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On the hyper order of solutions of a class of higher order linear differential equations

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Abstract

In this paper, we investigate the order and the hyper order of entire solutions of the higher order linear differential equation

$$f^{(k)}+A_{k-1}(z)\,e^{P_{k-1}(z)}f^{(k-1)}+...+A_1(z)\,e^{P_1(z)}f^{'}+A_0(z)\,e^{P_0(z)}f=0 \ \ (k\geq 2),$$

where $P_j(z)$ $(j=0,...,k-1)$ are nonconstant polynomials such that $\deg P_j = n$ $(j=0,...,k-1)$ and $A_j(z)\,(\not\equiv 0)$ $(j=0,...,k-1)$ are entire functions with $\rho(A_j)<n$ $(j=0,...,k-1)$. Under some conditions, we prove that every solution $f(z)\not\equiv 0$ of the above equation is of infinite order and $\rho_2(f)=n$.

1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [8], [12]). Let $\rho(f)$ denote the order of an entire function f and the hyper order $\rho_2(f)$ is defined by (see [9], [13])

$$\rho_2(f)=\overline{\lim_{r\to+\infty}}\frac{\log\log T(r,\ f)}{\log r}=\overline{\lim_{r\to+\infty}}\frac{\log\log\log M(r,\ f)}{\log r}, \tag{1.1}$$

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where $T(r,\ f)$ is the Nevanlinna characteristic function of f and $M(r,f) = \max_{|z|=r} |f(z)|$. See [8], [12], [13] for notations and definitions.

Several authors [2, 6, 9] have studied the second order linear differential equation

$$f^{''} + A_1(z) e^{P_1(z)} f^{'} + A_0(z) e^{P_0(z)} f = 0, \tag{1.2}$$

where $P_1(z), P_0(z)$ are nonconstant polynomials, $A_1(z), A_0(z) (\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P_1(z)$, $\rho(A_0) < \deg P_0(z)$. Gundersen showed in [6, p. 419] that, if $\deg P_1(z) \neq \deg P_0(z)$, then every nonconstant solution of (1.2) is of infinite order. If $\deg P_1(z) = \deg P_0(z)$, then (1.2) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f^{''} + e^z f^{'} - e^z f = 0$.

In [9], Kwon has investigated the case when $\deg P_1(z) = \deg P_0(z)$ and has proved the following:

Theorem A [9] *Let $P_1(z)$ and $P_0(z)$ be nonconstant polynomials such that*

$$P_1(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \tag{1.3}$$

$$P_0(z) = b_n z^n + b_{n-1} z^{n-1} + ... + b_1 z + b_0, \tag{1.4}$$

where a_i, b_i $(i = 0, 1, .., n)$ are complex numbers, $a_n \neq 0$, $b_n \neq 0$, let $A_1(z)$ and $A_0(z) (\not\equiv 0)$ be entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. Then the following four statements hold:
(i) *If either* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$, *then every nonconstant solution f of* (1.2) *has infinite order with $\rho_2(f) \geq n$.*
(ii) *Let $a_n = b_n$ and $deg(P_1 - P_0) = m \geq 1$, and let the orders of $A_1(z)$ and $A_0(z)$ be less than m. Then every nonconstant solution f of* (1.2) *has infinite order with $\rho_2(f) \geq m$.*
(iii) *Let $a_n = cb_n$ with $c > 1$ and $deg(P_1 - cP_0) = m \geq 1$. Suppose that $\rho(A_1) < m$ and $A_0(z)$ is an entire function with $0 < \rho(A_0) < 1/2$. Then every nonconstant solution f of* (1.2) *has infinite order with $\rho_2(f) \geq \rho(A_0)$.*
(iv) *Let $a_n = cb_n$ with $c \geq 1$ and $P_1(z) - cP_0(z)$ be a constant. Suppose that $\rho(A_1) < \rho(A_0) < 1/2$. Then every nonconstant solution f of* (1.2) *has infinite order with $\rho_2(f) \geq \rho(A_0)$.*

Recently in [3], [4], Chen and Shon have investigated the order of a class of higher order linear differential and have proved the following results:

Theorem B [3] *Let* $h_j(z)$ $(j = 0, 1, ..., k-1)$ $(k \geq 2)$ *be entire functions with* $\rho(h_j) < 1$, *and* $H_j(z) = h_j(z) e^{a_j z}$, *where* a_j $(j = 0, ..., k-1)$ *are complex numbers. Suppose that there exists* a_s *such that* $h_s \not\equiv 0$, *and for* $j \neq s$, *if* $H_j \not\equiv 0$, $a_j = c_j a_s$ $(0 < c_j < 1)$; *if* $H_j \equiv 0$, *we define* $c_j = 0$. *Then every transcendental solution* f *of the linear differential equation*

$$f^{(k)} + H_{k-1}(z) f^{(k-1)} + ... + H_s(z) f^{(s)} + ... + H_0(z) f = 0 \qquad (1.5)$$

is of infinite order.

Furthermore, if $\max\{c_1, ..., c_{s-1}\} < c_0$, *then every solution* $f(z) \not\equiv 0$ *of* (1.5) *is of infinite order.*

Theorem C [4] *Assume that* $H_j(z) = h_j(z) e^{a_j z}$ $(j = 0, ..., k-1)$ $(k \geq 2)$, *where* $h_j(z)$ $(j = 0, 1, ..., k-1)$ *are entire functions with* $\rho(h_j) < 1$. *Let* $a_j = d_j e^{i\theta_j} (d_j \geq 0,\ \theta_j \in [0, 2\pi))$ *be complex constants. If* $h_j \not\equiv 0$, *then* $a_j \neq 0$. *Suppose that in* $\{\theta_j\}$ $(j = 0, ..., k-1)$, *there are* s $(1 \leq s \leq k)$ *distinct values* $\theta_{t_1}, ..., \theta_{t_s}$ $(0 \leq t_1 < t_2 < ... < t_s \leq k-1)$. *Set* $A_m = \{a_j : \arg a_j = \theta_{t_m}\}$ $(m = 1, ..., s)$. *If there exists an* a_{t_m} *such that* $d_j < d_{t_m}$ *for* $a_j \in A_m$ $(j \neq t_m)$, *then every transcendental solution* f *of*

$$f^{(k)} + H_{k-1} f^{(k-1)} + ... + H_1 f^{'} + H_0 f = 0 \qquad (1.6)$$

is of infinite order.

Furthermore, if $t_1 = 0$, *then every solution* $f \not\equiv 0$ *of* (1.6) *is of infinite order and* $\rho_2(f) = 1$.

In this paper, we will extend and improve Theorem A(i), Theorem B and Theorem C to some higher order linear differential equations. In the following Theorem 1.1, we obtain the more precisely estimation " $\rho_2(f) = n$ " than in the Theorem B. In fact, we will prove:

Theorem 1.1 *Let* $P_j(z) = \sum\limits_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ *be nonconstant polynomials, where* $a_{0,j}$, ..., $a_{n,j}$ $(j = 0, 1, ..., k-1)$ *are complex numbers such that* $a_{n,j} a_{n,s} \neq 0$ $(j \neq s)$, *let* $A_j(z) (\not\equiv 0)$ $(j = 0, ..., k-1)$ *be entire functions. Suppose that* $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$, $\rho(A_j) < n$ $(j = 0, ..., k-1)$. *Then every transcendental solution* f *of*

$$f^{(k)} + A_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + A_s(z) e^{P_s(z)} f^{(s)} + ... + A_0(z) e^{P_0(z)} f = 0, \qquad (1.7)$$

where $k \geq 2$, *satisfies* $\rho(f) = \infty$ *and* $\rho_2(f) = n$.

Furthermore, if $\max\{c_1, ..., c_{s-1}\} < c_0$, *then every solution* $f(z) \not\equiv 0$ *of* (1.7) *satisfies* $\rho(f) = \infty$ *and* $\rho_2(f) = n$.

Theorem 1.2 *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ *be nonconstant polynomials, where* $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ *are complex numbers such that* $a_{n,j} a_{n,s} \neq 0$ $(j \neq s)$, *let* $A_j(z) (\not\equiv 0)$ $(j = 0, ..., k-1)$ *be entire functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,s}$ $(j \neq s)$, $\rho(A_j) < n$ $(j = 0, ..., k-1)$. *Then every transcendental solution* f *of* (1.7) *satisfies* $\rho(f) = \infty$ *and* $\rho_2(f) = n$.

Theorem 1.3 *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ *be nonconstant polynomials, where* $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ *are complex numbers. Let* $H_j(z) = h_j(z) e^{P_j(z)}$, *where* $h_j(z)$ $(j = 0, 1, ..., k-1)$ $(k \geq 2)$ *are entire functions with* $\rho(h_j) < n$. *Let* $a_{n,j} = d_j e^{i\theta_j}$ $(d_j > 0,\ \theta_j \in [0, 2\pi))$. *If* $h_j \not\equiv 0$, *then* $a_{n,j} \neq 0$. *Suppose that in* $\{\theta_j\}$, *there are* s $(1 \leq s \leq k)$ *distinct values* $\theta_{t_1}, ..., \theta_{t_s}$ $(0 \leq t_1 < ... < t_s \leq k-1)$. *Set* $A_m = \{a_{n,j} : \arg a_{n,j} = \theta_{t_m}\}$ $(m = 1, ..., s)$. *If there exists an* a_{n,t_m} *such that* $d_j < d_{t_m}$ *for* $a_{n,j} \in A_m$ $(j \neq t_m)$, *then every transcendental solution* f *of*

$$f^{(k)} + H_{k-1} f^{(k-1)} + ... + H_1 f' + H_0 f = 0 \tag{1.8}$$

satisfies $\rho(f) = \infty$. *If* $t_1 = 0$, *then every solution* $f \not\equiv 0$ *of* (1.8) *satisfies* $\rho(f) = \infty$ *and* $\rho_2(f) = n$.

2 Lemmas

Our proofs depend mainly upon the following Lemmas.

Lemma 2.1 [5] *Let* f *be a transcendental meromorphic function of finite order* ρ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ $(i = 1, ..., m)$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_0 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E_0$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho - 1 + \varepsilon)}. \tag{2.1}$$

Lemma 2.2 [3] *Let* $P(z) = (\alpha + i\beta) z^n + ...$ $(\alpha,\ \beta$ *are real numbers,* $|\alpha| + |\beta| \neq 0)$ *be a polynomial with degree* $n \geq 1$, *and let* $A(z) (\not\equiv 0)$ *be an entire function with* $\rho(A) < n$. *Set* $f(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_1 \subset [0, 2\pi)$ *which has linear measure zero, such that for any* $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, *where* $E_2 =$

$\{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, there is* $R_1 > 0$ *such that for* $|z| = r > R_1$, *we have*
(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\}, \tag{2.2}$$

(ii) *if* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leq \left|f\left(re^{i\theta}\right)\right| \leq \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\}. \tag{2.3}$$

Lemma 2.3 ([10], [7, Lemma 3]) *Let* $f(z)$ *be an entire function and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded on some ray* $\arg z = \theta$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, ...)$, *where* $r_n \to +\infty$, *such that* $f^{(k)}(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}(1 + o(1))|z_n|^{k-j} \quad (j = 0, ..., k-1). \tag{2.4}$$

Lemma 2.4 [3] *Let* $f(z)$ *be an entire function with* $\rho(f) = \rho < \infty$. *Suppose that there exists a set* $E_3 \subset [0, 2\pi)$ *that has linear measure zero, such that for any ray* $\arg z = \theta_0 \in [0, 2\pi) \setminus E_3$, $\left|f\left(re^{i\theta_0}\right)\right| \leq Mr^k$, *where* $M = M(\theta_0) > 0$ *is a constant and* $k(> 0)$ *is a constant independent of* θ_0. *Then* $f(z)$ *is a polynomial with* $\deg f \leq k$.

Lemma 2.5 [11, pp. 253-255] *Let* $P_0(z) = \sum_{i=0}^{n} b_i z^i$, *where* n *is a positive integer and* $b_n = \alpha_n e^{i\theta_n}, \alpha_n > 0, \theta_n \in [0, 2\pi)$. *For any given* ε $(0 < \varepsilon < \pi/4n)$, *we introduce* $2n$ *closed angles*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \quad (j = 0, 1, ..., 2n-1). \tag{2.5}$$

Then there exists a positive number $R_2 = R_2(\varepsilon)$ *such that for* $|z| = r > R_2$,

$$ReP_0(z) > \alpha_n r^n (1 - \varepsilon)\sin(n\varepsilon), \tag{2.6}$$

if $z = re^{i\theta} \in S_j$, *when* j *is even; while*

$$ReP_0(z) < -\alpha_n r^n (1 - \varepsilon)\sin(n\varepsilon), \tag{2.7}$$

if $z = re^{i\theta} \in S_j$, *when* j *is odd.*

Lemma 2.6 [2] *Let $f(z)$ be an entire function of order $\rho(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have*

$$\exp\left\{-r^{\alpha+\varepsilon}\right\} \leq |f(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{2.8}$$

Lemma 2.7 [5] *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_5 \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only on α and (m, n) (m, n positive integers with $m < n$) such that for all z satisfying $|z| = r \notin [0,1] \cup E_5$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left[\frac{T(\alpha r,\, f)}{r}\left(\log^{\alpha} r\right)\log T(\alpha r,\, f)\right]^{n-m}. \tag{2.9}$$

Lemma 2.8 [3] *Let $f(z)$ be a transcendental entire function. Then there is a set $E_6 \subset (1, +\infty)$ that has finite logarithmic measure, such that, for all z with $|z| = r \notin [0,1] \cup E_6$ at which $|f(z)| = M(r, f)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s \ \ (s \in \mathbf{N}). \tag{2.10}$$

Lemma 2.9 [3] *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions of finite order. If f is a solution of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{2.11}$$

then $\rho_2(f) \leq \max\{\rho(A_0), ..., \rho(A_{k-1})\}$.

Lemma 2.10 [1] *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ be nonconstant polynomials where $a_{0,j}, ..., a_{n,j}$ $(j = 0, 1, ..., k-1)$ are complex numbers such that $a_{n,j} a_{n,0} \neq 0$ $(j = 1, ..., k-1)$, let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, ..., k-1)$ be entire functions. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, ..., k-1)$ and $\rho(A_j) < n$ $(j = 0, ..., k-1)$. Then every solution $f(z) \not\equiv 0$ of the equation*

$$f^{(k)} + A_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + A_1(z) e^{P_1(z)} f' + A_0(z) e^{P_0(z)} f = 0, \tag{2.12}$$

is of infinite order and $\rho_2(f) = n$.

3 Proof of Theorem 1.1

Assume $f(z)$ is a transcendental solution of (1.7), we show that $\rho(f) = \infty$. Suppose that $\rho(f) = \rho < \infty$. Set $c = \max\{c_j : j \neq s\}$, then $0 < c < 1$. By Lemma 2.1, there exists a set $E_0 \subset [0, 2\pi)$ with linear measure zero, for $\theta \in [0, 2\pi) \setminus E_0$ there is a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq |z|^{(j-s)(\rho-1+\varepsilon)} \quad (j = s+1, ..., k). \tag{3.1}$$

Let $P_s(z) = a_{n,s}z^n + ...,\ (a_{n,s} = \alpha + i\beta \neq 0),\ \delta(P_s, \theta) = \alpha\cos n\theta - \beta\sin n\theta$. By Lemma 2.2, $A_s \not\equiv 0$ and $\rho(A_j) < n$ $(j = 0, ..., k-1)$ there exists a set $E_1 \subset [0, 2\pi)$ with linear measure zero such that for $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$, where $E_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0\}$, is a finite set, for any given ε $(0 < 3\varepsilon < 1 - c)$, we obtain for sufficiently large r :
(i) If $\delta(P_s, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P_s,\theta)r^n\} \leq \left|A_s(z)e^{P_s(z)}\right| \leq \exp\{(1+\varepsilon)\delta(P_s,\theta)r^n\} \tag{3.2}$$

and

$$\left|A_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\delta(P_s,\theta)cr^n\} \quad (j \neq s). \tag{3.3}$$

(ii) If $\delta(P_s, \theta) < 0$, then

$$\left|A_s(z)e^{P_s(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_s,\theta)r^{\ n}\}, \tag{3.4}$$

$$\left|A_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_s,\theta)c_j r^n\} \quad (j \neq s). \tag{3.5}$$

For any $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$, then $\delta(P_s, \theta) > 0$ or $\delta(P_s, \theta) < 0$. We divide it into two cases.
Case (i) : $\delta(P_s, \theta) > 0$. Now we prove that $\left|f^{\ (s)}\left(re^{i\theta}\right)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{\ (s)}\left(re^{i\theta}\right)\right|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_q = r_q e^{i\theta}$ $(q = 1, 2, ...)$ such that as $q \to +\infty$ we have $r_q \to +\infty$, $f^{\ (s)}(z_q) \to \infty$ and

$$\left|\frac{f^{\ (j)}(z_q)}{f^{\ (s)}(z_q)}\right| \leq \frac{1}{(s-j)!}(1+o(1))\,|z_q|^{s-j} \quad (j = 0, ..., s-1). \tag{3.6}$$

Substituting $(3.1) - (3.3)$ and (3.6) into (1.7), we obtain

$$\exp\left\{(1-\varepsilon)\delta(P_s,\theta)r_q^{\ n}\right\} \leq \left|A_s(z_q)e^{P_s(z_q)}\right|$$

$$\leq \left|\frac{f^{(k)}\left(z_q\right)}{f^{(s)}\left(z_q\right)}\right| + ... + \left|A_{s+1}\left(z_q\right)e^{P_{s+1}\left(z_q\right)}\frac{f^{(s+1)}\left(z_q\right)}{f^{(s)}\left(z_q\right)}\right|$$

$$+\left|A_{s-1}\left(z_q\right)e^{P_{s-1}\left(z_q\right)}\frac{f^{(s-1)}\left(z_q\right)}{f^{(s)}\left(z_q\right)}\right| + ... + \left|A_0\left(z_q\right)e^{P_0\left(z_q\right)}\frac{f\left(z_q\right)}{f^{(s)}\left(z_q\right)}\right|$$

$$\leq d_1 \exp\left\{(1+\varepsilon)\,\delta\left(P_s,\theta\right)cr_q^n\right\}\left|z_q\right|^{d_2}, \tag{3.7}$$

where $(d_1 > 0,\ d_2 > 0)$ are some constants. By (3.7), we obtain

$$\exp\left\{\frac{1}{3}\left(1-c\right)\delta\left(P_s,\theta\right)r_q{}^n\right\} \leq d_1 r_q^{d_2}. \tag{3.8}$$

This is a contradiction. Hence $\left|f^{\ (s)}\left(re^{i\theta}\right)\right| \leq M$ on $\arg z = \theta$. By s-fold iterated integration along the line segment $[0, z]$, we obtain

$$\left|f\left(re^{i\theta}\right)\right| \leq \left|f\left(0\right)\right| + \left|f^{'}\left(0\right)\right|\frac{r}{1!} + \left|f^{''}\left(0\right)\right|\frac{r^2}{2!} + ... + M\frac{r^s}{s!}, \tag{3.9}$$

on the ray $\arg z = \theta$.

Case (ii) : $\delta\left(P_s,\theta\right) < 0$. By (1.7), we get

$$-1 = A_{k-1}\left(z\right)e^{P_{k-1}(z)}\frac{f^{(k-1)}\left(z\right)}{f^{(k)}\left(z\right)} + ... + A_s\left(z\right)e^{P_s(z)}\frac{f^{(s)}\left(z\right)}{f^{(k)}\left(z\right)}$$

$$+ ... + A_0\left(z\right)e^{P_0(z)}\frac{f\left(z\right)}{f^{(k)}\left(z\right)}. \tag{3.10}$$

Now we prove that $\left|f^{\ (k)}\left(re^{i\theta}\right)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{\ (k)}\left(re^{i\theta}\right)\right|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_q = r_q e^{\ i\theta}$ $(q = 1, 2, ...)$ such that as $q \to +\infty$ we have $r_q \to +\infty$, $f^{\ (k)}\left(z_q\right) \to \infty$ and

$$\left|\frac{f^{(j)}\left(z_q\right)}{f^{(k)}\left(z_q\right)}\right| \leq \frac{1}{(k-j)!}\left(1+o\left(1\right)\right)\left|z_q\right|^{k-j} \ \ (j = 0, ..., k-1). \tag{3.11}$$

By (3.4) and (3.11), we have as $q \to +\infty$

$$\left|A_s\left(z_q\right)e^{P_s\left(z_q\right)}\frac{f^{(s)}\left(z_q\right)}{f^{(k)}\left(z_q\right)}\right|$$

$$\leq \frac{1}{(k-s)!}\left(1+o\left(1\right)\right)\exp\left\{(1-\varepsilon)\,\delta\left(P_s,\theta\right)r_q{}^n\right\}r_q{}^{k-s} \to 0. \tag{3.12}$$

By (3.5), (3.11) and $c_j > 0$, we have as $q \to +\infty$

$$\left| A_j(z_q) e^{P_j(z_q)} \frac{f^{(j)}(z_q)}{f^{(k)}(z_q)} \right|$$

$$\leq \frac{1}{(k-j)!}(1+o(1)) \exp\left\{(1-\varepsilon)\delta(P_s,\theta) c_j r_q^{\ n}\right\} r_q^{\ k-j} \to 0 \ \ (j \neq s). \quad (3.13)$$

Substituting (3.12) and (3.13) into (3.10), we obtain as $q \to +\infty$

$$1 \leq 0. \quad (3.14)$$

This is a contradiction. Hence $\left|f^{\ (k)}\left(re^{i\theta}\right)\right| \leq M_1$ on $\arg z = \theta$. Therefore,

$$\left|f\left(re^{i\theta}\right)\right| \leq |f(0)| + \left|f^{'}(0)\right| \frac{r}{1!} + \left|f^{''}(0)\right| \frac{r^2}{2!} + ... + M_1 \frac{r^k}{k!} \quad (3.15)$$

holds on $\arg z = \theta$. By Lemma 2.4, combining (3.9) and (3.15) and the fact that $E_0 \cup E_1 \cup E_2$ has linear measure zero, we know that $f(z)$ is a polynomial which contradicts our assumption, therefore $\rho(f) = \infty$.

Assume $\max\{c_1, ..., c_{s-1}\} < c_0$ and $f(z)$ is a polynomial solution of (1.7) that the degree of $f(z)$, $\deg f(z) = m$. If $m \geq s$, then we take $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$ satisfying $\delta(P_s, \theta) > 0$. For any given

$$\varepsilon_1 \ \left(0 < 3\varepsilon_1 < \min\left\{1-c, c_0 - c^{'}\right\} \ \left(c^{'} = \max\{c_1, ..., c_{s-1}\}\right) < c_0\right).$$

By (1.7) and Lemma 2.2, we have

$$\begin{aligned} \exp\{(1-\varepsilon_1)\delta(P_s,\theta) r^{\ n}\} d_3 r^{m-s} &\leq \left|A_s\left(re^{i\theta}\right) e^{P_s\left(re^{i\theta}\right)} f^{(s)}\left(re^{i\theta}\right)\right| \\ &\leq \sum_{j\neq s} \left|A_j\left(re^{i\theta}\right) e^{P_j\left(re^{i\theta}\right)} f^{\ (j)}\left(re^{i\theta}\right)\right| \\ &\leq d_4 r^m \exp\left((1+\varepsilon_1)\delta(P_s,\theta) c r^{\ n}\right), \quad (3.16) \end{aligned}$$

where $(d_3 > 0,\ d_4 > 0)$ are some constants. By (3.16), we get

$$\exp\left\{\frac{1}{3}(1-c)\delta(P_s,\theta) r^{\ n}\right\} \leq \frac{d_4}{d_3} r^s. \quad (3.17)$$

Hence, (3.17) is a contradiction. If $m < s$ taking θ as above, by (1.7) and Lemma 2.2, we have

$$\exp\{(1-\varepsilon_1)\delta(P_s,\theta) c_0 r^{\ n}\} d_5 r^{s-1} \leq \left|A_0\left(re^{i\theta}\right) e^{P_0\left(re^{i\theta}\right)} f\left(re^{i\theta}\right)\right|$$

$$\leq \sum_{j=1}^{s-1} \left| A_j\left(re^{i\theta}\right) e^{P_j\left(re^{i\theta}\right)} f^{(j)}\left(re^{i\theta}\right) \right|$$

$$\leq d_6 r^{s-2} \exp\left\{ (1+\varepsilon_1)\,\delta(P_s,\theta)\, c' r^{n} \right\}$$

and

$$\exp\left\{ \frac{1}{3}\left(c_0 - c'\right)\delta(P_s,\theta)\, r^{n} \right\} \leq \frac{d_6}{d_5 r}, \tag{3.18}$$

where $(d_5 > 0,\ d_6 > 0)$ are some constants. This is a contradiction. Therefore, when $\max\{c_1, ..., c_{s-1}\} < c_0$, every solution $f \not\equiv 0$ of (1.7) has infinite order.

Now we prove that $\rho_2(f) = n$. Put $c = \max\{c_j : j \neq s\}$, then $0 < c < 1$. Since $\deg P_s > \deg(P_j - c_j P_s)\ (j \neq s)$, by Lemma 2.5, there exist real numbers $b > 0$, λ, R_2 and $\theta_1 < \theta_2$ such that for all $r \geq R_2$ and $\theta_1 \leq \theta \leq \theta_2$, we have

$$Re P_s\left(re^{i\theta}\right) > br^n,\ Re\left(P_j\left(re^{i\theta}\right) - c_j P_s\left(re^{i\theta}\right)\right) < \lambda\ \ (j \neq s). \tag{3.19}$$

$$Re\left(P_j\left(re^{i\theta}\right) - cP_s\left(re^{i\theta}\right)\right) = Re\left(P_j\left(re^{i\theta}\right) - c_j P_s\left(re^{i\theta}\right)\right) + (c_j - c)\, Re P_s\left(re^{i\theta}\right) < \lambda\ \ (j \neq s). \tag{3.20}$$

Let $\max\{\rho(A_j)\ \ (j = 0, ..., k-1)\} = \beta < n$. Then by Lemma 2.6, there exists a set $E_3 \subset [1, +\infty)$ that has finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, for any given ε $(0 < \varepsilon < n - \beta)$, we have

$$\exp\left\{-r^{\beta+\varepsilon}\right\} \leq |A_j(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}\ \ (j = 0, ..., k-1). \tag{3.21}$$

By Lemma 2.7, there is a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq Br\,[T(2r,f)]^{j-s+1}\ \ (j = s+1, ..., k) \tag{3.22}$$

and

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq Br\,[T(2r,f)]^{j+1}\ \ (j = 1, ..., s-1). \tag{3.23}$$

It follows from (1.7) that

$$\left| A_s(z)\, e^{(1-c)P_s(z)} \right| \leq \left| e^{-cP_s(z)} \right| \left| \frac{f^{(k)}(z)}{f^{(s)}(z)} \right| + \left| A_{k-1}(z)\, e^{P_{k-1}(z) - cP_s(z)} \right| \left| \frac{f^{(k-1)}(z)}{f^{(s)}(z)} \right|$$

$$+\ldots+\left|A_{s+1}(z)e^{P_{s+1}(z)-cP_s(z)}\right|\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right|+\left|A_{s-1}(z)e^{P_{s-1}(z)-cP_s(z)}\right|\left|\frac{f^{(s-1)}(z)}{f^{(s)}(z)}\right|$$

$$+\ldots+\left|A_1(z)e^{P_1(z)-cP_s(z)}\right|\left|\frac{f'(z)}{f^{(s)}(z)}\right|+\left|A_0(z)e^{P_0(z)-cP_s(z)}\right|\left|\frac{f(z)}{f^{(s)}(z)}\right|$$

$$=\left|e^{-cP_s(z)}\right|\left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right|+\left|A_{k-1}(z)e^{P_{k-1}(z)-cP_s(z)}\right|\left|\frac{f^{(k-1)}(z)}{f^{(s)}(z)}\right|+\ldots$$

$$+\left|A_{s+1}(z)e^{P_{s+1}(z)-cP_s(z)}\right|\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right|$$

$$+\left|\frac{f(z)}{f^{(s)}(z)}\right|\left[\left|A_{s-1}(z)e^{P_{s-1}(z)-cP_s(z)}\right|\left|\frac{f^{(s-1)}(z)}{f(z)}\right|\right.$$

$$\left.+\ldots+\left|A_1(z)e^{P_1(z)-cP_s(z)}\right|\left|\frac{f'(z)}{f(z)}\right|+\left|A_0(z)e^{P_0(z)-cP_s(z)}\right|\right]. \tag{3.24}$$

By Lemma 2.8, there is a set $E_5 \subset (1,+\infty)$ that has finite logarithmic measure such that, for all z with $|z| = r \notin [0,1] \cup E_5$ at which $|f(z)| = M(r,f)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s \quad (s \in \mathbf{N}). \tag{3.25}$$

Hence by $(3.19)-(3.25)$, we get for all z with $|z| = r \notin [0,1] \cup E_3 \cup E_4 \cup E_5$, $r \geq R_2$, $\theta_1 \leq \theta \leq \theta_2$ at which $|f(z)| = M(r,f)$

$$\exp\left\{-r^{\beta+\varepsilon}\right\}\exp\left\{(1-c)br^n\right\}$$

$$\leq \left[\exp\left\{-cbr^n\right\}+(k-s-1)\exp\left\{r^{\beta+\varepsilon}\right\}\exp\{\lambda\}\right]Br\left[T(2r,f)\right]^{k-s+1}$$

$$+2sr^s\exp\{\lambda\}\exp\left\{r^{\beta+\varepsilon}\right\}Br\left[T(2r,f)\right]^s$$

$$\leq M_1 r^{s+1}\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^k,$$

where $M_1 > 0$ is a constant. Thus $n > \beta+\varepsilon$ implies $\rho_2(f) \geq n$. By Lemma 2.9, we have $\rho_2(f) = n$.

4 Proof of Theorem 1.2

Assume $f(z)$ is a transcendental solution of (1.7). Then it follows from Lemma 2.5 that there exists real number $\alpha > 0$, R_3 and $\theta_3 < \theta_4$, such that, for all $r \geq R_3$ and $\theta_3 \leq \theta \leq \theta_4$, we have

$$ReP_j\left(re^{i\theta}\right) < 0 \ \ (j \neq s) \ \text{ and } ReP_s\left(re^{i\theta}\right) > \alpha r^{\,n}. \tag{4.1}$$

We have from (1.7)

$$\begin{aligned}\left|A_s(z)e^{P_s(z)}\right| &\leq \left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)}\right|\left|\frac{f^{(k-1)}(z)}{f^{(s)}(z)}\right| + \ldots \\ &+ \left|A_{s+1}(z)e^{P_{s+1}(z)}\right|\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right| + \left|A_{s-1}(z)e^{P_{s-1}(z)}\right|\left|\frac{f^{(s-1)}(z)}{f^{(s)}(z)}\right| \\ &+ \ldots + \left|A_1(z)e^{P_1(z)}\right|\left|\frac{f'(z)}{f^{(s)}(z)}\right| + \left|A_0(z)e^{P_0(z)}\right|\left|\frac{f(z)}{f^{(s)}(z)}\right| \\ &= \left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)}\right|\left|\frac{f^{(k-1)}(z)}{f^{(s)}(z)}\right| + \ldots \\ &+ \left|A_{s+1}(z)e^{P_{s+1}(z)}\right|\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right| + \left|\frac{f(z)}{f^{(s)}(z)}\right|\left[\left|A_{s-1}(z)e^{P_{s-1}(z)}\right|\left|\frac{f^{(s-1)}(z)}{f(z)}\right|\right. \\ &\left. + \ldots + \left|A_1(z)e^{P_1(z)}\right|\left|\frac{f'(z)}{f(z)}\right| + \left|A_0(z)e^{P_0(z)}\right|\right].\end{aligned} \tag{4.2}$$

Hence by $(3.21)-(3.23)$, (3.25) and $(4.1)-(4.2)$, we get for all z with $|z| = r \notin [0,1] \cup E_3 \cup E_4 \cup E_5$, $r \geq R_3$, $\theta_3 \leq \theta \leq \theta_4$ at which $|f(z)| = M(r,f)$

$$\begin{aligned}\exp\left\{-r^{\beta+\varepsilon}\right\}\exp\left\{\alpha r^n\right\} &\leq \left(1+(k-s-1)\exp\left\{r^{\beta+\varepsilon}\right\}\right)Br\left[T(2r,f)\right]^{k-s+1} \\ &+2sr^s\exp\left\{r^{\beta+\varepsilon}\right\}Br\left[T(2r,f)\right]^s \\ &\leq Mr^{s+1}\exp\left\{r^{\beta+\varepsilon}\right\}\left[T(2r,f)\right]^k,\end{aligned} \tag{4.3}$$

where $M > 0$ is a constant. Thus $n > \beta + \varepsilon$ implies that $\rho(f) = \infty$ and $\rho_2(f) \geq n$. By Lemma 2.9, we have $\rho_2(f) = n$.

5 Proof of Theorem 1.3

Assume that $f(z)$ is a transcendental entire solution of (1.8) with $\rho(f) = \rho < \infty$. Set

$$E = \{\theta \in [0, 2\pi) : \cos(n\theta + \theta_{t_m}) = 0$$
$$\text{or } d_{t_m} \cos(n\theta + \theta_{t_m}) = d_{t_l} \cos(n\theta + \theta_{t_l}) \ \ (m \geq 0, \ l \leq s, \ m \neq l)\}.$$

Then, E is clearly a finite set. If $H_j \not\equiv 0$ $(j = 0, ..., k-1)$ then by Lemma 2.2, there exists a set $E_1 \subset [0, 2\pi)$ with linear measure zero such that, for any $\theta \in [0, 2\pi) \setminus (E \cup E_1)$ there exists $R > 0$, and when $|z| = r > R$, we have:
(i) if $\cos(n\theta + \theta_j) > 0$, then

$$\exp\{(1-\varepsilon) d_j r^n \cos(n\theta + \theta_j)\} \leq \left|H_j\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon) d_j r^n \cos(n\theta + \theta_j)\}; \tag{5.1}$$

(ii) if $\cos(n\theta + \theta_j) < 0$, then

$$\exp\{(1+\varepsilon) d_j r^n \cos(n\theta + \theta_j)\} \leq \left|H_j\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon) d_j r^n \cos(n\theta + \theta_j)\}. \tag{5.2}$$

Now, by Lemma 2.1 and $\rho(f) < \infty$ there exists a set $E_2 \subset [0, 2\pi)$ with linear measure zero such that for all z satisfying $\arg z = \theta \notin E_2$ and $|z| = r$ sufficiently large and for $d > j$ $(j, \ d \in \{0, ..., k-1\})$

$$\left|\frac{f^{(d)}(z)}{f^{(j)}(z)}\right| \leq |z|^{M'} \ \ \left(M' > 0\right). \tag{5.3}$$

For any $\theta \in [0, 2\pi) \setminus (E \cup E_1 \cup E_2)$, set $\delta'_m = d_{t_m} \cos(n\theta + \theta_{t_m})$. Then $\delta'_m \neq \delta'_l$ $(m \neq l)$ and $\delta'_m \neq 0$ by $\theta \notin E$ and $a_{n,j} \neq 0$. Set $\delta' = \max\left\{\delta'_m : m = 1, ..., s\right\}$. Then there exists $\delta'_l = \delta'$ $(l \in \{1, ..., s\})$ and $\delta' > \delta'_m$ $(m \in \{1, ..., s\} \setminus \{l\})$. We consider the following two cases:
Case 1: $\delta' > 0$. Set $\delta'' = \max\{0, d_j \cos(n\theta + \theta_j) \ : \{0 \leq j \leq k-1\} \cap \{j \neq t_l\}\}$. Then $\delta'' < \delta'$. For any given $\varepsilon \left(0 < \varepsilon < \frac{\delta' - \delta''}{3\delta'}\right)$, by (5.1) there exists an $R_1 > 0$, such that as $r > R_1$

$$\left|H_{t_l}\left(re^{i\theta}\right)\right| \geq \exp\left\{(1-\varepsilon)\delta' r^n\right\}. \tag{5.4}$$

And for $j \neq t_l$, if $\cos(n\theta + \theta_j) > 0$, then by (5.1) there exists an $R_2 > 0$, such that for $r > R_2$, we have

$$\left|H_j\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon) d_j r^n \cos(n\theta + \theta_j)\}$$

$$\leq \exp\left\{(1+\varepsilon)\,\delta^{''} r^n\right\} \leq \exp\left\{(1-2\varepsilon)\,\delta^{'} r^n\right\}. \tag{5.5}$$

If $\cos(n\theta+\theta_j) < 0$, then by (5.2) there exists a $R_3 > 0$, as $r > R_3$, we have

$$\left|H_j\left(re^{i\theta}\right)\right| \leq \exp\left\{(1-\varepsilon)\,d_j r^n \cos(n\theta+\theta_j)\,r^n\right\} < 1. \tag{5.6}$$

Now we prove that $\left|f^{\,(t_l)}\left(re^{i\theta}\right)\right|$ is bounded on the ray $\arg z = \theta \in [0,2\pi) \setminus (E \cup E_1 \cup E_2)$. If $\left|f^{(t_l)}\left(re^{i\theta}\right)\right|$ is unbounded on $\arg z = \theta$ then by Lemma 2.3 there exists an infinite sequence of points $z_q = r_q e^{i\theta}$ $(q = 1,2,...)$, $r_q \to +\infty$ such that $f^{\,(t_l)}(z_q) \to \infty$, and

$$\left|\frac{f^{(j)}(z_q)}{f^{(t_l)}(z_q)}\right| \leq \frac{1}{(t_l-j)!}\left|z_q\right|^{t_l-j}(1+o(1)) \quad (j=0,...,t_l-1). \tag{5.7}$$

Then by (5.3), we have

$$\left|\frac{f^{(d)}(z_q)}{f^{(t_l)}(z_q)}\right| \leq \left|z_q\right|^{M^{'}} \quad (d = t_l+1,...,k). \tag{5.8}$$

By (1.8) and (5.4) – (5.8), we obtain that

$$\exp\left\{(1-\varepsilon)\,\delta^{'} r_q^n\right\} \leq \left|H_{t_l}(z_q)\right|$$

$$\leq \left|\frac{f^{(k)}(z_q)}{f^{(t_l)}(z_q)}\right| + \left|H_{k-1}(z_q)\frac{f^{(k-1)}(z_q)}{f^{(t_l)}(z_q)}\right| + ... + \left|H_{t_l+1}(z_q)\frac{f^{(t_l+1)}(z_q)}{f^{(t_l)}(z_q)}\right|$$

$$+\left|H_{t_l-1}(z_q)\frac{f^{(t_l-1)}(z_q)}{f^{(t_l)}(z_q)}\right| + ... + \left|H_0(z_q)\frac{f(z_q)}{f^{(t_l)}(z_q)}\right|$$

$$\leq k\exp\left\{(1-2\varepsilon)\,\delta^{'} r_q^n\right\}\left|z_q\right|^{M^{''}} \quad \left(M^{''} > 0\right).$$

This is a contradiction. Hence on $\arg z = \theta$, we have $\left|f^{(t_l)}\left(re^{i\theta}\right)\right| \leq M$. By using the same argument as in the proof of Theorem 1.1, we obtain

$$\left|f\left(re^{i\theta}\right)\right| \leq |f(0)| + \left|f^{'}(0)\right|\frac{r}{1!} + \left|f^{''}(0)\right|\frac{r^2}{2!} + ... + M\frac{r^{t_l}}{t_l!}. \tag{5.9}$$

Case 2: $\delta^{'} < 0$. Then $d_j \cos(n\theta+\theta_j) \leq \delta^{'} < 0$, for all $H_j \not\equiv 0$. By (5.2), there exists an $R_4 > 0$, as $r > R_4$, we have

$$\left|H_j\left(re^{i\theta}\right)\right| \leq \exp\left\{(1-\varepsilon)\,d_j r^n \cos(n\theta+\theta_j)\right\} \leq \exp\left\{(1-\varepsilon)\,\delta^{'} r^n\right\}. \tag{5.10}$$

Now we prove that $\left|f^{(k)}\left(re^{i\theta}\right)\right|$ is bounded on the ray $\arg z=\theta\in[0,2\pi)\setminus(E\cup E_1\cup E_2)$. If $\left|f^{(k)}\left(re^{i\theta}\right)\right|$ is unbounded on $\arg z=\theta$, then by Lemma 2.3 there exists an infinite sequence of points $z_q=r_qe^{i\theta}$ $(q=1,2,...)$, $r_q\to+\infty$ such that $f^{(k)}(z_q)\to\infty$, and

$$\left|\frac{f^{(j)}(z_q)}{f^{(k)}(z_q)}\right|\leq\frac{1}{(k-j)!}\left|z_q\right|^{k-j}(1+o(1))\quad(j=0,...,k-1). \tag{5.11}$$

By (1.8) and (5.10), (5.11) we have

$$\begin{aligned}1&\leq\left|H_{k-1}(z_q)\frac{f^{(k-1)}(z_q)}{f^{(k)}(z_q)}\right|+...+\left|H_0(z_q)\frac{f(z_q)}{f^{(k)}(z_q)}\right|\\&\leq\exp\left\{(1-\varepsilon)\delta' r_q^n\right\}(1+o(1))\left|z_q\right|^k\to0\quad(q\to+\infty).\end{aligned}$$

This is a contradiction. Hence on $\arg z=\theta$, we have $\left|f^{(k)}\left(re^{i\theta}\right)\right|\leq M_1$. Therefore,

$$\left|f\left(re^{i\theta}\right)\right|\leq|f(0)|+\left|f'(0)\right|\frac{r}{1!}+\left|f''(0)\right|\frac{r^2}{2!}+...+M_1\frac{r^k}{k!}. \tag{5.12}$$

Combining the above two cases, by (5.9) and (5.12), we see that

$$\left|f\left(re^{i\theta}\right)\right|\leq M_2r^k\quad(M_2>0),$$

holds on $\arg z=\theta\in[0,2\pi)\setminus(E\cup E_1\cup E_2)$. Since $E\cup E_1\cup E_2$ is a set with linear measure zero and by Lemma 2.4, we see that $f(z)$ is a polynomial. This contradicts our assumption. Therefore $\rho(f)=\infty$. If $t_1=0$, then the additional hypotheses of Lemma 2.10 are also satisfied. Hence, every solution $f\not\equiv0$ of (1.8) satisfies $\rho_2(f)=n$.

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