Entire Functions Sharing Small Functions With Their Difference Operators

Zinelâabidine LATREUCH[1], Abdallah EL FARISSI[2] and Benharrat BELAÏDI[1]

[1]Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)
z.latreuch@gmail.com
belaidi@univ-mosta.dz
[2]Department of Mathematics and Informatics,
Faculty of Exact Sciences,
University of Bechar-(Algeria)
elfarissi.abdallah@yahoo.fr

Abstract. We investigate uniqueness problems for an entire function that shares two small functions of finite order with their difference operators. In particular, we give a generalization of a result in [2].

2010 *Mathematics Subject Classification*:30D35, 39A32.
Key words: Uniqueness, Entire functions, Difference operators.

1 Introduction and Main Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory ([7] , [9] , [12]). In addition, we will use $\rho(f)$ to denote the order of growth of f and $\tau(f)$ to denote the type of growth of f, we say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r,a) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z+c)$ (Resp. $f_0(z) = f(z)$) and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \ n \in \mathbb{N}, \ n \geq 2.$$

In particular, $\Delta_c^n f(z) = \Delta^n f(z)$ for the case $c = 1$.

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicity), provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities.

The problem of meromorphic functions sharing small functions with their differences is an important topic of uniqueness theory of meromorphic functions (see, $[1, 4-6]$). In 1986, Jank, Mues and Volkmann (see, [8]) proved:

Theorem A *Let f be a nonconstant meromorphic function, and let $a \neq 0$ be a finite constant. If f, f' and f'' share the value a CM, then $f \equiv f'$.*

In $[11]$, P. Li and C. C. Yang gives the following generalization of Theorem A.

Theorem B *Let f be a nonconstant entire function, let a be a finite nonzero constant, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f \equiv f'$.*

In $[2]$, B. Chen et al proved a difference analogue of result of Theorem A and obtained the following results:

Theorem C *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z) (\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f$ and $\Delta_c^2 f$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Theorem D *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z), b(z) (\not\equiv 0) \in S(f)$ be periodic entire functions with period c. If $f(z) - a(z)$, $\Delta_c f(z) - b(z)$ and $\Delta_c^2 f(z) - b(z)$ share 0 CM, then $\Delta_c f \equiv \Delta_c^2 f$.*

Recently in $[3]$, B. Chen and S. Li generalized Theorem C and proved the following results:

Theorem E *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c f$ and $\Delta_c^n f$ $(n \geq 2)$ share $a(z)$ CM, then $\Delta_c f \equiv \Delta_c^n f$.*

Theorem F *Let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^n f(z)$ share 0 CM, then $\Delta_c^n f(z) = C\Delta_c f(z)$, where C is a nonzero constant.*

It is interesting now to see what happening when $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM. The main of this paper is to give a difference analogue of result of Theorem B. In fact, we prove that the conclusion of Theorems E and F remains valid when we replace $\Delta_c f(z)$ by $\Delta_c^{n+1} f(z)$, and we obtain the following results.

Theorem 1.1 *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Example 1.1 Let $f(z) = e^{z \ln 2}$ and $c = 1$. Then, for any $a \in \mathbb{C}$, we notice that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share a CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$. This example satisfies Theorem 1.1.

Remark 1.1 In Example 1.1, we have $\Delta_c^m f(z) \equiv \Delta_c^n f(z)$ for any integer $m > n+1$. However, it remains open when $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^m f(z)$ $(m > n+1)$ share $a(z)$ CM, the claim $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ in Theorem 1.1 can be replaced by $\Delta_c^m f(z) \equiv \Delta_c^n f(z)$ in general.

Theorem 1.2 *Let $f(z)$be a nonconstant entire function of finite order, and let $a(z)$, $b(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$.*

Theorem 1.3 *Let $f(z)$be a nonconstant entire function of finite order. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then $\Delta_c^{n+1} f(z) \equiv C\Delta_c^n f(z)$, where C is a nonzero constant.*

Example 1.2 Let $f(z) = e^{az}$ and $c = 1$ where $a \neq 2k\pi i$ $(k \in \mathbb{Z})$, it is clear that $\Delta_c^n f(z) = (e^a - 1)^n e^{az}$ for any integer $n \geq 1$. So, $f(z)$, $\Delta_c^n f(z)$ and

$\Delta_c^{n+1} f(z)$ share 0 CM for all $n \in \mathbb{N}$ and we can easily see that $\Delta_c^{n+1} f(z) \equiv C\Delta_c^n f(z)$ where $C = e^a - 1$. This example satisfies Theorem 1.3.

2 Some lemmas

Lemma 2.1 [10] *Let f and g be meromorphic functions such that $0 < \rho(f), \rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then we have*
(i) *If $\rho(f) > \rho(g)$, then we obtain*

$$\tau(f+g) = \tau(fg) = \tau(f).$$

(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then we get*

$$\rho(f+g) = \rho(fg) = \rho(f) = \rho(g).$$

Lemma 2.2 [12] *Suppose $f_j(z)$ $(j = 1, 2, \cdots, n+1)$ and $g_j(z)$ $(j = 1, 2, \cdots, n)$ $(n \geq 1)$ are entire functions satisfying the following conditions:*
(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}(z)$;
(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n+1$, $1 \leq k \leq n$. And furthermore, the order of $f_j(z)$ is less than the order of $e^{g_h(z) - g_k(z)}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.*
Then $f_j(z) \equiv 0$, $(j = 1, 2, \cdots n+1)$.

Lemma 2.3 [5] *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f),$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

3 Proof of the Theorems

Proof of the Theorem 1.1. Suppose on the contrary to the assertion that $\Delta_c^n f(z) \not\equiv \Delta_c^{n+1} f(z)$. Note that $f(z)$ is a nonconstant entire function of finite order. By Lemma 2.3, for $n \geq 1$, we have

$$T\left(r, \Delta_c^n f\right) = m\left(r, \Delta_c^n f\right) \leq m\left(r, \frac{\Delta_c^n f}{f}\right) + m\left(r, f\right) \leq T\left(r, f\right) + S\left(r, f\right).$$

Since $f\left(z\right)$, $\Delta^n f\left(z\right)$ and $\Delta^{n+1} f\left(z\right)$ $\left(n \geq 1\right)$ share $a\left(z\right)$ CM, then

$$\frac{\Delta_c^n f\left(z\right) - a\left(z\right)}{f\left(z\right) - a\left(z\right)} = e^{P(z)} \tag{3.1}$$

and

$$\frac{\Delta_c^{n+1} f\left(z\right) - a\left(z\right)}{f\left(z\right) - a\left(z\right)} = e^{Q(z)}, \tag{3.2}$$

where P and Q are polynomials. Set

$$\varphi\left(z\right) = \frac{\Delta_c^{n+1} f\left(z\right) - \Delta_c^n f\left(z\right)}{f\left(z\right) - a\left(z\right)}. \tag{3.3}$$

From (3.1) and (3.2), we get $\varphi\left(z\right) = e^{Q(z)} - e^{P(z)}$. Then, by supposition and (3.3), we see that $\varphi\left(z\right) \not\equiv 0$. By Lemma 2.3, we deduce that

$$T\left(r, \varphi\right) = m\left(r, \varphi\right) \leq m\left(r, \frac{\Delta_c^{n+1} f}{f - a}\right) + m\left(r, \frac{\Delta_c^n f}{f - a}\right) + O\left(1\right) = S\left(r, f\right). \tag{3.4}$$

Note that $\frac{e^{Q(z)}}{\varphi(z)} - \frac{e^{P(z)}}{\varphi(z)} = 1$. By using the second main theorem and (3.4), we have

$$\begin{aligned} T\left(r, \frac{e^Q}{\varphi}\right) &\leq \overline{N}\left(r, \frac{e^Q}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^Q}\right) + \overline{N}\left(r, \frac{1}{\frac{e^Q}{\varphi} - 1}\right) + S\left(r, \frac{e^Q}{\varphi}\right) \\ &= \overline{N}\left(r, \frac{e^Q}{\varphi}\right) + \overline{N}\left(r, \frac{\varphi}{e^Q}\right) + \overline{N}\left(r, \frac{\varphi}{e^P}\right) + S\left(r, \frac{e^Q}{\varphi}\right) \\ &= S\left(r, f\right) + S\left(r, \frac{e^Q}{\varphi}\right). \end{aligned} \tag{3.5}$$

Thus, by (3.4) and (3.5), we have $T(r, e^Q) = S(r, f)$. Similarly, $T(r, e^P) = S(r, f)$. Setting now $g\left(z\right) = f\left(z\right) - a\left(z\right)$, we have from (3.1) and (3.2)

$$\Delta_c^n g\left(z\right) = g\left(z\right) e^{P(z)} + a\left(z\right) \tag{3.6}$$

and

$$\Delta_c^{n+1} g(z) = g(z) e^{Q(z)} + a(z). \tag{3.7}$$

By (3.6) and (3.7), we have

$$g(z) e^{Q(z)} + a(z) = \Delta_c (\Delta_c^n g(z)) = \Delta_c \left(g(z) e^{P(z)} + a(z) \right).$$

Thus

$$g(z) e^{Q(z)} + a(z) = g_c(z) e^{P_c(z)} - g(z) e^{P(z)},$$

which implies

$$g_c(z) = M(z) g(z) + N(z), \tag{3.8}$$

where $M(z) = e^{-P_c(z)} \left(e^{P(z)} + e^{Q(z)} \right)$ and $N(z) = a(z) e^{-P_c(z)}$. From (3.8), we have

$$g_{2c}(z) = M_c(z) g_c(z) + N_c(z) = M_c(z) (M(z) g(z) + N(z)) + N_c(z),$$

hence

$$g_{2c}(z) = M_c(z) M_0(z) g(z) + N^1(z),$$

where $N^1(z) = M_c(z) N_0(z) + N_c(z)$. By the same method, we can deduce that

$$g_{ic}(z) = \left(\prod_{k=0}^{i-1} M_{kc}(z) \right) g(z) + N^{i-1}(z) \quad (i \geq 1), \tag{3.9}$$

where $N^{i-1}(z)$ $(i \geq 1)$ is an entire function depending on $a(z), e^{P(z)}, e^{Q(z)}$ and their differences. Now, we can rewrite (3.6) as

$$\sum_{i=1}^{n} C_n^i (-1)^{n-i} g_{ic}(z) = \left(e^{P(z)} - (-1)^n \right) g(z) + a(z). \tag{3.10}$$

By (3.9) and (3.10), we have

$$\sum_{i=1}^{n} C_n^i (-1)^{n-i} \left(\left(\prod_{k=0}^{i-1} M_{kc}(z) \right) g(z) + N^{i-1}(z) \right) - \left(e^{P(z)} - (-1)^n \right) g(z) = a(z)$$

which implies

$$A(z) g(z) + B(z) = 0, \tag{3.11}$$

where

$$A(z) = \sum_{i=1}^{n} C_n^i (-1)^{n-i} \prod_{k=0}^{i-1} M_{kc}(z) - e^{P(z)} + (-1)^n$$

and

$$B(z)=\sum_{i=1}^{n}C_n^i(-1)^{n-i}N^{i-1}(z)-a(z).$$

It is clear that $A(z)$ and $B(z)$ are small functions with respect to $f(z)$. If $A(z)\not\equiv 0$, then (3.11) yields the contradiction

$$T(r,f)=T(r,g)=T\left(r,\frac{B}{A}\right)=S(r,f).$$

Suppose now that $A(z)\equiv 0$, rewrite the equation $A(z)\equiv 0$ as

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}\prod_{k=0}^{i-1}e^{-P_{(k+1)c}}\left(e^{P_{kc}}+e^{Q_{kc}}\right)=e^P-(-1)^n.$$

We can rewrite the left side of above equality as

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{-\sum\limits_{k=1}^{i}P_{kc}}\prod_{k=0}^{i-1}\left(e^{P_{kc}}+e^{Q_{kc}}\right)$$

$$=\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{-\sum\limits_{k=1}^{i}P_{kc}}e^{\sum\limits_{k=0}^{i-1}P_{kc}}\prod_{k=0}^{i-1}\left(1+e^{Q_{kc}-P_{kc}}\right)$$

$$=\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{P-P_{ic}}\prod_{k=0}^{i-1}\left(1+e^{Q_{kc}-P_{kc}}\right).$$

So

$$\sum_{i=1}^{n}C_n^i(-1)^{n-i}e^{P-P_{ic}}\prod_{k=0}^{i-1}\left(1+e^{h_{kc}}\right)=e^P-(-1)^n, \tag{3.12}$$

where $h_{kc}=Q_{kc}-P_{kc}$. On the other hand, let $\Omega_i=\{0,1,\cdots,i-1\}$ be a finite set of i elements, and

$$P(\Omega_i)=\{\varnothing,\{0\},\{1\},\cdots,\{i-1\},\{0,1\},\{0,2\},\cdots,\Omega_i\},$$

where $\varnothing$ is an empty set. It is easy to see that

$$\prod_{k=0}^{i-1}\left(1+e^{h_{kc}}\right)=1+\sum_{A\in P(\Omega_i)\setminus\{\varnothing\}}\exp\left(\sum_{j\in A}h_{jc}\right)$$

$$= 1+\left[e^{h}+e^{h_c}+\cdots+e^{h_{(i-1)c}}\right]+\left[e^{h+h_c}+e^{h+h_{2c}}+\cdots\right]+\cdots+\left[e^{h+h_c+\cdots+h_{(i-1)c}}\right]. \tag{3.13}$$

Dividing the proof on two parts:

Part (1). $h(z)$ is non-constant polynomial. Suppose that $h(z) = a_m z^m + \cdots + a_0$ $(a_m \neq 0)$, since $P(\Omega_i) \subset P(\Omega_{i+1})$, then by (3.12) and (3.13) we have

$$\sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P-P_{ic}} + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m} = e^P - (-1)^n$$

which is equivalent to

$$\alpha_0 + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m} = e^P, \tag{3.14}$$

where α_i $(i = 0, \cdots, n)$ are entire functions of order less than m. Moreover,

$$\alpha_0 = \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P-P_{ic}} + (-1)^n$$

$$= e^P \left(\sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{-P_{ic}} + (-1)^n e^{-P} \right) = e^P \Delta_c^n e^{-P}.$$

(i) If $\deg P > m$, then we obtain from (3.14) that

$$\deg P \leq m$$

which is a contradiction.

(ii) If $\deg P < m$, then by using Lemma 2.1 and (3.14) we obtain

$$\deg P = \rho\left(e^P\right) = \rho\left(\alpha_0 + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m}\right) = m,$$

which is also a contradiction.

(iii) If $\deg P = m$, then we suppose that $P(z) = dz^m + P^*(z)$ where $\deg P^* < m$. We have to study two subcases:

$(*)$ If $d \neq ia_m$ $(i = 1, \cdots, n)$, then we have

$$\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m} - e^{P^*} e^{dz^m} = -\alpha_0.$$

By using Lemma 2.2, we obtain $e^{P^*} \equiv 0$, which is impossible.

(∗∗) Suppose now that there exists at most $j \in \{1, 2, \cdots, n\}$ such that $d = ja_m$. Without loss of generality, we assume that $j = n$. Then we rewrite (3.14) as

$$\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \left(\alpha_n - e^{P^*}\right) e^{na_m z^m} = -\alpha_0.$$

By using Lemma 2.2, we have $\alpha_0 \equiv 0$, so $\Delta_c^n e^{-P} = 0$. Thus

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-P_{ic}} \equiv 0. \tag{3.15}$$

Suppose that $\deg P = \deg h = m > 1$ and

$$P(z) = b_m z^m + b_{m-1} z^{m-1} + ... + b_0, \ \ (b_m \neq 0).$$

Note that for $j = 0, 1, \cdots, n$, we have

$$P(z + jc) = b_m z^m + (b_{m-1} + mb_m jc) z^{m-1} + \beta_j(z),$$

where $\beta_j(z)$ are polynomials with degree less than $m - 1$. Rewrite (3.15) as

$$e^{-\beta_n(z)} e^{-b_m z^m - (b_{m-1} + mb_m nc) z^{m-1}} - ne^{-\beta_{n-1}(z)} e^{-b_m z^m - (b_{m-1} + mb_m (n-1)c) z^{m-1}}$$

$$+ \cdots + (-1)^n e^{-\beta_0(z)} e^{-b_m z^m - b_{m-1} z^{m-1}} \equiv 0. \tag{3.16}$$

For any $0 \leq l < k \leq n$, we have

$$\rho\left(e^{-b_m z^m - (b_{m-1} + mb_m lc) z^{m-1} - \left(-b_m z^m - (b_{m-1} + mb_m kc) z^{m-1}\right)}\right) = \rho\left(e^{-mb_m (l-k) cz^{m-1}}\right)$$

$$= m - 1,$$

and for $j = 0, 1, \cdots, n$, we see that

$$\rho\left(e^{\beta_j}\right) \leq m - 2.$$

By this, together with (3.16) and Lemma 2.2, we obtain $e^{-\beta_n(z)} \equiv 0$, which is impossible. Suppose now that $P(z) = \mu z + \eta$ $(\mu \neq 0)$ and $Q(z) = \alpha z + \beta$ because if $\deg Q > 1$, then we back to the case (ii). It easy to see that

$$\Delta_c^n e^{-P} = \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu(z+ic)-\eta} = e^{-P} \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu ic}$$

$$= e^{-P}\left(e^{-\mu c}-1\right)^n.$$

This together with $\Delta_c^n e^{-P} \equiv 0$ gives $\left(e^{-\mu c}-1\right)^n \equiv 0$, which yields $e^{\mu c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$

$$e^{P(z+jc)} = e^{\mu z+\mu jc+\eta} = \left(e^{\mu c}\right)^j e^{P(z)} = e^{P(z)}.$$

In order to prove that $e^{Q(z)}$ is also periodic entire function with period c, we suppose the contrary, which means that $e^{\alpha c} \neq 1$. Since $e^{P(z)}$ is of period c, then by (3.14), we get

$$\alpha_1 e^{(\alpha-\mu)z} + \alpha_2 e^{2(\alpha-\mu)z} + \cdots + \alpha_n e^{n(\alpha-\mu)z} = e^{\mu z+\eta}, \tag{3.17}$$

where α_i $(i = 1, \cdots, n)$ are constants. In particular,

$$\alpha_n = e^{n(\beta-\eta)+\alpha c\frac{n(n-1)}{2}}$$

and

$$\begin{aligned}\alpha_1 = &\left[\sum_{i=1}^{n} C_n^i(-1)^{n-i} + \sum_{i=2}^{n} C_n^i(-1)^{n-i} e^{\alpha c}\right.\\
&\left.+\sum_{i=3}^{n} C_n^i(-1)^{n-i} e^{2\alpha c} + \cdots + e^{(n-1)\alpha c}\right] e^{(\beta-\eta)}\\
= &[C_n^1(-1)^{n-1} + C_n^2(-1)^{n-2}\left(1+e^{\alpha c}\right) + C_n^3(-1)^{n-3}\left(1+e^{\alpha c}+e^{2\alpha c}\right)\\
&+\cdots + C_n^n(-1)^{n-n}\left(1+e^{\alpha c}+\cdots+e^{(n-1)\alpha c}\right)]e^{(\beta-\eta)}\\
= &[C_n^1(-1)^{n-1}\frac{e^{\alpha c}-1}{e^{\alpha c}-1} + C_n^2(-1)^{n-2}\frac{e^{2\alpha c}-1}{e^{\alpha c}-1} + C_n^3(-1)^{n-3}\frac{e^{3\alpha c}-1}{e^{\alpha c}-1}\\
&+\cdots + C_n^n(-1)^{n-n}\frac{e^{n\alpha c}-1}{e^{\alpha c}-1}]e^{(\beta-\eta)}\\
= &[C_n^1(-1)^{n-1}\left(e^{\alpha c}-1\right) + C_n^2(-1)^{n-2}\left(e^{2\alpha c}-1\right) + C_n^3(-1)^{n-3}\left(e^{3\alpha c}-1\right)\\
&+\cdots + C_n^n(-1)^{n-n}\left(e^{n\alpha c}-1\right)]\frac{e^{(\beta-\eta)}}{e^{\alpha c}-1}\\
= &\left[\sum_{i=0}^{n} C_n^i(-1)^{n-i} e^{i\alpha c} - (-1)^n - \sum_{i=1}^{n} C_n^i(-1)^{n-i}\right]\frac{e^{(\beta-\eta)}}{e^{\alpha c}-1}\\
= &\left(e^{\alpha c}-1\right)^{n-1} e^{(\beta-\eta)}.\end{aligned}$$

Rewrite (3.17) as

$$\alpha_1 e^{(\alpha-2\mu)z} + \alpha_2 e^{(2\alpha-3\mu)z} + \cdots + \alpha_n e^{(n\alpha-(n+1)\mu)z} = e^{\eta}, \tag{3.18}$$

it is clear that for each $1 \leq l < m \leq n$, we have

$$\rho\left(e^{(m\alpha-(m+1)\mu-l\alpha+(l+1)\mu)z}\right) = \rho\left(e^{(m-l)(\alpha-\mu)z}\right) = 1.$$

We have the following two cases:
(i_1) If $j\alpha - (j+1)\mu \neq 0$ for all $j \in \{1, 2, \cdots, n\}$, which means that

$$\rho\left(e^{(j\alpha-(j+1)\mu)z}\right) = 1, \ 1 \leq j \leq n$$

then, by applying Lemma 2.2 we obtain $e^{\eta} \equiv 0$, which is a contradiction.
(i_2) If there exists (at most one) an integer $j \in \{1, 2, \cdots, n\}$ such that $j\alpha - (j+1)\mu = 0$. Without loss of generality, assume that $e^{(n\alpha-(n+1)\mu)z} = 1$, the equation (3.18) will be

$$\alpha_1 e^{(\alpha-2\mu)z} + \alpha_2 e^{(2\alpha-3\mu)z} + \cdots + \alpha_{n-1} e^{((n-1)\alpha-n\mu)z} = e^{\eta} - e^{n(\beta-\eta)+\alpha c\frac{n(n-1)}{2}}$$

and by applying Lemma 2.2, we obtain $\alpha_1 = (e^{\alpha c} - 1)^{n-1} e^{(\beta-\eta)} \equiv 0$, which is impossible. So, by (i_1) and (i_2), we deduce that $e^{\alpha c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$ we have

$$e^{Q(z+jc)} = e^{\alpha z+\beta} \left(e^{\alpha c}\right)^j = e^{Q(z)},$$

which implies that e^Q is periodic of period c. Since $e^{P(z)}$ is of period c, then by (3.1), we obtain

$$\Delta_c^{n+1} f(z) = e^P \Delta_c f(z), \tag{3.19}$$

then $\Delta_c^{n+1} f(z)$ and $\Delta_c f(z)$ share 0 CM. Substituting (3.19) into the second equation (3.2), we get

$$e^{P(z)} \Delta_c f(z) = e^{Q(z)} (f(z) - a(z)) + a(z). \tag{3.20}$$

Since $e^{P(z)}$ and $e^{Q(z)}$ are of period c, then by (3.20), we obtain

$$\Delta_c^{n+1} f(z) = e^{Q-P} \Delta_c^n f(z). \tag{3.21}$$

So, $\Delta^{n+1} f(z)$ and $\Delta^n f(z)$ share $0, a(z)$ CM, combining (3.1), (3.2) and (3.21), we deduce that

$$\frac{\Delta^{n+1} f(z) - a(z)}{\Delta^n f(z) - a(z)} = \frac{\Delta^{n+1} f(z)}{\Delta^n f(z)},$$

and we get

$$\Delta^{n+1} f(z) = \Delta^n f(z)$$

which is a contradiction. Suppose now that $P = c_1$ and $Q = c_2$ are constants $(e^{c_1} \neq e^{c_2})$. By (3.8) we have

$$g_c(z) = \left(e^{c_2 - c_1} + 1\right) g(z) + a(z) e^{-c_1}$$

by the same

$$g_{2c}(z) = \left(e^{c_2 - c_1} + 1\right)^2 g(z) + a(z) e^{-c_1} \left(\left(e^{c_2 - c_1} + 1\right) + 1\right).$$

By induction, we obtain

$$g_{nc}(z) = \left(e^{c_2 - c_1} + 1\right)^n g(z) + a(z) e^{-c_1} \sum_{i=0}^{n-1} \left(e^{c_2 - c_1} + 1\right)^i$$

$$= \left(e^{c_2 - c_1} + 1\right)^n g(z) + a(z) e^{-c_2} \left(\left(e^{c_2 - c_1} + 1\right)^n - 1\right).$$

Rewrite the equation (3.6) as

$$\Delta_c^n g(z) = \sum_{i=0}^{n} C_n^i (-1)^{n-i} \left[\left(e^{c_2 - c_1} + 1\right)^i g(z) + a(z) e^{-c_2} \left(\left(e^{c_2 - c_1} + 1\right)^i - 1\right)\right]$$

$$= e^{c_1} g(z) + a(z).$$

Since $A(z) \equiv 0$, then we have

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} \left(e^{c_2 - c_1} + 1\right)^i = e^{c_1}$$

and

$$\sum_{i=0}^{n} C_n^i (-1)^{n-i} \left(\left(e^{c_2 - c_1} + 1\right)^i - 1\right) = e^{c_2}$$

which are equivalent to

$$e^{n(c_2 - c_1)} = e^{c_1}$$

and

$$e^{n(c_2 - c_1)} = e^{c_2}$$

which is a contradiction.

Part (2). $h(z)$ is a constant. We show first that $P(z)$ is a constant. If $\deg P > 0$, from the equation (3.12), we see

$$\deg P \leq \deg P - 1,$$

which is a contradiction. Then $P(z)$ must be a constant and since $h(z) = Q(z) - P(z)$ is a constant, we deduce that both of $P(z)$ and $Q(z)$ is constant. This case is impossible too (the last case in Part (1)), and we deduced that $h(z)$ can not be a constant. Thus, the proof of Theorem 1.1 is completed.

Proof of the Theorem 1.2. Setting $g(z) = f(z) + b(z) - a(z)$, we can remark that

$$g(z) - b(z) = f(z) - a(z),$$
$$\Delta_c^n g(z) - b(z) = \Delta_c^n f(z) - b(z)$$

and

$$\Delta_c^{n+1} g(z) - b(z) = \Delta_c^n f(z) - b(z), \ n \geq 2.$$

Since $f(z) - a(z)$, $\Delta_c^n f(z) - b(z)$ and $\Delta_c^{n+1} f(z) - b(z)$ share 0 CM, it follows that $g(z)$, $\Delta_c^n g(z)$ and $\Delta_c^{n+1} g(z)$ share $b(z)$ CM. By using Theorem 1.1, we deduce that $\Delta_c^{n+1} g(z) \equiv \Delta_c^n g(z)$, which leads to $\Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z)$ and the proof of Theorem 1.2 is completed.

Proof of the Theorem 1.3. Note that $f(z)$ is a nonconstant entire function of finite order. Since $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM, then we have

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{P(z)} \tag{3.22}$$

and

$$\frac{\Delta_c^{n+1} f(z)}{f(z)} = e^{Q(z)}, \tag{3.23}$$

where P and Q are polynomials. If $Q - P$ is a constant, then we can get easily from (3.22) and (3.23)

$$\Delta_c^{n+1} f(z) = e^{Q(z) - P(z)} \Delta_c^n f(z) :\equiv C \Delta_c^n f(z).$$

This complete our proof. If $Q - P$ is a not constant, with a similar arguing as in the proof of Theorem 1.1, we can deduce that the case $\deg P = \deg (Q - P) > 1$ is impossible. For the case $\deg P = \deg (Q - P) = 1$, we can obtain that $e^{P(z)}$ is periodic entire function with period c. This together with (3.22) yields

$$\Delta_c^{n+1} f(z) = e^{P(z)} \Delta_c f(z) \tag{3.24}$$

which means that $f(z)$, $\Delta_c f(z)$ and $\Delta_c^{n+1} f(z)$ share 0 CM. Thus, by Theorem F, we obtain

$$\Delta_c^{n+1} f(z) \equiv C \Delta_c f(z)$$

which is a contradiction with (3.22) and $\deg P = 1$. Theorem 1.3 is thus proved.

Acknowledgements. The authors are grateful to the referee for his/her valuable comments which lead to the improvement of this paper.

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