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Partial sums of a certain harmonic univalent meromorphic functions

M.K. Aouf, R.M. El-Ashwah,
J.Dziok, J. Stankiewicz

Abstract: In the present paper we determine sharp lower bounds of the real part of the ratios of harmonic univalent meromorphic functions to their sequences of partial sums.
Let Σ_H denote the class of functions f that are harmonic univalent and sense-preserving in $U^* =, \{z : |z| \ > 1\}$ which are of the form

$$f(z) = h(z) + \overline{g(z)}\ ,$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \ g(z) = \sum_{n=1}^{\infty} b_n z^{-n}.$$

Now, we define the sequences of partial sums of functions f of the form

$$\begin{aligned} f_s(z) &= z + \sum_{n=1}^{s} a_n z^{-n} + \overline{g(z)}, \\ \widetilde{f}_r(z) &= g(z) + \sum_{n=1}^{r} \overline{b_n z^{-n}}, \\ f_{s,r}(z) &= z + \sum_{n=1}^{s} a_n z^{-n} + \sum_{n=1}^{r} \overline{b_n z^{-n}.} \end{aligned}$$

In the present paper we will determine sharp lower bounds for $\text{Re}\left\{\frac{f(z)}{f_s(z)}\right\}$, $\text{Re}\left\{\frac{f_s(z)}{f(z)}\right\}$, $\text{Re}\left\{\frac{f(z)}{\widetilde{f}_r(z)}\right\}$, $\text{Re}\left\{\frac{\widetilde{f}_r(z)}{f(z)}\right\}$, $\text{Re}\left\{\frac{f(z)}{f_{s,r}(z)}\right\}$, $\text{Re}\left\{\frac{f_{s,r}(z)}{f(z)}\right\}$.

AMS Subject Classification: *30C45, 30C50*
Keywords and Phrases: *Harmonic function, meromorphic, univalent, sense-preserving.*

1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [4] that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$. There are numerous papers on univalent harmonic functions defined in a domain $U = \{z \in \mathbb{C} : |z| < 1\}$ (see [6,7], [14] and [15]). Hergartner and Schober [10] investigated functions harmonic in the exterior of the unit disc i.e $U^* = \{z \in \mathbb{C} : |z| > 1\}$. They showed that a complex valued, harmonic, sense preserving univalent function f, defined on U^* and satisfying $f(\infty) = \infty$ must admit the represntation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \quad (A \in \mathbb{C}), \tag{1.1}$$

where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \quad (z \in U^*,\ 0 \leq |\beta| < |\alpha|), \tag{1.2}$$

and $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfy $|a(z)| < 1$ for $z \in U^*$.
Let us denote by Σ_H the class of functions f that are harmonic univalent and sense-preserving in U^*, which are of the form

$$f(z) = h(z) + \overline{g(z)} \quad (z \in U^*)\,, \tag{1.3}$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \ g(z) = \sum_{n=1}^{\infty} b_n z^{-n}.$$

Now, we introduce a class $\Sigma_H(c_n, d_n, \delta)$ consisting of functions of the form (1.3) such that

$$\sum_{n=1}^{\infty} c_n |a_n| + \sum_{n=1}^{\infty} d_n |b_n| < \delta \quad (d_n \geq c_n \geq c_2 > 0; \delta > 0). \tag{1.4}$$

It is easy to see that various subclasses of Σ_H consisting of functions $f(z)$ of the form (1.3) can be represented as $\Sigma_H(c_n, d_n, \delta)$ for suitable choices of c_n, d_n and δ studies earlier by various authors.

- $\Sigma_H(n, n, 1) = H_0^*$ (see Jahangiri and Silverman. [8]);
- $\Sigma_H(n + \gamma, n - \gamma, 1 - \gamma) = \Sigma_H^*(\gamma)(0 \leq \gamma < 1, n \geq 1)$ (see Jahangiri [5]);

- $\Sigma_H(|(n+1)\lambda-1|, |(n-1)\lambda+1|, 1-\alpha) = \Sigma_H R(\alpha,\lambda)(0 \le \alpha < 1, \lambda \ge 0, n \ge 1)$ (see Ahuja and Jahangiri [1]);
- $\Sigma_H(n+\alpha-\alpha\lambda(n+1), n-\alpha-\alpha\lambda(n-1), 1-\alpha) = \Sigma_H S^*(\alpha,\lambda)(0 \le \alpha < 1, 0 \le \lambda \le 1, n \ge 1)$ (see Janteng and Halim [9]),
- $\Sigma_H(n(n+2)^m, n(n-2)^m, 1) = MH^*(m)(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,...\}, n \ge 1)$ (see Bostanci and Ozturk [2]);
- $\Sigma_H((n+\gamma)(n+2)^m, (n-\gamma)(n-2)^m, 1-\gamma) = MH^*(m,\gamma)(0 \le \gamma < 1, m \in \mathbb{N}_0, n \ge 1)$ (see Bostanci and Ozturk [3]).

Silvia [17] studied the partial sums of the convex functions of order α, later on Silverman [16] studied partial sum for starlike and convex functions. Very recentaly, Porwal [12], Porwal and Dixit [13] and Porwal [11] studied analogues interesting results on the partial sums of certain harmonic univalent functions.

Since to a certain extent the work in the harmonic univalent meromorphic functions case has paralleled that of the harmonic analytic univalent case, one is tempted to search results analogous to those of Porwal [11] for meromorphic harmonic univalent functions in U^*.

Now, we define the sequences of partial sums of functions f of the form (1.3) by

$$\begin{aligned} f_s(z) &= z + \sum_{n=1}^{s} a_n z^{-n} + \sum_{n=1}^{\infty} \overline{b_n z^{-n}}, \\ \widetilde{f}_r(z) &= z + \sum_{n=1}^{\infty} a_n z^{-n} + \sum_{n=1}^{r} \overline{b_n z^{-n}}, \\ f_{s,r}(z) &= z + \sum_{n=1}^{s} a_n z^{-n} + \sum_{n=1}^{r} \overline{b_n z^{-n}}, \end{aligned} \tag{1.5}$$

when the coefficients of f are sufficiently small to satisfy the condition (1.4).

In the present paper, motivated essentially by the work of Silverman [16] and Porwal [11], we will determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)}{f_s(z)}\right\}$, $\operatorname{Re}\left\{\frac{f_s(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f(z)}{\widetilde{f}_r(z)}\right\}$, $\operatorname{Re}\left\{\frac{\widetilde{f}_r(z)}{f(z)}\right\}$, $\operatorname{Re}\left\{\frac{f(z)}{f_{s,r}(z)}\right\}$ and $\operatorname{Re}\left\{\frac{f_{s,r}(z)}{f(z)}\right\}$.

2 Main Results

Theorem 1. *Let $s \in \mathbb{N}$ and let $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. Then*

(i) $$\operatorname{Re}\left\{\frac{f(z)}{f_s(z)}\right\} > 1 - \frac{\delta}{c_{s+1}} \quad (z \in U), \tag{2.1}$$

and

(ii) $$\operatorname{Re}\left\{\frac{f_s(z)}{f(z)}\right\} > \frac{c_{s+1}}{\delta + c_{s+1}} \quad (z \in U), \tag{2.2}$$

whenever

$$c_n \geq \begin{cases} \delta, & n = 2, 3, ..., s, \\ c_{s+1}, & n = s+1, s+2, ... \end{cases} \tag{2.3}$$

The estimates in (2.1) *and* (2.2) *are sharp for the function given by*

$$f(z) = z + \frac{\delta}{c_{s+1}} z^{-s-1} \quad (z \in U^*) \, . \tag{2.4}$$

Proof. (i) To obtain the sharp lower bound given by (2.1), let us put

$$g_1(z) = \frac{c_{s+1}}{\delta} \left\{ \frac{f(z)}{f_s(z)} - (1 - \frac{\delta}{c_{s+1}}) \right\}$$

$$= 1 + \frac{\frac{c_{s+1}}{\delta} \sum\limits_{n=s+1}^{\infty} a_n z^{-n}}{z + \sum\limits_{n=1}^{s} a_n z^{-n} + \sum\limits_{n=1}^{\infty} \overline{b_n z^{-n}}}. \tag{2.5}$$

Then, it is sufficient to show that $\operatorname{Re} g_1(z) > 0 \quad (z \in U^*)$ or equivalently

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq 1 \quad (z \in U^*) \, .$$

Since

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{\frac{c_{s+1}}{\delta} \sum\limits_{n=s+1}^{\infty} |a_n|}{2 - 2 \left(\sum\limits_{n=1}^{s} |a_n| + \sum\limits_{n=1}^{\infty} |b_n| \right) - \frac{c_{s+1}}{\delta} \sum\limits_{n=s+1}^{\infty} |a_n|}, \tag{2.6}$$

the last expression is bounded above by 1, if and only if

$$\sum_{n=1}^{s} |a_n| + \sum_{n=1}^{\infty} |b_n| + \frac{c_{s+1}}{\delta} \sum_{n=s+1}^{\infty} |a_n| \leq 1. \tag{2.7}$$

Then, it is sufficient to show that L.H.S. of (2.7) is bounded above by

$$\sum_{n=1}^{\infty} \frac{c_n}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n}{\delta} |b_n| \, ,$$

which is equivalent to the true inequality

$$\sum_{n=1}^{s} \frac{c_n - \delta}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n - \delta}{\delta} |b_n| + \sum_{n=s+1}^{\infty} \frac{c_n - c_{s+1}}{\delta} |a_n| \geq 0. \tag{2.8}$$

If we take

$$f(z) = z + \frac{\delta}{c_{s+1}} z^{-s-1}, \tag{2.9}$$

with $z = re^{\frac{i\pi}{s+2}}$ and let $r \to 1^+$, we obtain

$$\frac{f(z)}{f_s(z)} = 1 + \frac{\delta z^{-s-2}}{c_{s+1}} \to 1 - \frac{\delta}{c_{s+1}},$$

which shows that the bound in (2.1) is best possible.

(ii) Similarly, if we put

$$g_2(z) = \left(\frac{\delta + c_{s+1}}{\delta}\right)\left(\frac{f_s(z)}{f(z)} - \frac{c_{s+1}}{\delta + c_{s+1}}\right)$$

$$= 1 - \frac{\left(\frac{\delta + c_{s+1}}{\delta}\right)\left(\sum\limits_{n=s+1}^{\infty} |a_n|\, z^n + \sum\limits_{n=1}^{\infty} \overline{b_n z^{-n}}\right)}{z + \sum\limits_{n=1}^{\infty} |a_n|\, z^n + \sum\limits_{n=1}^{\infty} \overline{b_n z^{-n}}},$$

and make use of (2.3), we can deduce that

$$\left|\frac{g_2(z) - 1}{g_2(z) + 1}\right| \le \frac{\left(\frac{c_{s+1} + \delta}{\delta}\right)\left(\sum\limits_{n=s+1}^{\infty}|a_n| + \sum\limits_{n=1}^{\infty}|b_n|\right)}{2 - 2\left(\sum\limits_{n=1}^{\infty}|a_n| + \sum\limits_{n=1}^{\infty}|b_n|\right) - \left(\frac{c_{s+1} - \delta}{\delta}\right)\left(\sum\limits_{n=s+1}^{\infty}|a_n| + \sum\limits_{n=1}^{\infty}|b_n|\right)}. \tag{2.10}$$

This last expression is bounded above by 1, if and only if

$$\sum_{n=1}^{s} |a_n| + \sum_{n=1}^{\infty} |b_n| + \left(\frac{c_{s+1}}{\delta}\right) \sum_{n=s+1}^{\infty} |a_n| \le 1. \tag{2.11}$$

Since L.H.S. of (2.11) is bounded above by

$$\sum_{n=1}^{\infty} \frac{c_n}{\delta} |a_n| + \sum_{n=1}^{\infty} \frac{d_n}{\delta} |b_n|,$$

the bound in (2.2) follows and is sharp with the extremal function $f(z)$ given by (2.4). The proof of Theorem 1 is now complete.

Employing the techinques used in Theorem 1, we can prove the following theorems.

Theorem 2. *Let* $r \in \mathbb{N}$ *and let* $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. *Then*

(i) $\operatorname{Re}\left\{\frac{f(z)}{\widetilde{f}_r(z)}\right\} > 1 - \frac{\delta}{d_{r+1}} \quad (z \in U),$ (2.12)

and

(ii) $\operatorname{Re}\left\{\frac{\widetilde{f}_r(z)}{f(z)}\right\} > \frac{d_{r+1}}{\delta + d_{r+1}} \quad (z \in U),$ (2.13)

whenever

$$d_n \geq \begin{cases} \delta, & n = 2, 3, ..., r, \\ d_{r+1}, & n = r+1, r+2, ... \end{cases}$$

The estimates in (2.12) *and* (2.13) *are sharp for the function given by*

$$f(z) = z + \frac{\delta}{d_{r+1}} \overline{z}^{-r-1} \quad (z \in U^*). \tag{2.14}$$

Theorem 3. *Let* $s, r \in \mathbb{N}$ *and let* $f(z) = h(z) + \overline{g(z)} \in \Sigma_H$. *Then*

$$(i) \quad \operatorname{Re}\left\{\frac{f(z)}{f_{s,r}(z)}\right\} > 1 - \frac{\delta}{c_{s+1}} \quad (z \in U), \tag{2.15}$$

and

$$(ii) \quad \operatorname{Re}\left\{\frac{f_{s,r}(z)}{f(z)}\right\} > \frac{c_{s+1}}{\delta + c_{s+1}} \quad (z \in U), \tag{2.16}$$

whenever

$$\begin{aligned} c_n \quad &\geq \quad \begin{cases} \delta, & n = 2, 3, ..., s, \\ c_{s+1}, & n = s+1, s+2, ..., \end{cases} \\ d_n \quad &\geq \quad \begin{cases} \delta, & n = 2, 3, ..., s, \\ c_{s+1}, & n = s+1, s+2, ... \end{cases} \end{aligned} \tag{2.17}$$

Also,

$$(i) \quad \operatorname{Re}\left\{\frac{f(z)}{f_{s,r}(z)}\right\} > 1 - \frac{\delta}{d_{r+1}} \quad (z \in U), \tag{2.18}$$

and

$$(ii) \quad \operatorname{Re}\left\{\frac{f_{s,r}(z)}{f(z)}\right\} > \frac{d_{r+1}}{\delta + d_{r+1}} \quad (z \in U), \tag{2.19}$$

whenever

$$\begin{aligned} c_n \quad &\geq \quad \begin{cases} \delta, & n = 2, 3, ..., r, \\ d_{r+1}, & n = r+1, r+2, ..., \end{cases} \\ d_n \quad &\geq \quad \begin{cases} \delta, & n = 2, 3, ..., r, \\ d_{r+1}, & n = r+1, r+2, ... \end{cases} \end{aligned} \tag{2.20}$$

The estimates in (2.15), (2.16), (2.18) *and* (2.19) *respectively, are sharp for the function given by* (2.4) *and* (2.14), *respectively.*

Remark. By specializing the coefficients c_n, d_n and the parameters δ we obtain corresponding results for various subclasses mentioned in the introduction.

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Involutory BE–algebras

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ABSTRACT: This paper is devoted to the study of some structural properties of bounded and involutory BE–algebras and investigate the relationship between them. We construct a commutative monoid by definition of proper operation in an involutory BE–algebra. Some rules of calculus for BE–algebras with a semi-lattice structure are provided. Many results related to the natural order of a BE–algebras were found. Finally, we show that an involutory bounded BE–algebra X is semi-simple.

AMS Subject Classification: *06F35, 03G25*
Keywords and Phrases: *(bounded, involutory)BE–algebra, involution, semi-lattice, lattice, semi-simple.*

1 Introduction and Preliminaries

The study of BCK/BCI–algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic. There exist several generalization of BCK/BCI–algebras, such as BCH–algebras, d–algebras, B–algebras, BH–algebras, etc.

Especially, the notion of BE–algebras was introduced by H. S. Kim and Y. H. Kim [7], in which was deeply studied by S. S. Ahn and et. al., in [1, 2, 3], Walendziak in [15], A. Rezaei and et. al., in [12, 13, 14]. Lattice-valued logic is becoming a research filed strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. BE–algebras are important tools for certain investigations in algebraic logic since they can be consider as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "*true*". In this paper, we develop the theory BE–algebras with define a new structure as bounded and involutory BE–algebras and investigate the relationship between them and proved some theorems.

The paper has been organized in tree sections. In section 1, we give some definitions and some previous results and in section 2 we define bounded BE–algebras and define a congruence relation on this algebra with respect to a filter which this congruence relation allowed us to define a quotient algebra is also a bounded BE–algebra. In section 3 we discus on involutory BE–algebra because it is well known this structure has an important and vital role in investigating the structure of a logical system. Since quotient algebra is a basic tool for exploring the structures of algebras and there are close contacts among congruences and quotient algebras, we introduce a new congruence relation on X and construct quotient algebra via this congruence relation.

Definition 1.1. *[7] An algebra $(X;*,1)$ of type $(2,0)$ is called a BE–algebra if following axioms hold:*

$(BE1)$ $x*x=1$,

$(BE2)$ $x*1=1$,

$(BE3)$ $1*x=x$,

$(BE4)$ $x*(y*z)=y*(x*z)$, *for all* $x,y,z\in X$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$.

Proposition 1.2. *[7] Let X be a BE–algebra. Then*

(i) $x*(y*x)=1$,

(ii) $y*((y*x)*x)=1$, *for all* $x,y\in X$.

From now on, in this paper X is a BE–algebra, unless otherwise is stated.
A subset F of X is called a filter of X if $(F1)$ $1\in F$ and $(F2)$ $x\in F$ and $x*y\in F$ imply $y\in F$. We denote By $F(X)$ the set of all filters of X and $Max(X)$ the set of all maximal filters of X. Let A be a non-empty subset of X, then the set

$$<A>=\bigcap\{G\in F(X)|\ A\subseteq G\}$$

is called the filter generated by A, written $<A>$. If $A=\{a\}$, we will denote $<\{a\}>$, briefly by $<a>$, and we call it a principal filter of X. For $F\in F(X)$ and $a\in X$, we denote by F_a the filter generated by $F\cup\{a\}$. X is said to be self distributive if $x*(y*z)=(x*y)*(x*z)$, for all $x,y,z\in X$, (Example 8., [7]).

In a self distributive BE–algebra X, $F_a=\{x\in X: a*x\in F\}$,([3]). X is said to be transitive if $y*z\leq(x*y)*(x*z)$ for all $x,y,z\in X$, [1]. We say that X is commutative if $(x*y)*y=(y*x)*x$, for all $x,y\in X$. In [15], A. Walendziak, showed that every dual BCK–algebra is a BE–algebra and any commutative BE–algebra is a dual BCK–algebra.

We note that "$\leq$" is reflexive by $(BE1)$. If X is self distributive, then relation "$\leq$" is a transitive order set on X. Because if $x\leq y$ and $y\leq z$, then

$$x*z=1*(x*z)=(x*y)*(x*z)=x*(y*z)=x*1=1$$

and so $x \leq z$. If X is commutative, then by Proposition 3.3, [15], relation "$\leq$" is antisymmetric. Hence if X is a commutative self distributive BE–algebra, then "$\leq$" is a partial order set on X, (Example 3.4., [3]). We show that if I be an obstinate ideal of a self distributive BE–algebra X, then $(X/I; *, C_1)$ is also a BE-algebra, which is called to be the quotient algebra via I, and $C_1 = I$, (see Theorem 3.13, [12]).

Proposition 1.3. *[12] Let X be self distributive. If $x \leq y$, then*

(i) $z * x \leq z * y$ *and* $y * z \leq x * z$,

(ii) $y * z \leq (z * x) * (y * x)$, *for all* $x, y, z \in X$.

Theorem 1.4. *[13] A dual BCK–algebra X is commutative if and only if $(X; \leq)$ is an upper semi-lattice with $x \vee y = (y * x) * x$, for all $x, y \in X$.*

Proposition 1.5. *[13] Let X be a commutative BE–algebra. Then*

(i) *for each $a \in X$, the mapping $f_a : x \to x * a$ is an anti-tone involution on the section $[a, 1]$.*

(ii) *$(A, \leq)$ is a near-lattice with section anti-tone involutions and for every $a \in X$, the anti-tone involutions f_a on $[a, 1]$ is given by $f_a(x) = x * a$.*

Theorem 1.6. *[15, 13] Let X be commutative. Then it is a semi-lattice with respect to $\vee$.*

Definition 1.7. *[4] A filter F of X is called an obstinate filter if $x, y \notin F$ imply $x * y \in F$ and $y * x \in F$.*

Theorem 1.8. *[5] Let X be self distributive. $F \in F(X)$ and $F \neq X$. Then the following are equivalent:*

(i) *F is an obstinate filter,*

(ii) *if $x \notin F$, then $x * y \in F$, for all $y \in F$.*

2 On Bounded BE-algebras

Definition 2.1. *X is called bounded if there exists the smallest element 0 of X (i.e., $0 * x = 1$, for all $x \in X$).*

Example 2.2. (i). *The interval $[0, 1]$ of real numbers with the operation "$*$" defined by*

$$x * y = \min\{1 - x + y, 1\}, \ \ for \ all \ x, y \in X$$

is a bounded BE–algebra.

(ii). *Let $(X; *, 1)$ be a BE–algebra, $0 \notin X$ and $\bar{X} = X \cup \{0\}$. If we extensively define*

$$0 * x = 0 * 0 = 1 \ \ and \ \ x * 0 = 0 \ \ for \ all \ x \in X.$$

Then $(\bar{X};*,0,1)$ is a bounded BE–algebra with 0 as the smallest element.

(iii). Let $X := \{0,a,b,1\}$ be a set with the following table.

$*$	0	a	b	1
0	1	1	1	1
a	a	1	b	1
b	b	a	1	1
1	0	a	b	1

Then $(X;*,0,1)$ is a bounded BE–algebra with 0 as the smallest element.

(iv). Let $X := \{0,a,b,c,1\}$ be a set with the following table.

$*$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	a	1	c	1
c	0	1	b	1	1
1	0	a	b	c	1

Then $(X;*,0,1)$ is a bounded BE–algebra with 0 as the smallest element.

(v). Let $(X;\vee,\wedge,\neg,0,1)$ be a Boolean-lattice. Then $(X;*,1)$ is a bounded BE–algebra, where operation " $*$ " is defined by $x*y=(\neg x)\vee y$, for all $x,y\in X$.

Remark. The following example shows that the bounded BE–algebra is not a dual BCK–algebra and Hilbert algebra in general (see Definition 2.3, [15] and Definition 3.1, [14]).

Example 2.3. *Let $X := \{0,a,b,1\}$ be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	a	b	1

Then $(X;*,0,1)$ is a bounded BE–algebra with 0 as the smallest element but it is not a dual BCK–algebra, Hilbert algebra. Because

$$a*b=b*a=1 \quad \text{while} \quad a\neq b.$$

Also, it is not an implication algebra. Because

$$(a*b)*b=1*b=b\neq(b*a)*a=1*a=a.$$

Given a bounded BE–algebra X with 0 as the smallest element, we denote $x*0$ by Nx, then N can be regarded as a unary operation on X.

Proposition 2.4. *Let X be bounded with the smallest element 0. Then the following hold:*

(i) $N0 = 1$ *and* $N1 = 0$,

(ii) $x \leq NNx$,

(iii) $x * Ny = y * Nx$, *for all* $x, y \in X$.

Proof. (i). By $(BE1)$ and $(BE2)$ we have $N0 = 0 * 0 = 1$ and $N1 = 1 * 0 = 0$.

(ii). Since $x * (Nxx) = x * ((x * 0) * 0) = (x * 0) * (x * 0) = 1$, then $x \leq NNx$.

(iii). By $(BE4)$ we have $x * Ny = x * (y * 0) = y * (x * 0) = y * Nx$. □

Proposition 2.5. *Let* X *be a self distributive and bounded. Then*

(i) $y * x \leq Nx * Ny$,

(ii) $x \leq y$, *implies* $Ny \leq Nx$, *for all* $x, y \in X$.

Proof. (i). We have

$$\begin{aligned}(y * x) * (Nx * Ny) = Nx * ((y * x) * Ny) &= (x * 0) * ((y * x) * (y * 0)) \\ &= (x * 0) * (y * (x * 0)) \\ &= y * ((x * 0) * (x * 0)) \\ &= y * 1 = 1.\end{aligned}$$

Hence $y * x \leq Nx * Ny$.

(ii). By $(BE3)$ and assumption we have

$$\begin{aligned}Ny * Nx = (y * 0) * (x * 0) &= (y * 0) * (1 * (x * 0)) \\ &= (y * 0) * ((x * y) * (x * 0)) \\ &= (y * 0) * (x * (y * 0)) \\ &= x * ((y * 0) * (y * 0)) \\ &= x * 1 = 1.\end{aligned}$$

Hence $Ny \leq Nx$. □

In the following example we show that the self-distributivity condition in the above theorem is necessary.

Example 2.6. *Example 2.2(iii), is a bounded BE–algebra with* 0 *as the smallest element, while it is not self-distributive. Because*

$$b * (0 * a) = 2 * 1 = 1 \neq (b * 0) * (b * a) = b * a = a.$$

We can seen easily that, $b = a * b \nleq Nb * Na = b * a = a$.

Proposition 2.7. *Let* X *be bounded implicative self distributive. Then the following hold:*

(i) X *is commutative,*

(*ii*) $x = Nx * x$,

(*iii*) $x \vee y = y \vee x = Nx * y$, *for all* $x, y \in X$.

Proof. (*i*). See proof of Theorem 3.12([14]).

(*ii*). Assume that X is a bounded implicative. Then $Nx * x = (x * 0) * x = x$.

(*iii*). Let X be bounded implicative self distributive and $x, y \in X$. then by Proposition 1.3, $0 \leq y$ and $x * 0 \leq x * y$. Furthermore, by Propositions 1.2 and 1.3, we get

$$x \leq (x * y) * y \leq (x * 0) * y = Nx * y$$

Since by Proposition 1.2, $y \leq Nx * y$, then $Nx * y$ is an upper bound of x and y. Hence $x \vee y \leq Nx * y$. Also, we have

$$Nx * y \leq (y * x) * (Nx * x) = (y * x) * x.$$

Since X is commutative, then by Theorem 1.6, we have $(y * x) * x = x \vee y = y \vee x$ and so by Proposition 3.3([15]), the proof is complete. □

Corollary 2.8. *Let X be self distributive, $F \in F(X)$ and $F \neq X$. Then the following are equivalent:*

(*i*) *F is an obstinate filter,*

(*ii*) *if $x \notin F$, then $Nx \in F$.*

Definition 2.9. *Let X and Y be bounded. A homomorphism from X to Y is a function $f : X \to Y$ such that*

(*i*) $f(x * y) = f(x) * f(y)$,

(*ii*) $f(Nx) = N(f(x))$,

(*iii*) $f(0) = 0$, *for all* $x, y \in X$.

Example 2.10. *Consider X as Example 2.2(iii) and Y as Example 2.3. Define $f : X \to Y$ such that $f(1) = f(a) = f(b) = 1$ and $f(0) = 0$. Then f is a homomorphism.*

Theorem 2.11. *Let $f : X \to Y$ be a homomorphism. Then $\ker(f) = \{x \in X : f(x) = 1\}$ is a filter in X. Moreover, if $f(x) = f(y)$, then $x * y \in \ker(f)$ and $y * x \in \ker(f)$, for all $x, y \in X$. If Y is commutative, then the converse is valid.*

Proof. We have $f(1) = f(x * x) = f(x) * f(x) = 1$. Hence $1 \in \ker(f)$. Now, let $x \in \ker(f)$ and $x*y \in \ker(f)$. Then $f(x) = f(x*y) = 1$. But $f(x*y) = f(x)*f(y) = 1$. Hence $f(y) = 1 * f(y) = 1$. Therefore, $y \in \ker(f)$.

Now, let $f(x) = f(y)$. By using $(BE1)$, $f(x) * f(y) = 1$ and $f(y) * f(x) = 1$. But $1 = f(x) * f(y) = f(x * y)$ and $1 = f(y) * f(x) = f(y * x)$ implies $x * y \in \ker(f)$ and $y * x \in \ker(f)$.

Assume that Y is commutative, $x * y \in \ker(f)$ and $y * x \in \ker(f)$. Then $f(x * y) = f(y * x) = 1$ which implies that $f(x) * f(y) = f(y) * f(x) = 1$. Hence by Proposition 3.3([15]), $f(x) = f(y)$. □

Theorem 2.12. *Let X be bounded transitive, F be a filter and X/F be the corresponding quotient algebra. Then the map $f : X \to X/F$ which is defined by $f(a) = [a]$, for all $a \in X$, is a homomorphism and* $\ker(f) = F$.

Proof. By Propositions 5.4 and 5.7([11]), X/F is a quotient BE–algebra. Now, we have $f(0) = [0]$ and

$$f(Nx) = f(x * 0) = f(x) * f(0) = f(x) * [0] = N(f(x)).$$

Now, let $x \in \ker(f)$. Then $f(x) = [x] = [1]$ if and only if $1 = x * 1 \in F$ and $x = 1 * x \in F$ if and only if $1 \in F$ and $x \in F$. Therefore, $\ker(f) = F$. □

3 Involutory BE–algebras

If $NNx = x$, then x is called an involution of X. The smallest element 0 and the greatest element 1 are two involutions of X, because

$$NN0 = N(0 * 0) = N1 = 1 * 0 = 0,$$

$$NN1 = N(1 * 0) = N0 = 0 * 0 = 1.$$

Definition 3.1. *A bounded BE–algebra X is called involutory if any element of X is involution.*

Example 3.2. (i). *Examples 2.2(i), (iii), (v), are involutory.*
(ii). *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

Then $(X; *, 0, 1)$ is a bounded BE–algebra but it is not an involutory. Because

$$NNb = N(b * 0) = N0 = 0 * 0 = 1 \neq b.$$

(iii). *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	1	1	1
1	0	a	b	1

Then $(X; *, 0, 1)$ is an involutory BE–algebra but it is not an involutory dual BCK–algebra and involutory Hilbert algebra. Because

$$a * b = 1 \text{ and } b * a = 1 \text{ while, } a \neq b.$$

Also, it is not an involutory implication algebra. Because

$$(a * b) * b = 1 * b = b \neq (b * a) * a = 1 * a = a.$$

Proposition 3.3. *If X is a bounded commutative, then X is an involutory.*

Proof. By using the commutativity we get

$$NNx = (x * 0) * 0 = (0 * x) * x = 1 * x = x.$$

Hence X is an involutory. □

In the following example we show that the commutativity condition in the above theorem is necessary.

Example 3.4. *Example 3.2(ii), is not an involutory. Because it is not commutative.*

Proposition 3.5. *If X is an involutory, then*

(i) $x * y = Ny * Nx$,

(ii) $x \leq Ny$ *implies* $y \leq Nx$, *for all* $x, y \in X$.

Proof. (i). Since X is an involutory, then we have $NNx = x$, for all $x, y \in X$. Hence by Proposition 2.4(iii), $x * y = x * NNy = Ny * Nx$.

(ii). Since $x \leq Ny$, we get $x * Ny = 1$. Hence by Proposition 2.4(iii), $1 = x * Ny = y * Nx$. So, $y \leq Nx$. □

Lemma 3.6. *Let X be bounded self distributive and $x, y \in X$.*

(i) *if the smallest upper bound $x \vee y$ of x and y exists, then the greatest lower bound $Nx \wedge Ny$ of Nx and Ny exists and $Nx \wedge Ny = N(x \vee y)$.*

(ii) *if X is involutory and the greatest lower bound $x \wedge y$ exists, then the least upper bound $Nx \vee Ny$ exists and $Nx \vee Ny = N(x \wedge y)$.*

Proof. (i). Assume that the smallest upper bound $x \vee y$ of x and y exists. Since $x \leq x \vee y$, then by Proposition 1.3, $(x \vee y) * 0 \leq x * 0$, (i.e., $N(x \vee y) \leq Nx$). By the similar way $N(x \vee y) \leq Ny$. Hence $N(x \vee y)$ is a lower bound of Nx and Ny. Also, assume that u is any lower bound of Nx and Ny. Then $u \leq Nx$ and $u \leq Ny$. Hence by $(BE4)$, we have $x * (u * 0) = u * (x * 0) = u * Nx = 1$. Hence $x \leq Nu$ and by the similar way $y \leq Nu$. So, $x \vee y \leq Nu$. Now, by $(BE4)$, we have $(x \vee y) * (u * 0) = u * ((x \vee y) * 0) = 1$. So, $u \leq N(x \vee y)$. Hence $N(x \vee y)$ is a greatest lower bound of Nx and Ny. Therefore, the greatest lower bound $Nx \wedge Ny$ of Nx and Ny exists, and $Nx \wedge Ny = N(x \vee y)$.

(ii). Assume that $x \wedge y$ exists. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, then by Proposition 2.5, we have $N(x) \leq N(x \wedge y)$ and $N(y) \leq N(x \wedge y)$. Hence $N(x \wedge y)$ is an upper bound of Nx and Ny. Also, let u be any upper bound of Nx and Ny. Then $Nx \leq u$ and $Ny \leq u$. Since X is involutory, then by Proposition 2.5, we derive $Nu \leq NNx = x$ and $Nu \leq NNy = y$. So, $Nu \leq x \wedge y$. By Proposition 2.5, we have $N(x \wedge y) \leq NNu = u$. Hence $N(x \wedge y)$ is the smallest upper bound of Nx and Ny. Then the least upper bound $Nx \vee Ny$ exists, and $Nx \vee Ny = N(x \wedge y)$. □

Theorem 3.7. *Let X be involutory self distributive. Then the following are equivalent:*

(i) $(X;\leq)$ *is an upper semi-lattice,*

(ii) $(X;\leq)$ *is a lower semi-lattice,*

(iii) $(X;\leq)$ *is a lattice.*

Moreover, if $(X;\leq)$ is a lattice, then the following identities hold:

$$x \wedge y = N(Nx \vee Ny) \text{ and } x \vee y = N(Nx \wedge Ny).$$

Proof. $(i) \Rightarrow (ii)$. Since $(X;\leq)$ is an upper semi-lattice, then $Nx \vee Ny$ exists for all $x, y \in X$. By the first half part of Lemma 3.6, $NNx \wedge NNy$ exists. Also, since X is involutory, we have $NNx \wedge NNy = x \wedge y$. Then $x \wedge y$ exists. So, $(X;\leq)$ is a lower semi-lattice.

$(ii) \Rightarrow (iii)$. Since $(X;\leq)$ is a lower semi-lattice, $Nx \wedge Ny$ exists and using the second half part of Lemma 3.6, $NNx \vee NNy$ exists, for all $x, y \in X$. Also, since X is involutory, we have $NNx \vee NNy = x \vee y$. Then $x \vee y$ exists. So, $(X;\leq)$ is an upper semi-lattice.

$(iii) \Rightarrow (i)$. The proof is obvious.

Now, let $(X;\leq)$ is a lattice. Since by Lemma 3.6, X is involutory, then we have

$$x \wedge y = NNx \wedge NNy = N(Nx \vee Ny),$$

$$x \vee y = NNx \vee NNy = N(Nx \wedge Ny).$$

□

Theorem 3.8. *Let I be an obstinate ideal of involutory(bounded) self-distributive X. Then $(X/I;*,C_1)$ is involutory(bounded) self-distributive, too.*

Proof. By Theorems 3.13 and 3.16([12]), $(X/I;*,C_1)$ is a self-distributive BE–algebra. Let $x \in X$. Then $C_0 * C_x = C_{0*x} = C_1$. Hence X/I is a bounded BE–algebra. Now,

$$NNC_x = (C_x * C_0) * C_0 = C_{x*0} * C_0 = C_{(x*0)*0} = C_{NNx} = C_x.$$

Therefore, X/I is an involutory BE–algebra. □

Proposition 3.9. *Let X be involutory and operation "$\circ$" is defined on X by $x \circ y = Nx * y$, for all $x, y \in X$. Then $(X;\circ,0)$ is a commutative monoid.*

Proof. By Proposition 2.4(iii),

$$x \circ y = Nx * y = Nx * NNy = Ny * NNx = Ny * x = y \circ x$$

and so X is commutative. Now, by Proposition 2.4(iii), and ($BE4$) we have

$$\begin{aligned} x \circ (y \circ z) = Nx * (y \circ z) &= Nx * (z \circ y) \\ &= Nx * (Nz * y) \\ &= Nz * (Nx * y) \\ &= z \circ (Nx * y) \\ &= (Nx * y) \circ z \\ &= (x \circ y) \circ z. \end{aligned}$$

Hence " $\circ$ " is associative operation on X. Moreover, for any $x \in X$

$$x \circ 0 = Nx * 0 = NNx = x \ \ and \ \ 0 \circ x = N0 * x = 1 * x = x.$$

□

In the following example we show that the converse of the Proposition 3.9, is not valid in general.

Example 3.10. *Let $X := \{0, a, b, 1\}$ be a set with the following table.*

$*$	1	a
1	1	a
a	a	a

Then $(X; *, 1)$ is a commutative monoid, but it is not a BE–algebra. Because $a * a = a \neq 1$ and $a * 1 = a \neq 1$, (i.e., conditions ($BE1$) and ($BE2$) are not holds).

Lemma 3.11. *Let X be bounded. Then*

(i) *filter F of X is proper if and only if $0 \notin F$.*

(ii) *each proper filter F is contained in a maximal filter.*

Proof. (i). Let F be a proper filter of X and $0 \in F$. If $x \in X$, since $0 * x = 1 \in F$, which implies $x \in F$. Hence $X = F$, which is a contradiction. The converse is clear. (ii). The proof is obvious. □

Theorem 3.12. *Every bounded BE–algebra contains at least one maximal filter.*

Proof. Let X be a bounded BE–algebra. Since $\{1\}$ is a proper filter of X, then the proof is clear by Lemma 3.11. □

Definition 3.13. *Let X be bounded. Then the radical of X, written $Rad(X)$, is defined by*

$$Rad(X) = \cap\{F : \ F \in Max(X)\}.$$

In view of Theorem 3.12, $Rad(X)$ always exists for a bounded algebra X. Following a standard terminology in the contemporary algebra, we shall call an algebra X semi-simple if $Rad(X) = \{1\}$.

Example 3.14. *In Example 2.2(iv), $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, a, b, c\}$ and X are filters in X and F_3 is only maximal filter of X. Hence $Rad(X) = F_3$.*

Example 3.15. *In Example 2.2(iii), $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$ and X are filters in X and F_2, F_3 are maximal filters of X, also $F_2 \cap F_3 = \{1\}$. Hence $Rad(X) = \{1\}$ and therefore X is semi-simple.*

Lemma 3.16. *Let X be an involutory bounded BE–algebra. Then for every $x \in X$ with $x \neq 1$, there exists a maximal filter F of X such that $x \notin F$.*

Proof. Let $1 \neq x \in X$. We claim that $< Nx >$ is a proper filter of X. By contrary, if it is not, then $< Nx >= X$. Hence $0 \in< Nx >$ and therefore $Nx * 0 = NNx = 1$. Since X is involutory, then $x = NNx = 1$, which is a contradiction. By Lemma 3.11(ii), there is a maximal filter F of X such that $< Nx >\subseteq F$, and $x \notin F$. Suppose $x \in F$. Since $Nx = x * 0 \in F$, then $0 \in F$, which is contrary by Lemma 3.11(i). □

Theorem 3.17. *Let X be involutory and bounded. Then X is a semi-simple.*

Proof. By Lemma 3.16, the proof is clear. □

In this section we define a congruence relation "θ" on involutory bounded BE–algebra X and construct quotient algebra $(X/\theta; *, \theta_0, \theta_1)$ induced by the congruence relation "θ", where, we denote θ_x for the equivalence class $[x]$ containing x. Since "θ" is a congruence on X, then the operation " $*$ " on X/θ given by $\theta_x * \theta_y = \theta_{x*y}$ is well-defined, because "θ" satisfied of the substitution property. Then $(X/\theta; *, \theta_0, \theta_1)$ is an algebra of type $(2, 0, 0)$ where,

$$\theta_0 = \{x : N0 = Nx\} = \{x : Nx = 1\}$$

is the zero equivalence class containing 0 and

$$\theta_1 = \{x : N1 = Nx\} = \{x : Nx = 0\}$$

is the one equivalence class containing 1. Now, in the following theorem define and prove this results.

Theorem 3.18. *Let X be involutory and bounded. The relation "θ" defined on X by:*

$$(x, y) \in \theta \ \ if \ and \ only \ if \ \ Nx = Ny$$

is a congruence relation on X and the quotient algebra $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra.

Proof. It is clear that "θ" is an equivalence relation on X. Now, Let $(x, y) \in \theta$ and $(u, v) \in \theta$. Then $Nx = Ny$ and $Nu = Nv$. Hence,

$$Nx * Nu = (x * 0) * (u * 0) = u * ((x * 0) * 0) = u * NNx = u * x.$$

Thus $N(u * x) = N(Nx * Nu) = N(Ny * Nv) = N(v * y)$, and so $(u * x, v * y) \in \theta$. By the similarly way we have $(x * u, y * v) \in \theta$. Hence "$\theta$" is a congruence relation on X.

Let $\theta_x, \theta_y, \theta_z \in X/\theta$. Then
$(BE1)$ $\theta_x * \theta_x = \theta_{x*x} = \theta_1$,
$(BE2)$ $\theta_x * \theta_1 = \theta_{x*1} = \theta_1$,
$(BE3)$ $\theta_1 * \theta_x = \theta_{1*x} = \theta_x$,
$(BE4)$ $\theta_x * (\theta_y * \theta_z) = \theta_x * \theta_{y*z} = \theta_{x*(y*z)} = \theta_{y*(x*z)} = \theta_y * \theta_{x*z} = \theta_y * (\theta_x * \theta_z)$.

Now, since $\theta_0 * \theta_x = \theta_{0*x} = \theta_1$. Hence θ_0 is as the smallest element of X/θ. Also,

$$NN\theta_x = (\theta_x * \theta_0) * \theta_0 = \theta_{x*0} * \theta_0 = \theta_{(x*0)*0} = \theta_{NNx} = \theta_x.$$

Therefore, $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra. □

Example 3.19. *Consider Example 2.2(iv), $\theta_0 = \{0\}$ and $\theta_a = \theta_b = \theta_c = \theta_1 = \{a, b, c, 1\}$. Then $X/\theta = \{\theta_0, \theta_1\}$. Thus $(X/\theta; *, \theta_0, \theta_1)$ is an involutory bounded BE–algebra.*

Corollary 3.20. *Let X be an involutory bounded BE–algebra and $X_0 := \{Nx : x \in X\}$. Then $(X_0; *, N0)$ is a BE–algebra.*

Example 3.21. *In Example 2.2(ii), (iv), respectively, $X_0 = \{0, 1\}$ and $X_0 = \{0, a, b, 1\}$.*

Proposition 3.22. *Let X be involutory, bounded and self-distributive(commutative). Then X/θ is involutory, bounded and self-distributive(commutative), too.*

4 Conclusion and future research

In this paper, we introduced the notion of bounded and involutory BE–algebras and get some results. In addition, we have defined a congruence relation on involutory bounded BE–algebras and construct the quotient BE–algebra via this relations. In [10], J. Meng proved that implication algebras are dual to implicative BCK–algebras. Also R. Halaŝ in [9], showed commutative *Hilbert* algebras are implication algebras and A. Digo in [6], proved implication algebras are *Hilbert* algebras. Recently, A. Walendziak in [15], showed that an implication algebra is a BE–algebra and commutative BE–algebras are dual BCK–algebras. In [14], we showed that every *Hilbert* algebra is a self distributive BE–algebra and commutative self distributive BE–algebra is a *Hilbert* algebra. Then in the following diagram we summarize the results of this paper and we give the relations among such structures of involutory algebras.

"$A \to B$," means that A conclude B.

$$\begin{tikzcd}
 & \text{invo-}BE\text{–algebra} \arrow[dl, shift left] \arrow[d, shift left, "\text{imp-com}"] \arrow[dr, shift left] & \\
\text{invo-dual } BCK\text{–algebra} \arrow[ur, shift left, "\text{com}"] \arrow[r, shift left, "\text{imp}"] & \text{invo-implication algebra} \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & \text{invo- } Hilbert \text{ algebra} \arrow[l, shift left, "\text{com}"] \arrow[ul, shift left, "\text{self-com}"]
\end{tikzcd}$$

We think such results are very useful for study in this structure. In the future work we try assemble of calculus relative to different kinds of BE–algebras, as example, latticeal structure and Boolean lattices.

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On Generalised Quasi-ideals and Bi-ideals in Ternary Semigroups

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Abstract: In this paper, we introduce the notions of generalised quasi-ideals and generalised bi-ideals in a ternary semigroup. We also characterised these notions in terms of minimal quasi-ideals and minimal bi-ideals in a ternary semigroup.

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1 Introduction and Preliminaries

Good and Hughes[5] introduced the notion of bi-ideals and Steinfeld [2] introduced the notion of quasi-ideals in semigroups. In [1], Sioson studied the concept of quasi-ideals in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterised them by using the notion of quasi-ideals. In [7], Dixit and Dewan studied about the quasi-ideals and bi-ideals of ternary semigroups.

A nonempty set S with a ternary operation $S \times S \times S \longmapsto S$, written as $(x_1, x_2, x_3) \longmapsto [x_1x_2x_3]$ is called a ternary semigroup if it satisfies the following associative law: $[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$ for any $x_1, x_2, x_3, x_4, x_5 \in S$. In this paper, we denote $[x_1x_2x_3]$ by $x_1x_2x_3$.

A non-empty subset T of a ternary semigroup S is called a ternary subsemigroup if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$. A ternary subsemigroup I of a ternary semigroup S is called a left ideal of S if $SSI \subseteq I$, a lateral ideal if $SIS \subseteq I$, a right ideal of S if $ISS \subseteq I$, a two-sided ideal of S if I is both left and right ideal of S, and an ideal of S if I is a left, a right and a lateral ideal of S. An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$. Let S be a ternary semigroup. If there exists an element $0 \in S$ such that $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then "0" is called the zero element or simply the zero of the ternary semigroup S. In this case $S \cup \{0\}$ becomes a ternary semigroup with zero. For example, the set of all non-positive integers Z_0^{-1} forms a ternary semigroup with usual ternary multiplication and $'0'$ forms a ternary semigroup with zero element and also the zero element satisfy $(SS)^0S = S^0SS^0 = S(SS)^0 = S$. Throughout this paper S will always denote a ternary semigroup with zero. A ternary subsemigroup Q of a ternary semigroup S is called a quasi-ideal of S if $QSS \cap (SQS \cup SSQSS) \cap SSQ \subseteq Q$ and a ternary subsemigroup B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$. It is easy to see that every quasi-ideal in a ternary semigroup is a bi-ideal of S. An element a in a ternary semigroup S is called regular if there exists an element x in S such that $axa = a$. A ternary semigroup is called regular if all of its elements are regular. A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, lateral ideal M and left ideal L of S.

2 Generalised Quasi-ideals in Ternary Semigroup

In this section, we introduce the concept of generalised quasi-ideals in ternary semigroups and prove some results related to the same.

Definition 2.1. *A ternary subsemigroup Q of a ternary semigroup S is called a generalised quasi-ideal or $(m,(p,q),n)$-quasi-ideal of S if $Q(SS)^m \cap (S^pQS^q \cup S^pSQSS^q) \cap (SS)^nQ \subseteq Q$, where m,n,p,q are positive integers greater than 0 and $p+q = even$.*

Remark 2.1. *Every quasi-ideal of a ternary semigroup S is $(1,(1,1),1)$-quasi-ideal of S. But $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S need not be a quasi-ideal of S.*

Example 1. *Let $Z^- \setminus \{-1\}$ be the set of all negative integers excluding $\{0\}$. Then $Z^- \setminus \{-1\}$ is a ternary semigroup with usual ternary multiplication. Consider $Q = \{-3\} \cup \{k \in Z^- : k \leq -14\}$. Clearly Q is a non-empty ternary subsemigroup of S and also Q is $(2,(1,1),3)$-quasi-ideal of S. Now, $\{-12\} \in QSS \cap (SQS \cup SSQSS) \cap SSQ$. But $\{-12\} \notin Q$. Therefore $QSS \cap (SQS \cup SSQSS) \cap SSQ \not\subseteq Q$. Hence Q is not quasi-ideal of $Z^- \setminus \{-1\}$.*

Lemma 2.1. *Non-empty intersection of arbitrary collection of ternary subsemigroups of a ternary semigroup S is a ternary subsemigroup of S.*

Proof. Let T_i be a ternary subsemigroup of S for all $i \in I$ such that $\bigcap\limits_{i\in I} T_i \neq \emptyset$. Let $t_1, t_2, t_3 \in \bigcap\limits_{i\in I} T_i$. Then $t_1, t_2, t_3 \in T_i$ for all $i \in I$. Since T_i is a ternary subsemigroup of S for all $i \in I$, therefore $t_1t_2t_3 \in T_i$ for all $i \in I$. Therefore $t_1t_2t_3 \in \bigcap\limits_{i\in I} T_i$. Hence $\bigcap\limits_{i\in I} T_i$ is a ternary subsemigroup of S. □

Theorem 2.1. *Let S be a ternary semigroup and Q_i be an $(m,(p,q),n)$-quasi-ideal of S such that $\bigcap\limits_{i\in I} Q_i \neq \emptyset$. Then $\bigcap\limits_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Clearly $\bigcap\limits_{i\in I} Q_i$ is a ternary subsemigroup of S (by Lemma 2.1).

Let $x \in \left[\bigcap\limits_{i\in I} Q_i(SS)^m\right] \cap \left[S^p \bigcap\limits_{i\in I} Q_iS^q \cup S^pS \bigcap\limits_{i\in I} Q_iSS^q\right] \cap \left[(SS)^n \bigcap\limits_{i\in I} Q_i\right]$. Then $x \in$

$\bigcap_{i\in I} Q_i(SS)^m$, $x \in S^p \bigcap_{i\in I} Q_iS^q \cup S^pS \bigcap_{i\in I} Q_iSS^q$ and $x \in (SS)^n \bigcap_{i\in I} Q_i$. This implies $x \in Q_i(SS)^m$, $x \in [S^pQ_iS^q \cup S^pSQ_iSS^q]$ and $x \in (SS)^nQ_i$ for all $i \in I$. Therefore $x \in [Q_i(SS)^m] \cap [S^pQ_iS^q \cup S^pSQ_iSS^q] \cap [(SS)^nQ_i] \subseteq Q_i$ for all $i \in I$, since Q_i is an $(m,(p,q),n)$-quasi-ideal of S. Thus $x \in Q_i$ for all $i \in I$. Therefore $x \in \bigcap_{i\in I} Q_i$. Hence $\bigcap_{i\in I} Q_i$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Remark 2.2. *Let Z^- be the set of all negative integers under ternary multiplication and $Q_i = \{k \in Z^- : k \leq -i\}$ for all $i \in I$. Then Q_i is an $(2,(1,1),3)$-quasi-ideal of Z^- for all $i \in I$. But $\bigcap_{i\in I} Q_i = \emptyset$. So condition $\bigcap_{i\in I} Q_i \neq \emptyset$ is necessary.*

Definition 2.2. *Let S be a ternary semigroup. Then a ternary subsemigroup*

(i) R of S is called an m-right ideal of S if $R(SS)^m \subseteq R$.
(ii) M of S is called an (p,q)-lateral ideal of S if $S^pMS^q \cup S^pSMSS^q \subseteq M$,
(iii) L of S is called an n-left ideal of S if $(SS)^nL \subseteq L$,

where m,n,p,q are positive integers and $p+q$ is an even positive integer.

Theorem 2.2. *Every m-right, (p,q)-lateral and n-left ideal of a ternary semigroup S is an $(m,(p,q),n)$-quasi-ideal of S. But converse need not be true.*

Proof. One way is straight forward. Conversely, let $S = M_2(Z_0^-)$ be the ternary semigroup of 2×2 square matrices over Z_0^-. Consider $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^- \right\}$. Then Q is an $(2,(1,1),3)$-quasi-ideal of S. But it is not 2-right ideal, $(1,1)$-lateral ideal and 3-left ideal of S. □

Theorem 2.3. *Let S be a ternary semigroup. Then the following statements hold:*

(i) *Let R_i be an m-right ideal of S such that $\bigcap_{i\in I} R_i \neq \emptyset$. Then $\bigcap_{i\in I} R_i$ is an m-right ideal of S.*
(ii) *Let M_i be an (p,q)-lateral ideal of S such that $\bigcap_{i\in I} M_i \neq \emptyset$. Then $\bigcap_{i\in I} M_i$ is an (p,q)-lateral ideal of S.*
(iii) *Let L_i be an n-left ideal of S such that $\bigcap_{i\in I} L_i \neq \emptyset$. Then $\bigcap_{i\in I} L_i$ is an n-left ideal of S.*

Proof. Similar to the proof of Theorem 2.1 □

Theorem 2.4. *Let R be an m-right ideal, M be an (p,q)-lateral ideal and L be an n-left ideal of a ternary semigroup S. Then $R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Suppose $Q = R \cap M \cap L$. Since every m-right, (p,q)-lateral and n-left ideal of ternary semigroup S is an $(m,(p,q),n)$-quasi-ideal of S, therefore R, M and L are $(m,(p,q),n)$-quasi-ideals of S. Clearly, $R \cap M \cap L$ is non-empty. By Theorem 2.1, we have $Q = R \cap M \cap L$ is an $(m,(p,q),n)$-quasi-ideal of S. □

Lemma 2.2. *Let Q be an $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S. Then*

(i) $R = Q \cup Q(SS)^m$ *is an m-right ideal of S.*
(ii) $M = Q \cup (S^pQS^q \cup S^pSQSS^q)$ *is an (p,q)-lateral ideal of S.*
(iii) $L = Q \cup (SS)^nQ$ *is an n-left ideal of S.*

Proof. It is easy to show that R is ternary subsemigroup of S. Now to show that R is an m-right ideal of S.

$$\begin{aligned} R(SS)^m &= [(Q \cup Q(SS)^m](SS)^m \\ &= Q(SS)^m \cup Q(SS)^m(SS)^m \\ &= Q(SS)^m \cup Q(SSSS)^m \\ &\subseteq Q(SS)^m \cup Q(SS)^m \\ &= Q(SS)^m \subseteq R. \end{aligned}$$

Therefore R is an m-right ideal of S. Similarly, we can show that M is an (p,q)-lateral ideal of S and L is an n-left ideal of S. □

Theorem 2.5. *Every $(m,(p,q),n)$-quasi-ideal in a regular ternary semigroup S is the intersection of m-right, (p,q)-lateral and n-left ideal of S.*

Proof. Let S be regular ternary semigroup and Q be an $(m,(p,q),n)$-quasi-ideal of S. Then $R = Q \cup Q(SS)^m$, $M = Q \cup (S^pQS^q \cup S^pSQSS^q)$ and $L = Q \cup (SS)^nQ$ are m-right, (p,q)- lateral and n-left ideal of S respectively. Clearly $Q \subseteq R$, $Q \subseteq M$ and $Q \subseteq L$ implies $Q \subseteq R \cap M \cap L$. Since S is regular therefore $Q \subseteq Q(SS)^m$, $Q \subseteq S^pQS^q \cup S^pSQSS^q$ and $Q \subseteq (SS)^nQ$.
Thus $R = Q(SS)^m$, $M = S^pQS^q \cup S^pSQSS^q$ and $L = (SS)^nQ$. Now

$$R \cap M \cap L = Q(SS)^m \cap (S^pQS^q \cup S^pSQSS^q) \cap (SS)^nQ \subseteq Q$$

Hence, $Q = R \cap M \cap L$. □

3 Generalised Minimal Quasi-ideals

In this section, we study the concept of generalised minimal quasi-ideal or minimal $(m,(p,q),n)$-quasi-ideals of ternary semigroup S.

An $(m,(p,q),n)$-quasi-ideal Q of a ternary semigroup S is called minimal $(m,(p,q),n)$-quasi-ideal of S if Q does not properly contain any $(m,(p,q),n)$-quasi-ideal of S. Similarly, we can define minimal m-right ideals, minimal (p,q)-lateral ideals and minimal n-left ideals of a ternary semigroup.

Lemma 3.1. *Let S be a ternary semigroup and $a \in S$. Then the following statements hold:*

(i) $a(SS)^m$ *is an m-right ideal of S.*
(ii) $(S^paS^q \cup S^pSaSS^q)$ *is an (p,q)-lateral ideal of S.*

(iii) $(SS)^n a$ *is an* n*-left ideal of* S.

(iv) $a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a$ *is an* $(m,(p,q),n)$*-quasi-ideal of* S.

Proof. (i), (ii) and (iii) are obvious. (iv) follows from (i), (ii), (iii) and Theorem 2.4. □

Theorem 3.1. *Let* S *be a ternary semigroup and* Q *be an* $(m,(p,q),n)$*-quasi-ideal of* S. *Then* Q *is minimal iff* Q *is the intersection of some minimal* m*-right ideal* R, *minimal* (p,q)*-lateral ideal* M *and minimal* n*-left ideal* L *of* S.

Proof. Suppose Q is minimal $(m,(p,q),n)$-quasi-ideal of S. Let $a \in Q$. Then by above Lemma, we have $a(SS)^m$ is an m-right ideal, $(S^p a S^q \cup S^p S a S S^q)$ is an (p,q)-lateral ideal, $(SS)^n a$ is an n-left ideal and $a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a$ is an $(m,(p,q),n)$-quasi-ideal of S. Now,

$$\begin{aligned}
&a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a \\
&\quad \subseteq Q(SS)^m \cap (S^p Q S^q \cup S^p S Q S S^q) \cap (SS)^n Q \\
&\quad \subseteq Q.
\end{aligned}$$

Since Q is minimal therefore $a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a = Q$.
Now, to show that $a(SS)^m$ is minimal m-right ideal of S. Let R be an m-right ideal of S contained in $a(SS)^m$. Then

$$\begin{aligned}
&R \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a \\
&\quad \subseteq a(SS)^m \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a \\
&\quad = Q\,.
\end{aligned}$$

Since $R \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a$ is an $(m,(p,q),n)$-quasi-ideal of S and Q is minimal, therefore $R \cap (S^p a S^q \cup S^p S a S S^q) \cap (SS)^n a = Q$. This implies $Q \subseteq R$ and therefore

$$a(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$$

implies $R = a(SS)^m$. Thus m-right ideal $a(SS)^m$ is minimal. Similarly, we can prove that $(S^p a S^q \cup S^p S a S S^q)$ is minimal (p,q)-lateral ideal of S and $(SS)^n a$ is minimal n-left ideal of S.
Conversely, assume that $Q = R \cap M \cap L$ for some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L. So, $Q \subseteq R, Q \subseteq M$ and $Q \subseteq L$. Let Q' be an $(m,(p,q),n)$-quasi-ideal of S contained in Q. Then $Q'(SS)^m \subseteq Q(SS)^m \subseteq R(SS)^m \subseteq R$. Similarly, $(S^p Q' S^q \cup S^p S Q' S S^q) \subseteq M$ and $(SS)^n Q' \subseteq (SS)^n Q \subseteq L$.
Now $Q'(SS)^m$ is an m-right ideal of S, as $Q'(SS)^m(SS)^m \subseteq Q'(SS)^m$. Similarly, $(S^p Q' S^q \cup S^p S Q' S S^q)$ is an (p,q)-lateral ideal of S and $(SS)^n Q'$ is an n-left ideal of S. Since R, M and L are minimal m-right ideal, minimal (p,q)-lateral ideal and minimal n-left ideal of S respectively, therefore $Q'(SS)^m = R, S^p Q' S^q \cup S^p S Q' S S^q = M$ and $(SS)^n Q' = L$.
Thus $Q = R \cap M \cap L = Q'(SS)^m \cap (S^p Q' S^q \cup S^p S Q' S S^q) \cap (SS)^n Q' \subseteq Q'$. Hence $Q = Q'$. Thus Q is minimal $(m,(p,q),n)$-quasi-ideal of S. □

Note. *A ternary semigroup S need not contains a minimal $(m,(p,q),n)$-quasi-ideal of S.*
For example, let Z^- be the set of all negative integers. Then Z^- is a ternary semigroup with usual ternary multiplication. Let $Q = \{-2,-3,-4,\ldots\}$. Then Q is an $(2,(1,1),3)$-quasi-ideal of Z^-. Suppose Q is minimal $(2,(1,1),3)$-quasi-ideal of Z^-. Let $Q' = Q \setminus \{-2\}$. Then we can easily show that Q' is an $(2,(1,1),3)$-quasi-ideal of Z^-. But Q' is proper subset of Q. This is contradiction. Hence, Z^- does not contain a minimal $(m,(p,q),n)$-quasi-ideal.

Theorem 3.2. *Let S be a ternary semigroup. Then the following holds:*

(i) *An m-right ideal R is minimal iff $a(SS)^m = R$ for all $a \in R$.*
(ii) *An (p,q)-lateral ideal M is minimal iff $(S^p aS^q \cup S^p SaSS^q) = M$ for all $a \in M$.*
(iii) *An n-left ideal L is minimal iff $(SS)^n a = L$ for all $a \in L$.*
(iv) *An $(m,(p,q),n)$-quasi-ideal Q is minimal iff $a(SS)^m \cap (S^p aS^q \cup S^p SaSS^q) \cap (SS)^n a = Q$ for all $a \in Q$.*

Proof. (i) Suppose m-right ideal R is minimal. Let $a \in R$. Then $a(SS)^m \subseteq R(SS)^m \subseteq R$. By Lemma 3.1, we have $a(SS)^m$ is an m-right ideal of S. Since R is minimal m-right ideal of S therefore $a(SS)^m = R$.
Conversely, Suppose that $a(SS)^m = R$ for all $a \in R$. Let R' be an m-right ideal of S contained in R. Let $x \in R'$. Then $x \in R$. By assumption, we have $x(SS)^m = R$ for all $x \in R$. $R = x(SS)^m \subseteq R'(SS)^m \subseteq R'$. This implies $R \subseteq R'$. Thus, $R = R'$. Hence, R is minimal m-right ideal.
Similarly we can prove (ii), (iii) and (iv). □

4 Generalised Bi-ideals in Ternary Semigroup

In this section, we define generalised bi-ideals in a ternary semigroup and give their characterizations.

Definition 4.1. *A ternary subsemigroup B of a ternary semigroup S is called a generalised bi-ideal or $(m,(p,q),n)$ bi-ideal of S if $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B$, where m,n,p,q are positive integers greater than zero and p and q are odd.*

Remark. *Every bi-ideal of a ternary semigroup S is $(1,(1,1),1)$-bi-ideal of S. But every $(m,(p,q),n)$-bi-ideal of a ternary semigroup S need not be a bi-ideal of S which is illustrated by the following example.*

Example 2. *Let $Z^- \setminus \{-1\}$ be the set of all negative integers excluding $\{0\}$. Then $Z^- \setminus \{-1\}$ is a ternary semigroup with usual ternary multiplication. Consider $B = \{-3,-27\} \cup \{k \in Z^- : k \leq -110\}$. Clearly B is a non-empty ternary subsemigroup of S and also B is $(3,(1,1),4)$-bi-ideal of S. Now $-108 \in BSBSB$. But $-108 \notin B$. Therefore $BSBSB \nsubseteq B$. Hence B is not a bi-ideal of $Z^- \setminus \{-1\}$.*

Theorem 4.1. *Let S be a ternary semigroup and B_i be an $(m,(p,q),n)$-bi-ideals of S such that $\bigcap_{i\in I} B_i \neq \emptyset$. Then $\bigcap_{i\in I} B_i$ is an $(m,(p,q),n)$ bi-ideal of S.*

Proof. It is straight forward. □

Remark. *Let Z^- be the set of all negative integers. Then Z^- is a ternary semigroup under usual ternary multiplication and $B_i = \{k \in Z^- : k \leq -i\}$ for all $i \in I$. Then B_i is an $(3,(1,1),4)$-bi-ideal of Z^- for all $i \in I$. But $\bigcap_{i\in I} B_i = \emptyset$. So condition $\bigcap_{i\in I} B_i \neq \emptyset$ is necessary.*

Theorem 4.2. *Every $(m,(p,q),n)$-quasi-ideal of a ternary semigroup S is an $(m,(p,q),n)$-bi-ideal of S.*

Proof. Let Q be an $(m,(p,q),n)$-quasi-ideal of S. Then

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq Q(SS)^{m-1}S^pSS^q(SS)^{n-1}S \subseteq Q(SS)^m.$$

Similarly,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S(SS)^{m-1}(S^pQS^q)(SS)^{n-1}S \subseteq S^{p+1}QS^{q+1}.$$

Again $\{0\} \subseteq S^pQS^q$. So

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S^pQS^q \cup S^{p+1}QS^{q+1}.$$

Also,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq S(SS)^{m-1}S^pSS^q(SS)^{n-1}Q \subseteq (SS)^nQ.$$

Consequently,

$$Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q \subseteq Q(SS)^m \cap (S^pQS^q \cup S^{p+1}QS^{q+1}) \cap (SS)^nQ \subseteq Q.$$

Hence Q is an $(m,(p,q),n)$-bi-ideal of S. □

Remark. *Every $(m,(p,q),n)$-bi-ideal need not be an $(m,(p,q),n)$-quasi-ideal of S which is illustrated by the following example.*

Example 3. *Consider the ternary semigroup $S{=}Z^- \setminus \{-1\}$ with usual ternary multiplication and let $B = \{-3,-27\} \cup \{k \in Z^- : k \leq -194\}$. Clearly, B is non-empty ternary subsemigroup of S and also B is $(2,(1,1),3)$-bi-ideal of S. Now, $-192 \in B(SS)^2 \cap (SBS \cup SSBSS) \cap (SS)^3B$. But $-192 \notin B$. Therefore $B(SS)^2 \cap (SBS \cup SSBSS) \cap (SS)^3B \nsubseteq B$. Hence B is not $(2,(1,1),3)$-quasi-ideal of S.*

Theorem 4.3. *A ternary subsemigroup B of a regular ternary semigroup S is an $(m,(p,q),n)$-bi-ideal of S if and only if $B = BSB$.*

Proof. Suppose B is an $(m,(p,q),n)$-bi-ideal of a regular ternary semigroup S. Let $b \in B$. Then there exists $x \in S$ such that $b = bxb$. This implies that $b \in BSB$. Hence $B \subseteq BSB$. Now,

$$BSB \subseteq BSBSBSBSB \subseteq B(SS)(SBS)(SS)B \subseteq B.$$

Therefore $B = BSB$.

Conversely, if $B = BSB$, then

$$B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B(SS)^{m-1}S^pSS^q(SS)^{n-1}B \subseteq BSB = B.$$

Hence B is an $(m,(p,q),n)$-bi-ideal of S. □

Theorem 4.4. *Let S be a regular ternary semigroup. Then every $(m,(p,q),n)$-bi-ideal of S is an $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Let B be an $(m,(p,q),n)$-bi-ideal of S. Let $a \in B(SS)^m \cap (S^pBS^q \cup S^pSBSS^q) \cap (SS)^nB$. Then $a \in B(SS)^m, a \in (S^pBS^q \cup S^pSBSS^q)$ and $a \in (SS)^nB$. Thus $a = b(SS)^m = S^pb'S^q \cup S^pSb''SS^q = (SS)^nb'''$ for some $b, b', b'', b''' \in B$. Since S is regular, therefore for $a \in S$ there exists an element x in S such that $a = axa$. Then

$$\begin{aligned} a = axa &= axaxa \\ &= b(SS)^m x(S^pb'S^q \cup S^pSb''SS^q)x(SS)^nb''' \\ &\in B(SS)^mS(S^pBS^q \cup S^pSBSS^q)S(SS)^nB \\ &= [B(SS)^mSS^pBS^qS(SS)^nB] \cup [B(SS)^mSS^pSBSS^qS(SS)^nB] \\ &\subseteq B[(SS)^mSS^pSS^qS(SS)^n]B \cup B[(SS)^mSS^pSSSS^qS(SS)^n]B \\ &\subseteq BSB \cup BSB = B \cup B = B. \end{aligned}$$

Thus $a \in B$. Therefore $B(SS)^m \cap (S^pBS^q \cup S^pSBSS^q) \cap (SS)^nB \subseteq B$. Hence B is an $(m,(p,q),n)$-quasi-ideal of S. □

It is easy to prove the following propositions:

Proposition 4.5. *The intersection of an $(m,(p,q),n)$-bi-ideal B of a ternary semigroup S with a ternary subsemigroup T of S is either empty or an $(m,(p,q),n)$-bi-ideal of T.*

Proposition 4.6. *Let B be an $(m,(p,q),n)$-bi-ideal of a ternary semigroup S and T_1, T_2 are two ternary subsemigroups of S. Then BT_1T_2, T_1BT_2 and T_1T_2B are $(m,(p,q),n)$-bi-ideals of S.*

Proposition 4.7. *Let B_1, B_2 and B_3 are three $(m,(p,q),n)$-bi-ideals of a ternary semigroup S. Then $B_1B_2B_3$ is an $(m,(p,q),n)$-bi-ideal of S.*

Proposition 4.8. *Let Q_1, Q_2 and Q_3 are three $(m,(p,q),n)$-quasi-ideals of a ternary semigroup S. Then $Q_1Q_2Q_3$ is an $(m,(p,q),n)$-bi-ideal of S.*

Proposition 4.9. *Let R be an m-right, M be an (p,q)-lateral and L be an n-left ideal of a ternary semigroup S. Then the ternary subsemigroup $B = RML$ of S is an $(m,(p,q),n)$-bi-ideal of S.*

Theorem 4.10. *Let S be a regular ternary semigroup. If B is an $(m,(p,q),n)$-bi-ideal of S, then $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B = B$.*

Proof. Let B be an $(m,(p,q),n)$-bi-ideal of S. Let $a \in B$. Then $a \in S$. Since S is regular, therefore there exists $x \in S$ such that $a = axa$. Now $a = axa = a(xa)(xax)(ax)a \in B(SS)(SBS)(SS)B$. Similarly, by property of regularity it is easy to show that $a \in B(SS)^{m-1}S^pBS^q(SS)^{n-1}B$. Thus, $B \subseteq B(SS)^{m-1}S^pBS^q(SS)^{n-1}B$. Since B is an $(m,(p,q),n)$-bi-ideal of S, therefore $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B$. Hence $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B = B$ □

Corollary 4.1. *Let S be a regular ternary semigroup. If Q is an $(m,(p,q),n)$-quasi-ideal of S, then $Q(SS)^{m-1}S^pQS^q(SS)^{n-1}Q = Q$.*

Proof. Since every $(m,(p,q),n)$-quasi-ideal of S is an $(m,(p,q),n)$-bi-ideal of S, therefore result follows directly. □

5 Generalised Minimal Bi-ideals

In this section, we introduce the concept of generalised minimal bi-ideal or minimal $(m,(p,q),n)$-bi-ideals in ternary semigroups.

Definition 5.1. *An $(m,(p,q),n)$-bi-ideal B of a ternary semigroup S is called minimal $(m,(p,q),n)$-bi-ideal of S if B does not properly contain any $(m,(p,q),n)$-bi-ideal of S.*

Lemma 5.1. *Let S be a ternary semigroup and $a \in S$. Then the following holds:*

(i) *$a(SS)^{m-1}$ is an m-right ideal of S.*
(ii) *S^paS^q is an (p,q)-lateral ideal of S.*
(iii) *$(SS)^{n-1}a$ is an n-left ideal of S.*
(iv) *$a(SS)^{m-1}S^paS^q\ (SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal.*

Proof. (i), (ii) and (iii) are obvious and (iv) follows from (i), (ii), (iii). □

Theorem 5.1. *Let S be a ternary semigroup and B be an $(m,(p,q),n)$-bi-ideal of S. Then B is minimal if and only if B is the product of some minimal m-right ideal R, minimal (p,q)-lateral ideal M and minimal n-left ideal L of S.*

Proof. Suppose B is minimal $(m,(p,q),n)$-bi-ideal of S. Let $a \in B$. Then by above Lemma, $a(SS)^{m-1}$ is an m-right ideal, S^paS^q is an (p,q)-lateral ideal, $(SS)^{n-1}a$ is an n-left ideal and $a(SS)^{m-1}S^paS^q(SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal of S. Now $a(SS)^{m-1}S^paS^q(SS)^{n-1}a \subseteq B(SS)^{m-1}S^pB\ S^q(SS)^{n-1}B \subseteq B$. Since B is minimal, therefore $a(SS)^{m-1}S^paS^q(SS)^{n-1}a = B$. Now to show that $a(SS)^{m-1}$ is minimal m-right ideal of S. Let R be an m-right ideal of S contained in $a(SS)^{m-1}$. Then $R(S^paS^q)(SS)^{n-1}a \subseteq a(SS)^{m-1}(S^paS^q)(SS)^{n-1}a = B$. Since $R\ S^paS^q(SS)^{n-1}a$ is an $(m,(p,q),n)$-bi-ideal of S and B is minimal, therefore $R(S^paS^q)(SS)^{n-1}a = B$. This implies $B \subseteq R$. Therefore $a(SS)^{m-1} \subseteq B(SS)^{m-1} \subseteq R(SS)^{m-1} \subseteq R$. Thus $a(SS)^{m-1}$ is minimal. Similarly we can prove that S^paS^q is minimal (p,q)-lateral ideal of S and $(SS)^{n-1}a$ is minimal n-left ideal of S.

Conversely, assume that $B = RML$ for some minimal m-right ideal R, minimal (p, q)-lateral ideal M and minimal n-left ideal L. So $B \subseteq R$, $B \subseteq M$ and $B \subseteq L$. Let B' be an $(m, (p, q), n)$-bi-ideal of S contained in B. Then $B'(SS)^{m-1} \subseteq B(SS)^{m-1} \subseteq R(SS)^{m-1} \subseteq R$. Similarly, $S^pB'S^q \subseteq S^pBS^q \subseteq S^pMS^q \subseteq M$ and $(SS)^{n-1}B' \subseteq (SS)^{n-1}B \subseteq (SS)^{n-1}L \subseteq L$. Now, $B'(SS)^{m-1}(SS)^m \subseteq B'(SS)^{m-1}$. So $B'(SS)^{m-1}$ is an m-right ideal of S. Similarly $S^pB'S^q$ is an (p, q)-lateral ideal and $(SS)^{n-1}B'$ is an n-left ideal of S. Since R, M and L are minimal m-right ideal, minimal (p, q)-lateral ideal and minimal n-left ideal of S respectively, therefore $B'(SS)^{m-1} = R$, $S^pB'S^q = M$ and $(SS)^{n-1}B' = L$. Thus $B = RML = B'(SS)^{m-1}S^pB'S^q(SS)^{n-1}B' \subseteq B'$. Hence $B = B'$. Consequently, B is minimal $(m, (p, q), n)$-bi-ideal of S. □

Definition 5.2. *Let S be a ternary semigroup. Then S is called a bi-simple ternary semigroup if S is the unique $(m, (p, q), n)$-bi-ideal of S.*

Theorem 5.2. *Let S be a ternary semigroup and B be an $(m, (p, q), n)$-bi-ideal of S. Then B is a minimal $(m, (p, q), n)$-bi-ideal of S if and if B is a bi-simple ternary semigroup.*

Proof. Suppose B is a minimal $(m, (p, q), n)$-bi-ideal of S. Let C be an $(m, (p, q), n)$-bi-ideal of B. Then $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C \subseteq B$. By Proposition 4.9, BCC is an $(m, (p, q), n)$-bi-ideal of S. Therefore

$(BCC)(SS)^{m-1}S^p(BCC)S^q(SS)^{n-1}BCC \subseteq BCC \subseteq BBB \subseteq B$. Since B is minimal, therefore $BCC = B$. It is easy to show that $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C$ is an $(m, (p, q), n)$-bi-ideal of S.

Since B is minimal, therefore $C(BB)^{m-1}B^pCB^q(BB)^{n-1}C = B$. This implies $B = C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C$. Hence $C = B$. Consequently, B is a bi-simple ternary semigroup.

Conversely, suppose B is a bi-simple ternary semigroup. Let C be an $(m, (p, q), n)$-bi-ideal of S such that $C \subseteq B$. Then

$C(BB)^{m-1}B^pCB^q(BB)^{n-1}C \subseteq C(SS)^{m-1}S^pCS^q(SS)^{n-1}C \subseteq C$

which implies that C is an $(m, (p, q), n)$-bi-ideal of B. Since B is bi-simple ternary semigroup, therefore $C = B$. Hence B is minimal. □

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Remarkable identities

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Abstract: In the paper a number of identities involving even powers of the values of functions tangent, cotangent, secans and cosecans are proved. Namely, the following relations are shown:

$$\begin{aligned} \sum_{j=1}^{m-1} f^{2n}\left(\frac{\pi j}{2m}\right) &= w_f(m), \\ \sum_{j=0}^{m-1} f^{2n}\left(\frac{2j+1}{4m}\pi\right) &= v_f(m), \\ \sum_{j=1}^{m} f^{2n}\left(\frac{\pi j}{2m+1}\right) &= \tilde{w}_f(m), \end{aligned}$$

where m, n are positive integers, f is one of the functions: tangent, cotangent, secans or cosecans and $w_f(x)$, $v_f(x)$, $\tilde{w}_f(x)$ are some polynomials from $\mathbb{Q}[x]$.

One of the remarkable identities is the following:

$$\sum_{j=0}^{m-1} \sin^{-2}\frac{(2j+1)\pi}{2m} = m^2, \qquad \text{provided } m \geq 1.$$

Some of these identities are used to find, by elementary means, the sums of the series of the form $\sum_{j=1}^{\infty}\frac{1}{j^{2n}}$, where n is a fixed positive integer. One can also notice that Bernoulli numbers appear in the leading coefficients of the polynomials $w_f(x)$, $v_f(x)$ and $\tilde{w}_f(x)$.

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In [7] the following formulas have been proved

$$\sum_{j=1}^{m} \cot^2\frac{\pi j}{2m+1} = \frac{m(2m-1)}{3}, \qquad \sum_{j=1}^{m} \sin^{-2}\frac{\pi j}{2m+1} = \frac{2m(m+1)}{3}, \tag{1}$$

where $m \in \mathbb{N}_1$. By $\mathbb{N}_k$ for a positive integer k we mean $\mathbb{N} \setminus \{0, 1, 2 \ldots, k-1\}$. The above identities were then used in an elementary proof of the formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

In this paper we develop the ideas from [7] to prove more generalized identities than (1). Next we use some of them to find the sum of $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$, where $n \in \mathbb{N}_1$. The general identities given in this article yield, in particular, the following identity of uncommon beauty

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{2j+1}{2m}\pi = m^2, \qquad m \in \mathbb{N}_1.$$

Some elementary methods of finding the sums of the series of the form $\sum_{j=1}^{\infty} \frac{1}{j^{2n}}$ may be found for example in [1], [3], [5], [6], [8].

We start by recalling some basic facts on symmetric polynomials in m variables. Put

$$\sigma_n = \sum_{j=1}^{m} x_j^n \qquad \text{for } n \in \mathbb{N}_1,$$

$$\tau_k = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} x_{j_1} x_{j_2} \cdots x_{j_k} \qquad \text{for } k \in \{1, 2, \ldots, m\}.$$

Moreover, for the convenience set $\tau_k = 0$ for $k > m$.

The following lemma comes from [2].

Lemma 1 (Newton)**.** *Let $n \in \mathbb{N}_1$, then*

$$\sigma_n - \tau_1 \sigma_{n-1} + \tau_2 \sigma_{n-2} - \cdots + (-1)^{n-1} \tau_{n-1} \sigma_1 + (-1)^n n \tau_n = 0. \tag{2}$$

In view of Lemma 1 we have

$$\sigma_n = \det \begin{pmatrix} (-1)^{n+1} n \tau_n & -\tau_1 & \tau_2 & \ldots & (-1)^{n-2} \tau_{n-2} & (-1)^{n-1} \tau_{n-1} \\ (-1)^n (n-1) \tau_{n-1} & 1 & -\tau_1 & \ldots & (-1)^{n-3} \tau_{n-3} & (-1)^{n-2} \tau_{n-2} \\ (-1)^{n-1} (n-2) \tau_{n-2} & 0 & 1 & \ldots & (-1)^{n-4} \tau_{n-4} & (-1)^{n-3} \tau_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -2\tau_2 & 0 & 0 & \ldots & 1 & -\tau_1 \\ \tau_1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \tag{3}$$

for every $n \in \mathbb{N}_1$. Indeed, putting in (2) instead of n respectively $n-1, n-2, \ldots, 1$ we get, together with (2), the system of n equations in n variables: $\sigma_1, \ldots, \sigma_n$. Such a system is a Cramer's system and by the Cramer's rule we get (3).

From now on by $D_{\tan}$ and $D_{\cot}$ we denote the domains of the trigonometric functions tangent and cotangent, respectively.

Lemma 2. *The following identities hold true:*

(A) $\frac{\sin 2mx}{\cos^{2m} x} \cot x = \sum_{j=0}^{m} \binom{2m}{2j+1} (-1)^j \tan^{2j} x, \qquad (m, x) \in \mathbb{N} \times (D_{\tan} \cap D_{\cot})$;

(B) $\frac{\cos 2mx}{\cos^{2m} x} = \sum_{j=0}^{m} \binom{2m}{2j} (-1)^j \tan^{2j} x, \qquad (m, x) \in \mathbb{N} \times D_{\tan}$;

(C) $\frac{\sin(2m+1)x}{\cos^{2m+1}x}\cot x = \sum_{j=0}^{m}\binom{2m+1}{2j+1}(-1)^j\tan^{2j}x, \qquad (m,x)\in\mathbb{N}\times(D_{\tan}\cap D_{\cot})$;

(D) $\frac{\sin(2m+1)x}{\sin^{2m+1}x} = \sum_{j=0}^{m}\binom{2m+1}{2j+1}(-1)^j\cot^{2m-2j}x, \qquad (m,x)\in\mathbb{N}\times D_{\cot}$;

(E) $\frac{\cos(2m+1)x}{\cos^{2m+1}x} = \sum_{j=0}^{m}\binom{2m+1}{2j}(-1)^j\tan^{2j}x, \qquad (m,x)\in\mathbb{N}\times D_{\tan}$;

(F) $\frac{\cos(2m+1)x}{\sin^{2m+1}x}\tan x = \sum_{j=0}^{m}\binom{2m+1}{2j}(-1)^j\cot^{2m-2j}x, \qquad (m,x)\in\mathbb{N}\times(D_{\tan}\cap D_{\cot})$.

Proof. It is a known fact that

$$\sum_{j=0}^{k}\binom{k}{j}\cos^{k-j}x(\mathrm{i}\sin x)^j = (\cos x+\mathrm{i}\sin x)^k = \cos kx+\mathrm{i}\sin kx$$

for $k\in\mathbb{N}$ and $x\in\mathbb{R}$. Putting $k=2m$ in the above equation and comparing real and imaginary parts of the both sides we obtain (A) and (B). Similarly, with $k=2m+1$ we get (C), (D), (E) and (F). □

Now we prove the following result.

Theorem 1. *For every $m\in\mathbb{N}_2$ and any $n\in\mathbb{N}_1$,*

$$\sigma_{n,m}(A) = \sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m},$$

where $\sigma_{n,m}(A)$ denotes the determinant given by (3) *in which $\tau_j = \frac{\binom{2m}{2j+1}}{2m}$ for $j\in\{1,2,\dots,n\}$.*

Proof. Replace in the identity (A) of Lemma 2, $\tan^2 x$ by t and set

$$w_A(t) = \sum_{j=0}^{m}\binom{2m}{2j+1}(-1)^j t^j, \tag{4}$$

then $w_A(t)$ is a polynomial of order $m-1$ in the real variable t.

On the other hand, substituting $\frac{\pi l}{2m}$, where $l\in\{1,2,\dots,m-1\}$, for x in (A) we get

$$0 = \sum_{j=0}^{m}\binom{2m}{2j+1}(-1)^j\tan^{2j}\frac{\pi l}{2m}, \qquad l\in\{1,2,\dots,m-1\}.$$

Hence and by (4) we obtain

$$w_A(t) = (-1)^{m-1}\binom{2m}{2m-1}\prod_{j=1}^{m-1}\left(t-\tan^2\frac{\pi j}{2m}\right) = (-1)^{m-1}2m\prod_{j=1}^{m-1}\left(t-\tan^2\frac{\pi j}{2m}\right).$$

This and the Vieta's formulas give

$$\sum_{1\le k_1<k_2<\cdots<k_j\le m-1} \tan^2\frac{\pi k_1}{2m}\tan^2\frac{\pi k_2}{2m}\cdots\tan^2\frac{\pi k_j}{2m} = \frac{\binom{2m}{2j+1}}{2m}$$

and in view of (3) we have

$$\sigma_{n,m}(A) = \sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m}.$$

As $\tan\frac{\pi j}{2m} = \cot\frac{\pi(m-j)}{2m}$ for $j \in \{1,2,\ldots,m-1\}$ we get

$$\sum_{j=1}^{m-1}\tan^{2n}\frac{\pi j}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi(m-j)}{2m} = \sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m},$$

which completes the proof. □

Theorem 2. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(B) = \sum_{j=0}^{m-1}\tan^{2n}\frac{2j+1}{4m}\pi = \sum_{j=0}^{m-1}\cot^{2n}\frac{2j+1}{4m}\pi,$$

where $\sigma_{n,m}(B)$ denotes the determinant given by (3) in which $\tau_j = \binom{2m}{2j}$ for $j \in \{1,2,\ldots,n\}$.

Proof. Similarly as in the proof of Theorem 1, replace in the right hand side of the identity (B) of Lemma 2, $\tan^2 x$ by t and set

$$w_B(t) = \sum_{j=0}^{m}\binom{2m}{2j}(-1)^j t^j.$$

Next, substitute $\frac{2l+1}{4m}\pi$, where $l \in \{0,1,\ldots,m-1\}$, for x in (B). This yields

$$0 = \sum_{j=0}^{m}\binom{2m}{2j}(-1)^j\tan^{2j}\frac{2l+1}{4m}\pi, \qquad l \in \{0,1,\ldots,m-1\}.$$

Hence and by the definition of $w_B(t)$ we get

$$w_B(t) = (-1)^m\prod_{j=0}^{m-1}\left(t - \tan^2\frac{2j+1}{4m}\pi\right),$$

which in view of the Vieta's formulas gives

$$\sum_{1\le k_1<k_2<\cdots<k_j\le m-1}\tan^2\frac{2k_1+1}{4m}\tan^2\frac{2k_2+1}{4m}\cdots\tan^2\frac{2k_j+1}{4m} = \binom{2m}{2j}.$$

By this and (3),

$$\sigma_{n,m}(B) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{4m}\pi.$$

Using the same argument as in the proof of Theorem 1 we get

$$\sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2m}\pi = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{4m}\pi$$

and the proof is completed. $\square$

Using identities (C) and (D) of Lemma 2 and the same method as in proofs of Theorems 1 and 2 one may obtain

Theorem 3. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(C) = \sum_{j=1}^{m} \tan^{2n} \frac{\pi j}{2m+1},$$

where $\sigma_{n,m}(C)$ denotes the determinant given by (3) *in which $\tau_j = \binom{2m+1}{2j}$ for $j \in \{1, 2, \dots, n\}$.*

Theorem 4. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(D) = \sum_{j=1}^{m} \cot^{2n} \frac{\pi j}{2m+1},$$

where $\sigma_{n,m}(D)$ denotes the determinant given by (3) *in which $\tau_j = \frac{1}{2m+1}\binom{2m+1}{2j+1}$ for $j \in \{1, 2, \dots, n\}$.*

Finally, applying the same reasoning as in the proof of Theorem 1 from (E) and (F) of Lemma 2 we have

Theorem 5. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(E) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2(2m+1)}\pi,$$

where $\sigma_{n,m}(E)$ denotes the determinant given by (3) *in which $\tau_j = \frac{1}{2m+1}\binom{2m+1}{2j+1}$ for $j \in \{1, 2, \dots, n\}$.*

Theorem 6. *For every $m, n \in \mathbb{N}_1$ the following identity holds true:*

$$\sigma_{n,m}(F) = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{2(2m+1)}\pi,$$

where $\sigma_{n,m}(F)$ denotes the determinant given by (3) *in which $\tau_j = \binom{2m+1}{2j}$ for $j \in \{1, 2, \dots, n\}$.*

The following formulas

$$\cot^{2n} x = \left(\frac{1-\sin^2 x}{\sin^2 x}\right)^n, \qquad \tan^{2n} x = \left(\frac{1-\cos^2 x}{\cos^2 x}\right)^n$$

yield

Lemma 3. *The following identities hold true:*

(G) $\sum_{j=0}^{n-1} \binom{n}{j}(-1)^j \sin^{2j-2n} x = (-1)^{n-1} + \cot^{2n} x, \qquad (n,x) \in \mathbb{N}_1 \times D_{\cot};$

(H) $\sum_{j=0}^{n-1} \binom{n}{j}(-1)^j \cos^{2j-2n} x = (-1)^{n-1} + \tan^{2n} x, \qquad (m,x) \in \mathbb{N}_1 \times D_{\tan}.$

Lemma 4. *Assume that $n \in \mathbb{N}_1$ and $x \in D_{\cot}$, then*

$$\frac{1}{\sin^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \cot^{2n} x & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ (-1)^{n-2} + \cot^{2n-2} x & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ (-1)^{n-3} + \cot^{2n-4} x & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 + \cot^2 x & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Proof. Replacing n in (G) (Lemma 3) by $n-1, n-2, \dots, 1$, respectively we get, together with (G), the system of n equations in n variables:

$$\frac{1}{\sin^{2n} x}, \frac{1}{\sin^{2n-2} x}, \dots, \frac{1}{\sin^2 x}.$$

Such a system is a Cramer's system and the assertion follows by the Cramer's rule. $\square$

Using (H) in the same manner as in Lemma 4 we obtain

Lemma 5. *Let $n \in \mathbb{N}_1$ and $x \in D_{\tan}$, then*

$$\frac{1}{\cos^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \tan^{2n} x & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ (-1)^{n-2} + \tan^{2n-2} x & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ (-1)^{n-3} + \tan^{2n-4} x & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 + \tan^2 x & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

To shorten notation from now on we set

$$\mu(a_n, a_{n-1}, \dots, a_1) = \det \begin{pmatrix} a_n & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \dots & (-1)^{n-1}\binom{n}{n-1} \\ a_{n-1} & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \dots & (-1)^{n-2}\binom{n-1}{n-2} \\ a_{n-2} & 0 & 1 & -\binom{n-2}{1} & \dots & (-1)^{n-3}\binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

thus the identities of Lemmas 4 and 5 can be written as

$$\frac{1}{\sin^{2n} x} = \mu\left((-1)^{n-1} + \cot^{2n} x, (-1)^{n-2} + \cot^{2n-2} x, \ldots, 1 + \cot^2 x\right) \tag{5}$$

and

$$\frac{1}{\cos^{2n} x} = \mu\left((-1)^{n-1} + \tan^{2n} x, (-1)^{n-2} + \tan^{2n-2} x, \ldots, 1 + \tan^2 x\right), \tag{6}$$

respectively.

Theorem 7. *For every $m \in \mathbb{N}_2$ and each $n \in \mathbb{N}_1$ the following identity holds true:*

$$\begin{aligned} \sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cos^{-2n} \frac{\pi j}{2m} & \\ = \ \mu\left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A)\right., & \\ \left. \ldots, (m-1) + \sigma_{1,m}(A)\right), & \end{aligned} \tag{7}$$

where the numbers $\sigma_{k,m}(A)$ for $k \in \{1, 2, \ldots, n\}$ are defined in Theorem 1.

Proof. In view of (5) we can write

$$\sin^{-2n} \frac{\pi j}{2m} = \mu\left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}, (-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}, \ldots, 1 + \cot^2 \frac{\pi j}{2m}\right)$$

for $j \in \{1, 2, \ldots, m-1\}$. This by the definition of μ, properties od determinants and Theorem 1 gives

$$\begin{aligned} & \sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} \\ = \ & \mu\left(\sum_{j=1}^{m-1}\left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}\right), \sum_{j=1}^{m-1}\left((-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}\right),\right. \\ & \left. \ldots, \sum_{j=1}^{m-1}\left(1 + \cot^2 \frac{\pi j}{2m}\right)\right) \\ = \ & \mu\left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A)\right., \\ & \left. \ldots, (m-1) + \sigma_{1,m}(A)\right). \end{aligned}$$

The same reasoning applies to the second identity. $\square$

Analysis similar to that in the proof of Theorem 7 and the use of Theorems 2 – 6 give

Theorem 8. *For every $n, m \in \mathbb{N}_1$ the following identities holds true:*

$$\sum_{j=0}^{m-1} \sin^{-2n} \frac{2j+1}{4m}\pi = \sum_{j=0}^{m-1} \cos^{-2n} \frac{2j+1}{4m}\pi \tag{8}$$
$$= \mu\left((-1)^{n-1}m + \sigma_{n,m}(B), (-1)^{n-2}m + \sigma_{n-1,m}(B), \ldots, m + \sigma_{1,m}(B)\right),$$

$$\sum_{j=1}^{m} \sin^{-2n} \frac{\pi j}{2m+1} \tag{9}$$
$$= \mu\left((-1)^{n-1}m + \sigma_{n,m}(D), (-1)^{n-2}m + \sigma_{n-1,m}(D), \ldots, m + \sigma_{1,m}(D)\right),$$

$$\sum_{j=1}^{m} \cos^{-2n} \frac{\pi j}{2m+1} \tag{10}$$
$$= \mu\left((-1)^{n-1}m + \sigma_{n,m}(C), (-1)^{n-2}m + \sigma_{n-1,m}(C), \ldots, m + \sigma_{1,m}(C)\right),$$

$$\sum_{j=0}^{m-1} \sin^{-2n} \frac{(2j+1)\pi}{2(2m+1)} \tag{11}$$
$$= \mu\left((-1)^{n-1}m + \sigma_{n,m}(F), (-1)^{n-2}m + \sigma_{n-1,m}(F), \ldots, m + \sigma_{1,m}(F)\right),$$

$$\sum_{j=0}^{m-1} \cos^{-2n} \frac{(2j+1)\pi}{2(2m+1)} \tag{12}$$
$$= \mu\left((-1)^{n-1}m + \sigma_{n,m}(E), (-1)^{n-2}m + \sigma_{n-1,m}(E), \ldots, m + \sigma_{1,m}(E)\right),$$

where $\sigma_{k,m}(B)$, $\sigma_{k,m}(C)$, $\sigma_{k,m}(D)$, $\sigma_{k,m}(E)$, $\sigma_{k,m}(F)$ for $k \in \{1, 2, \ldots, n\}$ are defined in Theorems 2 – 6.

Now we show that the general identities from Theorems 1 – 8 yield some particular equalities, including the one considered by the authors as remarkable.

Theorem 9. *If $m \in \mathbb{N}$, then*

$$\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2-1}{3}, \qquad \textit{provided } m \geq 2, \tag{13}$$

$$\sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{m} = \frac{(m-1)(m-2)}{3}, \qquad \textit{provided } m \geq 2, \tag{14}$$

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{2m} = m^2, \qquad \textit{provided } m \geq 1, \tag{15}$$

Proof. According to Theorem 1 we have

$$\sum_{j=1}^{m-1} \tan^2 \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{2m} = \frac{1}{2m}\binom{2m}{3}. \tag{16}$$

On the other hand, in view of

$$\tan^2 x + \cot^2 x = \frac{4}{\sin^2 2x} - 2$$

we get

$$\sum_{j=1}^{m-1} \tan^{2n} \frac{\pi j}{2m} + \sum_{j=1}^{m-1} \cot^{2n} \frac{\pi j}{2m} = 4\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} - 2(m-1).$$

Combining this with (16) gives

$$\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2-1}{3}$$

for $m \geq 2$. This proves (13).

To prove (14) notice that the identity

$$\cot^2 x - \frac{1}{\sin^2 x} = -1$$

yields

$$\sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{m} - \sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = -(m-1), \qquad m \geq 2.$$

Thus by (13) we obtain (14).

Finally we show the remarkable (15). Theorem 8 leads to

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} = \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = m + \binom{2m}{2} = 2m^2 \tag{17}$$

for $m \geq 1$. Since

$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{4}{\sin^2 2x}$$

we have

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} + \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = 4\sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{2m}, \qquad m \geq 1,$$

which by (17) implies (15), and the theorem follows. □

Next we use the the identities proved here to find the sums of the series of the form $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$, where $n \in \mathbb{N}_1$. We begin with the following lemma.

Lemma 6. *Let $n \in \mathbb{N}_1$, then expression $\sigma_{n,m}(A)$, defined in Theorem 1, is a value of some polynomial from $\mathbb{Q}[x]$, where $x = m$. The order of such a polynomial does not exceed $2n$.*

Proof. The proof is by induction on n. For $n = 1$ we have

$$\sigma_{1,m}(A) = \frac{1}{2m}\binom{2m}{3} = \frac{2}{3}m^2 - m - \frac{1}{3},$$

and the assertion follows. Fix $n \geq 2$ Assuming Lemma 6 to hold for any $k \in \mathbb{N}_1$, $k \leq n-1$ we prove it for n. By (2),

$$\sigma_{n,m}(A) = \sum_{j=1}^{n-1}(-1)^{j-1}\tau_j\sigma_{n-j,m}(A) - (-1)^n n\tau_n,$$

where $\tau_j = \frac{1}{2m}\binom{2m}{2j+1}$ for $j \in \{1, 2, \ldots, n\}$. Hence by the inductive assumption $\sigma_{n,m}(A)$ is a value of some polynomial from $\mathbb{Q}[x]$ of order not greater than $2n$ with $x = m$, as claimed. □

Theorem 10. *For every $n \in \mathbb{N}_1$,*

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}} = \lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}},$$

where $\sigma_{n,m}(A)$ is defined in Theorem 1.

Proof. Observe that

$$0 < \cot x < \frac{1}{x} < \frac{1}{\sin x} \qquad x \in \left(0, \frac{\pi}{2}\right),$$

thus

$$\cot^{2n}\frac{\pi j}{2m} < \left(\frac{2m}{\pi j}\right)^{2n} < \frac{1}{\sin^{2n}\frac{\pi j}{2m}}$$

and in consequence

$$\sum_{j=1}^{m-1}\cot^{2n}\frac{\pi j}{2m} < \left(\frac{2m}{\pi}\right)^{2n}\sum_{j=1}^{m-1}\frac{1}{j^{2n}} < \sum_{j=1}^{m-1}\frac{1}{\sin^{2n}\frac{\pi j}{2m}}$$

for $n \in \mathbb{N}_1$, $m \in \mathbb{N}_2$ and $j \in \{1, 2, \ldots, n\}$. By the definitions of $\sigma_{n,m}(A)$ and the function μ we have

$$\begin{aligned}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}} < \sum_{j=1}^{m-1}\frac{1}{j^{2n}} < \frac{\pi^{2n}}{(2m)^{2n}}\mu((-1)^{n-1}&(m-1) + \sigma_{n,m}(A),\\ (-1)^{n-2}&(m-1) + \sigma_{n-1,m}(A),\\ &\ldots, m-1+\sigma_{1,m}(A)).\end{aligned} \tag{18}$$

The formula for μ and the properties of determinants give

$$\begin{aligned}
&\mu\left((-1)^{n-1}(m-1)+\sigma_{n,m}(A),(-1)^{n-2}(m-1)+\sigma_{n-1,m}(A),\ldots,m-1+\sigma_{1,m}(A)\right)\\
&\quad= (m-1)\mu((-1)^{n-1},(-1)^{n-2},\ldots,(-1)^{n-n})+\mu(\sigma_{n,m}(A),\sigma_{n-1,m}(A),\ldots,\sigma_{1,m}(A))\\
&\quad= (m-1)C_1+\sigma_{n,m}(A)+C_2\sigma_{n-1,m}(A)+\ldots+C_n\sigma_{1,m}(A),
\end{aligned}$$

where $C_1,\ldots,C_n$ are constants depending on n. Hence by Lemma 6 and inequality (18) we obtain

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}\le\sum_{j=1}^{\infty}\frac{1}{j^{2n}}\le\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}},$$

which establishes the formula. □

Remark 1. Note that in the proof Theorem 10 (the last step of the proof) we have actually proved more, namely that the order of the polynomial from Lemma 6 equals exactly $2n$. Indeed, if it was not true, we would have

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}=0$$

and consequently

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}}\le 0,$$

which is impossible.

Remark 2. Treating $\sigma_{n,m}(A)$ as a polynomial in m of order $2n$ we have

$$\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(A)}{(2m)^{2n}}=a_{2n}\frac{\pi^{2n}}{4^n},$$

where a_{2n} denotes the leading coefficient of $\sigma_{n,m}(A)$. On the other hand,

$$B_{2n}\frac{2^{2n-1}\pi^{2n}}{(2n)!}(-1)^{n-1}=\sum_{j=1}^{\infty}\frac{1}{j^{2n}},\qquad n\in\mathbb{N}_1,$$

where B_{2n} stands for the $2n$-th Bernoulli number (see [4], p.320). Thus we get the following relation between Bernoulli numbers and the coefficients of $\sigma_{n,m}(A)$

$$a_{2n}=B_{2n}\frac{2^{4n-1}}{(2n)!}(-1)^{n-1},\qquad n\in\mathbb{N}_1.$$

Remark 3. Similarly as Theorem 10 one can show that

$$\sum_{j=1}^{\infty}\frac{1}{j^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(B)}{(2m)^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(D)}{(2m)^{2n}}=\lim_{m\to\infty}\frac{\pi^{2n}\sigma_{n,m}(F)}{(2m)^{2n}},\qquad n\in\mathbb{N}_1.$$

where $\sigma_{n,m}(B)$, $\sigma_{n,m}(D)$ and $\sigma_{n,m}(F)$ are defined in Theorems 2, 4 and 6, respectively.

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On the zeros of an analytic function

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ABSTRACT: Kuniyeda, Montel and Toya had shown that the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k; a_0 \neq 0$, of degree n, does not vanish in

$$|z| \leq \{1 + (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p}\}^{-1/q},$$

where $p > 1$, $q > 1$, $(1/p) + (1/q) = 1$ and we had proved that $p(z)$ does not vanish in $|z| \leq \alpha^{1/q}$, where

$$\alpha = \text{unique root in } (0,1) \text{ of } D_n x^3 - D_n S x^2 + (1 + D_n S)x - 1 = 0,$$

$$D_n = (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p},$$

$$S = (|a_1| + |a_2|)^q (|a_1|^p + |a_2|^p)^{-(q-1)},$$

a refinement of Kuniyeda et al.'s result under the assumption

$$D_n < (2 - S)/(S - 1).$$

Now we have obtained a generalization of our old result and proved that the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, (\not\equiv \text{aconstant}); a_0 \neq 0,$$

analytic in $|z| \leq 1$, does not vanish in $|z| < \alpha_m^{1/q}$, where

$$\alpha_m = \text{unique root in } (0,1) \text{ of } D x^{m+1} - D M_m x^2 + (1 + D M_m)x - 1 = 0,$$

$$D = (\sum_{k=1}^{\infty} |a_k/a_0|^p)^{q/p},$$

$$M_m = (\sum_{k=1}^{m} |a_k|)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p},$$

$m =$ any positive integer with the characteristic that there exists a positive integer $k(\leq m)$ with $a_k \neq 0$.

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1 Introduction and statement of results

Let

$$P(z) = b_0 + b_1 z + \ldots + b_n z^n$$

be a polynomial of degree n. Then according to a classical result of Kuniyeda, Montel and Toya [3, p. 124] on the location of zeros of a polynomial we have

Theorem A. *All the zeros of the polynomial* $P(z)$ *lie in*

$$|z| < \{1 + (\sum_{j=0}^{n-1} |b_j/b_n|^p)^{q/p}\}^{1/q},$$

where

$$p > 1, \quad q > 1, \quad (1/p) + (1/q) = 1. \tag{1.1}$$

On applying Theorem A to the polynomial $z^n p(1/z)$, we have the following equivalent formulation of Theorem A.

Theorem B. *The polynomial*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n; a_0 \neq 0, \tag{1.2}$$

of degree n *does not vanish in*

$$|z| \leq (1 + D_n)^{-1/q}, \tag{1.3}$$

where p*,* q *are given in (1.1) and*

$$D_n = (\sum_{j=1}^{n} |a_j/a_0|^p)^{q/p}. \tag{1.4}$$

We [2] had obtained

Theorem C. *All the zeros of* $P(z)$ *lie in*

$$|z| < \chi^{1/q},$$

where χ *is the unique root of the equation*

$$x^3 - (1 + LM)x^2 + LMx - L = 0,$$

in $(1, \infty)$*,*

$$\begin{aligned} L &= (\sum_{j=0}^{n-1} |b_j/b_n|^p)^{q/p}, \\ M &= (|b_{n-1}| + |b_{n-2}|)^q (|b_{n-1}|^p + |b_{n-2}|^p)^{-(q-1)}. \end{aligned}$$

Theorem C is a refinement of Theorem A, under the assumption

$$L < (2 - M)/(M - 1).$$

The equivalent formulation of Theorem C, (similar to the formulation of Theorem B from Theorem A) is

Theorem D. *The polynomial*

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n; a_0 \neq 0,$$

of degree n does not vanish in

$$|z| \leq \alpha^{1/q},$$

where α is the unique root of the equation

$$D_n x^3 - D_n S x^2 + (1 + D_n S)x - 1 = 0,$$

in $(0, 1)$,

$$S = (|a_1| + |a_2|)^q (|a_1|^p + |a_2|^p)^{-(q-1)},$$

and D_n is as in Theorem B.

Theorem D is a refinememnt of Theorem B, under the assumption

$$D_n < (2 - S)/(S - 1).$$

In this paper we have obtained a generalization of Theorem D for the functions, analytic in $|z| \leq 1$. More precisely we have proved

Theorem 1. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, (\not\equiv \text{aconstant}); a_0 \neq 0, \tag{1.5}$$

be analytic in $|z| \leq 1$. Then $f(z)$ does not vanish in

$$|z| < \alpha_m^{1/q}, \tag{1.6}$$

where

$$\begin{array}{rcll} q & > & 1,\ \ p\ >\ 1,\ \ (1/p) + (1/q)\ =\ 1, & \\ m & = & \text{any positive integer with the characteristic that} & (1.7) \\ & & \text{there exists a positive integer } k(\leq m) \text{ with } a_k \neq 0, & \\ \alpha_m & = & \text{unique root in } (0,1),\ \text{of} & \\ & & \{g(x) \equiv\},\ Dx^{m+1} - DM_m x^2 + (1 + DM_m)x - 1 = 0, & (1.8) \\ D & = & (\sum_{k=1}^{\infty} |a_k/a_0|^p)^{q/p}, (> 0, \text{by}(1.5)), & (1.9) \\ M_m & = & (\sum_{k=1}^{m} |a_k|)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p}, (> 0, \text{by}(1.7)). & (1.10) \end{array}$$

From Theorem 1 we easily get

***Corollary* 1.** *Under the same hypothesis as in Theorem 1, $f(z)$ does not vanish in*

$$|z| < \sup_{m \geq M, q > 1} {\alpha_m}^{1/q},$$

where

$$M = \textit{least positive integer } k \ \ \textit{such that } a_k \neq 0.$$

2 Lemmas

For the proof of the theorem, we require the following lemmas.

***Lemma* 1.** *Let*

$$\begin{aligned} &\alpha_j > 0, \quad \beta_j > 0, \ \text{for } j = 1, 2, \ldots, n, \\ &q > 1, \quad p > 1, \quad (1/p) + (1/q) = 1, \\ &1 \leq m < n. \end{aligned}$$

Then

$$\sum_{j=1}^{n} \alpha_j \beta_j \leq ((\sum_{j=1}^{n} \beta_j^p)^{1/p} (\sum_{j=1}^{m} \beta_j^p)^{-1/p}) \left\{ (\sum_{j=1}^{m} \alpha_j \beta_j)^q + ((\sum_{j=1}^{m} \beta_j^p)^{q-1}) (\sum_{j=m+1}^{n} \alpha_j^q) \right\}^{1/q}. \tag{2.1}$$

This lemma is due to Beckenbach [1].
From Lemma 1 we easily obtain

***Lemma* 2.** *Inequality (2.1) is true even if*

$$\begin{aligned} \alpha_j \geq 0, \qquad j = 1, 2, \ldots, n, \\ \beta_j \geq 0, \qquad j = 1, 2, \ldots, n, \end{aligned}$$

with

$$\beta_j \neq 0, \mathrm{for at least one} j, 1 \leq j \leq m.$$

***Lemma* 3.** *The equation*

$$Dx^{m+1} - DM_m x^2 + (1 + DM_m)x - 1 = 0 \tag{2.2}$$

has a unique root α_m in $(0, 1)$ where m, D and M_m are as in Theorem 1.

Proof of Lemma 3. We firstly assume that

$$m > 1.$$

Now we consider the transformation

$$x = 1/t$$

in equation (2.2), thereby giving the transformed equation

$$t^{m+1} - (1 + DM_m)t^m + DM_m t^{m-1} - D = 0, \tag{2.3}$$

and then the transformation

$$t = 1 + y$$

in (2.3), thereby giving the transformed equation

$$(1 + y)^{m+1} - (1 + DM_m)(1 + y)^m + DM_m(1 + y)^{m-1} - D = 0, \tag{2.4}$$

i.e.

$$\begin{aligned} y^{m+1} &+ y^m((m/1) - DM_m) + ((m-1)/1!)((m/2) - DM_m)y^{m-1} \\ &+ ((m-1)(m-2)/2!)((m/3) - DM_m)y^{m-2} + \dots \\ &+ ((m-1)(m-2)\dots(m-j+1)/(j-1)!)((m/j) - DM_m)y^{m+1-j} + \dots \\ &+ ((m-1)(m-2)\dots(m-m+1)/(m-1)!)((m/m) - DM_m)y - D \\ =&0. \end{aligned} \tag{2.5}$$

By using Déscarte's rule of signs we can say that equation (2.5) (i.e. equation (2.4)) will have a unique positive root and accordingly the equation (2.3) will have a unique root in $(1, \infty)$. Hence the equation (2.2) will have a unique root α_m, (say), in $(0, 1)$, thereby proving Lemma 3 for the possibility under consideration.

For the possibility

$$m = 1,$$

the transformed equation, similar to equation (2.5), (i.e. equation (2.4)), is

$$y^2 + y(1 - DM_m) - D = 0.$$

Now Lemma 3 follows for this possibility, by using arguments similar to those used for proving Lemma 3 for the possibility

$$m > 1.$$

This completes the proof of Lemma 3.

3 Proof of Theorem 1

Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k, n = 1, 2, 3, \dots \quad .$$

Then for $|z|<1$ and $n>m$

$$\begin{aligned}
|f_n(z)| &\geq |a_0| - \sum_{k=1}^{n} |z|^k |a_k|, \\
&\geq |a_0| - \left\{ (\sum_{k=1}^{n} |a_k|^p)^{1/p} (\sum_{k=1}^{m} |a_k|^p)^{-1/p} \right\} \Bigg[(\sum_{k=1}^{m} |z|^k |a_k|)^q \\
&\quad + \left\{ (\sum_{k=1}^{m} |a_k|^p)^{q-1} \right\} (\sum_{k=m+1}^{n} |z|^{kq}) \Bigg]^{1/q}, \text{(by Lemma 2)}, \\
&\geq |a_0| - (\sum_{k=1}^{n} |a_k|^p)^{1/p} \Bigg[(\sum_{k=1}^{m} |a_k||z|^k)^q (\sum_{k=1}^{m} |a_k|^p)^{-q/p} \\
&\quad + (\sum_{k=m+1}^{n} |z|^{kq}) \Bigg]^{1/q}, \text{(by 1.1))}, \\
&\geq |a_0| - (\sum_{k=1}^{n} |a_k|^p)^{1/p} \left[M_m |z|^q + (\sum_{k=m+1}^{n} |z|^{kq}) \right]^{1/q}, \text{(by 1.10))},
\end{aligned}$$

which, by making

$$n \to \infty,$$

implies that

$$\begin{aligned}
|f(z)| &\geq |a_0| - (\sum_{k=1}^{\infty} |a_k|^p)^{1/p} \left[M_m |z|^q + (\sum_{k=m+1}^{\infty} |z|^{kq}) \right]^{1/q}, (\sum_{k=1}^{\infty} |a_k|^p \text{ will converge} \\
&\qquad \text{as } \sum_{k=1}^{\infty} |a_k| \text{ converges and } (\sum_{k=1}^{n} |a_k|^p)^{1/p} \leq \sum_{k=1}^{n} |a_k|, n = 1, 2, \ldots), \\
&= |a_0| \left[1 - \{ D(M_m |z|^q + (|z|^{(m+1)q} / (1 - |z|^q))) \}^{1/q} \right], \text{(by 1.9))}, \\
&> 0, \qquad (3.1)
\end{aligned}$$

if

$$D|z|^{(m+1)q} - DM_m |z|^{2q} + (1 + DM_m)|z|^q - 1 < 0. \qquad (3.2)$$

Now as

$$g(0) = -1, \text{(by(1.8))},$$

we can say by Lemma 3, (3.1) and (3.2) that

$$|f(z)| > 0,$$

if

$$|z|^q < \alpha_m,$$

thereby proving Theorem 1.

***Remark* 1.** *Theorem 1 gives better bound than that given by the result, that $f(z)$ does not vanish in*

$$|z| < \{1/(1+D)\}^{1/q},$$

obtained by using Hölder's inequality instead of Lemma 2 and following the method of proof of Theorem 1, provided

$$\begin{aligned} m=1 \quad &\& \quad M_m < m, \\ m \geq 2 \quad &\& \quad M_m \leq 1, \\ m \geq 2, 1 < M_m < m \quad &\text{and} \quad D < D_0, \end{aligned} \tag{3.3}$$

where D_0 is the unique positive root of the equation

$$\begin{aligned} (M_m-1)D^{m-1} &+ (m-1)(M_m-(m/(m-1)))D^{m-2} \\ &+ ((m-1)(m-2)/2)(M_m-(m/(m-2)))D^{m-3} \\ &+ \ldots + (m-1)(M_m-(m/2))D + (M_m-m) \\ &= 0, \quad (m \geq 2\& 1 < M_m < m), \end{aligned}$$

as for $m = 1\& M_m < m$

$$g(1/(1+D)) < 0,$$

and for $m \geq 2$

$$g(1/(1+D)) < 0,$$

is equivalent to

$$\begin{aligned} (M_m-1)D^{m-1} &+ (m-1)(M_m-(m/(m-1)))D^{m-2} \\ &+ ((m-1)(m-2)/2)(M_m-(m/(m-2)))D^{m-3} \\ &+ \ldots + (m-1)(M_m-(m/2))D + (M_m-m) \\ &< 0. \end{aligned}$$

The function

$$f(z) = 1 + z + (z/(2i))^3 + (z/(2i))^4 + (z/(2i))^5 + \ldots$$

satisfies (3.3) with

$$p = q = m = 2$$

and the corresponding $\alpha_m^{1/q}$ is .752.

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On circularly symmetric functions

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Abstract: Let $D \subset \mathbb{C}$ and $0 \in D$. A set D is circularly symmetric if for each $\varrho \in \mathbb{R}^+$ a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ. A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions we denote by X. The above definitions were given by Jenkins in [2].

In this paper besides X we also consider some of its subclasses: $X(\lambda)$ and $Y \cap S^*$ consisting of functions in X with the second coefficient fixed and univalent starlike functions respectively. According to the suggestion, in Abstract we add one more paragraph at the end of the section:

For $X(\lambda)$ we find the radii of starlikeness, starlikeness of order α, univalence and local univalence. We also obtain some distortion results. For $Y \cap S^*$ we discuss some coefficient problems, among others the Fekete-Szegö ineqalities.

AMS Subject Classification: *30C45*
Keywords and Phrases: *symmetric function, radius of starlikeness, zeros of polinomials*

1 The class of circularly symmetric functions and some its subclasses.

Let $\tilde{\mathcal{A}}$ denote the class of all functions analytic in $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and let $\mathcal{A}$ denote the class of all functions analytic in Δ normalized by $f(0) = f'(0) - 1 = 0$. Similar notation is applied to the class of typically real functions, i.e. functions satisfying the following condition: $\operatorname{Im} z \operatorname{Im} f(z) \geq 0$ for $z \in \Delta$. The set of all analytic and typically real functions is denoted by $\tilde{T}$; the subset of $\tilde{T}$ consisting of normalized functions is denoted by T. Hence $T = \tilde{T} \cap \mathcal{A}$. It follows from the definition of a typically real function that $z \in \Delta^+ \Leftrightarrow f(z) \in \mathbb{C}^+$ and $z \in \Delta^- \Leftrightarrow f(z) \in \mathbb{C}^-$. The symbols Δ^+, Δ^-, $\mathbb{C}^+$, $\mathbb{C}^-$ mean the following open sets: the upper and the lower half of the unit disk Δ and the upper and the lower halfplane.

In this paper we focus on so called circularly symmetric functions, which were defined by Jenkins in [2]. Let us start with the following definitions.

Let $D \subset \mathbb{C}$ and $0 \in D$.

Definition 1. A set D is circularly symmetric if for each $\varrho \in \mathbb{R}^+$ a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ.

Definition 2. A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions we denote by X.

In fact, Jenkins claimed more than it was stated in the above definition. He considered only these circularly symmetric functions which are univalent. This assumption is rather restrictive. Furthermore, there are no objections to reject it. The number of interesting problems appear while discussing non-univalent circularly symmetric functions. For these reasons we decided to define a circularly symmetric function as in Definition 2. In order to distinguish the classes of non-univalent and univalent circularly symmetric functions we will denote the latter by Y.

Besides X we will also consider some of its subclasses: $X(\lambda)$ and $Y \cap S^*$ consisting of functions in X with the fixed second coefficient of the Taylor series expansion and univalent starlike functions respectively. As it was shown in [2], for all $r \in (0,1)$ and for a circularly symmetric function f the expression $|f(re^{i\varphi})|$ is a nonincreasing function for $\varphi \in (0,\pi)$ and a nondecreasing function for $\varphi \in (\pi, 2\pi)$. From this fact and the equality

$$-\frac{\partial}{\partial\varphi}\left(\log|f(re^{i\varphi})|\right) = \mathrm{Im}\left(re^{i\varphi}\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right)$$

it follows that on the circle $|z| = r$ there is

$$\mathrm{Im}\,\frac{zf'(z)}{f(z)} \geq 0 \quad \text{if and only if} \quad \mathrm{Im}\, z \geq 0\ .$$

Hence

Theorem 1. *[2]*

$$f \in X \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T}\ .$$

The condition $\frac{zf'(z)}{f(z)} \in \tilde{T}$ is not sufficient for univalence of f. We have only

Theorem 2. *If $f \in Y$ then $\frac{zf'(z)}{f(z)} \in \tilde{T}$.*

According to Theorem 1, all coefficients of the Taylor expansion of $f \in X$ are real.

Some other results concerning Y one can find in [1] and [4].

Similar, but more general, functions were discussed by Libera in [3]. He considered so called disk-like functions. The functions f of this class have the property: there exists a number ϱ depending on f that for each fixed r, $r \in (\varrho, 1]$, there exist numbers φ_1, φ_2 depending on r that $|f(re^{i\varphi})|$ is decreasing if φ increases in some interval $I_1 = (\varphi_1, \varphi_2)$ and increasing in $I_2 = (\varphi_2, \varphi_1 + 2\pi)$. The class of these functions Libera denoted by $\mathcal{D}$. In particular, if f has real coefficients and $|f(re^{i\varphi})|$ is increasing on the lower half of the circle $|z| = r$ and is decreasing on the upper half of this circle, then f is a circularly symmetric function. Although $\mathcal{D}$ is more general than X, some of the results of the paper [3] are still valid for the class X.

Let us assume that f is of the form $f(z) = z + \lambda z^2 + \ldots$. From Theorem 1 it follows that a function

$$\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right)$$

is in T; let us denote it by $h(z)$. Hence

$$f(z) = z \exp\left(\lambda \int_0^z \frac{h(\zeta)}{\zeta} d\zeta\right) . \tag{1}$$

Applying the very well known relation between T and $CVR(i)$ consisting of functions with real coefficients g which are convex in the direction of the imaginary axis and normalized by $g(0) = g'(0) - 1 = 0$, we obtain

Corollary 1.

$$f \in X \Leftrightarrow f(z) = z \exp\{\lambda g(z)\} \ , \ g \in CVR(i) \ , \ \lambda > 0 \ . \tag{2}$$

The conclusion similar to the above corollary one can find in the paper of Libera (corollary on page 253).

Basing on the equivalence (2) we can define the subclass of X containing these circularly symmetric functions for which the second coefficient is fixed and equal to $\lambda \geq 0$. We denote this class by $X(\lambda)$. For $\lambda = 0$ the set $X(0)$ has only one element - the identity function. We shall present the properties of $X(\lambda)$ in next section.

2 Properties of $X(\lambda)$.

Theorem 3. *The radius of starlikeness for $X(\lambda)$ is equal to $r_{S^*}(X(\lambda)) = r_\lambda$, where $r_\lambda = \frac{1}{4}\left(\sqrt{\lambda+4} - \sqrt{\lambda}\right)^2$. The extremal function is $f_\lambda(z) = z \exp\left(\lambda \frac{z}{1+z}\right)$.*

Proof
It follows from (1) that $\frac{zf'(z)}{f(z)} = 1 + \lambda z g'(z) = 1 + \lambda h(z)$, where $g \in CVR(i)$, $h \in T$. The well-known estimate of the real part of a typically real function leads to

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} \geq 1 - \lambda \frac{r}{(1-r)^2} \ .$$

Therefore, $\mathrm{Re}\, \frac{zf'(z)}{f(z)} \geq 0$ if and only if $r \leq \frac{2}{2+\lambda+\sqrt{\lambda^2+4\lambda}}$, or equivalently, if $r \leq r_\lambda$. Equality in the above estimate holds for $h(z) = \frac{z}{(1+z)^2}$ and $z = -r$. It means that the extremal function is f_λ. $\blacksquare$

The result of Theorem 3 can be generalized in order to finding the radius of starlikeness of order α, $\alpha \in [0, 1)$. It sufficies to replace the inequality $1 - \lambda \frac{r}{(1-r)^2} \geq 0$ by $1 - \lambda \frac{r}{(1-r)^2} \geq \alpha$. Hence

Theorem 4. *The radius of starlikeness of order α, $\alpha \in [0,1)$ for $X(\lambda)$ is equal to* $r_{S^*(\alpha)}(X(\lambda)) = \frac{1}{4}\left(\sqrt{\frac{\lambda}{1-\alpha}+4}-\sqrt{\frac{\lambda}{1-\alpha}}\right)^2$. *The extremal function is* $f_\lambda(z) = z\exp\left(\lambda\frac{z}{1+z}\right)$.

Observe that for all $f \in X(\lambda)$ the condition $\frac{zf'(z)}{f(z)} \neq 0$ holds if only $z \in \Delta_{r_\lambda}$. Moreover, for f_λ and $z = -r_\lambda$ there is

$$\left.\frac{zf'(z)}{f(z)}\right|_{z=-r_\lambda} = \left.\left(1+\lambda\frac{z}{(1+z)^2}\right)\right|_{z=-r_\lambda} = 1-\lambda\frac{r_\lambda}{(1-r_\lambda)^2} = 0\ .$$

This results in

Theorem 5. *The radius of local univalence for $X(\lambda)$ is equal to $r_{LU}(X(\lambda)) = r_\lambda$.*

Because of $r_{S^*} \leq r_S \leq r_{LU}$, which is true for any class of analytic functions, we obtain

Corollary 2. *The radius of univalence for $X(\lambda)$ is equal to $r_S(X(\lambda)) = r_\lambda$.*

It is known that the second coefficients of the Taylor expansion of functions in the following subclasses of $\mathcal{A}$ consisting of: convex functions, univalent functions and locally univalent functions have the upper bounds: 1, 2, 4 respectively. For this reason it is worth observing that

$$r_S(X(1)) = \frac{1}{2}(3-\sqrt{5})\ ,\ r_S(X(2)) = 2-\sqrt{3}\ ,\ r_S(X(4)) = (\sqrt{2}-1)^2\ .$$

Theorem 6. *If $f \in X(\lambda)$ and $r = |z| \in (0,1)$ then*

$$r\exp\left(\frac{-\lambda r}{1-r}\right) \leq |f(z)| \leq r\exp\left(\frac{\lambda r}{1-r}\right)\ , \tag{3}$$

Equalities in the above estimates hold for $f(z) = z\exp\left(\frac{\lambda z}{1+z}\right)$, $z = -r$ and $f(z) = z\exp\left(\frac{\lambda z}{1-z}\right)$, $z = r$ respectively.

Proof

For $g \in CVR(i)$ the exact estimate holds (see for example [5])

$$|\mathrm{Re}\, g(z)| \leq \frac{r}{1-r}\ , \tag{4}$$

with equality for $g(z) = \frac{z}{1+z}$, $z = -r$ and $g(z) = \frac{z}{1-z}$, $z = r$ respectively. From Corollary 1 it follows that

$$|f(z)| = |z|\exp\left(\lambda\,\mathrm{Re}\, g(z)\right)\ .$$

Combining it with (4) completes the proof. ■

Theorem 7. *If $f \in X(\lambda)$ and $r = |z| \in (0,1)$ then*

$$|f'(z)| \le \left(1 + \lambda \frac{r}{(1-r)^2}\right) \exp\left(\frac{\lambda r}{1-r}\right) \ , \tag{5}$$

Equality in the above estimate holds for $f(z) = z \exp\left(\frac{\lambda z}{1-z}\right)$ and $z = r$.

Proof
From Corollary 1 and (1) we have

$$|f'(z)| = \left|\frac{f(z)}{z}\right| |1 + \lambda h(z)| \ ,$$

where $h \in T$. Applying Theorem 6 and the estimate of the modulus of a function in T in the above equality leads to the assertion. ■

According to Theorems 6 and 7, both $|f(z)|$ and $|f'(z)|$ can be arbitrarily large while considering functions in the whole class X, not only functions with the second coefficient fixed.

3 Properties of $Y \cap S^*$.

In the paper of Szapiel [4] one can find the relation between the class of circurally symmetric functions which are starlike with the class of typically real functions T:

Theorem 8.

$$f \in Y \cap S^* \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T} \cap P_R \ .$$

Szapiel also proved the representation formula for functions in the class $\mathcal{R}^2 = \{q \in \mathcal{A} : q = p^2, p \in \tilde{T} \cap P_R\}$. Namely, $q \in \mathcal{R}^2$ if and only if

$$q(z) = \int_{-1}^{1} \frac{(1+z)^2}{1 - 2zt + z^2} d\mu(t) \ . \tag{6}$$

From this formula one can establish the relationship between $\mathcal{R}^2$ and T:

$$q \in \mathcal{R}^2 \Leftrightarrow g \in T \ , \tag{7}$$

where

$$q(z) = (1+z)^2 \frac{g(z)}{z} \ .$$

From the above we get

Corollary 3.

$$f \in Y \cap S^* \Leftrightarrow \frac{zf'(z)}{f(z)} = (1+z)\sqrt{\frac{g(z)}{z}} \ , \ g \in T. \tag{8}$$

Examples. Putting functions of the class T into (8) we obtain associated functions from $Y \cap S^*$:
1. If $g(z) = z$, then $\frac{zf'(z)}{f(z)} = 1 + z$ and hence $f(z) = ze^z$.
2. If $g(z) = \frac{z}{(1+z)^2}$, then $\frac{zf'(z)}{f(z)} = 1$ and hence $f(z) = z$.
3. If $g(z) = \frac{z}{(1-z)^2}$, then $\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$ and hence $f(z) = \frac{z}{(1-z)^2}$.

Many of the properties of $Y \cap S^*$ follow directly from obvious inclusion $Y \cap S^* \subset S^*$ and the fact that the Koebe function $f(z) = \frac{z}{(1-z)^2}$, which is starlike, belongs also to $Y \cap S^*$. This observation gives us the following sharp results:

1. If $f \in Y \cap S^*$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \le n$.
2. If $f \in Y \cap S^*$ and $r = |z| \in (0,1)$, then $\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}$.
3. If $f \in Y \cap S^*$ and $r = |z| \in (0,1)$, then $\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}$.
4. Every function in $Y \cap S^*$ is convex in the disk $|z| < 2 - \sqrt{3}$.
5. Every function in $Y \cap S^*$ is strongly starlike of order α in the disk $|z| < \tan(\alpha\frac{\pi}{4})$.

Now we shall find the lower bounds of the second and the third coefficients in $Y \cap S^*$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Y \cap S^*$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in T$. From (8) we conclude

$$2a_2 = b_2 + 2 \ , \tag{9}$$

$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \ . \tag{10}$$

Let us denote by $A_{2,3}(A)$ a set $\{(a_2(f), a_3(f)) : f \in A\}$. This set for T is known: $A_{2,3}(T) = \{(x,y) : -2 \le x \le 2, x^2 - 1 \le y \le 3\}$. This results in the following bound for a function in $Y \cap S^*$:

$$0 \le a_2 \le 2 \ . \tag{11}$$

Taking into account (9) and (10) in $A_{2,3}(T)$ we obtain

Theorem 9.

$$A_{2,3}(Y \cap S^*) = \left\{ (x,y) : 0 \le x \le 2, \frac{1}{4}\left(5x^2 - 4x\right) \le y \le \frac{1}{4}\left(x^2 + 4x\right) \right\} \ .$$

Consequently

Corollary 4. *Let $f \in Y \cap S^*$ have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $-\frac{1}{5} \le a_3 \le 3$.*

From (10) and $A_{2,3}(T)$ it follows that

$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \ge {b_2}^2 + 2b_2 \ge -1$$

and

$$4a_3 - {a_2}^2 = b_3 + 2b_2 + 1 \le {b_2}^2 + 4 \le 8 \ .$$

Hence

Theorem 10. *Let* $f \in Y \cap S^*$ *have the Taylor series expansion* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then* $-\frac{1}{4} \le a_3 - \frac{1}{4}{a_2}^2 \le 2$.

The points of intersection of two parabolas from Theorem 9 coincide with two pairs of coefficients (a_2, a_3) of the functions $f_1(z) = z$ and $f_2(z) = \frac{z}{(1-z)^2}$. From Theorem 10 it follows that the class $Y \cap S^*$ is not a convex set because the set $A_{2,3}(Y \cap S^*)$ is not convex.

Basing on Theorem 9 one can derive so called the Fekete-Szegö ineqalities for $Y \cap S^*$.

Theorem 11. *Let* $f \in Y \cap S^*$ *be of the form* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then*

$$\begin{cases} \frac{1}{4\mu-5} & \mu \le 1 \\ 3 - 4\mu & \mu \ge 1 \end{cases} \le a_3 - \mu {a_2}^2 \le \begin{cases} 3 - 4\mu & \mu \le \frac{1}{2} \\ \frac{1}{4\mu-1} & \mu \ge \frac{1}{2} \end{cases} .$$

Proof

Assume that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Y \cap S^*$. Let us denote by Q a function $Q(a_2, a_3) = a_3 - \mu {a_2}^2$. With a fixed $\mu \in \mathbb{R}$ the function Q achieves its extremal value on the boundary of the set $A_{2,3}(Y \cap S^*)$.

Let us consider two functions

$$Q_1(x) = Q(x, \frac{1}{4}x^2 + x) = x^2 \left(\frac{1}{4} - \mu \right) + x$$

and

$$Q_2(x) = Q(x, \frac{5}{4}x^2 - x) = x^2 \left(\frac{5}{4} - \mu \right) - x \; .$$

For $x \in [0, 2]$ the inequality

$$Q_1(x) \ge Q_2(x)$$

holds; hence

$$\begin{aligned} (12) \quad & \max\{Q : f \in Y \cap S^*\} \\ & = \max\{Q(x,y) : (x,y) \in A_{2,3}(Y \cap S^*)\} \\ & = \max\{Q_1(x) : x \in [0,2]\} \end{aligned}$$

and

$$\begin{aligned} (13) \quad & \min\{Q : f \in Y \cap S^*\} \\ & = \min\{Q(x,y) : (x,y) \in A_{2,3}(Y \cap S^*)\} \\ & = \min\{Q_2(x) : x \in [0,2]\} \; . \end{aligned}$$

The function Q_1 for $\mu \le \frac{1}{2}$ is strictly increasing in $[0, 2]$; thus $\max\{Q_1(x) : x \in [0, 2]\} = Q_1(2)$. For $\mu > \frac{1}{2}$ the function Q_1 increases in $(0, x_1)$ and decreases in $(x_1, 2)$, where $x_1 = \frac{2}{4\mu-1}$. This results in $\max\{Q_1(x) : x \in [0, 2]\} = Q_1(x_1)$.

Similarly, the function Q_2 for $\mu < 1$ decreases in $(0, x_2)$ and increases in $(x_2, 2)$, where $x_2 = \frac{2}{5-4\mu}$. Hence $\min\{Q_2(x) : x \in [0,2]\} = Q_2(x_2)$. For $\mu \geq 1$ the function Q_2 is strictly decreasing in $[0,2]$, so $\min\{Q_2(x) : x \in [0,2]\} = Q_2(2)$. ∎

Taking $\mu = 0$ or $\mu = \frac{1}{4}$ we obtain previously obtained results from Corollary 4 and from Theorem 10.

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Properties of higher order differential polynomials generated by solutions of complex differential equations in the unit disc

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Abstract: The main purpose of this paper is to study the controllability of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0.$$

In fact, we study the growth and oscillation of higher order differential polynomial with meromorphic coefficients in the unit disc $\Delta = \{z : |z| < 1\}$ generated by solutions of the above k^{th} order differential equation.

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1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z : |z| < 1\}$ (see [13], [14], [18], [20]). We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (see [10], [11]).

Definition 1.1 ([10], [11]) Let f be a meromorphic function in Δ, and

$$D(f) := \limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = b.$$

If $b < \infty$, we say that f is of finite b degree (or is non-admissible). If $b = \infty$, we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r, f)$.

Definition 1.2 ([10], [11]) Let f be an analytic function in Δ, and

$$D_M(f) := \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{\log \frac{1}{1-r}} = a < \infty \ \ (or\ a = \infty),$$

then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r, f) = \max_{|z|=r} |f(z)|$.

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [4], [17], [18]). Let us define inductively, for $r \in [0, 1)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, 1)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r$, $\log_0 r = r, \exp_{-1} r = \log_1 r, \log_{-1} r = \exp_1 r$.

Definition 1.3 [5, 6] The iterated $p-$order of a meromorphic function f in Δ is defined by

$$\rho_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log \frac{1}{1-r}} \ \ (p \geq 1).$$

For an analytic function f in Δ, we also define

$$\rho_{M,p}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log \frac{1}{1-r}} \ \ (p \geq 1).$$

Remark 1.1 It follows by M. Tsuji in [25] that if f is an analytic function in Δ, then

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1.$$

However, it follows by Proposition 2.2.2 in [18]

$$\rho_{M,p}(f) = \rho_p(f), (p \geq 2).$$

Definition 1.4 [5] The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N}, \rho_j(f) < \infty\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function$f$ in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{j \in \mathbb{N}, \rho_{M,j}(f) < \infty\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_{M,j}(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.5 [3, 15, 20] The iterated $p-$type of a meromorphic function f of iterated $p-$order ρ $(0 < \rho < \infty)$ in Δ is defined by

$$\tau_p(f) = \limsup_{r \to 1^-} (1-r)^{\rho_p(f)} \log_{p-1}^+ T(r, f).$$

Definition 1.6 [7] Let f be a meromorphic function in Δ. Then the iterated $p-$convergence exponent of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| \leq r\}$. Similarly, the iterated $p-$convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.7 [7] The growth index of the convergence exponent of the sequence of the zeros of $f(z)$ in Δ is defined by

$$i_\lambda(f) = \begin{cases} 0, & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min\{j \in \mathbb{N}, \lambda_j(f) < \infty\}, & \text{if some } j \in \mathbb{N} \text{ with } \lambda_j(f) < \infty, \\ +\infty, & \text{if } \lambda_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Similarly, we can define the growth index of the convergence exponent of the sequence of distinct zeros $i_{\overline{\lambda}}(f)$ of $f(z)$ in Δ.

Consider the complex differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

and the k^{th} order differential polynomial

$$g_k = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f, \tag{1.2}$$

where A_j $(j = 0, 1, \cdots, k-1)$ and d_i $(i = 0, 1, \cdots, k)$ are meromorphic functions in Δ.

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. If $\mathbf{G} = \boldsymbol{\Delta}$, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\boldsymbol{\Delta})$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic: } \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant. In [7], T. B. Cao, H. Y. Xu and C. X. Zhu studied the complex oscillation of differential polynomial generated meromorphic solutions of second order linear differential equations with meromorphic coefficients and obtained the following results.

Theorem A [7] *Let A be an admissible meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ $(1 \leq p < \infty)$ in the unit disc Δ such that $\delta(\infty, A) = \liminf_{r \to 1^-} \frac{m(r,A)}{T(r,A)} = \delta > 0$, and let f be a non-zero meromorphic solution of the differential equation*

$$f'' + A(z) f = 0,$$

such that $\delta(\infty, f) > 0$. Moreover, let

$$P[f] = \sum_{j=0}^{k} p_j f^{(j)}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does not vanish identically. If $\varphi \in \mathcal{L}_{p+1,\rho}$ is a non-zero meromorphic function in Δ, and neither $P[f]$ nor $P[f] - \varphi$ vanishes identically, then we have

$$i(f) = i_{\overline{\lambda}}(P[f] - \varphi) = p + 1$$

and

$$\overline{\lambda}_p(P[f] - \varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$$

if $p > 1$, while

$$\rho_p(A) \leq \overline{\lambda}_{p+1}(P[f] - \varphi) \leq \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$.

Remark 1.2 The idea of the proofs of Theorem A is borrowed from the paper of Laine, Rieppo [19] with the modifications reflecting the change from the complex plane $\mathbb{C}$ to the unit disc $\mathbf{\Delta}$.

Before we state our results, we define the sequence of meromorphic functions $\alpha_{i,j}$ $(j = 0, \cdots, k-1)$ in Δ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases} \tag{1.3}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \text{ for } i = 0, \cdots, k-1 \tag{1.4}$$

we define also h_k by

$$h_k = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix}$$

and $\psi_k(z)$ by

$$\psi_k(z) = C_0\varphi + C_1\varphi' + \cdots + C_{k-1}\varphi^{(k-1)},$$

where C_j $(j = 0, \cdots, k-1)$ are finite iterated $p-$order meromorphic functions in Δ depending on $\alpha_{i,j}$, and $\varphi \not\equiv 0$ is a meromorphic function in Δ with $\rho_p(\varphi) < \infty$.

The main purpose of this paper is to study the controllability of solutions of the differential equation (1.1). In the fact we study the growth and oscillation of higher order differential polynomial with meromorphic coefficients in the unit disc Δ generated by solutions of equation (1.1).

Theorem 1.1 *Let $A_i(z)$ $(i = 0, 1, \cdots, k-1)$ be meromorphic functions in Δ of finite iterated $p-$order. Let $d_j(z)$ $(j = 0, 1, \cdots, k)$ be finite iterated $p-$order meromorphic functions in Δ that are not all vanishing identically such that $h \not\equiv 0$. If $f(z)$ is an infinite iterated $p-$order meromorphic solution in Δ of* (1.1) *with $\rho_{p+1}(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho_p(g_k) = \rho_p(f) = \infty$$

and

$$\rho_{p+1}(g_k) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f is a finite iterated $p-$order meromorphic solution in Δ such that

$$\rho_p(f) > \max\{\rho_p(A_i) \quad (i = 0, 1, \cdots, k-1), \rho_p(d_j) \quad (j = 0, 1, \cdots, k)\}, \tag{1.5}$$

then

$$\rho_p(g_k) = \rho_p(f).$$

Remark 1.3 In Theorem 1.1, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 1.1 cannot hold. For example, if we take $d_i = d_k A_i$ $(i = 0, \cdots, k-1)$, then $h \equiv 0$. It follows that $g_k \equiv 0$ and $\rho_p(g_k) = 0$. So, if $f(z)$ is an infinite iterated $p-$order meromorphic solution of (1.1), then $\rho_p(g_k) = 0 < \rho_p(f) = \infty$, and if f is a finite iterated $p-$order meromorphic solution of (1.1) such that (1.5) holds, then $\rho_p(g_k) = 0 < \rho_p(f)$.

Theorem 1.2 *Under the hypotheses of Theorem 1.1, let $\varphi(z) \not\equiv 0$ be a meromorphic function in Δ with finite iterated $p-$order such that $\psi_k(z)$ is not a solution of* (1.1). *If $f(z)$ is an infinite iterated $p-$order meromorphic solution in Δ of* (1.1) *with $\rho_{p+1}(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_p(g_k - \varphi) = \lambda_p(g_k - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_k - \varphi) = \lambda_{p+1}(g_k - \varphi) = \rho.$$

Furthermore, if f is a finite iterated $p-$order meromorphic solution in Δ such that

$$\rho_p(f) > \max\{\rho_p(A_i) \quad (i = 0, 1, \cdots, k-1), \rho_p(d_j) \quad (j = 0, 1, \cdots, k), \rho_p(\varphi)\}, \tag{1.6}$$

then

$$\overline{\lambda}_p\left(g_k-\varphi\right)=\lambda_p\left(g_k-\varphi\right)=\rho_p\left(f\right).$$

From Theorems 1-2, we obtain the following corollaries which have been proved in [23].

Corollary 1.1 [23] *Suppose that $A(z)$ is admissible meromorphic function in Δ such that $i(A)=p$ $(1\leq p<\infty)$ and $\delta(\infty,A)=\delta>0$. Let $d_j(z)$ $(j=0,1,\cdots,k)$ be finite iterated $p-$order meromorphic functions in Δ that are not all vanishing identically such that $h\not\equiv 0$, and let f be a nonzero meromorphic solution of*

$$f^{(k)}+A(z)f=0. \tag{1.7}$$

If $\delta(\infty,f)>0$, then the differential polynomial g_k satisfies $i(g_k)=p+1$ and $\rho_{p+1}(g_k)=\rho_{p+1}(f)=\rho_p(A)$ if $p>1$, while

$$\rho_p(A)\leq\rho_{p+1}(g_k)=\rho_{p+1}(f)\leq\rho_p(A)+1$$

if $p=1$.

Corollary 1.2 [23] *Under the hypotheses of Corollary 1.1, let $\varphi(z)\not\equiv 0$ be meromorphic function in Δ with finite iterated $p-$order such that $\psi_k(z)\not\equiv 0$. Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_{p+1}(g_k-\varphi)=\lambda_{p+1}(g_k-\varphi)=\rho_{p+1}(f)=\rho_p(A)$$

if $p>1$, while

$$\rho_p(A)\leq\overline{\lambda}_{p+1}(g_k-\varphi)=\lambda_{p+1}(g_k-\varphi)=\rho_{p+1}(f)\leq\rho_p(A)+1$$

if $p=1$.

Remark 1.4 The present article may be understood as an extension and improvement of the recent article of the authors [23] from equation (1.7) to equation (1.1). The method used in the proofs of our theorems is simple and quite different from the method used in the papers of Laine and Rieppo [19] and Cao, Xu and Zhu [7].

We consider now the differential equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{1.8}$$

where $A_1(z),A_0(z)$ are analytic functions of finite iterated $p-$order in the unit disc Δ. In the following we will give sufficient conditions on A_1 and A_0 which satisfied the results of Theorem 1.1 and Theorem 1.2 without the conditions " $h_k\not\equiv 0$ " and " $\psi_k(z)$ is not a solution of (1.1) " where $k=2$.

Corollary 1.3 *Let $A_1(z),A_0(z)$ $(\not\equiv 0)$ be analytic functions in Δ such that $\rho_p(A_0)=\rho$ $(0<\rho<\infty)$, $\tau_p(A_0)=\tau$ $(0<\tau<\infty)$, and let $\rho_p(A_1)<\rho_p(A_0)$ or $\tau_p(A_1)<$*

$\tau_p(A_0)$ *if* $\rho_p(A_0) = \rho_p(A_1)$. *Let* d_2, d_1, d_0 *be analytic functions in* Δ *such that at least one of* d_2, d_1, d_0 *does not vanish identically with* $\max\{\rho_p(d_j)\ \ (j = 0, 1, 2)\} < \rho_p(A_0)$. *If* $f \not\equiv 0$ *is a solution of* (1.8), *then the differential polynomial* $g_2 = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\rho_{p+1}(g_2) = \rho_{p+1}(f) = \rho_p(A_0)$ *if* $p > 1$, *while*

$$\rho_p(A_0) \leq \rho_{p+1}(g_2) = \rho_{p+1}(f) \leq \max\{\rho_M(A_j)\ \ (j = 0, 1)\}$$

if $p = 1$.

Corollary 1.4 *Let* $A_1(z)$, $A_0(z)\ (\not\equiv 0)$ *be analytic functions in* Δ *such that* $\rho_p(A_0) = \rho\ (0 < \rho < \infty)$, $\tau_p(A_0) = \tau\ (0 < \tau < \infty)$, *and let* $\rho_p(A_1) < \rho_p(A_0)$ *or* $2\tau_p(A_1) < \tau_p(A_0)$ *if* $\rho_p(A_0) = \rho_p(A_1)$. *Let* d_2, d_1, d_0 *be analytic functions in* Δ *such that at least one of* d_2, d_1, d_0 *does not vanish identically with* $\max\{\rho_p(d_j)\ \ (j = 0, 1, 2)\} < \rho_p(A_1)$, *and let* $\varphi(z) \not\equiv 0$ *be analytic function in* Δ *of finite iterated* $p-$*order such that* $\psi_2(z) \not\equiv 0$. *If* $f \not\equiv 0$ *is a solution of* (1.8), *then the differential polynomial* $g_2 = d_2 f'' + d_1 f' + d_0 f$ *satisfies*

$$\overline{\lambda}_{p+1}(g_2 - \varphi) = \lambda_{p+1}(g_2 - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$$

if $p > 1$, *while*

$$\rho_p(A_0) \leq \overline{\lambda}_{p+1}(g_2 - \varphi) = \lambda_{p+1}(g_2 - \varphi) = \rho_{p+1}(f) \leq \max\{\rho_M(A_j)\ \ (j = 0, 1)\}$$

if $p = 1$.

Remark 1.5 For some papers related in the complex plane see [19, 22, 24] and in the unit disc see [7, 9, 12].

2 Auxiliary lemmas

Lemma 2.1 [8] *Let* $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ *be meromorphic functions in* Δ, *and let* f *be a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z) \tag{2.1}$$

such that $i(f) = p\ (1 \leq p < \infty)$. *If either*

$$\max\{i(A_j)\ \ (j = 0, 1, \cdots, k-1), i(F)\} < p$$

or

$$\max\{\rho_p(A_j)\ \ (j = 0, 1, \cdots, k-1), \rho_p(F)\} < \rho_p(f),$$

then

$$i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p$$

and

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f).$$

Using the same arguments as in the proof of Lemma 2.1 (see, the proof of Lemma 2.5 in [8]), we easily obtain the following lemma.

Lemma 2.2 *Let A_0, $A_1, \cdots$, A_{k-1}, $F \not\equiv 0$ be finite iterated $p-$order meromorphic functions in the unit disc Δ. If f is a meromorphic solution with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < \infty$ of equation (2.1), then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Lemma 2.3 [5] *Let $p \geq 1$ be an integer, and let $A_0(z), \cdots, A_{k-1}(z)$ be analytic functions in Δ such that $i(A_0) = p$. If*

$$\max\{i(A_j) : j = 1, \cdots, k-1\} < p$$

or

$$\max\{\rho_p(A_j) : j = 1, \cdots, k-1\} < \rho_p(A_0),$$

then every solution $f \not\equiv 0$ of equation (1.1) satisfies $i(f) = p+1$ and $\rho_p(f) = \infty$, $\rho_p(A_0) \leq \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \max\{\rho_{M,p}(A_j) : j = 0, 1, \cdots, k-1\}$.

Lemma 2.4 [3] *Let f and g be meromorphic functions in the unit disc Δ such that $0 < \rho_p(f), \rho_p(g) < \infty$ and $0 < \tau_p(f), \tau_p(g) < \infty$. Then we have*
(i) *If $\rho_p(f) > \rho_p(g)$, then we obtain*

$$\tau_p(f+g) = \tau_p(fg) = \tau_p(f).$$

(ii) *If $\rho_p(f) = \rho_p(g)$ and $\tau_p(f) \neq \tau_p(g)$, then we get*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f) = \rho_p(g).$$

Lemma 2.5 ([14]) *Let f be a meromorphic function in the unit disc and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 2.6 [2] *Let f be a meromorphic function in the unit disc for which $i(f) = p \geq 1$ and $\rho_p(f) = \beta < \infty$, and let $k \in \mathbb{N}$. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left(\log \frac{1}{1-r}\right)^{\beta+\varepsilon}\right)$$

for all r outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$.

Lemma 2.7 *Let f be a meromorphic function in Δ with iterated order $\rho_p(f) = \rho$ $(0 < \rho < \infty)$ and iterated type $\tau_p(f) = \tau$ $(0 < \tau < \infty)$. Then for any given $\beta < \tau$, there exists a subset E_3 of $[0,1)$ that has an infinite logarithmic measure such that $\log_{p-1} T(r,f) > \beta \left(\frac{1}{1-r}\right)^{\rho}$ holds for all $r \in E_3$.*

Proof. When $p = 1$, the lemma is proved in [21]. Thus we assume $p \geq 2$. By definitions of iterated order and iterated type, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty} \subset [0,1)$ $(r_m \to 1^-)$ satisfying $\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m < r_{m+1}$ and

$$\lim_{m\to\infty} \frac{\log_{p-1} T(r_m, f)}{\left(\frac{1}{1-r_m}\right)^{\rho}} = \tau_p(f).$$

Then there exists a positive integer m_0 such that for all $m > m_0$ and for any given $0 < \varepsilon < \tau_p(f) - \beta$, we have

$$\log_{p-1} T(r_m, f) > (\tau_p(f) - \beta) \left(\frac{1}{1-r_m}\right)^{\rho}. \tag{2.2}$$

For any given $\beta < \tau_p(f) - \varepsilon$, there exists a positive integer m_1 such that for all $m > m_1$ we have

$$\left(1 - \frac{1}{m}\right)^{\rho} > \frac{\beta}{\tau_p(f) - \varepsilon}. \tag{2.3}$$

Take $m \geq m_2 = \max\{m_0, m_1\}$. By (2.2) and (2.3), for any $r \in [r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m]$, we have

$$\begin{aligned} \log_{p-1} T(r,f) &\geq \log_{p-1} T(r_m, f) > (\tau_p(f) - \beta) \left(\frac{1}{1-r_m}\right)^{\rho} \\ &\geq (\tau_p(f) - \beta) \left(1 - \frac{1}{m}\right)^{\rho} \left(\frac{1}{1-r}\right)^{\rho} > \beta \left(\frac{1}{1-r}\right)^{\rho}. \end{aligned}$$

Set $E_3 = \cup_{m=m_2}^{\infty} \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m\right]$, then there holds

$$m_l E_3 = \sum_{m=m_2}^{\infty} \int_{r_m}^{\frac{1}{m} + \left(1-\frac{1}{m}\right) r_m} \frac{dt}{1-t} = \sum_{m=m_2}^{\infty} \log \frac{m}{m-1} = \infty.$$

Lemma 2.8 [16] *Let f be a solution of equation (1.1) where the coefficients $A_j(z)$ $(j = 0, \cdots, k-1)$ are analytic functions in the disc $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$. Let $n_c \in \{1, \cdots, k\}$ be the number of nonzero coefficients $A_j(z)$ $(j = 0, \cdots, k-1)$, and let $\theta \in [0, 2\pi[$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in \Delta_R$ is such that $A_j(z_\theta) \neq 0$ for some $j = 0, \cdots, k-1$, then for all $\nu < r < R$,*

$$\left|f\left(re^{i\theta}\right)\right| \leq C \exp\left(n_c \int_{\nu}^{r} \max_{j=0,\cdots,k-1} \left|A_j\left(te^{i\theta}\right)\right|^{\frac{1}{k-j}} dt\right), \tag{2.4}$$

where $C > 0$ is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,\cdots,k-1} \left(\frac{\left| f^{(j)}(z_\theta) \right|}{(n_c)^j \max\limits_{n=0,\cdots,k-1} |A_n(z_\theta)|^{\frac{j}{k-n}}} \right).$$

Lemma 2.9 [1,14] *Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_4 \subset [0,1)$ for which $\int_{E_4} \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

Lemma 2.10 *Let $A_1(z)$ and $A_0(z)$ be analytic functions in Δ such that $\rho_p(A_0) = \rho$ $(0 < \rho < \infty)$, $\tau_p(A_0) = \tau$ $(0 < \tau < \infty)$, and let $\rho_p(A_1) < \rho_p(A_0)$ and $\tau_p(A_1) < \tau_p(A_0)$ if $\rho_p(A_1) = \rho_p(A_0)$. If $f \not\equiv 0$ is a solution of (1.8) then $\rho_p(f) = \infty$, $\rho_{p+1}(f) = \rho_p(A_0)$ if $p > 1$, while*

$$\rho_p(f) = \infty,\ \rho_p(A_0) \leq \rho_{p+1}(f) \leq \max\{\rho_M(A_j), (j=0,1)\}$$

if $p = 1$.

Proof. If $\rho_p(A_1) < \rho_p(A_0)$ then the result can easily deduced by Lemma 2.3. We prove only the case when $\rho_p(A_0) = \rho_p(A_1) = \rho$ and $\tau_p(A_1) < \tau_p(A_0)$. Since $f \not\equiv 0$, then by (1.8) we have

$$A_0 = -\left(\frac{f''}{f} + A_1 \frac{f'}{f}\right). \tag{2.5}$$

Suppose that f is of finite $p-$iterated order, then by Lemma 2.6

$$T(r, A_0) \leq T(r, A_1) + O\left(\exp_{p-2}\left(\log \frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \quad (\rho_p(f) = \beta < \infty) \tag{2.6}$$

which implies the contradiction

$$\tau_p(A_0) \leq \tau_p(A_1).$$

Hence $\rho_p(f) = \infty$. By using inequality (2.4), we have

$$\rho_{p+1}(f) \leq \max\{\rho_p(A_1), \rho_p(A_0)\} = \rho_p(A_0). \tag{2.7}$$

On the other hand, by Lemma 2.5

$$T(r, A_0) \leq T(r, A_1) + O\left(\log^+ T(r,f) + \log \frac{1}{1-r}\right) \tag{2.8}$$

holds possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$. By $\tau_p(A_1) < \tau_p(A_0)$ we choose α_0, α_1 satisfying $\tau_p(A_1) < \alpha_1 < \alpha_0 < \tau_p(A_0)$ such that for $r \to 1^-$, we have

$$T(r, A_1) \leq \exp_{p-1}\left\{\alpha_1 \left(\frac{1}{1-r}\right)^{\rho}\right\}. \tag{2.9}$$

By Lemma 2.7, there exists a subset $E_2 \subset [0,1)$ of infinite logarithmic measure such that

$$T(r, A_0) > \exp_{p-1}\left\{\alpha_0 \left(\frac{1}{1-r}\right)^{\rho}\right\}. \tag{2.10}$$

By (2.8) – (2.10) we obtain for all $r \in E_2 - E_1$

$$(1 - o(1)) \exp_{p-1}\left\{\alpha_0 \left(\frac{1}{1-r}\right)^{\rho}\right\} \leq O\left(\log^{+} T(r, f) + \log \frac{1}{1-r}\right). \tag{2.11}$$

By using (2.11) and Lemma 2.9, we obtain

$$\rho_{p+1}(f) \geq \rho_p(A_0). \tag{2.12}$$

From (2.7) and (2.12) we get $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0)$.

3 Proof of the Theorems and the Corollaries

Proof of Theorem 1.1 Suppose that f is an infinite iterated p-ordder meromorphic solution in Δ of (1.1). By (1.1), we have

$$f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)} \tag{3.1}$$

which implies

$$\begin{aligned} g_k &= d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f \\ &= \sum_{i=0}^{k-1} (d_i - d_k A_i) f^{(i)}. \end{aligned} \tag{3.2}$$

We can write (3.2) as

$$g_k = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.3}$$

where $\alpha_{i,0}$ are defined in (1.4). Differentiating both sides of equation (3.3) and replacing $f^{(k)}$ with $f^{(k)} = -\sum\limits_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$\begin{aligned} g_k' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} \end{aligned}$$

$$= \left(\alpha'_{0,0} - \alpha_{k-1,0}A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha'_{i,0} + \alpha_{i-1,0} - \alpha_{k-1,0}A_i\right) f^{(i)}. \tag{3.4}$$

We can rewrite (3.4) as

$$g'_k = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha'_{i,0} + \alpha_{i-1,0} - \alpha_{k-1,0}A_i, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,0} - A_0\alpha_{k-1,0}, \text{ for } i = 0. \end{cases} \tag{3.6}$$

Differentiating both sides of equation (3.5) and replacing $f^{(k)}$ with $f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$g''_k = \sum_{i=0}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)}$$

$$= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)}$$

$$= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)}$$

$$= \left(\alpha'_{0,1} - \alpha_{k-1,1}A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha'_{i,1} + \alpha_{i-1,1} - A_i\alpha_{k-1,1}\right) f^{(i)} \tag{3.7}$$

which implies that

$$g''_k = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha'_{i,1} + \alpha_{i-1,1} - A_i\alpha_{k-1,1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,1} - A_0\alpha_{k-1,1}, \text{ for } i = 0. \end{cases} \tag{3.9}$$

By using the same method as above we can easily deduce that

$$g_f^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \ j = 0, 1, \cdots, k-1, \tag{3.10}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i\alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0\alpha_{k-1,j-1}, \text{ for } i = 0 \end{cases} \tag{3.11}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \text{ for all } i = 0, 1, \cdots, k-1. \tag{3.12}$$

By (3.3) – (3.12) we obtain the system of equations

$$\left\{\begin{array}{r} g_k = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g'_k = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g''_k = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_k^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{array}\right. \tag{3.13}$$

By Cramer's rule, and since $h_k \not\equiv 0$ we have

$$f = \frac{\begin{vmatrix} g_k & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g'_k & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_k^{(k-1)} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h}. \tag{3.14}$$

Then

$$f = C_0 g_k + C_1 g'_k + \cdots + C_{k-1} g_k^{(k-1)}, \tag{3.15}$$

where C_j are finite iterated $p-$order meromorphic functions in Δ depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ is defined in (3.11).

If $\rho_p(g_k) < +\infty$, then by (3.15) we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_k) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_k) = \rho_{p+1}(f) = \rho$. By (3.2), we get $\rho_{p+1}(g_k) \leq \rho_{p+1}(f)$ and by (3.15) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_k)$. This yield $\rho_{p+1}(g_k) = \rho_{p+1}(f) = \rho$.

Furthermore, if f is a finite iterated $p-$order meromorphic solution in Δ of equation (1.1) such that

$$\rho_p(f) > \max\{\rho_p(A_i) \quad (i = 0, \cdots, k-1), \rho_p(d_j) \quad (j = 0, 1, \cdots, k)\}, \tag{3.16}$$

then

$$\rho_p(f) > \max\{\rho_p(\alpha_{i,j}) : i = 0, \cdots, k-1, j = 0, \cdots, k-1\}. \tag{3.17}$$

By (3.2) and (3.16) we have $\rho_p(g_k) \leq \rho_p(f)$. Now, we prove $\rho_p(g_k) = \rho_p(f)$. If $\rho_p(g_k) < \rho_p(f)$, then by (3.15) and (3.17) we get

$$\rho_p(f) \leq \max\{\rho_p(C_j) \quad (j = 0, \cdots, k-1), \rho_p(g_k)\} < \rho_p(f)$$

and this is a contradiction. Hence $\rho_p(g_k) = \rho_p(f)$.

Remark 3.1 From (3.13), it follows that the condition $h \not\equiv 0$ is equivalent to the condition $g_k, g'_k, \cdots, g_k^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite iterated $p-$order.

Proof of Theorem 1.2 Suppose that f is an infinite iterated $p-$order meromorphic solution in Δ of equation (1.1) with $\rho_{p+1}(f)=\rho$. Set $w(z)=g_k-\varphi$. Since $\rho_p(\varphi)<\infty$, then by Theorem 1.1 we have $\rho_p(w)=\rho_p(g_k)=\infty$ and $\rho_{p+1}(w)=\rho_{p+1}(g_k)=\rho$. To prove $\overline{\lambda}_p(g_k-\varphi)=\lambda_p(g_k-\varphi)=\infty$ and $\overline{\lambda}_{p+1}(g_k-\varphi)=\lambda_{p+1}(g_k-\varphi)=\rho$ we need to prove $\overline{\lambda}_p(w)=\lambda_p(w)=\infty$ and $\overline{\lambda}_{p+1}(w)=\lambda_{p+1}(w)=\rho$. By $g_k=w+\varphi$, and using (3.15), we get

$$f=C_0w+C_1w'+\cdots+C_{k-1}w^{(k-1)}+\psi_k(z), \tag{3.18}$$

where

$$\psi_k(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)}. \tag{3.19}$$

Substituting (3.18) into (1.1), we obtain

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=-\left(\psi_k^{(k)}+A_{k-1}(z)\,\psi_k^{(k-1)}+\cdots+A(z)\,\psi_k\right)=H, \tag{3.20}$$

where C_{k-1}, ϕ_j $(j=0,\cdots,2k-1)$ are meromorphic functions in Δ with finite iterated $p-$order. Since $\psi_k(z)$ is not a solution of (1.1), it follows that $H\not\equiv 0$. Then by Lemma 2.2, we obtain $\overline{\lambda}_p(w)=\lambda_p(w)=\infty$ and $\overline{\lambda}_{p+1}(w)=\lambda_{p+1}(w)=\rho$, i. e., $\overline{\lambda}_p(g_k-\varphi)=\lambda_p(g_k-\varphi)=\infty$ and $\overline{\lambda}_{p+1}(g_k-\varphi)=\lambda_{p+1}(g_k-\varphi)=\rho$.

Suppose that f is a finite iterated $p-$order meromorphic solution in Δ of equation (1.1) such that (1.6) holds. Set $w(z)=g_k-\varphi$. Since $\rho_p(\varphi)<\rho_p(f)$, then by Theorem 1.1 we have $\rho_p(w)=\rho_p(g_k)=\rho_p(f)$. To prove $\overline{\lambda}_p(g_k-\varphi)=\lambda_p(g_k-\varphi)=\rho_p(f)$ we need to prove $\overline{\lambda}_p(w)=\lambda_p(w)=\rho_p(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=-\left(\psi_k^{(k)}+A_{k-1}(z)\,\psi_k^{(k-1)}+\cdots+A(z)\,\psi_k\right)=F,$$

where C_{k-1}, ϕ_j $(j=0,\cdots,2k-1)$ are meromorphic functions in Δ with finite iterated $p-$order $\rho_p(C_{k-1})<\rho_p(f)$, $\rho_p(\phi_j)<\rho_p(f)$ $(j=0,\cdots,2k-1)$, and

$$\psi_k(z)=C_0\varphi+C_1\varphi'+\cdots+C_{k-1}\varphi^{(k-1)},\ \rho_p(F)<\rho_p(f).$$

Since $\psi_k(z)$ is not a solution of (1.1), it follows that $F\not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}_p(w)=\lambda_p(w)=\rho_p(f)$, i. e., $\overline{\lambda}_p(g_k-\varphi)=\lambda_p(g_k-\varphi)=\rho_p(f)$.

Proof of Corollary 1.3 Suppose that f is a nontrivial solution of (1.8). Then by Lemma 2.3, we have

$$\rho_p(A_0)\le\rho_{p+1}(f)\le\max\{\rho_{M,p}(A_j)\ \ (j=0,1)\}\ \ (p\ge 1).$$

By the same reasoning as before we obtain that

$$\begin{cases} g_2=\alpha_{0,0}f+\alpha_{1,0}f', \\ g_2'=\alpha_{0,1}f+\alpha_{1,1}f', \end{cases}$$

where

$$\alpha_{0,0} = d_0 - d_2 A_0,\ \alpha_{1,1} = d_2 A_1^2 - (d_2 A_1)' - d_1 A_1 - d_2 A_0 + d_0 + d_1'$$

and

$$\alpha_{0,1} = d_2 A_0 A_1 - (d_2 A_0)' - d_1 A_0 + d_0',\ \alpha_{1,0} = d_1 - d_2 A_1.$$

First, we suppose that $d_2 \not\equiv 0$. We have

$$h_2 = \begin{vmatrix} \alpha_{1,0} & \alpha_{0,0} \\ \alpha_{1,1} & \alpha_{0,1} \end{vmatrix}$$

$$\begin{aligned} = \quad & -d_2^2 A_0^2 - d_0 d_2 A_1^2 + \left(-d_2 d_1 + d_1' d_2 + 2 d_0 d_2 - d_1^2\right) A_0 \\ & + \left(d_2' d_0 - d_2 d_0' + d_0 d_1\right) A_1 + d_1 d_2 A_0 A_1 - d_1 d_2 A_0' + d_0 d_2 A_1' \\ & + d_2^2 A_0' A_1 - d_2^2 A_0 A_1' + d_0' d_1 - d_0 d_1' - d_0^2. \end{aligned}$$

By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.4, we have $\rho_p(h) = \rho_p(A_0)$. Hence $h \not\equiv 0$. Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0, d_1 \equiv 0$ and $d_0 \not\equiv 0$, then by using a similar reasoning as above we get $h \not\equiv 0$, and we obtain

$$f = \frac{\alpha_{1,0} g_f' - \alpha_{1,1} g_f}{h_2}. \tag{3.21}$$

It is clear that $\rho_p(g_2) \le \rho_p(f)$ $(\rho_{p+1}(g_2) \le \rho_{p+1}(f))$ and by (3.21) we have $\rho_p(f) \le \rho_p(g_2)$ $(\rho_{p+1}(f) \le \rho_{p+1}(g_2))$. Hence $\rho_p(g_2) = \rho_p(f)$ $(\rho_{p+1}(g_2) = \rho_{p+1}(f))$.

Proof of Corollary 1.4 Set $w(z) = d_2 f'' + d_1 f' + d_0 f - \varphi$. Then, by $\rho_p(\varphi) < \infty$, we have $\rho_p(w) = \rho_p(g_2) = \rho_p(f)$ and $\rho_{p+1}(w) = \rho_{p+1}(g_2) = \rho_{p+1}(f)$. In order to prove $\overline{\lambda}_{p+1}(g_2 - \varphi) = \lambda_{p+1}(g_2 - \varphi) = \rho_{p+1}(f)$, we need to prove only $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(f)$. Using $g_2 = w + \varphi$, we get from (3.21)

$$f = \frac{\alpha_{1,0} w' - \alpha_{1,1} w}{h_2} + \psi_2, \tag{3.22}$$

where

$$\psi_2(z) = \frac{\alpha_{1,0} \varphi' - \alpha_{1,1} \varphi}{h_2}. \tag{3.23}$$

Substituting (3.22) into equation (1.8), we obtain

$$\begin{aligned} & \frac{\alpha_{1,0}}{h_2} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w \\ = & -\left(\psi_2'' + A_1(z)\psi_2' + A_0(z)\psi_2\right) = A, \end{aligned} \tag{3.24}$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions in Δ with $\rho_p(\phi_j) < \infty$ $(j = 0,1,2)$. First, we prove that $\psi_2 \not\equiv 0$. Suppose that $\psi_2 \equiv 0$, then by (3.23) we obtain

$$\alpha_{1,1} = \alpha_{1,0} \frac{\varphi'}{\varphi}. \tag{3.25}$$

It follows that by using Lemma 2.6

$$m(r,\alpha_{1,1}) \leq m(r,\alpha_{1,0}) + O\left(\exp_{p-2}\left(\log\frac{1}{1-r}\right)^{\beta+\varepsilon}\right), \quad \rho_p(\varphi)=\beta<\infty. \tag{3.26}$$

(i) If $d_2 \not\equiv 0$, then by using Lemma 2.4 we obtain the contradiction

$$\begin{cases} \rho_p(A_0) \leq \rho_p(A_1), & \text{if } \rho_p(A_0) > \rho_p(A_1), \\ \tau_p(A_0) \leq \tau_p(A_1), & \text{if } \rho_p(A_0) = \rho_p(A_1). \end{cases}$$

(ii) If $d_2 \equiv 0$ and $d_1 \not\equiv 0$, we obtain the contradiction

$$\rho_p(A_1) \leq \rho_p(d_1).$$

(iii) If $d_2 = d_1 \equiv 0$ and $d_0 \not\equiv 0$, we have by (3.25)

$$d_0 = \alpha_{1,1} = \alpha_{1,0}\frac{\varphi'}{\varphi} = 0 \times \frac{\varphi'}{\varphi} \equiv 0,$$

which is a contradiction. Hence $\psi_2 \not\equiv 0$. It is clear now that $\psi_2 \not\equiv 0$ cannot be a solution of (1.8) because $\rho_p(\psi_2) < \infty$. Then, by Lemma 2.1, we obtain $\overline{\lambda}_{p+1}(g_2-\varphi) = \lambda_{p+1}(g_2-\varphi) = \rho_{p+1}(f)$, i. e., $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(f)$.

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On differential sandwich theorems of analytic functions defined by certain generalized linear operator

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ABSTRACT: In this paper, we obtain some applications of first order differential subordination and superordination results involving certain linear operator and other linear operators for certain normalized analytic functions. Some of our results improve and generalize previously known results.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \ (a \in C). \tag{1.1}$$

For simplicity $H[a] = H[a, 1]$. Also, let $\mathcal{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If $f,\ g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g.,[6], [16] and [17]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{1.3}$$

then $p(z)$ is a solution of the differential subordination (1.3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp^{'}(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{1.4}$$

then $p(z)$ is a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.4) is called the best subordinant. Using the results of Miller and Mocanu [17], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [25] obtained a sufficient condition for starlikeness of f in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [24] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [24] also obtained results for functions defined by using Carlson-Shaffer operator [7], Ruscheweyh derivative [20] and Sălăgean operator [22].

For functions f given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.5}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For functions $f, g \in \mathcal{A}$, we define the linear operator $D_\lambda^n : \mathcal{A} \to \mathcal{A}$ $(\lambda \geq 0, l \geq 0; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$ by:

$$D_{\lambda,l}^0 (f * g)(z) = (f * g)(z)\,,$$

$$\begin{aligned} D_{\lambda,l}^1 (f * g)(z) &= D_{\lambda,l}(f * g)(z) = (1-\lambda)(f * g)(z) + \frac{\lambda}{(l+1)\, z^{l-1}} (z^l\,(f * g)(z))'\,, \\ &= z + \sum_{k=2}^{\infty} \left[\frac{l+1+\lambda(k-1)}{l+1}\right] a_k b_k z^k \quad (\lambda \geq 0; l \geq 0)\,, \end{aligned} \tag{1.6}$$

and (in general)

$$\begin{aligned} D_{\lambda,l}^n (f * g)(z) &= D_{\lambda,l}(D_{\lambda,l}^{n-1}(f * g)(z)) \\ &= z + \sum_{k=2}^{\infty} \left[\frac{l+1+\lambda(k-1)}{l+1}\right]^n a_k b_k z^k \end{aligned} \tag{1.7}$$

$$(\lambda \geq 0; l \geq 0; n \in \mathbb{N}_0)\,.$$

From (1.7), we can easily deduce that

$$\lambda z \left(D_{\lambda,l}^n (f * g)(z)\right)' = (l+1)\, D_\lambda^{n+1}(f * g)(z) - (l+1-\lambda) D_{\lambda,l}^n (f * g)(z) \tag{1.8}$$

$$(\lambda > 0; l \geq 0; n \in \mathbb{N}_0).$$

We observe that the linear operator $D_{\lambda,l}^n (f * g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of n, λ, l and the function g :

(i) $D_{\lambda,0}^n (f * g)(z) = D_\lambda^n (f * g)(z)$, where $D_\lambda^n (f * g)(z)$ is linear operator which was defined by Aouf and Mostafa [3] ;

(ii) For $g\,(z) = \frac{z}{1-z}$, we have $D_{\lambda,l}^n (f * g)(z) = I(n, \lambda, l) f(z)$, where $I(n, \lambda, l)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [8] ;

(iii) For $\lambda = 1$ and $g\,(z) = \frac{z}{1-z}$, we see that $D_{1,l}^n (f * g)(z) = I(n, l) f\,(z)$, where $I(n, l) f\,(z)$ is the multiplier transformation (see [9]);

(iv) For $l = 0$ and $g\,(z) = \frac{z}{1-z}$, we see that $D_{\lambda,0}^n (f * g)(z) = D_\lambda^n f(z)$ where D_λ^n is the generalized Sălăgean operator (or Al-Oboudi operator [2]) which yield Sălăgean operator D^n for $\lambda = 1$ introduced and studied by Sălăgean [22];

(v) For $l = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \Gamma_k\,[a_1; b_1]\, z^k, \tag{1.9}$$

$$\Gamma_k\,[a_1; b_1] = \frac{(a_1)_{k-1} ... (a_q)_{k-1}}{(b_1)_{k-1} ... (b_s)_{k-1} (1)_{k-1}} \tag{1.10}$$

$$\left(a_i \in \mathbb{C}; i = 1, ..., q; b_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\}\,; j = 1, ..., s; q \leq s+1; q, s \in \mathbb{N}_0\right),$$

where

$$(x)_k = \begin{cases} 1 & (k = 0; x \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ x(x+1)...(x+k-1) & (k \in \mathbb{N}; x \in \mathbb{C}), \end{cases}$$

we have $D^n_{\lambda,0}(f * g)(z) = D^n_\lambda(a_1, b_1)f(z)$, where $D^n_\lambda(a_1, b_1)$ is the linear operator which was introduced and studied by Selvaraj and Karthikeyan [23]. The operator $D^n_\lambda(a_1, b_1)f(z)$, contains in turn many interesting operators such as, Dziok-Srivastava operator [10] (see also [11]), Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [7] and [21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [4], [14] and [15]) and Owa-Srivastava fractional derivative operator (see [19]);

(iv) For $g(z)$ of the form (1.9), we obtain

$$D^n_{\lambda,l}(f * g)(z) = I^{n,l}_{q,s,\lambda}(a_1, b_1)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{l+1+\lambda(k-1)}{l+1}\right]^n \Gamma_k [a_1; b_1] z^k, \quad (1.11)$$

where the operator $I^{n,l}_{q,s,\lambda}(a_1, b_1)f(z)$ is introduced and studied by El-Ashwah and Aouf [12].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $D^n_{\lambda,l}(f * g)$ and some of its special chooses of n, l, λ and the function $g(z)$.

2. Definitions and Preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 1 [17]. Denote by Q, the set of all functions f that are analytic and injective on $U\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$.

Lemma 1 [24]. Let $q(z)$ be univalent in U with $q(0) = 1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^*$, further assume that

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\alpha}{\gamma}\right)\right\}. \quad (2.1)$$

If $p(z)$ is analytic in U, and

$$\alpha p(z) + \gamma z p'(z) \prec \alpha q(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2 [24]. Let $q(z)$ be convex univalent in U, $q(0)=1$. Let $\alpha\in\mathbb{C}$; $\gamma\in\mathbb{C}^*$ and $\Re\left(\frac{\alpha}{\gamma}\right)>0$. If $p(z)\in H[q(0),1]\cap Q$, $\alpha p(z)+\gamma zp^{'}(z)$ is univalent in U and

$$\alpha q(z)+\gamma zq^{'}(z)\prec \alpha p(z)+\gamma zp^{'}(z),$$

then $q(z)\prec p(z)$ and $q(z)$ is the best subordinant.

3. Sandwich Results

Unless otherwise mentioned, we assume throughout this paper that $l\geq 0,\lambda>0,n\in\mathbb{N}_0$ and $g(z)$ is given by (1.5).

Theorem 1. Let $q(z)$ be univalent in U with $q(0)=1$, and $\gamma\in\mathbb{C}^*$. Further, assume that

$$\Re\left\{1+\frac{zq^{''}(z)}{q^{'}(z)}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{3.1}$$

If $f,g\in\mathcal{A}$ satisfy the following subordination condition:

$$\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}+\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z)D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\}$$

$$\prec q(z)+\gamma zq^{'}(z), \tag{3.2}$$

then

$$\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}\prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z)=\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}\qquad (z\in U). \tag{3.3}$$

Then the function $p(z)$ is analytic in U and $p(0)=1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.8) in the resulting equation, we have

$$\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}+\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z)D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\}=p(z)+\gamma zp^{'}(z),$$

that is,

$$p(z)+\gamma zp^{'}(z)\prec q(z)+\gamma zq^{'}(z).$$

Therefore, Theorem 1 now follows by applying Lemma 1. ■

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. Let $\gamma \in \mathbb{C}^*$ and

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f, g \in \mathcal{A}$ satisfy the following subordination condition:

$$\begin{aligned} \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)} &+ \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{D^n_{\lambda,l}(f*g)(z)D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\} \\ &\prec \frac{1+Az}{1+Bz} + \gamma\frac{(A-B)z}{(1+Bz)^2}, \end{aligned}$$

then

$$\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)} \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $g(z) = \frac{z}{1-z}$ in Theorem 1,we obtain the following subordination result for the generalized multiplier transformation $I(n, \lambda, l)$.

Corollary 2. Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:

$$\begin{aligned} \frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)} &+ \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{I(n,\lambda,l)f(z)I(n+2,\lambda,l)f(z)}{[I(n+1,\lambda,l)f(z)]^2}\right\} \\ &\prec q(z) + \gamma z q'(z), \end{aligned}$$

then

$$\frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $g(z)$ of the form (1.9) in Theorem 1, we obtain the following subordination result for the operator $I^{n,l}_{q,s,\lambda}(a_1; b_1)$.

Corollary 3. Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:

$$\begin{aligned} \frac{I^{n,l}_{q,s,\lambda}(a_1,b_1)f(z)}{I^{n+1,l}_{q,s,\lambda}(a_1,b_1)f(z)} &+ \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{I^{n,l}_{q,s,\lambda}(a_1,b_1)f(z)I^{n+2,l}_{q,s,\lambda}(a_1,b_1)f(z)}{\left[I^{n+1,l}_{q,s,\lambda}(a_1,b_1)f(z)\right]^2}\right\} \\ &\prec q(z) + \gamma z q'(z), \end{aligned}$$

then

$$\frac{I_{q,s,\lambda}^{n,l}(a_1,b_1)f(z)}{I_{q,s,\lambda}^{n+1,l}(a_1,b_1)f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $l = 0, \lambda = 1$ and $g(z) = \dfrac{z}{1-z}$ in Theorem 1, we obtain the following subordination result for Sălăgean operator which improves the result of Shanmugam et al. [24, Theorem 5.1] and obtained by Nechita [18].

Corollary 4 [18, Corollary 7]. Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that (3.1) holds. If $f \in \mathcal{A}$ satisfies the following subordination condition:

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{1 - \frac{D^n f(z) D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2}\right\} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{D^n f(z)}{D^{n+1} f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Now, by appealing to Lemma 2 it can be easily prove the following theorem.

Theorem 2. Let $q(z)$ be convex univalent in U with $q(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$. If $f, g \in \mathcal{A}$ such that $\frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)} \in H[q(0),1] \cap Q$,

$$\frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)} + \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{D_{\lambda,l}^{n}(f*g)(z).D_{\lambda,l}^{n+2}(f*g)(z)}{\left[D_{\lambda,l}^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z) + \gamma z q'(z) \prec \frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)} + \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{D_{\lambda,l}^{n}(f*g)(z).D_{\lambda,l}^{n+2}(f*g)(z)}{\left[D_{\lambda,l}^{n+1}(f*g)(z)\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)}$$

and $q(z)$ is the best subordinant.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we obtain the following corollary.

Corollary 5. Let $\gamma \in \mathbb{C}$ with $\Re(\bar{\gamma}) > 0$. If $f, g \in \mathcal{A}$ such that $\frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)} \in H[q(0),1] \cap Q$,

$$\frac{D_{\lambda,l}^{n}(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)} + \frac{\gamma(l+1)}{\lambda}\left\{1 - \frac{D_{\lambda,l}^{n}(f*g)(z)D_{\lambda,l}^{n+2}(f*g)(z)}{\left[D_{\lambda,l}^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^2} \prec \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)} +\frac{\gamma\,(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z)D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz}\prec\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}$$

and $q\,(z)$ is the best subordinant.

Taking $g\,(z)=\frac{z}{1-z}$ in Theorem 2, we obtain the following superordination result for the generalized multiplier transformation $I(n,\lambda,l)$.

Corollary 6. Let $q\,(z)$ be convex univalent in U with $q\,(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re\,(\bar{\gamma})>0$. If $f,g\in\mathcal{A}$ such that $\frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)}\in H\,[q\,(0)\,,1]\cap Q$,

$$\frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)}+\frac{\gamma\,(l+1)}{\lambda}\left\{1-\frac{I(n,\lambda,l)f(z).I(n+2,\lambda,l)f(z)}{[I(n+1,\lambda,l)f(z)]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q\,(z)+\gamma zq^{'}\,(z)\prec\frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)}+\frac{\gamma\,(l+1)}{\lambda}\left\{1-\frac{I(n,\lambda,l)f(z).I(n+2,\lambda,l)f(z)}{[I(n+1,\lambda,l)f(z)]^2}\right\}$$

holds, then

$$q\,(z)\prec\frac{I(n,\lambda,l)f(z)}{I(n+1,\lambda,l)f(z)}$$

and $q\,(z)$ is the best subordinant.

Taking $g\,(z)$ of the form (1.9) in Theorem 2, we obtain the following superordination result for the operator $I^{n,l}_{q,s,\lambda}(a_1;b_1)$.

Corollary 7. Let $q\,(z)$ be convex univalent in U with $q\,(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re\,(\bar{\gamma})>0$. If $f,g\in\mathcal{A}$ such that $\frac{I^{n,l}_{q,s,\lambda}(a_1,b_1)f(z)}{I^{n+1,l}_{q,s,\lambda}(a_1,b_1)f(z)}\in H\,[q\,(0)\,,1]\cap Q$,

$$\frac{I^{n,l}_{q,s,\lambda}(a_1,b_1)f(z)}{I^{n+1,l}_{q,s,\lambda}(a_1,b_1)f(z)}+\frac{\gamma\,(l+1)}{\lambda}\left\{1-\frac{I^{n,l}_{q,s,\lambda}(a_1,b_1)f(z)I^{n+2,l}_{q,s,\lambda}(a_1,b_1)f(z)}{\left[I^{n+1,l}_{q,s,\lambda}(a_1,b_1)f(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq'(z) \prec \frac{I_{q,s,\lambda}^{n,l}(a_1,b_1)f(z)}{I_{q,s,\lambda}^{n+1,l}(a_1,b_1)f(z)} + \frac{\gamma(l+1)}{\lambda}\left\{1-\frac{I_{q,s,\lambda}^{n,l}(a_1,b_1)f(z)I_{q,s,\lambda}^{n+2,l}(a_1,b_1)f(z)}{\left[I_{q,s,\lambda}^{n+1,l}(a_1,b_1)f(z)\right]^2}\right\}$$

holds, then

$$q(z) \prec \frac{I_{q,s,\lambda}^{n,l}(a_1,b_1)f(z)}{I_{q,s,\lambda}^{n+1,l}(a_1,b_1)f(z)}$$

and $q(z)$ is the best subordinant.

Taking $l=0,\lambda=1$ and $g(z)=\dfrac{z}{1-z}$ in Theorem 2, we obtain the following superordination result for Sălăgean operator which improves the result of Shanmugam et al. [24, Theorem 5.2] and obtained by Nechita [18]..

Corollary 8 [18, Corollary 12]. Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f\in\mathcal{A}$ such that $\frac{D^n f(z)}{D^{n+1}f(z)}\in H[q(0),1]\cap Q$,

$$\frac{D^n f(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^n f(z).D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq'(z)\prec\frac{D^n f(z)}{D^{n+1}f(z)}+\gamma\left\{1-\frac{D^n f(z).D^{n+2}f(z)}{\left[D^{n+1}f(z)\right]^2}\right\}$$

holds, then

$$q(z)\prec\frac{D^n f(z)}{D^{n+1}f(z)}$$

and $q(z)$ is the best subordinant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for the linear operator $D_{\lambda,l}^n(f*g)$.

Theorem 3. Let $q_1(z)$ be convex univalent in U with $q_1(0)=1$, $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0, q_2(z)$ be univalent in U with $q_2(0)=1$, and satisfies (3.1). If $f,g\in\mathcal{A}$ such that $\frac{D_{\lambda,l}^n(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)}\in H[q_2(0),1]\cap Q$,

$$\frac{D_{\lambda,l}^n(f*g)(z)}{D_{\lambda,l}^{n+1}(f*g)(z)}+\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D_{\lambda,l}^n(f*g)(z).D_{\lambda,l}^{n+2}(f*g)(z)}{\left[D_{\lambda,l}^{n+1}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and

$$\begin{aligned} q_1(z)+\gamma z q_1'(z) &\prec \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}+\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z).D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\} \\ &\prec q_2(z)+\gamma z q_2'(z) \end{aligned}$$

holds, then

$$q_1(z)\prec \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_i(z)=\frac{1+A_iz}{1+B_iz}$ $(i=1,2;-1\le B_2\le B_1<A_1\le A_2\le 1)$ in Theorem 3, we obtain the following corollary.

Corollary 9. Let $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0$. If $f,g\in\mathcal{A}$ such that $\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}\in H[q(0),1]\cap Q$,

$$\frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}+\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z).D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\}$$

is univalent in U, and

$$\begin{aligned} \frac{1+A_1z}{1+B_1z}+\gamma\frac{(A_1-B_1)z}{(1+B_1z)^2} &\prec \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)} \\ &\quad +\frac{\gamma(l+1)}{\lambda}\left\{1-\frac{D^n_{\lambda,l}(f*g)(z).D^{n+2}_{\lambda,l}(f*g)(z)}{\left[D^{n+1}_{\lambda,l}(f*g)(z)\right]^2}\right\} \\ &\prec \frac{1+A_2z}{1+B_2z}+\gamma\frac{(A_2-B_2)z}{(1+B_2z)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec \frac{D^n_{\lambda,l}(f*g)(z)}{D^{n+1}_{\lambda,l}(f*g)(z)}\prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are, respectively, the best subordinant and the best dominant.

Taking $l=0,\lambda=1$ and $g(z)=\dfrac{z}{1-z}$ in Theorem 3, we obtain the following sandwich result for Sălăgean operator which improves the result of Shanmugam et al. [24, Theorem 5.3].

Corollary 10. Let $q_1(z)$ be convex univalent in U with $q_1(0)=1$, $\gamma\in\mathbb{C}$ with $\Re(\bar{\gamma})>0, q_2(z)$ be univalent in U with $q_2(0)=1$, and satisfies (3.1). If $f\in\mathcal{A}$ such that $\frac{D^n f(z)}{D^{n+1} f(z)} \in H[q_2(0), 1] \cap Q$,

$$\frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\}$$

is univalent in U, and

$$q_1(z) + \gamma z q_1^{'}(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} + \gamma \left\{ 1 - \frac{D^n f(z).D^{n+2} f(z)}{\left[D^{n+1} f(z)\right]^2} \right\} \prec q_2(z) + \gamma z q_2^{'}(z)$$

holds, then

$$q_1(z) \prec \frac{D^n f(z)}{D^{n+1} f(z)} \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Remarks (i) Combining Corollary 2 and Corollary 6, we obtain similar sandwich theorem for the generalized multiplier transformation $I(n, \lambda, l)$;

(ii) Combining Corollary 3 and Corollary 7, we obtain similar sandwich theorems for the operator $I_{q,s,\lambda}^{n,l}(a_1, b_1)$;

(iii) Taking $l = 0$ and $g(z) = \dfrac{z}{1-z}$ in Theorems 1, 2 and 3, respectively, we obtain the results obtained by Nechita [18, Theorems 5, 10 and Corollary 13, respectively];

(iv) Taking $n = l = 0, \lambda = 1$ and $g(z) = \dfrac{z}{1-z}$ in Theorems 1, 2 and 3, respectively, we obtain the results obtained by Shanmugam et al. [24, Theorems 3.1, 3.2 and Corollary 3.3, respectively].

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Certain subordination results on the convolution of analytic functions

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Abstract: In this paper, certain subordination results on the convolution of finite number of analytic functions are derived. Our results include a sufficiency condition for convexity of the convolution of analytic functions f_i satisfying $f_i' \in \mathcal{P}(\alpha_i)$ $(\alpha_i \leq 1, i = 1, 2, ..., n)$.

AMS Subject Classification: *30C45*
Keywords and Phrases: *Analytic function, Convolution (Hadamard product), Subordination, Convex functions, Zeta function*

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote a class of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, $j \in \mathbb{N}$, let

$$\mathcal{H}[a, j] = \left\{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_j z^j + a_{j+1} z^{j+1} + ... \right\}.$$

We denote the special class of $\mathcal{H}[0, 1]$ by $\mathcal{A}$ whose members are of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k \, z^k, \;\; z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{K}$ denote a subclass of $\mathcal{A}$ whose members are convex (univalent) in $\mathbb{U}$ and satisfying

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \;\; z \in \mathbb{U}.$$

For two functions p, $q \in \mathcal{H}(\mathbb{U})$, we say p is *subordinate* to q, or q is *superordinate* to p in $\mathbb{U}$ and write $p(z) \prec q(z), z \in \mathbb{U}$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$, and $|\omega(z)| < 1, z \in \mathbb{U}$ such that $p(z) = q(\omega(z)), z \in \mathbb{U}$. Furthermore, if the function q is univalent in $\mathbb{U}$, then we have following equivalence:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.2}$$

Let $\mathcal{P}_\alpha$ denote a class of functions $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and

$$p(z) \prec q_\alpha(z) := \frac{1 + (1-2\alpha)\, z}{1-z}, \alpha \leq 1, z \in \mathbb{U}. \tag{1.3}$$

Convolution (or Hadamard product) $*$ of the functions $g_1(z)$ and $g_2(z)$ of the form:

$$g_1\,(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g_2\,(z) = \sum_{k=0}^{\infty} b_k z^k, \tag{1.4}$$

is defined by

$$g_1\,(z) * g_2\,(z) = (g_1 * g_2)\,(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (g_2 * g_1)\,(z)\,. \tag{1.5}$$

In 1973, Rusheweyh and Sheil-Small [3] proved the Pòlya-Schoenberg conjecture which shows that the convolution of two convex functions is again a convex function. Due to this convexity preserving property, attempts are made to involve and study convolutions in the Geometric Function Theory.

In this paper, we derive certain subordination results on the convolution of any finite number of analytic functions. Mainly, by applying the subordination principle, a sufficiency condition for convexity of $\phi(z) := (f_1 * f_2 * ... * f_n)\,(z)$ which is a convolution of analytic functions $f_i \in \mathcal{A}$ $(i = 1, 2, ..., n)$ such that $f_i' \in \mathcal{P}(\alpha_i)$ $(\alpha_i \leq 1, i = 1, 2, ..., n)$ is derived.

In order to obtain our results, we use following theorem of Rusheweyh and Stankiewicz [4]:

Theorem 1 *Let $F, G \in \mathcal{H}(\mathbb{U})$ be any convex univalent functions in $\mathbb{U}$. If $f \prec F$ and $g \prec G$, then*

$$f * g \prec F * G \text{ in } \mathbb{U}.$$

Also, we use a result of Stankiewicz and Stankiewicz [6] which is as follows:

Theorem 2 *If $\alpha \leq 1$ and $\beta \leq 1$, then*

$$P_\alpha * P_\beta = P_\delta$$

where $\delta = 1 - 2\,(1-\alpha)\,(1-\beta)\,.$

2 Main Results

We may easily generalize Theorem 2 for the classes $\mathcal{P}(\alpha_i)$ $(i = 1, 2, ..., n)$ and get the following lemma:

Lemma 1 *If $\alpha_i \leq 1$ $(i = 1, 2, ..., n)$, then*

$$P_{\alpha_1} * P_{\alpha_2} * ... * P_{\alpha_n} = P_\delta$$

where

$$\delta = 1 - 2^{n-1}\,(1-\alpha_1)\,(1-\alpha_2)\,...\,(1-\alpha_n)\,. \tag{2.1}$$

Theorem 3 *Let for each* $i = 1, 2, ..., n,\ f_i \in \mathcal{A}$ *and* $\alpha_i \leq 1$. *If* $f_i' \in \mathcal{P}(\alpha_i)$ *for each* $i = 1, 2, ..., n,$ *and*

$$\phi(z) = (f_1 * f_2 * ... * f_n)(z)$$

then

$$\phi'(z) \prec h(z),\ z \in \mathbb{U}$$

where

$$h(z) = 1 + 2^n (1 - \alpha_1)(1 - \alpha_2) ... (1 - \alpha_n) \left[\frac{z}{2^{n-1}} + \frac{z^2}{3^{n-1}} + ... \right] \tag{2.2}$$

is convex univalent in $\mathbb{U}$.

Proof. Let $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, ..., n$. Then, in view of (1.3), we have for $\alpha_i \leq 1, i = 1, 2, ..., n,\ z \in \mathbb{U}$,

$$f_i'(z) \prec q_{\alpha_i}(z) := \frac{1 + (1 - 2\alpha_i) z}{1 - z} = 1 + 2(1 - \alpha_i) \sum_{k=1}^{\infty} z^k \tag{2.3}$$

where the superordinate functions $q_{\alpha_i}(z)$ for each $i = 1, 2, ..., n$ map the disk $\mathbb{U}$ onto convex univalent regions in the positive half plane. By Theorem 1, we get that

$$f_1'(z) * f_2'(z) * ... * f_n'(z) \prec q_{\alpha_1}(z) * q_{\alpha_2}(z) * ... * q_{\alpha_n}(z),\ z \in \mathbb{U} \tag{2.4}$$

where

$$\begin{aligned} q_{\alpha_1}(z) * q_{\alpha_2}(z) * ... * q_{\alpha_n}(z) &= 1 + 2^n (1 - \alpha_1)(1 - \alpha_2) ... (1 - \alpha_n) \sum_{k=1}^{\infty} z^k \\ &=: \frac{1 + (1 - 2\delta) z}{1 - z}, z \in \mathbb{U} \end{aligned}$$

is convex univalent in $\mathbb{U}$ and in view of Lemma 1, δ is given by (2.1).

We know that the function

$$h_1(z) = -\frac{2}{z} [z + \ln(1 - z)] = \sum_{k=1}^{\infty} \frac{2}{k+1} z^k,\ z \in \mathbb{U}$$

belongs to the class $\mathcal{K}$ and for $f \in \mathcal{A}$

$$(f * h_1)(z) = \frac{2}{z} \int_0^z f(t) \mathrm{d}t.$$

Therefore, the function

$$h_2(z) = 1 + h_1(z),\ z \in \mathbb{U}$$

being a translation of $h_1(z)$, is convex univalent in $\mathbb{U}$ and for $p \in \mathcal{H}[1, 1]$

$$(p * h_2)(z) = -1 + \frac{2}{z} \int_0^z p(t) \mathrm{d}t. \tag{2.5}$$

On applying, Theorem 1, to the subordination (2.4) sequentially, $n-1$ times with the usual subordination: $h_2(z) \prec h_2(z),\ z \in \mathbb{U}$, we get

$$f_1' * f_2' * ... * f_n' * \underbrace{h_2 * h_2 * ... * h_2}_{n-1 \text{ times}} \prec q_{\alpha_1} * q_{\alpha_2} * ... * q_{\alpha_n} * \underbrace{h_2 * h_2 * ... * h_2}_{n-1 \text{ times}}$$

in $\mathbb{U}$, which can also be written as

$$(f_1' * h_2) * (f_2' * h_2) * ... * \left(f_{n-1}' * h_2\right) * f_n' \prec (q_{\alpha_1} * h_2) * (q_{\alpha_2} * h_2) * ... * \left(q_{\alpha_{n-1}} * h_2\right) * q_{\alpha_n}. \tag{2.6}$$

On suitably choosing series expansions of f_i' 's and q_{α_i}'s, in view of (2.5), we observe that the subordination (2.6) reduces to

$$\begin{aligned} & \frac{f_1(z)}{z} * \frac{f_2(z)}{z} * ... * \frac{f_{n-1}(z)}{z} * f_n'(z) \qquad (2.7) \\ \prec\ & \frac{1}{z}\int_0^z q_{\alpha_1}(t)\mathrm{d}t * \frac{1}{z}\int_0^z q_{\alpha_2}(t)\mathrm{d}t * ... * \frac{1}{z}\int_0^z q_{\alpha_{n-1}}(t)\mathrm{d}t * q_{\alpha_n}(z) \\ =\ & h(z),\ z \in \mathbb{U} \end{aligned}$$

where $h(z)$ is convex univalent in $\mathbb{U}$ and is of the form (2.2). The left hand side (2.7) of above subordination is

$$\left(f_1 * f_2 * ... * f_n\right)'(z) = \phi'(z).$$

This proves Theorem 3. ■

As the function $h(z)$ given by (2.2) is convex univalent with real coefficients, we may easily get following result from Theorem 3:

Corollary 1 *Let for each $i = 1, 2, ..., n,\ f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, ..., n$, and*

$$\phi(z) = \left(f_1 * f_2 * ... * f_n\right)(z)$$

then

$$h(-1) \leq \mathrm{Re}\left\{\phi'(z)\right\} \leq h(1),\ z \in \mathbb{U}$$

where $h(z)$ is given by (2.2).

In terms of Zeta function [[7], Ex.5, p.201], we may also find following result from Theorem 3:

Corollary 2 *Let for each $i = 1, 2, ..., n,\ f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, ..., n$, and*

$$\phi(z) = \left(f_1 * f_2 * ... * f_n\right)(z)$$

then for $n > 2$,

$$\mathrm{Re}\left\{\phi'(z)\right\} \geq 1 + 2^n\left(1-\alpha_1\right)\left(1-\alpha_2\right)...\left(1-\alpha_n\right)\left[\left(1-2^{2-n}\right)\zeta\left(n-1\right) - 1\right]$$

and

$$\mathrm{Re}\left\{\phi'(z)\right\} \leq 1 + 2^n (1-\alpha_1)(1-\alpha_2)...(1-\alpha_n)\left[\zeta(n-1)-1\right], \; z \in \mathbb{U}$$

where ζ is well known Zeta Function.

Taking $n = 3$ and writing $\zeta(2) = \frac{\pi^2}{6}$, Corollary 2 provides following result of Sokół [5]:

Corollary 3 *Let for each $i = 1, 2, 3$, $f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, 3$, and*

$$\phi(z) = (f_1 * f_2 * f_3)(z)$$

then

$$\mathrm{Re}\left\{\phi'(z)\right\} \geq 1 + 8(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left[\frac{\pi^2}{12}-1\right], \; z \in \mathbb{U}$$

and

$$\mathrm{Re}\left\{\phi'(z)\right\} \leq 1 + 8(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\left[\frac{\pi^2}{6}-1\right], \; z \in \mathbb{U}$$

To prove our next result, we prove first a lemma which is as follows:

Lemma 2 *Let for each $i = 1, 2, ..., n$, $f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, ..., n$, and*

$$\phi(z) = (f_1 * f_2 * ... * f_n)(z)$$

then there exist some positive integers $\lambda_1, \lambda_2, ..., \lambda_{n-2}$ depending upon n such that

$$\begin{aligned} & \phi'(z) + \lambda_1 z\phi''(z) + \lambda_2 z^2\phi'''(z) + ... + \lambda_{n-2} z^{n-2}\phi^{(n-1)}(z) + z^{n-1}\phi^{(n)}(z) \\ = \; & (f_1' * f_2' * ... * f_n')(z). \end{aligned}$$

Proof. Let $f_i \in \mathcal{A}$ be of the form

$$f_i(z) = z + \sum_{k=2}^{\infty} a_k^i \; z^k, \; z \in \mathbb{U}. \tag{2.8}$$

Then

$$(f_1' * f_2' * ... * f_n')(z) = 1 + \sum_{k=2}^{\infty} k^n d_k \; z^{k-1} \tag{2.9}$$

where

$$d_k := a_k^1 a_k^2 ... a_k^n, k \geq 2. \tag{2.10}$$

We note that for the functions $f_i(z), i = 1, 2, ..., n$, of the form (2.8), the r-th ($r \in \mathbb{N}$) derivative of $\phi(z)$ is given by

$$\phi^{(r)}(z) = \sum_{k=1}^{\infty} k(k-1)...(k-r+1) d_k \; z^{k-r},$$

where $d_1 = 1$ and for $k \geq 2, d_k$ is given by (2.10).

For some positive integers $\lambda_1, \lambda_2, ..., \lambda_{n-2}$ depending only upon n, we have for $k, n \in \mathbb{N}$, an identity :

$$\begin{aligned} k^n &\equiv k + \lambda_1 k(k-1) + \lambda_2 k(k-1)(k-2) + ... \\ &\quad + \lambda_{n-2} k(k-1)...(k-n+2) + k(k-1)...(k-n+1). \end{aligned} \tag{2.11}$$

For the positive integers $\lambda_1, \lambda_2, ..., \lambda_{n-2}$, appear in the identity (2.11) and with the use of this identity, we get

$$\begin{aligned} &\phi'(z) + \lambda_1 z\phi''(z) + \lambda_2 z^2 \phi'''(z) + ... + \lambda_{n-2} z^{n-2} \phi^{(n-1)}(z) + z^{n-1}\phi^{(n)}(z) \\ = &\sum_{k=1}^{\infty} k^n d_k \ z^{k-1}, \text{ where } d_1 = 1. \end{aligned}$$

This is the right hand side of (2.9). This proves Lemma 2. ■

Theorem 4 *Let for each $i = 1, 2, ..., n,\ f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, ..., n,$ and*

$$\phi(z) = (f_1 * f_2 * ... * f_n)(z)$$

then $\phi \in \mathcal{K}$ whenever for $n > 2$,

$$(1-\alpha_1)(1-\alpha_2)...(1-\alpha_n) \leq \frac{3}{2^{n+2}\left[1-(1-2^{2-n})\zeta(n-1)\right]} \tag{2.12}$$

where ζ is well known Zeta Function.

Proof. Let $p(z) = \phi'(z)$, then by Lemma 2 and by (2.4), we get

$$\begin{aligned} &\psi\left(p(z), zp'(z), ..., z^{n-1}p^{(n-1)}(z)\right) &(2.13) \\ = &\ p(z) + \lambda_1 zp'(z) + \lambda_2 z^2 p''(z) + ... + \lambda_{n-2} z^{n-2} p^{(n-2)}(z) + z^{n-1} p^{(n-1)}(z) \\ = &\ (f_1' * f_2' * ... * f_n')(z) \prec \frac{1+(1-2\delta)z}{1-z}, \ z \in \mathbb{U} \end{aligned}$$

where δ is given by (2.1). From Theorem 3, we have a possible solution of the above n-th order Euler-type differential subordination (2.13), as follows

$$\phi'(z) = p(z) \prec h(z), \ z \in \mathbb{U} \tag{2.14}$$

where $h(z)$ is given by (2.2).

The r-th ($r \in \mathbb{N}$) derivative of $h(z)$ is given by

$$h^{(r)}(z) = 2(1-\delta) \sum_{k=1}^{\infty} \frac{k(k-1)...(k-r+1)}{(k+1)^{n-1}} \ z^{k-r}.$$

For the positive integers $\lambda_1, \lambda_2, ..., \lambda_{n-2}$, appearing in the identity (2.11), we observe that

$$\begin{aligned} & \psi\left(h(z), zh'(z), ..., z^{n-1}h^{(n-1)}(z)\right) \\ = \;& h(z) + \lambda_1 zh'(z) + \lambda_2 z^2 h''(z) + ... + \lambda_{n-2} z^{n-2} h^{(n-2)}(z) + z^{n-1} h^{(n-1)}(z) \\ = \;& 1 + 2(1-\delta)\sum_{k=1}^{\infty} z^k = \frac{1 + (1-2\delta)\, z}{1-z}, \;\; z \in \mathbb{U} \end{aligned}$$

where δ is given by (2.1). This verifies the admissiblity condition for $p(z)$ in (2.14) to be a solution of the subordination (2.13).

Now, the function $\phi \in \mathcal{K}$ if

$$1 + \frac{z\phi''(z)}{\phi'(z)} \prec \frac{1+z}{1-z}, \;\; z \in \mathbb{U}$$

or

$$1 + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}, \;\; z \in \mathbb{U}. \tag{2.15}$$

By [[1], Theorem 2.6b, p.60] the condition (2.15) implies

$$p(z) \prec \frac{1}{(1-z)^2}, \;\; z \in \mathbb{U}. \tag{2.16}$$

Thus, in view of (2.14), the function $\phi \in \mathcal{K}$ if

$$h(z) \prec \frac{1}{(1-z)^2}, \;\; z \in \mathbb{U}$$

that is if

$$\min_{z \in \mathbb{U}} \Re\left\{h(z)\right\} = h(-1) \geq \frac{1}{4}$$

which is the given condition (2.12) if we write the expression of $h(-1)$ (as it is written in Corollary 2) in terms of Zeta Function [[7], Ex.5, p.201]. This proves the result of Theorem 4. ■

Taking $n = 3$ in Theorem 4 and on writing $\zeta(2) = \frac{\pi^2}{6}$, we get following result.

Corollary 4 *Let for each $i = 1, 2, 3$, $f_i \in \mathcal{A}$ and $\alpha_i \leq 1$. If $f_i' \in \mathcal{P}(\alpha_i)$ for each $i = 1, 2, 3$, then $(f_1 * f_2 * f_3)(z) \in \mathcal{K}$ whenever*

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \leq \frac{9}{8(12-\pi^2)} \approx 0.53. \tag{2.17}$$

Remark 1 *We remark that Corollary 4 improves the result of Sokół obtained in [[5], Theorem 2, 124].*

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Some inequalities for the polar derivative of a polynomial with restricted zeros

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Abstract: Let $p(z)$ be a polynomial of degree n and for any complex number α, $D_\alpha p(z) = np(z)+(\alpha-z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α. In this paper, we obtain new results concerning maximum modulus of the polar derivative of a polynomial with restricted zeros. Our result generalize certain well-known polynomial inequalities.

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1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree n, then according to Bernstein's inequality on the derivative of a polynomial, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1.1}$$

equality holds in (1.1) if $p(z)$ has all its zeros at the origin.
The inequality (1.1) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, in fact, P. Erdös conjectured and later Lax [9] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is best possible and equality holds in (1.2) for a polynomial which has all its zeros on $|z| = 1$.
If the polynomial $p(z)$ has all its zeros in $|z| \leq 1$, then it was proved by Turan [12] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{1.3}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.
For a polynomial $p(z)$ of degree at most n which having no zeros in $|z| < k,\ k \geq 1$, inequality (1.2) was generalized by Malik [10] who proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.4}$$

The inequality (1.4) is sharp and equality holds for $p(z) = (z+k)^n$.
If the polynomial $p(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then it was proved by Govil[7] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{1.5}$$

The result is best possible and equality holds in (1.5) for $p(z) = z^n + k^n$.
Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, also $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} [\frac{D_\alpha p(z)}{\alpha}] = p'(z). \tag{1.6}$$

In order to extend inequality (1.5) for the polar derivative, Aziz and Rather[1] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha p(z)| \geq n\frac{|\alpha|-k}{1+k^n} \max_{|z|=1} |p(z)|. \tag{1.7}$$

The bounds are obtained depends only on the zero of largest modulus and not on the other zeros even if some of them are close to the origin. Therefore, it would be interesting to obtain a bound, which depends on the location of all the zeros of a polynomial. In this connection we use some known ideas in the literature and obtain the following interesting results.

Theorem 1.1 *Let*

$$p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - z_\nu),\ a_n \neq 0,$$

be a polynomial of degree n, $|z_\nu| \leq k_\nu,\ 1 \leq \nu \leq n$, and $k = \max(k_1, k_2, ..., k_n) \geq 1$.

Then for every real or complex number α *with* $|\alpha| \geq k$*, we have*

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k+k_\nu} [2 \max_{|z|=1} |p(z)| + \frac{k^n-1}{k^n} \min_{|z|=k} |p(z)| \\ + \frac{4|a_{n-1}|}{k(n+1)} (\frac{k^n-1-n(k-1)}{n}) \\ + \frac{4|a_{n-2}|}{k^2} ((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ + \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}} |na_0 + \alpha a_1| + \frac{1}{k^{n-1}} [\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}] |(n-1)a_1 + 2\alpha a_2|,\end{aligned} \tag{1.8}$$

for $n > 3$
and

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k+k_\nu} [2 \max_{|z|=1} |p(z)| + \frac{k^n-1}{k^n} \min_{|z|=k} |p(z)| \\ + \frac{4|a_{n-1}|}{k(n+1)} (\frac{(k^n-1)-n(k-1)}{n}) + \frac{4|a_{n-2}|(k-1)^n}{k^2 n(n-1)}] \\ + \frac{(k^2-1)}{2k^{n-1}} |na_0 + \alpha a_1| + \frac{(k-1)^2}{2k^{n-1}} |(n-1)a_1 + 2\alpha a_2|,\end{aligned} \tag{1.9}$$

for $n = 3$*.*

Since $\frac{k}{k+k_\nu} \geq \frac{1}{2}$ for $1 \leq \nu \leq n$, the above theorem gives the following result which is an improvement of the inequality (1.7).

Corollary 1.2 *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k,\ k \geq 1$*, then for every real or complex number* α *with* $|\alpha| \geq k$*, we have*

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{1+k^n} [\max_{|z|=1} |p(z)| + \frac{k^n-1}{2k^n} \min_{|z|=k} |p(z)| \\ + \frac{2|a_{n-1}|}{(n+1)k} (\frac{k^n-1-n(k-1)}{n}) \\ + \frac{2|a_{n-2}|}{k^2} ((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ + \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}} |na_0 + \alpha a_1| + \frac{1}{k^{n-1}} [\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}] |(n-1)a_1 + 2\alpha a_2|,\end{aligned} \tag{1.10}$$

for $n > 3$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{1+k^n}[&\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ &+ \frac{2|a_{n-1}|}{(n+1)k}(\frac{(k^n-1)-n(k-1)}{n}) + \frac{2|a_{n-2}|(k-1)^n}{k^2n(n-1)}] \\ &+ \frac{(k^2-1)}{2k^{n-1}}|na_0+\alpha a_1| + \frac{(k-1)^2}{2k^{n-1}}|(n-1)a_1+2\alpha a_2|,\end{aligned} \tag{1.11}$$

for $n = 3$.

Dividing both sides of the inequalities (1.10) and (1.11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement of the inequality (1.5).

Corollary 1.3 *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n*, having all its zeros in* $|z| \leq k,\ k \geq 1$*, then for every real or complex number* α *with* $|\alpha| \geq k$*, we have*

$$\begin{aligned}&\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ &+ \frac{2|a_{n-1}|}{(n+1)k}(\frac{k^n-1-n(k-1)}{n}) \\ &+ \frac{2|a_{n-2}|}{k^2}((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ &+ \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}]|2a_2|,\end{aligned} \tag{1.12}$$

for $n > 3$
and

$$\begin{aligned}\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}[&\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ &+ \frac{2|a_{n-1}|}{(n+1)k}(\frac{(k^n-1)-n(k-1)}{n}) + \frac{2|a_{n-2}|(k-1)^n}{k^2n(n-1)}] \\ &+ \frac{(k^2-1)}{2k^{n-1}}|a_1| + \frac{(k-1)^2}{2k^{n-1}}|2a_2|,\end{aligned} \tag{1.13}$$

for $n = 3$.

As an application of Theorem 1.1 we prove the following result.

Theorem 1.4 *Let*

$$p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z-z_\nu),\ a_n \neq 0,$$

be a polynomial of degree n, $|z_\nu| \geq k_\nu$, $1 \leq \nu \leq n$, and $k = \min(k_1, k_2, ..., k_n) \leq 1$. Then for every real or complex number δ with $|\delta| \leq k$, we have

$$\max_{|z|=1} |D_\delta p(z)| \geq \frac{(k-|\delta|)k^{n-1}}{1+k^n} \sum_{\nu=1}^{n} \frac{k_\nu}{k+k_\nu} [2 \max_{|z|=1} |p(z)| + \frac{1-k^n}{k^n} \min_{|z|=k} |p(z)|$$
$$+ \frac{4|a_1|k}{(n+1)} (\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) + 4|a_2|k^2 ((\frac{(1-k^n) - n(k^{n-1}-k^n)}{n(n-1)k^n}) -$$
$$(\frac{(1-k^{n-2}) - (n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \tag{1.14}$$
$$+ \frac{2(1-k^{n-1})}{(n+1)} |na_n + \alpha a_{n-1}| +$$
$$k^{n-1} [\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] |(n-1)a_{n-1} + 2\alpha a_{n-2}|,$$

for $n > 3$
and

$$\max_{|z|=1} |D_\delta p(z)| \geq \frac{(k-|\delta|)k^{n-1}}{1+k^n} \sum_{\nu=1}^{n} \frac{k_\nu}{k+k_\nu} [2 \max_{|z|=1} |p(z)| + \frac{1-k^n}{k^n} \min_{|z|=k} |p(z)|$$
$$+ \frac{4|a_1|k}{(n+1)} (\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) + 4|a_2|k^2 (1-k)^n] \tag{1.15}$$
$$+ \frac{1-k^2}{2} |na_n + \alpha a_{n-1}| + \frac{(1-k)^2}{2} |(n-1)a_{n-1} + 2\alpha a_{n-2}|,$$

for $n = 3$.

Since $\frac{k_\nu}{k+k_\nu} \geq \frac{1}{2}$ for $1 \leq \nu \leq n$, then Theorem 1.4 gives the following result.

Corollary 1.5 *Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n does not vanish in $|z < k$ where $k \leq 1$. Then for every real or complex number δ with $|\delta| \leq k$, we have*

$$\max_{|z|=1} |D_\delta p(z)| \geq \frac{n(k-|\delta|)k^{n-1}}{1+k^n} [\max_{|z|=1} |p(z)| + \frac{1-k^n}{2k^n} \min_{|z|=k} |p(z)|$$
$$+ \frac{2|a_1|k}{(n+1)} (\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) + 2|a_2|k^2 ((\frac{(1-k^n) - n(k^{n-1}-k^n)}{n(n-1)k^n}) -$$
$$(\frac{(1-k^{n-2}) - (n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \tag{1.16}$$
$$+ \frac{2(1-k^{n-1})}{(n+1)} |na_n + \alpha a_{n-1}| +$$
$$k^{n-1} [\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] |(n-1)a_{n-1} + 2\alpha a_{n-2}|,$$

for $n > 3$
and

$$\max_{|z|=1}|D_\delta p(z)| \geq \frac{n(k-|\delta|)k^{n-1}}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{1-k^n}{2k^n}\min_{|z|=k}|p(z)|$$
$$+\frac{2|a_1|k}{(n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) + 2|a_2|k^2(1-k)^n] \tag{1.17}$$
$$+\frac{1-k^2}{2}|na_n+\alpha a_{n-1}| + \frac{(1-k)^2}{2}|(n-1)a_{n-1}+2\alpha a_{n-2}|,$$

for $n = 3$.

2 Lemmas

For proof of the theorems, we need the following lemmas. The first lemma is due to Dewan, Kaur and Mir [4].

Lemma 2.1 *If* $p(z)$ *is a polynomial of degree* n*, then for* $R \geq 1$*,*

$$\max_{|z|=R}|p(z)| \leq R^n\max_{|z|=1}|p(z)| - \frac{2(R^n-1)}{n+2}|p(0)|$$
$$-[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}]|p'(0)| \tag{2.1}$$

if $n > 2$*, and*

$$\max_{|z|=R}|p(z)| \leq R^n\max_{|z|=1}|p(z)| - \frac{R-1}{2}[(R+1)|p(0)| + (R-1)|p'(0)|] \tag{2.2}$$

if $n = 2$.

Lemma 2.2 *If* $p(z)$ *is a polynomial of degree* n*, having no zeros in* $|z| < 1$*, then for* $R \geq 1$ *,*

$$\max_{|z|=R}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)| -$$
$$\frac{2}{n+1}[\frac{(R^n-1)}{n} - (R-1)]|p'(0)| - \tag{2.3}$$
$$[\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}]|p''(0)|$$

if $n > 3$*, and*

$$\max_{|z|=R}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)|$$
$$-\frac{2}{n+1}[\frac{R^n-1}{n} - (R-1)]|p'(0)| \tag{2.4}$$
$$-\frac{(R-1)^n}{n(n-1)}|p''(0)|]$$

if $n = 3$.

This lemma is due to Dewan, Singh and Mir [5].

Lemma 2.3 *If* $p(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu),\ a_n \neq 0$, *is a polynomial of degree* n, *such that* $|z_\nu| \leq 1,\ 1 \leq \nu \leq n$, *then*

$$\max_{|z|=1} |p'(z)| \geq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \max_{|z|=1} |p(z)|. \tag{2.5}$$

This lemma is due to Giroux, Rahman and Schmeisser [8].

Lemma 2.4 *If* $p(z)$ *is a polynomial of degree* n *and* α *is any real or complex number with* $|\alpha| \neq 0$, *then for* $|z| = 1$

$$|D_\alpha q(z)| = |n\overline{\alpha}p(z) + (1 - \overline{\alpha}z)p'(z)|, \tag{2.6}$$

where $q(z) = z^n\overline{p(\frac{1}{\overline{z}})}$.

This lemma is due to Aziz [2].

3 Proofs of the theorems

Proof of the Theorem 1.1. Let $G(z) = p(kz)$. Since all the zeros of $p(z)$ lie in $|z| \leq k$, then all the zeros of $G(z)$ lie in $|z| \leq 1$. Now on applying Lemma 2.3 to the polynomial $G(z)$, we get

$$\max_{|z|=1} |G'(z)| \geq \sum_{\nu=1}^{n} \frac{1}{1 + \frac{|z_\nu|}{k}} \max_{|z|=1} |G(z)|. \tag{3.1}$$

Let $H(z) = z^n\overline{G(1/\overline{z})}$. Then it can be easily verified that for $|z| = 1$

$$|H'(z)| = |nG(z) - zG'(z)|. \tag{3.2}$$

The polynomial $G(z)$ has all its zeros in $|z| \leq 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$, therefore, by Gauss-Lucas theorem for $|z| = 1$, we have

$$|H'(z)| \leq |G'(z)|. \tag{3.3}$$

Now for every real or complex number α with $|\alpha| \geq k$, we have

$$|D_{\frac{\alpha}{k}}G(z)| = |nG(z) - zG'(z) + \frac{\alpha}{k}G'(z)| \geq |\frac{\alpha}{k}||G'(z)| - |nG(z) - zG'(z)|. \tag{3.4}$$

This gives with the help of (3.2) and (3.3) that

$$\max_{|z|=1} |D_{\frac{\alpha}{k}}G(z)| \geq \frac{|\alpha| - k}{k} \max_{|z|=1} |G'(z)|. \tag{3.5}$$

Using (3.1) in (3.4), we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}G(z)| \geq \frac{|\alpha|-k}{k}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|G(z)|. \tag{3.6}$$

Replacing $G(z)$ by $p(kz)$, we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}p(kz)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=1}|p(kz)|. \tag{3.7}$$

which implies

$$\max_{|z|=1}|np(kz)+(\frac{\alpha}{k}-z)kp'(kz)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=1}|p(kz)|.$$

which gives

$$\max_{|z|=k}|D_\alpha p(z)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=k}|p(z)|. \tag{3.8}$$

The polynomial $p(z)$ is of degree $n>3$ and so $D_\alpha p(z)$ is the polynomial of degree $n-1$, where $n-1>2$, hence applying Lemma 2.1 to the polynomial $D_\alpha p(z)$, we get

$$\begin{aligned}\max_{|z|=k}|D_\alpha p(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha p(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1| \\ -[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|.\end{aligned} \tag{3.9}$$

Combining (3.9) and (3.8), we get

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{k^{n-1}}[\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=k}|p(z)|]+ \\ &\frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1|+\frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|.\end{aligned} \tag{3.10}$$

Since the polynomial $p(z)$ has all zeros in $|z|\leq k,\ k\geq 1$, then $q(z)=z^n p(1/z)$ has no zero in $|z|<1/k$, hence the polynomial $q(z/k)$ having no zeros in $|z|<1$, therefore on applying Lemma 2.2 to the polynomial $q(z/k)$, we get

$$\begin{aligned}&\max_{|z|=k}|q(z/k)| \leq \\ &[\frac{k^n+1}{2}]\max_{|z|=1}|q(z/k)-(\frac{k^n-1}{2})\min_{|z|=1}|q(z/k)|-\frac{2|a_{n-1}|}{(n+1)k}[\frac{k^n-1}{n}-(k-1)] \\ &-\frac{2|a_{n-2}|}{k^2}[(\frac{(k^n-1)-n(k-1)}{n(n-1)})-(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)})].\end{aligned} \tag{3.11}$$

Since $\max_{|z|=1}|q(z/k)| = (1/k^n)\max_{|z|=k}|p(z)|$, and $\min_{|z|=1}|q(z/k)| = (1/k^n)\min_{|z|=k}|p(z)|$, then (3.11) is equivalent to

$$\begin{aligned}\max_{|z|=k}|p(z)| &\geq (\frac{2k^n}{k^n+1})\max_{|z|=1}|p(z) + (\frac{k^n-1}{k^n+1})\min_{|z|=k}|p(z)|\\ &+ \frac{4k^{n-1}|a_{n-1}|}{(n+1)(k^n+1)}[\frac{k^n-1}{n}-(k-1)]\\ &+ \frac{4k^{n-2}|a_{n-2}|}{k^n+1}[(\frac{(k^n-1)-n(k-1)}{n(n-1)})-(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)})].\end{aligned}\tag{3.12}$$

Combining (3.10) and (3.12), we get

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{|\alpha|-k}{k^{n-1}}\sum_{\nu=1}^{n}\frac{1}{k+k_\nu}[(\frac{2k^n}{k^n+1})\max_{|z|=1}|p(z)|\\ &+(\frac{k^n-1}{k^n+1})\min_{|z|=k}|p(z)| + \frac{4k^{n-1}|a_{n-1}|}{(n+1)(k^n+1)}(\frac{k^n-1}{n}-(k-1))\\ &+\frac{4k^{n-2}|a_{n-2}|}{k^n+1}((\frac{(k^n-1)-n(k-1)}{n(n-1)})-(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))]\\ &+\frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}]\\ &\times |(n-1)a_1+2\alpha a_2|.\end{aligned}\tag{3.13}$$

which completes the proof of (1.10). The proof of the Theorem 1.1 in the case $n=3$ follows along the same lines as the proof of (1.10) but instead of inequalities (2.1) and (2.3), we use inequalities (2.2) and (2.4), respectively. $\square$

Proof of the Theorem 1.4. Let $q(z) = z^n\overline{p(1/\overline{z})}$ then $1/|z_\nu| \leq 1/k_\nu$ for $1 \leq \nu \leq n$ such that $1/k = \max(1/k_1, 1/k_2, \cdots, 1/k_n) \geq 1$. On applying Theorem 1.1 to the polynomial $q(z)$, we get

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq (|\alpha|-1/k)k^{n-1}\sum_{\nu=1}^{n}\frac{1}{1/k+1/k_\nu}[(\frac{2/k^n}{1/k^n+1})\max_{|z|=1}|q(z)|\\ &+(\frac{1/k^n-1}{1/k^n+1})\min_{|z|=1/k}|q(z)| + \frac{4|a_1|}{(1+1/k^n)k^{n-1}(n+1)}(\frac{(1/k^n-1)-n(1/k-1)}{n})\\ &+\frac{4|a_2|}{k^{n-2}(1+1/k^n)}((\frac{(1/k^n-1)-n(1/k-1)}{n(n-1)})-(\frac{(1/k^{n-2}-1)-(n-2)(1/k-1)}{(n-2)(n-3)}))]\\ &+\frac{2k^{n-1}(1/k^{n-1}-1)}{(n+1)}|na_n+\alpha a_{n-1}| + k^{n-1}[\frac{(1/k^{n-1}-1)}{n-1}-\frac{(1/k^{n-3}-1)}{n-3}]\\ &\times |(n-1)a_{n-1}+2\alpha a_{n-2}|\end{aligned}\tag{3.14}$$

Now from lemma 2.4 it follows that for $|z| = 1$, $|D_\alpha q(z)| = |\alpha||D_{\frac{1}{\overline{\alpha}}} p(z)|$ Using the above equality in (3.14), we get for $|\alpha| \geq 1/k$,

$$\begin{aligned}
|\alpha| \max_{|z|=1} |D_{\frac{1}{\overline{\alpha}}} p(z)| &\geq (|\alpha| - 1/k)k^{n-1} \sum_{\nu=1}^{n} \frac{kk_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
&+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) \\
&+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n)-n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2})-(n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
&+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
&\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|,
\end{aligned} \tag{3.15}$$

Replacing $\frac{1}{\overline{\alpha}}$ by δ, so that $|\delta| \leq k$, we get from (3.15)

$$\begin{aligned}
|\frac{1}{\delta}| \max_{|z|=1} |D_\delta p(z)| &\geq (|\frac{1}{\delta}| - 1/k)k^{n-1} \sum_{\nu=1}^{n} \frac{kk_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
&+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) \\
&+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n)-n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2})-(n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
&+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
&\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|.
\end{aligned} \tag{3.16}$$

Or

$$\begin{aligned}
\max_{|z|=1} |D_\delta p(z)| &\geq (k - |\delta|)k^{n-1} \sum_{\nu=1}^{n} \frac{k_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
&+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) \\
&+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n)-n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2})-(n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
&+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
&\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|.
\end{aligned} \tag{3.17}$$

Which is (1.14). The proof of the Theorem 1.4 in the case $n = 3$ follows along the same lines as the proof of Theorem 1.1. Hence the proof of Theorem 1.4 is complete. □

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On efficient evaluation of integrals entering boundary equations of 3D potential and elasticity theory

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Abstract: The paper presents recurrent formulae for efficient evaluation of all the integrals needed for solving static 3D potential and elasticity problems by the boundary elements method. The power-type asymptoties for the density at edges of the boundary are accounted for explicitly.

AMS Subject Classification: *65M38, 33E05*
Keywords and Phrases: *potential and elasticity problems, singular and hypersingular boundary integral equations, boundary element method, higher order approximations, edge element, elliptic integrals*

1 Introduction

The purpose of the paper is to suggest an efficient general method for evaluation of the influence coefficients of the 3D boundary element method accounting for both smooth behaviour of the densities at internal parts of the boundary and power-type asymptotic behaviour near edges of the boundary.

Inspection of the boundary integrals equations of static 3D potential and elasticity theory [4] shows that it is sufficient to consider the function

$$\int\limits_{S^q} \frac{f\left(y\right)}{R} dS_y, \tag{1.1}$$

and its spatial derivatives $\partial/\partial x_i$, $\partial^2/\partial x_i \partial x_j$, $\partial^3/\partial x_i \partial x_j \partial x_k$.

Herein, S^q is the surface of a boundary element; $f(y)$ is a function to be properly approximated on the element; $R = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$, where x_1, x_2, x_3 and y_1, y_2, y_3 are global coordinates of the field and integration point, respectively.

2 Approximation of the boundary and density

We shall assume that, as usual (e.g. [1]), a curvilinear, in general, surface element is transformed into a plane element. The global coordinates are transformed to the local Cartesian coordinates of the plane element with the local axes y_2', y_3' in the element plane; the origin O' is in the plane of the transformed element. Besides, we assume that the entries of Jacobian matrix, its determinant and the expression for R are expanded into power series in $x_i' - y_i'$ and truncated. From now on, to simplify notation, we shall drop the prime in the transformed coordinates and refer (1.1) to a plane element in its local coordinates. Then $y_1 = 0$ and the function $f(y)$ is the product of the density depending on the local coordinates y_2, y_3 and powers of y_2 and y_3, which result from the truncated expressions mentioned.

Furthermore, we assume the plane element to be a trapezoid. This type of boundary elements includes as particular cases commonly used triangular, parallelogram, rectangular and square elements. Without loss of generality, we direct the y_2-axis along the trapezoid base, the y_3-axis orthogonal to it and we locate the origin in the lower left apex of the trapezoid (Fig. 1). For an edge element, we choose its edge as the base of the trapezoid. Then if the density near the edge has the power-type behaviour, it is described by the factor y_3^α with $0 \leqslant \alpha < 1$. The general approximation of the function $f(y)$ in (1.1) is of the form:

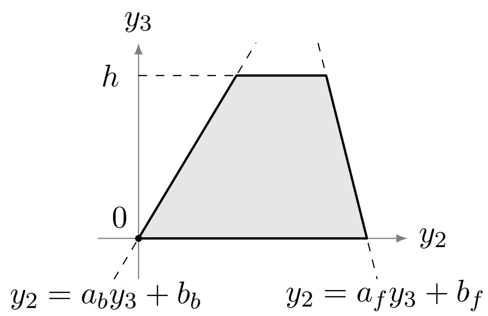

Figure 1: Trapezoidal element in local coordinates, $b_b = 0$

$$f(y) = y_3^\alpha \sum_{k+l=0}^{m_p} c_{kl} y_2^{k+s} y_3^{l+q}, \quad 0 \leqslant \alpha < 1, \tag{2.1}$$

where m_p is the degree of a polynomial approximating the density, c_{kl} are coefficients of approximation, s and q are degrees arising from the coordinate transformation (for initially plane parts of the boundary $s = q = 0$).

The two most important cases are: (i) $\alpha = 0$ what corresponds to smooth behaviour of the density, and (ii) $\alpha = 1/2$ what corresponds to square-root asymptotics typical for problem of linear fracture mechanics. Still, other exponents α may arise in approximations. For instance, $\alpha = 2/3$ for fracturing impermeable rock by a Newtonian fluid. Therefore, it is reasonable to specify a particular value of α at the end of the discussion.

Using (2.1) in (1.1) with S^q being the plane trapezoid of the height h (Fig. 1) implies that it is sufficient to consider the main integrals of the form:

$$A_\alpha^{kl}(x_1, x_2, x_3) = \int\limits_0^h \left(y_3^{l+\alpha} \int\limits_{y_2=a_b y_3+b_b}^{y_2=a_f y_3+b_f} \frac{(x_2-y_2)^k}{R} dy_2 \right) dy_3. \tag{2.2}$$

and their partial derivatives $\partial/\partial x_i$, $\partial^2/\partial x_i \partial x_j$, $\partial^3/\partial x_i \partial x_j \partial x_k$.

3 Evaluation of the main integrals

The integrals (2.2) are evaluated recurrently by using starting integrals for $k = 0$ and $k = 1$:

$$A_\alpha^{0j}(x_1, x_2, x_3) = -\int\limits_0^h \left[y_3^{j+\alpha} \ln[(x_2 - b_\xi - a_\xi y_3) + R_\xi] \right]_{\xi=b}^{\xi=f} dy_3, \tag{3.1}$$

$$A_\alpha^{1j}(x_1, x_2, x_3) = -\int\limits_0^h \left[y_3^{j+\alpha} R_\xi \right]_{\xi=b}^{\xi=f} dy_3, \tag{3.2}$$

where where $R_\xi = \sqrt{x_1^2 + (x_2 - b_\xi - a_\xi y_3)^2 + (x_3 - y_3)^2}$; the symbol $[\ ]_{x=a}^{x=b}$ means the double substitution: $\left[f(x)\right]_{x=a}^{x=b} = f(b) - f(a)$.

For $k \geqslant 2$, the recurrent equations are:

$$A_\alpha^{kl} = -\frac{1}{k}\left(\int\limits_0^h y_3^{\alpha+l} \left[(x_2 - b_\xi - a_\xi y_3)^{k-1} R_\xi \right]_{\xi=b}^{\xi=f} dy_3 + \right.$$

$$\left. +(k-1)\left(x_1^2 + x_3^2\right) A_\alpha^{(k-2)l} - 2x_3(k-1) A_\alpha^{(k-2)(l+1)} + (k-1) A_\alpha^{(k-2)(l+2)} \right). \tag{3.3}$$

Note that the integral A_α^{1j} in (3.2) is a particular case of the integrals on the r.h.s. of (3.3) when $k = 1$. Therefore, it remains to consider the integral on the r.h.s. of the (3.3) and the integral A_α^{0j} defined by (3.1). For both of them, an analysis shows that they are promptly expressed as linear combinations of three standard terms:

$$\left[\left[y_3^{l+1+\alpha} \ln[(x_2 - b_\xi - a_\xi y_3) + R_\xi] \right]_{\xi=b}^{\xi=f} \right]_{y_3=0}^{y_3=h}, \tag{3.4}$$

$$\left[u_s^\xi \int\limits_0^h \frac{y_3^\alpha y_3^s}{R_\xi} dy_3 \right]_{\xi=b}^{\xi=f}, \tag{3.5}$$

$$\left[\int\limits_0^h \frac{y_3^\alpha\left(\widetilde{A}y_3+\widetilde{B}\right)}{R_0^2 R_\xi}dy_3\right]_{\xi=b}^{\xi=f}, \tag{3.6}$$

where u_s^ξ, $\widetilde{A}$ and $\widetilde{B}$ are known coefficients depending on ξ, $R_0^2 = x_1^2 + (x_3 - y_3)^2$.

From (3.4)–(3.6) it follows that the problem is reduced to calculation of the integrals (3.5), (3.6) and their partial derivatives of the first, second and third order. Differentiation of (3.4), being trivial, we focus on the derivatives of the integrals (3.5) and (3.6).

4 Main integrals defining the first, second and third derivatives of standard terms

Evaluation of the first, second and third derivatives of the standard term (3.5) shows that it results in two new standard terms:

$$\left[v_i^\xi\int\limits_0^h \frac{y_3^\alpha}{\left(y_3+z_\xi\right)^i R_\xi}dy_3\right]_{\xi=b}^{\xi=f}, \quad \left[\bar{v}_i^\xi\int\limits_0^h \frac{y_3^\alpha}{\left(y_3+\bar{z}_\xi\right)^i R_\xi}dy_3\right]_{\xi=b}^{\xi=f}, \tag{4.1}$$

where v_i^ξ are known, in general complex, coefficients ($i = 1,2,3$); z_ξ is the complex root of the polynomial R_ξ^2, so that $\left(1+a_\xi^2\right)(y_3+z_\xi)(y_3+\bar{z}_\xi) = R_\xi^2$; the overbar denotes complex conjugation.

Similar analysis of the partial derivatives of the standard term (3.6) also yields two new standard terms:

$$\left[w_j^\xi\int\limits_0^h \frac{y_3^\alpha dy_3}{\left(y_3+z_0\right)^j R_\xi}\right]_{\xi=b}^{\xi=f}, \quad \left[\bar{w}_j^\xi\int\limits_0^h \frac{y_3^\alpha dy_3}{\left(y_3+\bar{z}_0\right)^j R_\xi}\right]_{\xi=b}^{\xi=f}, \tag{4.2}$$

where w_j^ξ are known, in general complex, coefficients; $z_0 = -x_3 + \mathrm{i}x_1$ is the root of R_0^2, $j = 1,2,3,4$, when $x_1 \neq 0$; in the case $x_1 = 0$ we have $z_0 = \bar{z}_0 = -x_3$ and then $j = 1,2,\ldots,8$. Actually (4.2) are particular cases of (4.1) when the root z_ξ of R_ξ^2 is changed to the root z_0 of R_0^2.

Noting that the second expression in (4.1) and (4.2) are the conjugated first ones, we come to the conclusion that the problem is reduced to evaluation of three types of integrals, at most:

$$\left[u_\xi\int\limits_0^h \frac{y_3^\alpha y_3^s}{R_\xi}dy_3\right]_{\xi=b}^{\xi=f}, \quad \left[v_\xi^i\int\limits_0^h \frac{y_3^\alpha}{\left(y_3+z_\xi\right)^i R_\xi}dy_3\right]_{\xi=b}^{\xi=f}, \quad \left[w_\xi^j\int\limits_0^h \frac{y_3^\alpha dy_3}{\left(y_3+z_0\right)^j R_\xi}\right]_{\xi=b}^{\xi=f}, \tag{4.3}$$

where $i = 1,2,3$, $j = 1,2,3,4$ for $x_1 \neq 0$ and $j = 1,2,\ldots,8$ for $x_1 = 0$. Emphasise that when representing the exponent α as a proper rational fracture $\alpha = n/m$, $(n < m)$, the integrals (4.3) may be evaluated recurrently. Below we give the explicit formulae for the cases most important for application: $\alpha = 0$ and $\alpha = 1/2$. Before presenting them, we distinguish three cases which suggest simplifications.

(i) Differentiation with respect to x_2. In this case, we may avoid using the recursive equation (3.3) by the method suggested in the paper [5]. Specifically, by the relation $\partial A^{kl}/\partial x_2 = -\partial A^{kl}/\partial y_2$ we obtain

$$\frac{\partial A^{kl}}{\partial x_2} = -\int\limits_0^h \left[y_3^{l+\alpha} \frac{(x_2 - b_\xi - a_\xi y_3)^k}{R_\xi} \right]_{\xi=b}^{\xi=f} dy_3. \tag{4.4}$$

This shows that differentiation with respect to x_2 immediately leads to arithmetic operations with the expressions (4.1).

(ii) Differentiation with respect to x_1. By differentiating equation (3.3) with respect to x_1, we obtain:

$$\frac{\partial A_\alpha^{kl}}{\partial x_1} = -\frac{1}{k} \left[\int\limits_0^h \frac{x_1 y_3^{\alpha+l} (x_2 - b_\xi - a_\xi y_3)^{k-1}}{R_\xi} dy_3 \right]_{\xi=b}^{\xi=f} +$$

$$-\frac{k-1}{k} \left(2x_1 A_\alpha^{(k-2)l} + (x_1^2 + x_3^2) \frac{\partial A_\alpha^{(k-2)l}}{\partial x_1} - 2x_3 \frac{\partial A_\alpha^{(k-2)(l+1)}}{\partial x_1} + \frac{\partial A_\alpha^{(k-2)(l+2)}}{\partial x_1} \right). \tag{4.5}$$

The derivative of the starting integral $\frac{\partial A_\alpha^{1l}}{\partial x_1}$ has the form of the first integral on the right hand side of the formula (4.5). Thus it is enough to consider the derivative of the starting integral A_α^{0l}.

$$\frac{\partial A_\alpha^{0l}}{\partial x_1} = x_1 \int\limits_0^h \left[\frac{y_3^{l+\alpha} (x_2 - b_\xi - a_\xi y_3)}{R_0^2 R_\xi} \right]_{\xi=b}^{\xi=f} dy_3 - x_1 \int\limits_0^h \left[\frac{y_3^{l+\alpha}}{R_0^2} \right]_{\xi=b}^{\xi=f} dy_3. \tag{4.6}$$

The first integral after decomposition into a sum of real partial fractions is evaluted by arithmetic operations with the integrals (3.5) and (3.6). The second integral does not depend on ξ, therefore it is zero. We see that evaluation of partial derivatives, containing differentiation with respect to x_1, is reduced to evaluation of expressions of the forms (4.1) for $i = 1, 2$ and (4.2) for $j = 1, 2, 3$.

(iii) Double differentiation with respect to x_3. Since the function $1/R$ satisfies the Laplace equation when $R \neq 0$, we may avoid repeated differentiation with respect to x_3 by using the equation

$$\frac{\partial^2 A_\alpha^{kl}}{\partial x_3^2} = -\left(\frac{\partial^2 A_\alpha^{kl}}{\partial x_1^2} + \frac{\partial^2 A_\alpha^{kl}}{\partial x_2^2} \right). \tag{4.7}$$

Then simplifications of points (i) and (ii) become available.

5 Case of smooth density ($\alpha = 0$)

In this case, all the integrals are evaluated analytically. Specifically, the integrals (3.5), (4.1) and (4.2) become respectively:

$$I_s = \int_0^h \frac{y_3^s}{\sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}} dy_3, \quad J_s = \int_0^h \frac{1}{(y_3+z_\xi)^s \sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}} dy_3,$$

$$\text{and} \quad K_s = \int_0^h \frac{dy_3}{(y_3+z_0)^s \sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}}. \tag{5.1}$$

Each of them is evaluated recurrently, with starting expressions:

$$\begin{aligned} I_0 = J_0 = K_0 &= 2\left[\ln\left(\sqrt{y_3+z_\xi}+\sqrt{y_3+\bar{z}_\xi}\right)\right]_{y_3=0}^{y_3=h}, \\ I_1 &= \left[\sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}\right]_{y_3=0}^{y_3=h} - \operatorname{Re}(z_\xi)\, I_0, \\ K_1 &= \frac{2\left[\arctan\left(\frac{\sqrt{y_3+z_\xi}\sqrt{\bar{z}_\xi-z_0}}{\sqrt{y_3+\bar{z}_\xi}\sqrt{z_0-z_\xi}}\right)\right]_{y_3=0}^{y_3=h}}{\sqrt{z_0-z_\xi}\sqrt{\bar{z}_\xi-z_0}}. \end{aligned} \tag{5.2}$$

The recursive formulas are:

$$I_s = \frac{1}{s}\left(\left[y_3^{s-1}\sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}\right]_{y_3=0}^{y_3=h} - (2s-1)\operatorname{Re}(z_\xi)\, I_{s-1} - (s-1)\,|z_\xi|^2\, I_{s-2}\right), \tag{5.3}$$

$$J_s = \frac{1}{\left(s-\frac{1}{2}\right)(z_\xi-\bar{z}_\xi)}\left(\left[\frac{\sqrt{y_3+\bar{z}_\xi}}{(y_3+z_\xi)^{s-\frac{1}{2}}}\right]_{y_3=0}^{y_3=h} + (s-1)\, J_{s-1}\right), \tag{5.4}$$

$$K_s = \frac{1}{(s-1)(z_\xi-z_0)(\bar{z}_\xi-z_0)}\left(-\left[\frac{\sqrt{(y_3+z_\xi)(y_3+\bar{z}_\xi)}}{(y_3+z_0)^{s-1}}\right]_{y_3=0}^{y_3=h} + \right.$$

$$\left. + \left(s-\frac{3}{2}\right)(2z_0-z_\xi-\bar{z}_\xi)\, K_{s-1} - (s-2)\, K_{s-2}\right). \tag{5.5}$$

Note that in the considered case, the representation of the trapezoid as a sum of right triangles and a rectangle, allows us to use also the efficient method suggested in [5].

6 The case of the density with square-root asymptotics near the element edge ($\alpha = 1/2$)

In this case, the starting integrals for evaluation of the integrals

$$I_s = \int_0^h \frac{y_3^s q dy_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}}, \quad J_s = \int_0^h \frac{dy_3}{(y_3 + z_\xi)^s \sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}},$$

$$K_s = \int_0^h \frac{dy_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}\,(y_3 + z_0)^s},$$

are:

$$I_0 = J_0 = K_0 = \int_0^h \frac{dy_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}}, \tag{6.1}$$

$$I_1 = \int_0^h \frac{y_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}} dy_3, \tag{6.2}$$

$$J_1 = \int_0^h \frac{dy_3}{(y_3 + z_\xi)\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}}, \tag{6.3}$$

$$K_1 = \int_0^h \frac{dy_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}\,(y_3 + z_0)}, \tag{6.4}$$

$$K_2 = \int_0^h \frac{dy_3}{\sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)}\,(y_3 + z_0)^2}. \tag{6.5}$$

The recursive formulae are:

$$I_s = \frac{1}{(2s-1)} \left(2 \left[y_3^{s-2} \sqrt{y_3 (y_3 + z_\xi)(y_3 + \bar{z}_\xi)} \right]_{y_3=0}^{y_3=h} + \right.$$

$$\left. - 4\,\mathrm{Re}\,(z_\xi)(s-1) I_{s-1} - |z_\xi|^2 (2s-3) I_{s-2} \right), \tag{6.6}$$

$$J_s = \frac{1}{\left(s - \frac{1}{2}\right)(\bar{z}_\xi - z_\xi) z_\xi} \left[\left[\frac{\sqrt{(y_3 + \bar{z}_\xi) y_3}}{(y_3 + z_\xi)^{s-\frac{1}{2}}} \right]_{y_3=0}^{y_3=h} + \right.$$

$$\left. + (s-1)(\bar{z}_\xi - 2z_\xi) J_{s-1} + \left(s - \frac{3}{2}\right) J_{s-2} \right], \tag{6.7}$$

$$K_s = \frac{1}{2\,(s-1)\,(z_0 - z_\xi)\,(z_0 - \bar{z}_\xi)\,z_0}\left(\left[\frac{2\sqrt{y_3\,(y_3+z_\xi)\,(y_3+\bar{z}_\xi)}}{(y_3+z_0)^{s-1}}\right]_{y_3=0}^{y_3=h} + \right.$$

$$\left. + (2s-5)\,K_{s-3} - 2\,(s-2)\,(3z_0 - \bar{z}_\xi - z_\xi)\,K_{s-2}\right) +$$

$$+ \frac{(2s-3)}{2\,(s-1)}\left(\frac{1}{z_0} + \frac{1}{z_0 - z_\xi} + \frac{1}{z_0 - \bar{z}_\xi}\right)K_{s-1}. \tag{6.8}$$

Remark 6.1 *For $z_0 = 0$ (i.e. $x_1 = 0, x_3 = 0$),*

$$K_s = \int_0^h \frac{dy_3}{\sqrt{y_3\,(y_3+z_\xi)\,(y_3+\bar{z}_\xi)}\,(y_3)^s},$$

and the recursive formula becomes:

$$K_s = -\frac{1}{\left(s-\frac{1}{2}\right)|z_\xi|^2}\left(\left[y_3^{-\left(s-\frac{1}{2}\right)}\sqrt{(y_3+z_\xi)\,(y_3+\bar{z}_\xi)}\right]_{y_3=0}^{y_3=h} + \right.$$

$$\left. + \left(s - \frac{3}{2}\right)K_{s-2} + 2\,\mathrm{Re}\,(z_\xi)\,(s-1)\,K_{s-1}\right), \tag{6.9}$$

with starting integrals:

$$K_{-1} = I_1, \quad K_0 = I_0. \tag{6.10}$$

Evaluation of the starting elliptic integrals $I_0,\ I_1,\ J_1,\ K_1$ and K_2 is efficiently performed by proper adjusting the Carlson algorithms as explained in the next section.

7 Efficient evaluation of standard elliptic integrals for problems involving cracks ($\alpha = 1/2$)

The conventional methods of evaluation the elliptic integrals employ Gauss and Landen transformations [6]. They converge quadratically and work well for elliptic integrals of the first and second kind. However, as emphasised in [6] and confirmed by our experience, they suffer from lost of significant digits for the integrals of the third kind needed for our purpose. In contrast, the Carlson algorithm provides a unified method for all the three kinds of integrals with extremely high efficiency. To use this algorithm, we introduce the new variable t defined by equation:

$$y_3 = \frac{1}{t + \frac{1}{h}}. \tag{7.1}$$

Then the starting integrals become:

$$I_0 = \frac{1}{|z_\xi|}\int\limits_0^\infty \frac{dt}{\sqrt{\left(t+\frac{1}{h}\right)\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)\left(t+\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)}}, \tag{7.2}$$

$$I_1 = \frac{1}{|z_\xi|}\int\limits_0^\infty \frac{dt}{\left(t+\frac{1}{h}\right)^{\frac{3}{2}}\sqrt{\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)\left(t+\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)}}, \tag{7.3}$$

$$J_1 = \frac{1}{z_\xi\,|z_\xi|}\int\limits_0^\infty \frac{\left(t+\frac{1}{h}\right)dt}{\sqrt{\left(t+\frac{1}{h}\right)\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)\left(t+\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)}\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)}, \tag{7.4}$$

$$K_1 = \frac{1}{|z_\xi|\,z_0}\int\limits_0^\infty \frac{\left(t+\frac{1}{h}\right)dt}{\sqrt{\left(t+\frac{1}{h}\right)\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)\left(t+\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)}\left(t+\frac{1}{h}+\frac{1}{z_0}\right)}, \tag{7.5}$$

$$K_2 = \frac{1}{|z_\xi|\,z_0^2}\int\limits_0^\infty \frac{\left(t+\frac{1}{h}\right)^2 dt}{\sqrt{\left(t+\frac{1}{h}\right)\left(t+\frac{1}{h}+\frac{1}{z_\xi}\right)\left(t+\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)}\left(t+\frac{1}{h}+\frac{1}{z_0}\right)^2}. \tag{7.6}$$

They are promptly expressed in terms of Carlson integrals R_F, R_D and R_J of the first, second and third kind, respectively, defined as:

$$\begin{aligned}
R_F(x,y,z) &= \tfrac{1}{2}\int\limits_0^\infty \left[(t+x)(t+y)(t+z)\right]^{-\frac{1}{2}}dt,\\
R_D(z,y,z) = R_J(x,y,z,z) &= \tfrac{3}{2}\int\limits_0^\infty \left[(t+x)(t+y)\right]^{-\frac{1}{2}}(t+z)^{-\frac{3}{2}}dt,\\
R_J(x,y,z,p) &= \tfrac{3}{2}\int\limits_0^\infty \left[(t+x)(t+y)(t+z)\right]^{-\frac{1}{2}}(t+p)^{-1}dt.
\end{aligned} \tag{7.7}$$

In terms of the Carlson integrals, the starting integrals are:

$$I_0 = \frac{2}{|z_\xi|}R_F\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right), \tag{7.8}$$

$$I_1 = \frac{2}{3\,|z_\xi|}R_D\left(\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}\right), \tag{7.9}$$

$$J_1 = \frac{I_0}{z_\xi} - \frac{2}{3z_\xi^2\,|z_\xi|}R_D\left(\frac{1}{h},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_\xi}\right), \tag{7.10}$$

$$K_1 = \frac{I_0}{z_0} - \frac{2}{3\,|z_\xi|\,z_0^2}R_J\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_0}\right), \tag{7.11}$$

$$K_2 = -\frac{I_0}{2z_0^2}\left(1+\frac{z_\xi}{z_0-z_\xi}+\frac{\bar{z}_\xi}{z_0-\bar{z}_\xi}\right)-\frac{I_1}{2z_0^3}+$$

$$+\frac{K_1}{2z_0}\left(3+\frac{z_\xi}{z_0-z_\xi}+\frac{\bar{z}_\xi}{z_0-\bar{z}_\xi}\right)+\frac{1}{|z_\xi|\,z_0^4}\frac{\sqrt{h}}{\left(\frac{1}{h}+\frac{1}{z_0}\right)\left|\frac{1}{h}+\frac{1}{z_\xi}\right|}+$$

$$-\frac{1}{3\,|z_\xi|\,z_0^3}\left(\frac{z_\xi}{z_\xi-z_0}R_D\left(\frac{1}{h},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_\xi}\right)+\frac{\bar{z}_\xi}{\bar{z}_\xi-z_0}R_D\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right)\right). \tag{7.12}$$

Finally, we need to evaluate five Carlson integrals only:

$$\begin{array}{ll} R_D\left(\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}\right), & R_D\left(\frac{1}{h},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_\xi}\right), \\ R_D\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi}\right), & R_F\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\right), \\ R_J\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_0}\right). & \end{array} \tag{7.13}$$

The integrals R_D and R_F are evaluated very fast and accurately by algorithms presented by Carlson in the paper [3]. The same also refers to the integrals R_J when its last argument in not a negative real number. The case, when the last argument $\left(\frac{1}{h}+\frac{1}{z_0}\right)$ of the integral R_J is a real negative number, is special. It occurs when the field point is within the strip $x_1 = 0$, $0 < x_3 < h$. Then the integral R_J is a singular real Cauchy integral:

$$R_J\left(\frac{1}{h},\frac{1}{h}+\frac{1}{z_\xi},\frac{1}{h}+\frac{1}{\bar{z}_\xi},\frac{1}{h}+\frac{1}{z_0}\right)=\int_0^\infty \frac{dt}{\sqrt{t+\frac{1}{h}}\sqrt{f+gt+t^2}\left(t+\frac{1}{h}+\frac{1}{z_0}\right)}, \tag{7.14}$$

where $f = |z_\xi|^2$, $g = 2\,\mathrm{Re}\, z_\xi$. In the paper [2] Carlson provides equations serving for efficient of this integral:

$$R_J = \frac{2c_{11}}{3c_{44}}\Bigg[-4x_3\left(c_{14}^2+\sqrt{c_{11}^2c_{44}^2}\right)R_J(M^2,L_-^2,L_+^2,W_+^2)+$$

$$-6R_F(M^2,L_-^2,L_+^2)+3R_C(U^2,W^2)-2R_C(P^2,Q^2)\Bigg], \tag{7.15}$$

where

$$\begin{aligned} c_{11}^2 &= 2\left(f-\tfrac{g}{h}+\tfrac{1}{h^2}\right), \\ c_{14}^2 &= 2\left(f-g\left(\tfrac{1}{h}+\tfrac{1}{2z_0}\right)+\tfrac{1}{h}\left(\tfrac{1}{h}+\tfrac{1}{z_0}\right)\right), \\ c_{44}^2 &= 2\left(f-g\left(\tfrac{1}{h}+\tfrac{1}{z_0}\right)+\left(\tfrac{1}{h}+\tfrac{1}{z_0}\right)^2\right), \end{aligned} \tag{7.16}$$

and

$$
\begin{array}{rcl}
M^2 & = & 2\sqrt{f} + g, \\
U^2 & = & \frac{1}{h}, \\
W^2 & = & U^2 + \frac{1}{2} z_0 c_{11}^2, \\
W_+^2 & = & M^2 + z_0 \left(c_{14}^2 + c_{11} c_{44} \right), \\
L_\pm^2 & = & M^2 + \left(\frac{2}{h} - g \right) \pm c_{11} \sqrt{2}, \\
Q^2 & = & W^2 \left(1 + \frac{h}{z_0} \right), \\
P^2 & = & Q^2 - \frac{1}{2} z_0 c_{44}^2, \\
R_C(a,b) & = & R_F(a,b,b).
\end{array}
\tag{7.17}
$$

With using these equations, evaluation of the elliptic integrals, needed for problems involving cracks, becomes extremely efficient. Our experience shows that calculations of influence coefficients for square-root edge elements ($\alpha = 1/2$) are performed as accurate and fast as those for ordinary elements ($\alpha = 0$).

We believe that similar, highly efficient algorithms may be developed for any proper fraction $\alpha = m/n$ $(m < n)$.

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