

On Picard Value Problem of Some Difference Polynomials

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Abstract. In this paper, we study the value distribution of zeros of certain nonlinear difference polynomials of entire functions of finite order.

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1 Introduction and Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory ([10], [13]). In addition, we will use $\rho(f)$ to denote the order of growth of f , we say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f) = o(T(r, f))$, as $r \rightarrow +\infty$ outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z + c)$.

In 1959, Hayman proved in [11] that if f is a transcendental entire function, then $f^n f'$ assume every nonzero complex number infinitely many times, provided that $n \geq 3$. Later, Hayman [12] conjectured that this result remains to be valid when $n = 1$ and $n = 2$. Then Mues [18] confirmed the case when $n = 2$ and Bergweiler-Eremenko [2] and Chen-Fang [3] confirmed

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the case when $n = 1$, independently. Since then, there are many research publications (see [17]) regarding this type of Picard-value problem. In 1997, Bergweiler obtained the following result.

Theorem A. ([1]) *If f is a transcendental meromorphic function of finite order and q is a not identically zero polynomial, then $ff' - q$ has infinitely many zeros.*

In 2007, Laine and Yang studied the difference analogue of Hayman's theorem and proved the following result.

Theorem B. ([14]) *Let $f(z)$ be a transcendental entire function of finite order, and c be a nonzero complex constant. Then for $n \geq 2$, $f^n(z) f(z+c)$ assume every non-zero value $a \in \mathbb{C}$ infinitely often.*

In the same paper, Laine and Yang showed that Theorem B does not remain valid for the case $n = 1$. Indeed, take $f(z) = e^z + 1$. Then

$$f(z) f(z + \pi i) - 1 = (1 + e^z)(1 - e^z) - 1 = -e^{2z}.$$

After their, a stream of studies on the value distribution of nonlinear difference polynomials in f has been launched and many related results have been obtained, see e.g. [5, 14, 15, 16]. For example, Liu and Yang improved the previous result and obtained the following.

Theorem C. ([15]) *Let $f(z)$ be a transcendental entire function of finite order, and c be a nonzero complex constant. Then for $n \geq 2$, $f^n(z) f(z+c) - p(z)$ has infinitely many zeros, where $p(z) \not\equiv 0$ is a polynomial in z .*

Hence, it is natural to ask: *What can be said about the value distribution of $f(z) f(z+c) - q(z)$, when f is a transcendental meromorphic function and q be a not identically zero small function of f ?* In this paper, as an attempt in resolving this question, we obtain the following results.

Theorem 1.1 *Let f be a transcendental entire function of finite order, let c_1, c_2 be two nonzero complex numbers such that $f(z+c_1) \not\equiv f(z+c_2)$ and q be not identically zero polynomial. Then $f(z) f(z+c_1) - q(z)$ and $f(z) f(z+c_2) - q(z)$ at least one of them has infinitely many zeros.*

The following corollary arises directly from Theorem 1.1 and Theorem C.

Corollary 1.1 *Let $n \geq 1$ be an integer and let c_1, c_2 ($c_1 c_2 \neq 0$) be two distinct complex numbers. Let α, β, p_1, p_2 and q ($\neq 0$) be nonconstant polynomials. If f is a finite order transcendental entire solution of*

$$\begin{cases} f^n(z) f(z + c_1) - q(z) = p_1(z) e^{\alpha(z)} \\ f^n(z) f(z + c_2) - q(z) = p_2(z) e^{\beta(z)} \end{cases},$$

then, $n = 1$ and f must be a periodic function of period $c_1 - c_2$.

2 Some lemmas

The following lemma is an extension of the difference analogue of the Clunie lemma obtained by Halburd and Korhonen [8].

Lemma 2.1 [4] *Let $f(z)$ be a non-constant, finite order meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f), Q(z, f)$ are difference polynomials in $f(z)$ with meromorphic coefficients $a_j(z)$ ($j = 1, \dots, s$), and let $\delta < 1$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most n , then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) + O\left(\sum_{j=1}^s m(r, a_j)\right).$$

for all r outside an exceptional set of finite logarithmic measure.

Lemma 2.2 [6] *Let $f(z)$ be a non-constant, finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f).$$

Lemma 2.3 [7] *Let $f(z)$ be a transcendental meromorphic function of finite order ρ , and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_0 \subset$*

$(1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_0 \cup [0, 1]$, and for all k, j , $0 \leq j < k$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

The following lemma is the lemma of the logarithmic derivative.

Lemma 2.4 [10] *Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside a set $E_1 \subset [0, +\infty)$ of a finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

The following lemma is a difference analogue of the lemma of the logarithmic derivative for finite order meromorphic functions.

Lemma 2.5 [6, 8, 9] *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.6 *Let $f(z)$ be a transcendental meromorphic solution of the system*

$$\begin{cases} f(z) f(z + c_1) - q(z) = p_1(z) e^{\alpha(z)}, \\ f(z) f(z + c_2) - q(z) = p_2(z) e^{\beta(z)}, \end{cases} \quad (2.1)$$

where α, β are polynomials and p_1, p_2, q are not identically zero rational functions. If $N(r, f) = S(r, f)$, then

$$\deg \alpha = \deg \beta = \deg(\alpha + \beta) = \rho(f) > 0.$$

Proof. First, we prove that $\deg \alpha = \rho(f)$ and by the same we can deduce that $\deg \beta = \rho(f)$. It's clear from (2.1) that $\deg \alpha \leq \rho(f)$. Suppose that $\deg \alpha < \rho(f)$, this means that

$$f(z) f(z + c_1) := F = q(z) + p_1(z) e^{\alpha(z)} \in S(f). \quad (2.2)$$

Applying Lemma 2.1 and Lemma 2.2 into (2.2), we obtain $T(r, f_c) = T(r, f) = S(r, f)$ which is a contradiction. Assume now that $\deg(\alpha + \beta) < \rho(f)$, this leads to $p_1 p_2 e^{\alpha + \beta} \in S(f)$. From this and (2.1) we have

$$f^2 P(z, f) = p_1 p_2 e^{\alpha + \beta} + q^2,$$

where

$$P(z, f) = a(z) f^2 - b(z)$$

and

$$a = \frac{f_{c_1} f_{c_2}}{f f}, \quad b = q \left(\frac{f_{c_1}}{f} + \frac{f_{c_2}}{f} \right).$$

It's clear that $P(z, f) \not\equiv 0$, and by using Lemma 2.1, we get

$$m(r, P(z, f)) = S(r, f)$$

which leads to

$$2T(r, f) = m \left(r, \frac{b(z) + P(z, f)}{a(z)} \right) = S(r, f)$$

which is a contradiction. Hence, $\deg(\alpha + \beta) = \deg \alpha = \deg \beta$. Finally, by using Lemma 2.1, it's easy to see that both of α and β are nonconstant polynomials.

3 Proof of Theorem 1.1

We shall prove this theorem by contradiction. Suppose contrary to our assertion that both of $f(z) f(z + c_1) - q(z)$ and $f(z) f(z + c_2) - q(z)$ have finitely many zeros. Then, there exist four polynomials α, β, p_1 and p_2 such that

$$f(z) f(z + c_1) - q(z) = p_1(z) e^{\alpha(z)} \tag{3.1}$$

and

$$f(z) f(z + c_2) - q(z) = p_2(z) e^{\beta(z)}. \tag{3.2}$$

By differentiating (3.1) and eliminating e^α , we get

$$A_1 f f_{c_1} - f' f_{c_1} - f f'_{c_1} = B_1, \tag{3.3}$$

where $A_1 = \frac{p'_1}{p_1} + \alpha'$, $B_1 = \left(\frac{p'_1}{p_1} + \alpha'\right) q - q'$. By Lemma 2.6 we have

$$\deg \alpha = \deg \beta = \deg(\alpha + \beta) = \rho(f) > 0.$$

Now, we prove that $A_1 \neq 0$. To show this, we suppose the contrary. Then, there exists a constant A such that $A = p_1(z) e^\alpha$, which implies the contradiction $\deg \alpha = \rho(f) = 0$. By the same, we can prove that $B_1 \neq 0$. By the same arguments as above, (3.2) gives

$$A_2 f f_{c_2} - f' f_{c_2} - f f'_{c_2} = B_2, \quad (3.4)$$

where $A_2 = \frac{p'_2}{p_2} + \beta'$ and $B_2 = \left(\frac{p'_2}{p_2} + \beta'\right) q - q'$. Obviously, $A_2 \neq 0$ and $B_2 \neq 0$. Dividing both sides of (3.3) and (3.4) by f^2 , we get for each $\varepsilon > 0$

$$\begin{aligned} 2m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{f_{c_i}}{f}\right) + m\left(r, \frac{f' f_{c_i}}{f}\right) + m\left(r, \frac{f'_{c_i} f_{c_i}}{f}\right) + O(\log r) \\ &= O(r^{\rho-1+\varepsilon}) + O(\log r) = S(r, f). \end{aligned}$$

So, by the first fundamental theorem, we deduce that

$$T(r, f) = N\left(r, \frac{1}{f}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r). \quad (3.5)$$

It's clear from (3.3) and (3.4) that any multiple zero of f is a zero of B_i ($i = 1, 2$). Hence

$$N_{(2)}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{B_i}\right) = O(\log r),$$

where $N_{(2)}\left(r, \frac{1}{f}\right)$ denotes the counting function of zeros of f whose multiplicities are not less than 2. It follows by this and (3.5) that

$$T(r, f) = N_{(1)}\left(r, \frac{1}{f}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r), \quad (3.6)$$

where $N_{(1)}\left(r, \frac{1}{f}\right)$ is the counting function of zeros, where only the simple zeros are considered. From (3.3) and (3.4), for every zero z_0 such that $f'(z_0) \neq 0$ which is not zero or pole of B_1 and B_2 , we have

$$(f' f_{c_1} + B_1)(z_0) = 0 \quad (3.7)$$

and

$$(f'f_{c_2} + B_2)(z_0) = 0. \quad (3.8)$$

By (3.7) and (3.8), we obtain

$$(B_2f_{c_1} - B_1f_{c_2})(z_0) = 0 \quad (3.9)$$

which means that the function $\frac{B_2f_{c_1} - B_1f_{c_2}}{f}$ has at most a finite number of simple poles. We consider two cases:

Case 1. $B_2f_{c_1} - B_1f_{c_2} \not\equiv 0$. Set

$$h(z) = \frac{B_2f_{c_1} - B_1f_{c_2}}{f(z)}. \quad (3.10)$$

Then, from the lemma of logarithmic differences, we have $m(r, h) = O(r^{\rho-1+\varepsilon}) + O(\log r)$. On the other hand

$$\begin{aligned} N(r, h) &= N\left(r, \frac{B_2f_{c_1} - B_1f_{c_2}}{f}\right) = N_1\left(r, \frac{B_2f_{c_1} - B_1f_{c_2}}{f}\right) \\ &\quad + O(r^{\rho-1+\varepsilon}) + O(\log r) = S(r, f). \end{aligned}$$

Thus, $T(r, h) = O(r^{\rho-1+\varepsilon}) + O(\log r) = S(r, f)$. From the equation (3.10), we have

$$f_{c_1}(z) = \frac{B_1}{B_2}f_{c_2}(z) + \frac{h}{B_2}f(z). \quad (3.11)$$

By differentiating (3.11), we get

$$f'_{c_1}(z) = \left(\frac{h}{B_2}\right)' f(z) + \frac{h}{B_2}f'(z) + \left(\frac{B_1}{B_2}\right)' f_{c_2}(z) + \frac{B_1}{B_2}f'_{c_2}(z). \quad (3.12)$$

Substituting (3.11) and (3.12) into (3.3)

$$\begin{aligned} &\left[\frac{A_1h}{B_2} - \left(\frac{h}{B_2}\right)'\right] f^2 + \left[-\frac{2h}{B_2}\right] ff' \\ &+ \left[\frac{A_1B_1}{B_2} - \left(\frac{B_1}{B_2}\right)'\right] ff_{c_2} - \frac{B_1}{B_2}f'f_{c_2} - \frac{B_1}{B_2}ff'_{c_2} = B_1. \end{aligned} \quad (3.13)$$

Equation (3.4), can be rewritten as

$$-\frac{B_1A_2}{B_2}ff_{c_2} + \frac{B_1}{B_2}f'f_{c_2} + \frac{B_1}{B_2}ff'_{c_2} = -B_1.$$

By adding this to (3.13), we get

$$\left[\frac{A_1 h}{B_2} - \left(\frac{h}{B_2} \right)' \right] f + \left[-\frac{2h}{B_2} \right] f' + \left[\frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \right] f_{c_2} = 0. \quad (3.14)$$

Its clear that $-\frac{2h}{B_2} \neq 0$. In order to complete the proof of our theorem, we need to prove

$$\frac{A_1 h}{B_2} - \left(\frac{h}{B_2} \right)' \neq 0 \text{ and } \frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \neq 0.$$

Suppose contrary to our assertion that $\frac{A_1 h}{B_2} - \left(\frac{h}{B_2} \right)' \equiv 0$. Then, by the definition of A_1 and by simple integration, we get

$$p_1 e^\alpha = C_1 \frac{h}{B_2},$$

where C_1 is a nonzero constant. This implies that $\deg \alpha = \rho(f) - 1$, which is a contradiction. Hence, $\frac{A_1 h}{B_2} - \left(\frac{h}{B_2} \right)' \neq 0$. Next, we shall prove $\frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \neq 0$. Suppose that $\frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \equiv 0$. Then we obtain

$$\frac{p_1}{p_2} e^{\alpha-\beta} = C_2 \frac{B_1}{B_2} := \gamma,$$

where C_2 is a nonzero constant and γ is a small function of f . From (3.1) and (3.2) we get

$$f(f_{c_1} - \gamma f_{c_2}) = (1 - \gamma)q. \quad (3.15)$$

If $\gamma \neq 1$, then by applying Clunie's lemma to (3.15), we obtain

$$m(r, f_{c_1} - \gamma f_{c_2}) = T(r, f_{c_1} - \gamma f_{c_2}) = S(r, f).$$

By this and (3.15), we have

$$T(r, f) = T\left(r, \frac{(1 - \gamma)q}{f_{c_1} - \gamma f_{c_2}}\right) = S(r, f)$$

which is a contradiction. If $\gamma \equiv 1$, then we obtain the contradiction $f_{c_1}(z) \equiv f_{c_2}(z)$. Thus, $\frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \neq 0$. From the above discussion and (3.14), we have

$$f_{c_2}(z) = M(z) f(z) + N(z) f'(z) \quad (3.16)$$

and

$$f_{c_1}(z) = \varphi(z) f(z) + \psi(z) f'(z), \quad (3.17)$$

where

$$M = \frac{\left(\frac{h}{B_2}\right)' - A_1 \frac{h}{B_2}}{(A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2}\right)'}, \quad N = \frac{\frac{2h}{B_2}}{(A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2}\right)'}$$

and

$$\varphi(z) = \frac{B_1}{B_2} M + \frac{h}{B_2}, \quad \psi = \frac{B_1}{B_2} N.$$

Differentiation of (3.16), gives

$$f'_{c_2} = M' f + (M + N') f' + N f''. \quad (3.18)$$

Substituting (3.16) and (3.18) into (3.4), we get

$$[M' - A_2 M] f^2 + [N' - A_2 N + 2M] f' f + N \left((f')^2 + f f'' \right) = -B_2. \quad (3.19)$$

Differentiating (3.19), we get

$$\begin{aligned} & [M' - A_2 M]' f^2 + (2[M' - A_2 M] + [N' - A_2 N + 2M]') f' f \\ & + (2N' - A_2 N + 2M) \left((f')^2 + f f'' \right) + N (3f' f'' + f f''') = -B_2'. \end{aligned} \quad (3.20)$$

Suppose z_0 is a simple zero of f and not a zero or pole of B_2 . Then from (3.19) and (3.20), we have

$$\left(N f' + \frac{B_2}{f'} \right) (z_0) = 0,$$

$$\left[(2N' - A_2 N + 2M) f' + 3N f'' + \frac{B_2'}{f'} \right] (z_0) = 0.$$

It follows that z_0 is a zero of $[B_2(2N' - A_2 N + 2M) - B_2' N] f' + 3B_2 N f''$. Therefore the function

$$H = \frac{[2B_2 N' - B_2 A_2 N + 2B_2 M - B_2' N] f' + 3B_2 N f''}{f}$$

satisfies $T(r, H) = S(r, f)$ and

$$f'' = \frac{H}{3B_2N}f + \frac{[-2B_2N' + B_2A_2N - 2B_2M + B_2'N]}{3B_2N}f'. \quad (3.21)$$

Substituting (3.21) into (3.19), we get

$$q_1f^2 + q_2f'f + q_3(f')^2 = -B_2, \quad (3.22)$$

where

$$q_1 = M' - A_2M + \frac{H}{3B_2},$$

$$q_2 = \frac{1}{3}N' + \frac{1}{3}\left(\frac{B_2'}{B_2} - 2A_2\right)N + \frac{4}{3}M, \quad q_3 = N.$$

We prove first $q_2 \neq 0$. Suppose the contrary. Then

$$\frac{q_2}{q_3} = \frac{2}{3}\frac{N'}{N} - \frac{1}{3}\frac{B_2'}{B_2} - \frac{2}{3}(A_1 + A_2) + \frac{2}{3}\frac{h'}{h} = 0$$

which leads to

$$\alpha' + \beta' = \frac{N'}{N} - 2\frac{B_2'}{B_2} + \frac{h'}{h} - \frac{p_1'}{p_1} - \frac{p_2'}{p_2}.$$

By simple integration of both sides of the above equation, we get

$$p_1p_2e^{\alpha+\beta} = c\frac{N}{B_2^2}h, \quad (3.23)$$

where c is a nonzero constant, this leads to the contradiction $\deg(\alpha + \beta) < \deg \alpha = \deg \beta$. Hence, $q_2 \neq 0$. Differentiating (3.22), we obtain

$$q_1'f^2 + (2q_1 + q_2')f'f + (q_2 + q_3')(f')^2 + q_2f''f + 2q_3f'f'' = -B_2'. \quad (3.24)$$

Let z_0 be a simple zero of f which is not a zero or pole of B_2 . Then from (3.22) and (3.24) we have

$$\left(q_3f' + \frac{B_2'}{f'}\right)(z_0) = 0,$$

$$\left[(q_2 + q_3')f' + 2q_3f'' + \frac{B_2'}{f'}\right](z_0) = 0.$$

Therefore z_0 is a zero of $(B_2 (q_2 + q'_3) - B'_2 q_3) f' + 2B_2 q_3 f''$. Hence the function

$$R = \frac{(B_2 (q_2 + q'_3) - B'_2 q_3) f' + 2B_2 q_3 f''}{f}.$$

satisfies $T(r, R) = S(r, f)$ and

$$f'' = \frac{R}{2B_2 q_3} f + \frac{B'_2 q_3 - B_2 (q_2 + q'_3)}{2B_2 q_3} f'. \quad (3.25)$$

Substituting (3.25) into (3.24)

$$\begin{aligned} \left[q'_1 + \frac{q_2 R}{2B_2 q_3} \right] f^2 + \left[2q_1 + q'_2 + \frac{1}{2} \frac{B'_2}{B_2} q_2 - \frac{1}{2} (q_2 + q'_3) \frac{q_2}{q_3} + \frac{R}{B_2} \right] f' f \\ + \frac{B'_2 q_3}{B_2} (f')^2 = -B'_2. \end{aligned} \quad (3.26)$$

Combining (3.26) and (3.22), we obtain

$$\left[q'_1 + \frac{q_2 R}{2B_2 q_3} - \frac{B'_2}{B_2} q_1 \right] f + \left[2q_1 + q'_2 - \frac{1}{2} \frac{B'_2}{B_2} q_2 - \frac{1}{2} (q_2 + q'_3) \frac{q_2}{q_3} + \frac{R}{B_2} \right] f' = 0. \quad (3.27)$$

From (3.27), we deduce that

$$q'_1 + \frac{q_2 R}{2B_2 q_3} - \frac{B'_2}{B_2} q_1 = 0$$

and

$$2q_1 + q'_2 - \frac{1}{2} \frac{B'_2}{B_2} q_2 - \frac{1}{2} (q_2 + q'_3) \frac{q_2}{q_3} + \frac{R}{B_2} = 0.$$

By eliminating R from the above two equations, we obtain

$$q_3 (4q_1 q_3 - q_2^2) \frac{B'_2}{B_2} + q_2 (4q_1 q_3 - q_2^2) - q_3 (4q_1 q_3 - q_2^2)' + q'_3 (4q_1 q_3 - q_2^2) = 0. \quad (3.28)$$

Thus, equation (3.25) can be rewritten as

$$f'' = \left(\frac{B'_2 q_1}{B_2 q_2} - \frac{q'_1}{q_2} \right) f + \frac{1}{2} \left(\frac{B'_2}{B_2} - \frac{q_2}{q_3} - \frac{N'}{N} \right) f'. \quad (3.29)$$

Subcase 1. If $4q_1q_3 - q_2^2 \neq 0$, then from (3.28) we have

$$\frac{q_2}{q_3} = \frac{(4q_1q_3 - q_2^2)'}{(4q_1q_3 - q_2^2)} - \frac{B_2'}{B_2} - \frac{q_3'}{q_3}.$$

On the other hand

$$\frac{q_2}{q_3} = \frac{1}{3} \frac{N'}{N} + \frac{1}{3} \frac{B_2'}{B_2} - \frac{2}{3} (A_1 + A_2) + \frac{2}{3} \frac{\left(\frac{h}{B_2}\right)'}{\frac{h}{B_2}}.$$

Hence

$$2(A_1 + A_2) = -3 \frac{(4q_1q_3 - q_2^2)'}{(4q_1q_3 - q_2^2)} + 4 \frac{N'}{N} + 4 \frac{B_2'}{B_2} + 2 \frac{\left(\frac{h}{B_2}\right)'}{\frac{h}{B_2}}.$$

By the definition of A_i ($i = 1, 2$) and simple integration, we deduce that

$$\deg(\alpha + \beta) < \deg \alpha = \deg \beta$$

which is a contradiction.

Subcase 2. If $4q_1q_3 \equiv q_2^2$, then from (3.29) and (3.21) we have

$$\frac{B_2' q_1}{B_2 q_2} - \frac{q_1'}{q_2} = \frac{H}{3B_2 N}. \quad (3.30)$$

On the other hand

$$\frac{q_1}{q_3} - \frac{M' - A_2 M}{N} = \frac{H}{3B_2 N}. \quad (3.31)$$

Combining (3.30) and (3.31), we obtain

$$\begin{aligned} & \frac{5 B_2'}{4 B_2} \frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'} \\ & + \frac{1}{6} \left(\frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'} \right)' - \left(\frac{1}{2} A_1 + A_2 \right) \frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'} \end{aligned}$$

$$\begin{aligned}
& \frac{5 h'}{3 h} \frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)} - \frac{5}{6} (A_1 + A_2) \frac{h'}{h} + \frac{23 B_2' h'}{12 B_2 h} - \frac{5}{4} \left(\frac{h'}{h} \right)^2 \\
& - \frac{1}{9} (A_1 + A_2) \frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)} + \frac{1 B_2'}{9 B_2} \frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)} \\
& + \frac{2 B_2'}{9 B_2} (A_1 + A_2) - \frac{7}{9} \left(\frac{B_2'}{B_2} \right)^2 - \frac{19}{36} \left(\frac{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)'}{\left((A_1 - A_2) \frac{B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' \right)} \right)^2 \\
& - \frac{1}{6} \left(\frac{B_2'}{B_2} \right)' - \frac{1}{2} A_1' + \frac{1}{3} (A_1' + A_2') + \frac{1}{3} (A_1 + A_2) \frac{B_2'}{B_2} + \frac{1}{2} A_2 A_1 \\
& = \frac{1}{9} (A_1 + A_2)^2.
\end{aligned}$$

Dividing both sides of the above equation by $\frac{(A_1+A_2)^2}{2}$ and since $\lim_{z \rightarrow \infty} \frac{R'(z)}{R(z)} = 0$ if R is a nonzero rational function, we obtain

$$\left| \frac{A_2 A_1}{(A_1 + A_2)^2} - \frac{2}{9} \right| \leq \frac{5}{3} \frac{\left| \frac{h'}{h} \right|}{|A_1 + A_2|} + \frac{23}{6} \left| \frac{B_2'}{B_2} \right| \frac{\left| \frac{h'}{h} \right|}{|A_1 + A_2|^2} + \frac{5}{2} \frac{\left| \frac{h'}{h} \right|^2}{|A_1 + A_2|^2} + o(1) \quad (3.32)$$

On the other hand, since $\rho(h) \leq \rho(f) - 1$ and by Lemma 2.3

$$\left| \frac{h'(z)}{h(z)} \right| \leq |z|^{\rho(f)-2+\varepsilon} \quad (3.33)$$

for all z satisfying $|z| \notin E_0 \cup [0, 1]$, where $E_0 \subset (1, \infty)$ is a set of finite logarithmic measure. By combining (3.32) and (3.33), we deduce

$$\lim_{\substack{z \rightarrow \infty \\ |z| \notin E_0 \cup [0, 1]}} \frac{A_2 A_1}{(A_1 + A_2)^2} = \lim_{\substack{z \rightarrow \infty \\ |z| \notin E_0 \cup [0, 1]}} \frac{\alpha' \beta'}{(\alpha' + \beta')^2} = \frac{2}{9}.$$

By setting $\alpha(z) = a_m z^m + \cdots + a_0$ and $\beta(z) = b_m z^m + \cdots + b_0$, we deduce

$$\lim_{\substack{z \rightarrow \infty \\ |z| \notin E_0 \cup [0, 1]}} \frac{\alpha' \beta'}{(\alpha' + \beta')^2} = \frac{a_m b_m}{(a_m + b_m)^2} = \frac{2}{9}$$

which implies that $\frac{am}{b_m} = 2$ or $\frac{1}{2}$. We consider first the case $\frac{am}{b_m} = \frac{1}{2}$, we get from (3.1) and (3.17)

$$\varphi f^2 + \psi f' f - q = Ae^{\frac{1}{2}b_m z^m} \quad (3.34)$$

and

$$Mf^2 + Nf' f - q = Be^{b_m z^m}, \quad (3.35)$$

where $A = p_1 e^{a_{m-1}z^{m-1} + \dots + a_0}$ and $B = p_2 e^{b_{m-1}z^{m-1} + \dots + b_0}$. From (3.34) and (3.35), we get

$$\varphi f^2 + \psi f' f = q + A \left(\frac{Mf^2 + Nf' f - q}{B} \right)^{\frac{1}{2}}.$$

Hence

$$\varphi f + \psi f' = \frac{q}{f} + A \left(\frac{Mf^2 + Nf' f - q}{Bf^2} \right)^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} T(r, \varphi f + \psi f') &= m(r, \varphi f + \psi f') + S(r, f) \\ &= \frac{1}{2\pi} \int_{E_1} \log^+ |\varphi(re^{i\theta}) f(re^{i\theta}) + \psi(re^{i\theta}) f'(re^{i\theta})| d\theta \\ &\quad + \frac{1}{2\pi} \int_{E_2} \log^+ |\varphi(re^{i\theta}) f(re^{i\theta}) + \psi(re^{i\theta}) f'(re^{i\theta})| d\theta + S(r, f), \end{aligned}$$

where $E_1 = \{\theta : |f(re^{i\theta})| \leq 1\}$ and $E_2 = \{\theta : |f(re^{i\theta})| > 1\}$. Now

$$\begin{aligned} &\frac{1}{2\pi} \int_{E_1} \log^+ |\varphi(re^{i\theta}) f(re^{i\theta}) + \psi(re^{i\theta}) f'(re^{i\theta})| d\theta \\ &\leq \frac{1}{2\pi} \int_{E_1} \log^+ |f'(re^{i\theta})| d\theta + S(r, f) \\ &\leq \frac{1}{2\pi} \int_{E_1} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta + S(r, f) = S(r, f). \end{aligned}$$

On the other hand

$$\frac{1}{2\pi} \int_{E_2} \log^+ |\varphi(re^{i\theta}) f(re^{i\theta}) + \psi(re^{i\theta}) f'(re^{i\theta})| d\theta$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{E_2} \log^+ \left| \frac{q(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \\
&+ \frac{1}{4\pi} \int_{E_2} \log^+ \left| \frac{M(re^{i\theta})}{B(re^{i\theta})} + \frac{N(re^{i\theta})}{B(re^{i\theta})} \frac{f'(re^{i\theta})}{f(re^{i\theta})} - \frac{q(re^{i\theta})}{f^2(re^{i\theta})} \right| d\theta + S(r, f) = S(r, f).
\end{aligned}$$

Hence

$$T(r, f_{c_1}) = T(r, \varphi f + \psi f') = S(r, f)$$

which is a contradiction. If $\frac{a_m}{b_m} = 2$, then by the same argument we have

$$Mf^2 + Nf'f = q + B \left(\frac{\varphi f^2 + \psi f'f - q}{A} \right)^{\frac{1}{2}}$$

which implies the contradiction

$$T(r, f_{c_2}) = T(r, Mf + Nf') = S(r, f).$$

Case 2. $B_2 f_{c_1} - B_1 f_{c_2} \equiv 0$, by using the same arguments as in the proof of (3.14), we obtain that

$$\frac{A_1 B_1}{B_2} - \left(\frac{B_1}{B_2} \right)' - \frac{B_1 A_2}{B_2} \equiv 0$$

which leads to

$$\frac{p_1}{p_2} e^{\alpha-\beta} = k \frac{B_1}{B_2} = k \frac{f_{c_1}}{f_{c_2}}, \quad (3.36)$$

where k is a nonzero complex constant. By this (3.1) and (3.2), we have

$$(1-c) f f_{c_1} f_{c_2} = q(f_{c_2} - k f_{c_1}). \quad (3.37)$$

If $k \neq 1$, then by applying Clunie lemma to (3.37), we deduce the contradiction $T(r, f_{c_i}) = S(r, f)$. Hence, $k = 1$ and from the equation (3.36), we conclude that $f_{c_1} \equiv f_{c_2}$ which exclude the hypothesis of our theorem. This shows that at least one of $f(z) f(z + c_1) - q(z)$ and $f(z) f(z + c_2) - q(z)$ has infinitely many zeros.

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