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On the [p, q]-Order of Meromorphic Solutions of Linear Differential Equations

Benharrat Belaïdi
Department of Mathematics, Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB), B. P. 227 Mostaganem-Algeria
belaidi@univ-mosta.dz

Abstract

In this article we study the growth of meromorphic solutions of high order linear differential equations with meromorphic coefficients of $[p, q]$-order. We extend some previous results due to Cao-Xu-Chen, Kinnunen, Liu-Tu-Shi, Li-Cao and others.

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1 Introduction and main results

Consider for $k \geq 2$ the linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z), \cdots, A_{k-1}(z), F(z)$ are meromorphic functions. In [11, 12] Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. In [16], in order to maintain accordance with general definitions of the entire function f of iterated p-order [13, 14], Liu-Tu-Shi gave a minor modification of the original definition of the $[p, q]$-order given in [11, 12] . With this new concept of $[p, q]$-order, Liu, Tu and Shi [16] have considered equations (1.1), (1.2) with entire coefficients and obtained different results concerning the growth of their solutions. In this paper, we continue to consider this subject and investigate the complex linear differential equations (1.1) and (1.2) when the coefficients $A_0, A_1, \cdots, A_{k-1}, F$ are meromorphic functions of $[p, q]$ $-$order.

In this paper, it is assumed that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [9, 14, 20]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1. ([13]) Let $p \geq 1$ be an integer. The iterated $p-$order of a meromorphic function $f(z)$ is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log r},$$

where $T(r,\ f)$ is the Nevanlinna characteristic function of f.

Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order, where p, q are positive integers satisfying $p \geq q \geq 1$ or $2 \leq q = p+1$. In order to keep accordance with Definition 1, we will give a minor modification to the original definition of $[p,q]$-order (e.g. see, [11, 12]).

Definition 2. ([15]) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If $f(z)$ is a rational, then $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 2, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.

Definition 3. ([15]) A transcendental meromorphic function $f(z)$ is said to have index-pair $[p,q]$ if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.

Definition 4. ([15]) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. The $[p,q]$ convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. Similarly, the $[p,q]$ convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$.

Remark 5. ([15]) If $f(z)$ is a meromorphic function satisfying $0 < \rho_{[p,q]}(f) < \infty$, then
(i) $\rho_{[p-n,q]} = \infty$ $(n < p)$, $\rho_{[p,q-n]} = 0$ $(n < q)$, $\rho_{[p+n,q+n]} = 1$ $(n < p)$ for $n = 1, 2, 3, \cdots$
(ii) If $[p_1, q_1]$ is any pair of integers satisfying $q_1 = p_1 + q - p$ and $p_1 < p$, then $\rho_{[p_1,q_1]} = 0$ if $0 < \rho_{[p,q]} < 1$ and $\rho_{[p_1,q_1]} = \infty$ if $1 < \rho_{[p,q]} < \infty$.
(iii) $\rho_{[p_1,q_1]} = \infty$ for $q_1 - p_1 > q - p$ and $\rho_{[p_1,q_1]} = 0$ for $q_1 - p_1 < q - p$.

Remark 6. ([15]) Suppose that f_1 is a meromorphic function of $[p,q]$-order ρ_1 and f_2 is a meromorphic function of $[p_1,q_1]$-order ρ_2, let $p \leq p_1$. We can easily deduce the result about their comparative growth:
(i) If $p_1 - p > q_1 - q$, then the growth of f_1 is slower than the growth of f_2.
(ii) If $p_1 - p < q_1 - q$, then f_1 grows faster than f_2.

(iii) If $p_1 - p = q_1 - q > 0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2 \geq 1$, and the growth of f_1 is faster than the growth of f_2 if $\rho_2 < 1$.
(iv) Especially, when $p_1 = p$ and $q_1 = q$ then f_1 and f_2 are of the same index-pair $[p, q]$. If $\rho_1 > \rho_2$, then f_1 grows faster than f_2; and if $\rho_1 < \rho_2$, then f_1 grows slower than f_2. If $\rho_1 = \rho_2$, Definition 2 does not show any precise estimate about the relative growth of f_1 and f_2.

We recall the following definitions. The linear measure of a set $E \subset (0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset (0, +\infty)$ is defined by

$$\overline{dens}E = \limsup_{r \longrightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r \longrightarrow +\infty} \frac{lm(F \cap [1, r])}{\log r}.$$

Proposition 7. *For all $H \subset [1, +\infty)$ the following statements hold :*
(i) If $lm(H) = \infty$, then $m(H) = \infty$;
(ii) If $\overline{dens}H > 0$, then $m(H) = \infty$;
(iii) If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.

Proof. (i) Since we have $\frac{\chi_H(t)}{t} \leq \chi_H(t)$ for all $t \in H \subset [1, +\infty)$, then

$$m(H) \geq lm(H).$$

So, if $lm(H) = \infty$, then $m(H) = \infty$. We can easily prove the results (ii) and (iii) by applying the definition of the limit and the properties $m(H \cap [0, r]) \leq m(H)$ and $lm(H \cap [1, r]) \leq lm(H)$. □

Definition 8. ([9, 20]) For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as

$$\delta(a, f) = \liminf_{r \to +\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Extensive work in recent years has been concerned with the growth of solutions of $[p, q]$-order of complex linear differential equations in the complex plane and in the unit disc. Many results have been obtained [2, 3, 4, 10, 15, 16, 17, 18, 19]. Examples of such results are the following two theorems:

Theorem 9. *([16]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be entire functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \alpha$. Suppose that there exists a positive constant β satisfying $\beta < \alpha$ such that for any given ε $(0 < \varepsilon < \alpha - \beta)$, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{(\alpha - \varepsilon) \log_q r\right\}$$

and

$$|A_j(z)| \leq \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \cdots, k-1)$$

for $z \in H$. Then, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \alpha$.

Theorem 10. *([15]) Let $H \subset (1, \infty)$ be a set satisfying $\overline{\log dens}\{|z| : |z| \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$, $F \not\equiv 0$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 1, 2, \cdots, k-1\} < \alpha$, where α is a constant. Suppose that there exists a constant β satisfying $\beta < \alpha$ such that for any given ε $(0 < \varepsilon < \alpha - \beta)$, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{(\alpha - \varepsilon)\log_q r\right\}$$

and

$$|A_j(z)| \leq \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \cdots, k-1)$$

as $|z| \in H$. Then the following statements hold:
(i) If $\rho_{[p+1,q]}(F) \geq \alpha$, then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of equation (1.2) satisfy $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) If $\rho_{[p+1,q]}(F) < \alpha$, then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of equation (1.2) satisfy $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \alpha$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \alpha$.

The main purpose of this paper is to consider the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite $[p, q]$-order in the complex plane. We obtain the following results which generalize and improve Theorem 9 and Theorem 10.

Theorem 11. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \rho$ $(0 < \rho < \infty)$. Suppose that there exist two real numbers satisfying $0 \leq \beta < \alpha$ such that, we have*

$$|A_0(z)| \geq \exp_p\left\{\alpha\left[\log_{q-1} r\right]^{\rho}\right\} \tag{1.3}$$

and

$$|A_j(z)| \leq \exp_p\left\{\beta\left[\log_{q-1} r\right]^{\rho}\right\} \quad (j = 1, \cdots, k-1) \tag{1.4}$$

as $|z| \to +\infty$ for $z \in H$. Then the following statements hold:
(i) If $p \geq q \geq 1$ or $3 \leq q = p + 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.
(ii) If $p = 1$, $q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$.

Theorem 12. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \rho$ $(0 < \rho < \infty)$. Suppose that there exist two positive constants α, β such that, we have*

$$m(r, A_0) \geq \exp_{p-1}\left\{\alpha\left[\log_{q-1} r\right]^{\rho}\right\} \tag{1.5}$$

and

$$m(r, A_j) \leq \exp_{p-1}\left\{\beta\left[\log_{q-1} r\right]^{\rho}\right\} \quad (j = 1, \cdots, k-1) \tag{1.6}$$

as $|z| \to +\infty$ for $z \in H$. Then the following statements hold:
(i) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.

(ii) If $3 \leq q = p+1$, $0 \leq \beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,p+1]}(f) = \rho$.
(iii) If $p = 1$, $q = 2$, $0 \leq (k-1)\beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$.

Corollary 13. *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions. Suppose that H, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ satisfy the hypotheses in Theorem 11. Then we have the following statements:*
(i) Let $p \geq q \geq 1$. If $\rho_{[p+1,q]}(F) \leq \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,q]}(F) > \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $3 \leq q = p+1$ and $\rho > 1$. If $\rho_{[p+1,p+1]}(F) \leq \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(f) = \rho$, with at most one exceptional solution f_0 satisfying $\rho_{[p+1,p+1]}(f_0) < \rho$; if $\rho_{[p+1,p+1]}(F) > \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Corollary 14. *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions. Suppose that H, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ satisfy the hypotheses in Theorem 12. Then we have the following statements:*
(i) Let $p \geq q \geq 2$, $0 \leq \beta < \alpha$. If $\rho_{[p+1,q]}(F) \leq \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,q]}(F) > \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $3 \leq q = p+1$, $0 \leq \beta < \alpha$ and $\rho > 1$. If $\rho_{[p+1,p+1]}(F) \leq \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,p+1]}(f_0) < \rho$; if $\rho_{[p+1,p+1]}(F) > \rho$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.2) satisfies $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Recently, the author [2, 3, 4], J. Tu and Z. X. Xuan [17] and J. Tu and H. X. Huang [18] have investigated the growth of solutions of differential equations (1.1) and (1.2) with analytic coefficients of $[p, q]$-order in the unit disc. So, it is also interesting to consider the growth of meromorphic solutions of differential equations with coefficients of $[p, q]$-order in the unit disc?

2 Some preliminary lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 15. *([1]) Let $g : (0, \infty) \to \mathbb{R}$, $h : (0, \infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_1 of finite linear measure. Then, for any $\lambda > 1$, there exists $r_1 > 0$ such that $g(r) \leq h(\lambda r)$ for all $r > r_1$.*

Lemma 16. *([8]) Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \le \psi(r)$ for all $r \notin E_2 \cup [0, 1]$, where $E_2 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_2 = r_2(\gamma) > 0$ such that $\varphi(r) \le \psi(\gamma r)$ for all $r > r_2$.*

Lemma 17. *([9]) Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside of an exceptional set $E_3 \subset (0, +\infty)$ with finite linear measure, and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 18. *([7]) Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_4 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \le i < j \le k)$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B\left\{\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\right\}^{j-i}.$$

Lemma 19. *([5]) Let f be a meromorphic solution of (1.1), assuming that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T(r) = \sum_{j=0}^{k-1} T(r, A_j)$, we have*

$$\log m(r, f) < T(r)\{(\log r)\log T(r)\}^\gamma, \text{ if } s = 0,$$
$$\log m(r, f) < r^{2s+\gamma-1}T(r)\{\log T(r)\}^\gamma, \text{ if } s > 0$$

outside of an exceptional set E_s with $\int_{E_s} t^{s-1}dt < +\infty$.

Remark 20. We note that in the above lemma, $s = 1$ corresponds to Euclidean measure and $s = 0$ to logarithmic measure.

Lemma 21. *Let $A_0(z), \cdots, A_{k-1}(z)$ be nonconstant meromorphic functions of $[p, q]-$order. Assume the existence of the meromorphic solutions of (1.1). Then the following statements hold:*
(i) If $p \ge q \ge 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) \le \max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}$.
(ii) If $3 \le q = p + 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,p+1]}(f) \le \max\{\rho_{[p,p+1]}(A_j)\ (j = 0, 1, \cdots, k-1)\}$.

Proof. We prove only (ii). For the proof of (i) see [15, 19]. From (1.1), we know that the poles of $f(z)$ can only occur at the poles of $A_0(z), \cdots, A_{k-1}(z)$. Since the multiplicities of poles of f are uniformly bounded, we have

$$N(r, f) \le M_1\overline{N}(r, f) \le M_1\sum_{j=0}^{k-1}\overline{N}(r, A_j)$$

$$\leq M \max \{N(r, A_j) : j = 0, 1, \cdots, k-1\}, \tag{2.1}$$

where M_1 and M are some suitable positive constants. This gives

$$T(r, f) = m(r, f) + O(\max \{N(r, A_j) : j = 0, 1, \cdots, k-1\}). \tag{2.2}$$

Set $\delta(\infty, f) := \eta > 0$, for sufficiently large r, we have

$$m(r, f) \geq \frac{\eta}{2} T(r, f). \tag{2.3}$$

From Lemma 19 and (2.2) or (2.3), we obtain

$$\log T(r, f) \leq \log m(r, f) + O(\log T(r)) \leq O(T(r)\{(\log r)\log T(r)\}^{\gamma}) \tag{2.4}$$

or

$$\log T(r, f) \leq \log\left(\frac{2}{\eta} m(r, f)\right) \leq O(T(r)\{(\log r)\log T(r)\}^{\gamma}) \tag{2.5}$$

outside of an exceptional set E_0 with finite logarithmic measure. From (2.4) or (2.5), we get for $p \geq 2$

$$\log_{p+1} T(r, f) \leq \max \left\{\log_p T(r),\ \log_{p+1} r\right\} \tag{2.6}$$

outside of an exceptional set E_0 with finite logarithmic measure. If at least one of the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) is transcendental, then by using Lemma 16 and (2.6), we obtain

$$\rho_{[p+1,p+1]}(f) \leq \max \left\{\rho_{[p,p+1]}(A_j) \quad (j = 0, 1, \cdots, k-1), 1\right\}$$

$$= \max \left\{\rho_{[p,p+1]}(A_j) \quad (j = 0, 1, \cdots, k-1)\right\}.$$

If all the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) are rational functions, then by using Lemma 16 and (2.6), we obtain

$$\rho_{[p+1,p+1]}(f) \leq \max \left\{\rho_{[p,p+1]}(A_j) \quad (j = 0, 1, \cdots, k-1), 1\right\} = 1$$

$$= \max \left\{\rho_{[p,p+1]}(A_j) \quad (j = 0, 1, \cdots, k-1)\right\}.$$

□

Lemma 22. *([15]) Let $1 \leq q \leq p$ or $2 \leq q = p + 1$ and let f be a meromorphic function with $0 \leq \rho_{[p,q]}(f) = \rho \leq \infty$. Then there exists a set $E_5 \subset [1, +\infty)$ with infinite logarithmic measure such that*

$$\lim_{\substack{r \to +\infty \\ r \in E_5}} \frac{\log_p T(r, f)}{\log_q r} = \rho.$$

Lemma 23. *Let $1 \leq q \leq p$ or $2 \leq q = p + 1$ and let f_1 and f_2 be meromorphic functions of $[p, q]$ −order satisfying $\rho_{[p,q]}(f_1) > \rho_{[p,q]}(f_2)$. Then there exists a set $E_6 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_6$, we have*

$$\lim_{r \to \infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Proof. Set $\rho_1 = \rho_{[p,q]}(f_1)$, $\rho_2 = \rho_{[p,q]}(f_2)$. By using Lemma 22, there exists a set E_6 with infinite logarithmic measure such that for any given $0 < \varepsilon < \frac{\rho_1 - \rho_2}{2}$ and all sufficiently large $r \in E_6$

$$T(r, f_1) > \exp_p\left\{(\rho_1 - \varepsilon)\log_q r\right\}$$

and for all sufficiently large r, we have

$$T(r, f_2) < \exp_p\left\{(\rho_2 + \varepsilon)\log_q r\right\}.$$

From this we can get

$$\frac{T(r, f_2)}{T(r, f_1)} < \frac{\exp_p\left\{(\rho_2 + \varepsilon)\log_q r\right\}}{\exp_p\left\{(\rho_1 - \varepsilon)\log_q r\right\}}$$

$$= \exp\left\{\exp_{p-1}\left\{(\rho_2 + \varepsilon)\log_q r\right\} - \exp_{p-1}\left\{(\rho_1 - \varepsilon)\log_q r\right\}\right\}, \ r \in E_6.$$

Since $0 < \varepsilon < \frac{\rho_1 - \rho_2}{2}$, then we have

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0, \ r \in E_6.$$

□

Lemma 24. *Let A_j $(j = 0, \cdots, k-1)$, $F \not\equiv 0$ be meromorphic functions. Then the following statements hold:*
(i) If $p \geq q \geq 1$, then every meromorphic solution f of equation (1.2) such that $\max\{\rho_{[p,q]}(A_j)$ $(j = 0, 1, \cdots, k-1), \rho_{[p,q]}(F)\} < \rho_{[p,q]}(f)$ satisfies $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then every meromorphic solution f of equation (1.2) such that $\max\{\rho_{[p,p+1]}(A_j)\ (j = 0, 1, \cdots, k-1), \rho_{[p,p+1]}(F), 1\} < \rho_{[p,p+1]}(f)$ satisfies $\overline{\lambda}_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

Proof. We prove only (ii). For the proof of (i) see [15]. By (1.2), if f has a zero at z_0 of order $\alpha\,(> k)$ and if $A_0, A_1, \cdots, A_{k-1}$ are all analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \leq k\,\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} n(r, A_{k-j})$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\,\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} N(r, A_{k-j}). \tag{2.7}$$

Now (1.2) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.8}$$

By Lemma 17 and (2.8), we have

$$m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k} m(r, A_{k-j}) + m\left(r, \frac{1}{F}\right) + O(1)$$

$$= \sum_{j=1}^{k} m\left(r, A_{k-j}\right) + m\left(r, \frac{1}{F}\right) + O\left(\log T\left(r, f\right) + \log r\right) \tag{2.9}$$

holds for all r outside a set $E_3 \subset (0, +\infty)$ with a finite linear measure $m\left(E_3\right) = \delta < +\infty$. By (2.7) and (2.9), we get

$$T\left(r, f\right) = T\left(r, \frac{1}{f}\right) + O\left(1\right)$$

$$\leq k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{k} T\left(r, A_{k-j}\right) + T\left(r, F\right) + O\left(\log T\left(r, f\right) + \log r\right) \quad \left(|z| = r \notin E_3\right). \tag{2.10}$$

Since $\max\{\rho_{[p,p+1]}\left(A_j\right)\ (j = 0, 1, \cdots, k-1), \rho_{[p,p+1]}\left(F\right)\} < \rho_{[p,p+1]}\left(f\right)$, then by Lemma 23, there exists a set $E_6 \subset [1, +\infty)$ with infinite logarithmic measure such that

$$\max\left\{\frac{T\left(r, A_j\right)}{T\left(r, f\right)}\ \left(j = 0, \cdots, k-1\right), \frac{T\left(r, F\right)}{T\left(r, f\right)}\right\} \to 0,\ r \to +\infty,\ r \in E_6. \tag{2.11}$$

Thus, by (2.10) and (2.11), we have for all $r \in E_6 \backslash E_3$

$$\left(1 - o\left(1\right)\right) T\left(r, f\right) \leq k\ \overline{N}\left(r, \frac{1}{f}\right) + O\left(\log T\left(r, f\right) + \log r\right).$$

Then, we obtain $\rho_{[p,p+1]}\left(f\right) \leq \overline{\lambda}_{[p,p+1]}\left(f\right) \leq \lambda_{[p,p+1]}\left(f\right)$. Therefore, by

$$\overline{\lambda}_{[p,p+1]}\left(f\right) \leq \lambda_{[p,p+1]}\left(f\right) \leq \rho_{[p,p+1]}\left(f\right)$$

we have $\overline{\lambda}_{[p,p+1]}\left(f\right) = \lambda_{[p,p+1]}\left(f\right) = \rho_{[p,p+1]}\left(f\right)$. □

Lemma 25. *Let f be a meromorphic function of $[p, q]-$order. Then the following statements hold:*
(i) If $p \geq q \geq 1$, then $\rho_{[p,q]}\left(f\right) = \rho_{[p,q]}\left(f'\right)$.
(ii) If $3 \leq q = p+1$, then $\rho_{[p,p+1]}\left(f'\right) \leq \max\left\{\rho_{[p,p+1]}\left(f\right), 1\right\}$ and $\rho_{[p,p+1]}\left(f\right) \leq \max\left\{\rho_{[p,p+1]}\left(f'\right), 1\right\}$.
(iii) If $p = 1, q = 2$, then $\rho_{[1,2]}\left(f'\right) \leq \max\left\{\rho_{[1,2]}\left(f\right), 1\right\}$ and $\rho_{[1,2]}\left(f\right) \leq 1 + \rho_{[1,2]}\left(f'\right)$.

Proof. (i) – (ii) By Lemma 17, we have

$$T\left(r, f'\right) = m\left(r, f'\right) + N\left(r, f'\right) \leq m\left(r, f\right) + m\left(r, \frac{f'}{f}\right) + 2N\left(r, f\right)$$

$$\leq 2T\left(r, f\right) + m\left(r, \frac{f'}{f}\right) \leq 2T\left(r, f\right) + O\left(\log T\left(r, f\right) + \log r\right) \tag{2.12}$$

holds outside of an exceptional set $E_3 \subset (0, +\infty)$ with finite linear measure. By (2.12) and Lemma 15, it is easy to see $\rho_{[p,q]}\left(f'\right) \leq \rho_{[p,q]}\left(f\right)$ $\left(p \geq q \geq 1\right)$ and $\rho_{[p,p+1]}\left(f'\right) \leq \max\left\{\rho_{[p,p+1]}\left(f\right), 1\right\}$ if $3 \leq q = p+1$. On the other hand, [6], ([20], p. 35), we have for $r \to +\infty$

$$T\left(r, f\right) < O\left(T\left(2r, f'\right) + \log r\right). \tag{2.13}$$

Hence, by using (2.13) we obtain $\rho_{[p,q]}\left(f'\right) = \rho_{[p,q]}\left(f\right)$ if $p \geq q \geq 1$ and $\rho_{[p,p+1]}\left(f\right) \leq \max\left\{\rho_{[p,p+1]}\left(f'\right), 1\right\}$ if $3 \leq q = p+1$. We can easily obtain the conclusion (iii) by using (2.12) and (2.13). □

3 Proof of Theorem 11

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1). From the conditions of Theorem 11, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (1.3) and (1.4) as $|z| \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$. By Lemma 18, we know that there exists a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^{j+1} \ (j = 1, \cdots, k). \tag{3.1}$$

By (1.1), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_0(z)| \left|\frac{f'}{f}\right|. \tag{3.2}$$

It follows by (1.3), (1.4), (3.1) and (3.2) that

$$\exp_p\left\{\alpha\left[\log_{q-1} r\right]^{\rho}\right\} \leq |A_0(z)| \leq kB \exp_p\left\{\beta\left[\log_{q-1} r\right]^{\rho}\right\}\left[T(2r, f)\right]^{k+1} \tag{3.3}$$

holds for all z satisfying $|z| = r \in H_1 \backslash ([0, 1] \cup E_4)$ as $|z| \to +\infty$. If $p \geq q \geq 1$ or $3 \leq q = p + 1$, then by (3.3) and Lemma 16, we obtain $\rho \leq \rho_{[p+1,q]}(f)$. On the other hand, by Lemma 21 (i) – (ii), we have

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\} \leq \rho,$$

if $p \geq q \geq 1$ or $3 \leq q = p + 1$. Hence every meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$ if $p \geq q \geq 1$ or $3 \leq q = p + 1$.
(ii) If $p = 1$, $q = 2$, then from (3.3), we have

$$\exp\left\{\alpha\left[\log r\right]^{\rho}\right\} \leq |A_0(z)| \leq kB \exp\left\{\beta\left[\log r\right]^{\rho}\right\}\left[T(2r, f)\right]^{k+1} \tag{3.4}$$

holds for all z satisfying $|z| = r \in H_1 \backslash ([0, 1] \cup E_4)$ as $|z| \to +\infty$. By (3.4) and Lemma 16, every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$. □

4 Proof of Theorem 12

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1). By (1.1), we can write

$$A_0(z) = -\left(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}\right). \tag{4.1}$$

From the conditions of Theorem 12, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (1.5) and(1.6) as $|z| \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then H_1 is a set of r with $\int_{H_1} \frac{dr}{r} = \infty$. It follows by (1.5), (1.6), (4.1) and Lemma 17 that

$$\exp_{p-1}\left\{\alpha\left[\log_{q-1} r\right]^{\rho}\right\} \leq m(r, A_0)$$

$$\leq \sum_{j=1}^{k-1} m\left(r, A_j\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O\left(1\right)$$

$$\leq \left(k-1\right) \exp_{p-1}\left\{\beta\left[\log_{q-1} r\right]^{\rho}\right\} + O\left(\log T\left(r,f\right) + \log r\right) \tag{4.2}$$

holds for all z satisfying $|z| = r \in H_1 \backslash E_3$ as $|z| \to +\infty$, where $E_3 \subset (0, +\infty)$ is a set with a finite linear measure. If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then by (4.2) and Lemma 15, we obtain $\rho \leq \rho_{[p+1,q]}\left(f\right)$. On the other hand, by Lemma 21 (i), we have $\rho_{[p+1,q]}\left(f\right) \leq \max\left\{\rho_{[p,q]}\left(A_j\right) : j = 0, 1, \cdots, k-1\right\} \leq \rho$. Hence every meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta\left(\infty, f\right) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}\left(f\right) = \rho$.
(ii) If $3 \leq q = p+1$, $0 \leq \beta < \alpha$ and $\rho > 1$, by the similar proof in case (i) and Lemma 21 (ii), we can obtain the conclusion.
(iii) If $p = 1$, $q = 2$, $0 \leq \left(k-1\right)\beta < \alpha$ and $\rho > 1$, then from (4.2), we have

$$\alpha\left[\log r\right]^{\rho} \leq m\left(r, A_0\right) \leq \left(k-1\right)\beta\left[\log r\right]^{\rho} + O\left(\log T\left(r,f\right) + \log r\right) \tag{4.3}$$

holds for all z satisfying $|z| = r \in H_1 \backslash E_3$ as $|z| \to +\infty$. By (4.3) and Lemma 16, every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}\left(f\right) \geq \rho$. □

5 Proof of Corollary 13

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta\left(\infty, f\right) > 0$ of equation (1.1).
(a) Suppose that $1 \leq q \leq p$ and $\rho_{[p+1,q]}\left(F\right) \leq \rho$. We assume that f is a solution of (1.2) and $\{f_1, f_2, \cdots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (1.2). By Theorem 11, we know that $\rho_{[p+1,q]}\left(f_j\right) = \rho\ (j = 1, 2, \cdots, k)$. Then f can be expressed in the form

$$f\left(z\right) = B_1\left(z\right) f_1\left(z\right) + B_2\left(z\right) f_2\left(z\right) + \cdots + B_k\left(z\right) f_k\left(z\right), \tag{5.1}$$

where $B_1\left(z\right), \cdots, B_k\left(z\right)$ are suitable meromorphic functions determined by

$$\begin{array}{c} B_1'\left(z\right) f_1\left(z\right) + B_2'\left(z\right) f_2\left(z\right) + \cdots + B_k'\left(z\right) f_k\left(z\right) = 0, \\ B_1'\left(z\right) f_1'\left(z\right) + B_2'\left(z\right) f_2'\left(z\right) + \cdots + B_k'\left(z\right) f_k'\left(z\right) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_1'\left(z\right) f_1^{(k-1)}\left(z\right) + B_2'\left(z\right) f_2^{(k-1)}\left(z\right) + \cdots + B_k'\left(z\right) f_k^{(k-1)}\left(z\right) = F\left(z\right). \end{array} \tag{5.2}$$

Since the Wronskian $W\left(f_1,\ f_2, \cdots,\ f_k\right)$ is a differential polynomial in $f_1,\ f_2, \cdots,\ f_k$ with constant coefficients, it is easy by using Theorem 11 to deduce that

$$\rho_{[p+1,q]}\left(W\right) \leq \max\left\{\rho_{[p+1,q]}\left(f_j\right) : j = 1, 2, \cdots, k\right\} = \rho. \tag{5.3}$$

From (5.2), we get

$$B_j' = F.G_j\left(f_1, f_2, \cdots, f_k\right).\left(W\left(f_1, f_2, \cdots, f_k\right)\right)^{-1}\ \ (j = 1, 2, \cdots, k), \tag{5.4}$$

where $G_j\left(f_1, f_2, \cdots, f_k\right)$ are differential polynomials in $f_1, f_2, \cdots, f_k$ with constant coefficients. Thus

$$\rho_{[p+1,q]}\left(G_j\right) \leq \max\left\{\rho_{[p+1,q]}\left(f_j\right) : j = 1, 2, \cdots, k\right\} = \rho\ \ (j = 1, 2, \cdots, k). \tag{5.5}$$

Since $\rho_{[p+1,q]}(F) \leq \rho$, then by using Lemma 25 (i), (5.3) and (5.5), we have from (5.4) for $j = 1, 2, \cdots, k$

$$\rho_{[p+1,q]}(B_j) = \rho_{[p+1,q]}(B'_j) \leq \max\left\{\rho_{[p+1,q]}(F), \rho\right\} = \rho. \tag{5.6}$$

Then, by (5.6), we get from (5.1)

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \cdots, k\right\} = \rho. \tag{5.7}$$

Now, we assert that every meromorphic solution f whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of (1.2) satisfies $\rho_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$. In fact, if f^* is another meromorphic solution with $\rho_{[p+1,q]}(f^*) < \rho$ of equation (1.2), then $\rho_{[p+1,q]}(f_0 - f^*) < \rho$. But $f_0 - f^*$ is a meromorphic solution of the corresponding homogeneous equation (1.1) of (1.2). This contradicts Theorem 11. Then $\rho_{[p+1,q]}(f) = \rho$ holds for all meromorphic solutions of (1.2) with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$. By Lemma 24 (i), we know that every meromorphic solution f whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ with $\rho_{[p+1,q]}(f) = \rho$ satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 25 (i), (5.3) and (5.5), we have from (5.4) for $j = 1, 2, \cdots, k$

$$\begin{aligned} \rho_{[p+1,q]}(B_j) &= \rho_{[p+1,q]}(B'_j) \\ &\leq \max\left\{\rho_{[p+1,q]}(F), \rho_{[p+1,q]}(f_j) : j = 1, 2, \cdots, k\right\} = \rho_{[p+1,q]}(F). \end{aligned} \tag{5.8}$$

Then from (5.8) and (5.1), we get

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \cdots, k\right\} \leq \rho_{[p+1,q]}(F). \tag{5.9}$$

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (1.2) that a simple consideration of $[p, q]$ −order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (5.9) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.

(ii) For $3 \leq q = p + 1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold. □

6 Proof of Corollary 14

Proof. By using the same reasoning of Corollary 13 we can obtain Corollary 14. □

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