \left(\varphi^{-1}\left(\log \frac{\gamma^{\rho_2}}{r^{\rho_2}}\right)\right)^{2k} \varphi^{-1}\left(\log \frac{\beta_1}{r^{\rho}}\right).$$

$$\leqslant \lambda k \frac{1}{r^{2k}} \left(\varphi^{-1} \left(\log \frac{\beta_1}{r^\rho} \right) \right)^{2k+1}.$$

Applying the logarithm on both sides, we find

$$\frac{2k \log r}{(2k+1)\log \varphi^{-1}\left(\log \frac{\beta_1}{r^\rho}\right)}$$

$$+ \frac{\log \varphi^{-1}\left(\log \frac{\beta_2}{r^\rho}\right)}{(2k+1)\log \varphi^{-1}\left(\log \frac{\beta_1}{r^\rho}\right)} \leqslant \frac{\log(\lambda k)}{(2k+1)\log \varphi^{-1}\left(\log \frac{\beta_1}{r^\rho}\right)} + 1. \tag{2.1.17}$$

As we did before

$$\lim_{r\to 0} \frac{\log r}{\log \varphi^{-1}\left(\log \frac{\beta_1}{r^\rho}\right)} = 0$$

and

$$\lim_{r\to 0} \frac{\log \varphi^{-1}\left(\log \frac{\beta_2}{r^\rho}\right)}{(2k+1)\log \varphi^{-1}\left(\log \frac{\beta_1}{r^\rho}\right)} = +\infty$$

because $\log \frac{\beta_2}{r^\rho} > \log \frac{\beta_1}{r^\rho}$. Since the right hand side of the inequality (2.1.17) is bounded by 1, thus taking limits yielding $+\infty \leqslant 1$ which is a contradiction, hence the conclusion. As $\beta_2 < \beta_1$ and $\varphi \in \phi$, we get $\rho_\varphi^1(f, z_0) \geqslant \rho$. Finally, by applying Lemma 1.3.8, the desired theorem will be proved.

Proof of Theorem 1.2.6. For this section, suppose that the dominant coefficient is unique and runs over the set $\{0, 1, ..., k-1\}$. In other words, there exists $s \in \{0, 1, ..., k-1\}$ such that

$$\max\left\{\widetilde{\rho}_\varphi^0(A_j, z_0) : j \neq s\right\} < \widetilde{\rho}_\varphi^0(A_s, z_0).$$

The equation (1.2.1) yields

$$m_{z_0}(r, A_s) \leqslant \sum_{j\neq s} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{j\neq s} m_{z_0}(r, A_j) + \log k. \tag{2.1.18}$$

By Lemma 1.3.10, there exists a set $E_1 \subset (0, r_0]$ for fixed $r_0 \in (0,1)$ which has finite logarithmic measure such that for all $|z - z_0| = r \in (0, r_0] \setminus E_1$

$$T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right).$$

Consequently

$$T_{z_0}(r, f^{(j)}) < O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right).$$

Then, it follows

$$\sum_{j\neq s} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leqslant O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{2.1.19}$$

By Lemma 1.3.11, there exists a set $E_2 \subset (0, r_0]$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_2$

$$\lim_{r\to 0} \frac{T_{z_0}(r, A_j)}{T_{z_0}(r, A_s)} = 0, \quad j \neq s,$$

so for any given $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$

$$m_{z_0}(r, A_j) \leqslant \varepsilon m_{z_0}(r, A_s),\ j \neq s. \tag{2.1.20}$$

By (2.1.18) , (2.1.19) and (2.1.20), we conclude that for all $|z - z_0| = r \in E_2 \setminus E_1$,

$$\frac{1}{2} m_{z_0}(r, A_s) \leqslant O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right) + O(1).$$

using this, we get

$$\limsup_{r\to 0} \frac{\varphi(\exp(m_{z_0}(r, A_s)))}{\log\frac{1}{r}} \leqslant \limsup_{r\to 0} \frac{\varphi(\exp(c\left(T_{z_0}(r, f) + \log\frac{1}{r}\right))}{\log\frac{1}{r}},$$

where $\varepsilon > 0$ is some constant. Using the fact that φ is slowly growing we get

$$\limsup_{r\to 0} \frac{\varphi(\exp(m_{z_0}(r, A_s)))}{\log\frac{1}{r}} \leqslant \limsup_{r\to 0} \frac{\varphi(e^{T_{z_0}(r,f)})}{\log\frac{1}{r}} = \rho_\varphi^0(f, z_0).$$

Therefore,

$$\rho_\varphi^0(A_s, z_0) \leqslant \rho_\varphi^0(f, z_0).$$

It remains to show that $\rho_\varphi^1(f, z_0) \leqslant \rho_\varphi^0(A_s, z_0)$. By Lemma 1.3.8, it follows

$$\rho_\varphi^1(f, z_0) \leqslant \rho_\varphi^0(A_s, z_0).$$

So we have the double inequality

$$\rho_\varphi^1(f, z_0) \leqslant \rho_\varphi^0(A_s, z_0) \leqslant \rho_\varphi^0(f, z_0)$$

and by Reamark 1.2.1 this leads

$$\widetilde{\rho}_\varphi^1(f, z_0) \leq \widetilde{\rho}_\varphi^0(A_s, z_0) \leq \widetilde{\rho}_\varphi^0(f, z_0).$$

Chapter 3

Examples

Here we provide some examples that illustrate all what we did before.

Example 3.0.1 *Consider the equation*

$$f'' - \left(\frac{1}{(z-z_0)^2} + \frac{2}{z_0 - z}\right) f' + \frac{1}{(z-z_0)^4} e^{\frac{2}{z_0 - z}} f = 0. \tag{3.0.1}$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{z-z_0}\right)$ *which is analytic in* $\overline{\mathbb{C}} - \{z_0\}$ *is a solution for* (3.0.1)*. Notice that, the function* $\varphi(t) = \log\log t = \log_2 t$ *is a function of* ϕ*, that is* φ *is unbounded, increasing and* $\psi(t) = \varphi(e^t) = \log t$ *is clearly slowly growing. A hand wavy calculations give*

$$\widetilde{\rho}^0_\varphi(A_1, z_0) = 0,\ \widetilde{\rho}^0_\varphi(A_0, z_0) = 1$$

Lossly speaking A_0 *is a dominant coefficient so by Theorem 1.2.4 we conclude that*

$$\widetilde{\rho}^1_\varphi(f, z_0) = \widetilde{\rho}^0_\varphi(A_0, z_0) = 1.$$

On the other hand, a simple computation gives

$$M_{z_0}(r, f) = e^{e^{\frac{1}{r}}}.$$

Therefore

$$\widetilde{\rho}^1_\varphi(f, z_0) = \limsup_{r\to 0} \frac{\varphi(\log M_{z_0}(r, f))}{\log \frac{1}{r}} = \limsup_{r\to 0} \frac{\varphi(\log\log\log e^{e^{\frac{1}{r}}})}{\log \frac{1}{r}} = 1.$$

This emphasizes the conclusion of the first Theorem 1.2.4.

Example 3.0.2 *Consider the equation*

$$f'' + \left(\left(1 - \frac{1}{z^2} \right) e^{\frac{1}{z}} - \frac{2z+1}{z^2} \right) f' + \frac{e^{\frac{2}{z}}}{z^2} f = 0. \tag{3.0.2}$$

It is not hard to see that $f(z) = \exp\left(\exp \frac{1}{z}\right)$ *which is analytic in* $\overline{\mathbb{C}} \setminus \{0\}$ *is a solution for* (3.0.2). *Notice that, the function* $\varphi(t) = \log\log t = \log_2 t$ *is a function of* ϕ*, that is* φ *is unbounded, increasing and* $\psi(t) = \varphi(e^t) = \log t$ *is clearly slowly growing. A hand wavy calculations give*

$$\widetilde{\rho}_{\varphi}^{0}(A_1, 0) = \widetilde{\rho}_{\varphi}^{0}(A_0, 0) = 1,$$

and

$$\widetilde{\tau}_{\varphi}^{0}(A_1, 0) = 1 < \widetilde{\tau}_{\varphi}^{0}(A_0, 0) = 2.$$

Lossly speaking A_0 *is a dominant coefficient so by Theorem 1.2.5 we conclude that*

$$\widetilde{\rho}_{\varphi}^{1}(f, 0) = \widetilde{\rho}_{\varphi}^{0}(A_0, 0) = 1.$$

This emphasizes the conclusion of the second Theorem 1.2.5.

Example 3.0.3 *Consider the equation*

$$f''' + e^{-\frac{1}{z}} f'' + \left(\frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4} \right) f' + \left(\frac{2}{z^3} + \frac{1}{z^4} \right) f = 0.$$

This equation accepts the analytic function f *in* $\overline{\mathbb{C}} \setminus \{0\}$ *given by* $f(z) = e^{\frac{1}{z}} - 1$. *By letting* $\varphi = \log_2 \in \phi$ *and setting*

$$A_0(z) = \frac{2}{z^3} + \frac{1}{z^4},$$
$$A_1(z) = \frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4},$$
$$A_2(z) = \exp\left(-\frac{1}{z}\right)$$

We see that $\widetilde{\rho}_{\varphi}^{0}(A_0, 0) = \widetilde{\rho}_{\varphi}^{0}(A_1, 0) = 0$ *and* $\widetilde{\rho}_{\varphi}^{0}\ (A_2, 0) = 1$. *So the coefficient* A_2 *is the dominant. Therefore, by the Theorem 1.2.6 one gets,*

$$\widetilde{\rho}_{\varphi}^{1}(f, 0) \leqslant 1 \leqslant \widetilde{\rho}_{\varphi}^{0}(f, 0).$$

While simple calculations give

$$\widetilde{\rho}_{\varphi}^{0}(f, 0) = \limsup_{r \to 0} \frac{\log\log\left(e^{\frac{1}{r}} - 1\right)}{\log \frac{1}{r}} = 1,$$

$$\widetilde{\rho}_{\varphi}^{1}(f,0)=\limsup_{r\to 0}\frac{\log\log\log\left(e^{\frac{1}{r}}-1\right)}{\log\frac{1}{r}}=0.$$

Consequently the conclusion of the third Theorem 1.2.6 holds.

CONCLUSION

We have seen throughout this work that the growth of the solutions of some given linear differential equations with complex analytic coefficients $\{A_j\}_{0\leqslant j\leqslant k-1}$ in the whole extended punctured plan denoted by $\overline{\mathbb{C}}-\{z_0\}$, where z_0 is an essential singularity is linked to the nature of these coefficients. In other words, we found that the growth of the nontrivial solutions can be determined or estimated by the one of the dominant coefficient which has the greatest growth among all the other coefficients. In particular, we demostrated firstly that if A_0 is the unique dominant coefficient then the $\widetilde{\rho}_{\varphi}^{1}$-order of any nontrivial solution which is assumed to be analytic in $\overline{\mathbb{C}}-\{z_0\}$ is equal to the order of A_0 and we write $\widetilde{\rho}_{\varphi}^{1}(f,z_0)=\widetilde{\rho}_{\varphi}^{0}(A_0,z_0)$. Then we have discussed the case when A_0 isn't the unique dominant coefficient so that we added an hypothesis that A_0 still dominant by considering its type of growth and as a result $\widetilde{\rho}_{\varphi}^{1}(f,z_0)=\widetilde{\rho}_{\varphi}^{0}(A_0,z_0)$ for nontrivial analytic solutions.

Finally we treated the case when the dominant coefficient A_s is unique and runs over the set $\{0,1,...,k-l\}$ and here we get the double unequality $\rho_{\varphi}^{1}(f,z_0)\leqslant\rho_{\varphi}^{0}(A_s,z_0)\leqslant\rho_{\varphi}^{0}(f,z_0)$ for nontrivial solutions.

Toward establishing the theorems we made use of some strong lemmas and we worked by the equivalence $A=B$ if only if $A\leqslant B$ and $A\geqslant B$ which can be seen as a principal argument in the proofs. By last, we would like to say that this work and other similar researches helped to understand the behavior of the complicated solutions of our differential equation which can't be given in some explicate formulas and as a perspective we can arrive to derive similar results by considering the lower limit as we can define,

$$\widetilde{\mu}_{\varphi}^{0}(f,z_0):=\liminf_{r\rightarrow 0}\frac{\varphi(M_{z_0}(r,f))}{\log\frac{1}{r}}.$$

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