ROMAI J., v.11, no.1(2015), 33–46

ITERATED ORDER OF MEROMORPHIC SOLUTIONS OF HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Benharrat Belaïdi

Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), Mostaganem, Algeria

belaidi@univ-mosta.dz

Abstract In this paper, we investigate the iterated order of meromorphic solutions of homogeneous and non-homogeneous to higher order linear differential equations

$$f^{(k)}+\sum_{j=1}^{k-1}A_jf^{(j)}+A_0f=0 \quad (k\geqslant 2),$$

$$f^{(k)}+\sum_{j=1}^{k-1}A_jf^{(j)}+A_0f=F \quad (k\geqslant 2),$$

where $A_j(z)$ $(j=0,1,\cdots,k-1)$ and $F(z)$ are meromorphic functions with finite iterated $p-$order. Under some conditions on the coefficients, we show that all meromorphic solutions $f\not\equiv 0$ of the above equations have an infinite iterated $p-$order and infinite iterated lower $p-$order. Furthermore, we give some estimates of iterated convergence exponent. We improve the results due to Chen; Shen and Xu; He, Zheng and Hu and others.

Keywords: linear differential equations, meromorphic functions, iterated order, iterated exponent of convergence of zeros.

2010 MSC: 34M10, 30D35.

1. INTRODUCTION

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [11], [19]). For the definition of the iterated order of a meromorphic function, we use the same definition as in [13], [5, p. 317], [15, p. 129].

For all $r\in\mathbb{R}$, we define $\exp_1 r:=e^r$ and $\exp_{p+1} r:=\exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We also define for all r sufficiently large $\log_1 r:=\log r$ and $\log_{p+1} r:=\log\left(\log_p r\right)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r:=r$, $\log_0 r:=r$, $\log_{-1} r:=\exp_1 r$ and $\exp_{-1} r:=\log_1 r$.

Definition 1.1. (*see* [13], [15]) *Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined as*

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{log_p T(r, f)}{log\ r} \quad (p \geqslant 1 \ \textit{is an integer}),$$

where $T(r, f)$ is the Nevanlinna characteristic function of f (see, [11], [19]). *For $p = 1$, this notation is called order and for $p = 2$ hyper-order.*

Definition 1.2. (*see* [6]) *Let f be a meromorphic function. Then the iterated lower $p-$order $\mu_p(f)$ of f is defined as*

$$\mu_p(f) = \liminf_{r\to+\infty} \frac{log_p T(r, f)}{log\ r} \quad (p \geqslant 1 \ \textit{is an integer}).$$

Definition 1.3. (*see* [13]) *The finiteness degree of the order of a meromorphic function f is defined as*

$$i(f) = \begin{cases} 0, & \textit{for } f \textit{ rational,} \\ \min\left\{p \in \mathbb{N} : \rho_p(f) < +\infty\right\}, & \textit{for } f \textit{ transcendental for which} \\ & \textit{some } p \in \mathbb{N} \textit{ with } \rho_p(f) < +\infty \textit{ exists,} \\ +\infty, & \textit{for } f \textit{ with } \rho_p(f) = +\infty \textit{ for all } p \in \mathbb{N}. \end{cases}$$

Remark 1.1. *Similarly, we can define the finiteness degree of the lower order $i_\mu(f)$ of a meromorphic function f.*

Definition 1.4. (*see* [13]) *Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as*

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log r} \quad (p \geqslant 1 \ \textit{is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leqslant r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of zeros and for $p = 2$ hyper-exponent of convergence of the sequence of zeros. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r} \quad (p \geqslant 1 \ \textit{is an integer}),$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leqslant r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of *distinct zeros and for $p = 2$ hyper-exponent of convergence of the sequence of distinct zeros.*

First, we recall the following definitions. The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset [0, +\infty)$ is defined by

$$\overline{dens}E = \limsup_{r \longrightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

Proposition 1.1. *For all $H \subset [1, +\infty)$ the following statements hold :*
i) If $lm(H) = \infty$, then $m(H) = \infty$;
ii) If $\overline{dens}H > 0$, then $m(H) = \infty$;
iii) If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.

Proof. i) Since we have $\frac{\chi_H(t)}{t} \leqslant \chi_H(t)$ for all $t \in H \subset [1, +\infty)$, then

$$m(H) \geqslant lm(H).$$

So, if $lm(H) = \infty$, then $m(H) = \infty$. We can easily prove the results ii) and iii) by applying the definition of the limit and the properties $m(H \cap [0, r]) \leqslant m(H)$ and $lm(H \cap [1, r]) \leqslant lm(H)$. ∎

In this paper, we consider for $k \geqslant 2$ the homogeneous and the non-homogeneous linear differential equations

$$f^{(k)} + \sum_{j=1}^{k-1} A_j f^{(j)} + A_0 f = 0, \tag{1.1}$$

$$f^{(k)} + \sum_{j=1}^{k-1} A_j f^{(j)} + A_0 f = F, \tag{1.2}$$

where $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ and $F(z)$ $(A_0 \not\equiv 0$ and $F \not\equiv 0)$ are meromorphic functions with finite iterated $p-$order. In [3], the author extended the results of Kwon [14], Chen and Yang [7] from second order to higher order linear differential equations and obtained the following two results.

Theorem A [3] *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be entire functions such that for real constants α, β, μ, where $0 \leqslant \beta < \alpha$ and $\mu > 0$, we have*

$$|A_0(z)| \geqslant e^{\alpha |z|^{\mu}}$$

and

$$\left|A_j(z)\right| \leqslant e^{\beta|z|^{\mu}},\ j = 1, \cdots, k-1$$

as $z \to \infty$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (1.1) *has infinite order and* $\rho_2(f) \geqslant \mu$.

Theorem B [3] *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z), \cdots, A_{k-1}(z)$ *be entire functions with*

$$\max\{\rho\left(A_j\right) : j = 1, \cdots, k-1\} \leqslant \rho(A_0) = \rho < +\infty$$

such that for real constants α, β $(0 \leqslant \beta < \alpha)$, *we have for any given* $\varepsilon > 0$

$$|A_0(z)| \geqslant e^{\alpha|z|^{\rho-\varepsilon}}$$

and

$$\left|A_j(z)\right| \leqslant e^{\beta|z|^{\rho-\varepsilon}},\quad j = 1, \cdots, k-1$$

as $z \to \infty$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (1.1) *has infinite order and* $\rho_2(f) = \rho(A_0)$.

In [8], Chen improved the previous results in [7, 14] by studying the zeros and the growth of meromorphic solutions of the homogeneous and the non-homogeneous equations $f'' + A(z)f' + B(z)f = 0$, $f'' + A(z)f' + B(z)f = F$ when $A(z)$, $B(z)$, $F(z)$ are meromorphic functions. In [16], Shen and Xu extended and genralized the results of Chen [8] to higher order linear differential equations with meromorphic coefficients. Recently, He, Zheng and Hu improved and extended the above results from usual order to iterated order as follows.

Theorem C [12] *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z), \cdots, A_{k-1}(z)$ *be meromorphic functions of finite iterated* $p-$*order such that for real constants* $\alpha_2 > \alpha_1 \geqslant 0$ *and* $\mu > 0$, *we have*

$$|A_0(z)| \geqslant \exp_p\{\alpha_2 |z|^{\mu}\}$$

and

$$\left|A_j(z)\right| \leqslant \exp_p\{\alpha_1 |z|^{\mu}\},\ j = 1, \cdots, k-1$$

as $z \to \infty$ *for* $z \in H$. *If the equation* (1.1) *have meromorphic solutions, then every meromorphic solution* $f \not\equiv 0$ *satisfies* $\rho_{p+1}(f) \geqslant \mu$.

Furthermore, if $\max\left\{\left|A_j(z)\right| : j = 0, \cdots, k-1\right\} \leqslant \exp_p\{\beta|z|^{\mu}\}$as $z \to 0$, *where* $\beta > 0$ *is a constant, then every meromorphic solution* $f \not\equiv 0$ *with* $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \mu$.

Theorem D [12] *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and $F(z) \not\equiv 0$ be a meromorphic function with $|F(z)| \leqslant \exp_q\{\alpha |z|^\mu\}$ as $z \to \infty$ or $\rho_q(F) \leq \mu$ $(0 < q \leqslant p < \infty)$. Let $A_0(z), \cdots, A_{k-1}(z)$ be meromorphic functions of finite iterated $p-$order satisfying the following conditions:*
(i) *for real constants $\alpha_2 > \alpha_1 \geqslant 0$ and $\mu > 0$, we have*

$$|A_0(z)| \geqslant \exp_p\{\alpha_2 |z|^\mu\}$$

and

$$\left|A_j(z)\right| \leqslant \exp_p\{\alpha_1 |z|^\mu\}, \; j = 1, \cdots, k-1$$

as $z \to \infty$ for $z \in H$;
(ii) $\max\left\{\left|A_j(z)\right| : j = 0, \cdots, k-1\right\} \leqslant \exp_p\{\beta |z|^\mu\}$ as $z \to \infty$, *where $\beta > 0$ is a constant.*
If the equation (1.2) *have meromorphic solutions, then every meromorphic solution $f \not\equiv 0$ with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = \mu$, with at most one exceptional solution $f_0(z)$ with $i(f_0) < p+1$ or $\rho_{p+1}(f_0) < \mu$.*

The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers. Section 3 is for some lemmas and basic theorems. The last sections are for the proofs of our main results.

2. MAIN RESULTS

In the present paper, we investigate the zeros and growth of meromorphic solutions of equations (1.1) and (1.2). We improve the results due to Chen; Shen and Xu; He, Zheng and Hu. The present article may be understood as an extension and improvement of the recent article of Andasmas and the author [1] from usual order to iterated $p-$order. In fact we will prove the following results.

Theorem 2.1. *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $\rho = \max\{\rho_p\left(A_j\right) : j = 1, 2, \cdots, k-1\} < \sigma$ and $|A_0(z)| \geqslant \exp_p\{\alpha r^\sigma\}$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution $f \not\equiv 0$ of equation* (1.1) *satisfies*

$$\mu_p(f) = \rho_p(f) = \infty, \; \rho_{p+1}(f) \geqslant \sigma.$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \infty$, then $i(f) = p+1$ and

$$\sigma \leqslant \rho_{p+1}(f) \leqslant \rho_p(A_0).$$

Theorem 2.2. *Let $H \subset [0,+\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0,1,\cdots,k-1)$ and $F(z) \not\equiv 0$ be meromorphic functions with finite iterated $p-$order. If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $\rho = \max\left\{\rho_p\left(A_j\right)(j = 1,2,\cdots,k-1), \rho_p(F)\right\} < \sigma$ and $|A_0(z)| \geqslant \exp_p\{\alpha r^{\sigma}\}$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution f with $\lambda_p\left(\frac{1}{f}\right) < \sigma$ of equation* (1.2) *satisfies*

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty,\ \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \min\left\{\mu_p(f), \sigma\right\}$, then $i(f) = p+1$ and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leqslant \rho_p(A_0).$$

Remark 2.1. *It is clear that $\rho_p(A_0) = \beta \geqslant \sigma$ in Theorems 2.1 and 2.2. Indeed, suppose that $\rho_p(A_0) = \beta < \sigma$. Then, by using Lemma 3.4 of this paper, there exists a set $E_3 \subset (1,+\infty)$ that has finite linear measure such that when $|z| = r \notin E_3$, we have for any given ε $(0 < \varepsilon < \sigma - \beta)$*

$$|A_0(z)| \leqslant \exp_p\left\{r^{\beta+\varepsilon}\right\}. \tag{2.1}$$

On the other hand, by the hypotheses of Theorems 2.1 and 2.2, there exist positive constants $\sigma > 0$, $\alpha > 0$ such that

$$|A_0(z)| \geqslant \exp_p\{\alpha r^{\sigma}\} \tag{2.2}$$

as $|z| = r \in H$, $r \to +\infty$, where $H \subset [0,+\infty)$ is a set with a positive upper density (and so with infinite linear measure $m(H) = \infty$). From (2.1) *and* (2.2), *we obtain for $|z| = r \in H \backslash E_3$, $r \to +\infty$*

$$\exp_p\{\alpha r^{\sigma}\} \leqslant |A_0(z)| \leqslant \exp_p\left\{r^{\beta+\varepsilon}\right\}$$

and by ε $(0 < \varepsilon < \sigma - \beta)$ this is a contradiction as $r \to +\infty$. Hence $\rho_p(A_0) = \beta \geqslant \sigma$.

3. LEMMAS FOR THE PROOFS OF THE THEOREMS

Lemma 3.1. ([9]) *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$, $\varepsilon > 0$ be given constants. Then there exists a set $E_1 \subset [0,\infty)$ that has finite linear measure and there exists a constant $c > 0$, such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant c\left[T(\alpha r, f) r^{\varepsilon} \log T(\alpha r, f)\right]^j \quad (j \in \mathbb{N}).$$

Let $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r) = \max\{|a_n| r^n;\ n = 0,1,\cdots\}$ the maximum term of g, and define by $\nu_g(r) = \max\{m;\ \mu(r) = |a_m| r^m\}$ the central index of g.

Lemma 3.2. [6] *Let $p,q \geqslant 1$ be integers and let $f(z)$ be an entire function with $i(f) = p+1$, $\rho_{p+1}(f) = \rho$, $i_\mu(f) = q+1$ and $\mu_{q+1}(f) = \mu$. Let $\nu_f(r)$ be the central index of $f(z)$. Then*

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_f(r)}{\log r} = \rho$$

and

$$\liminf_{r\to+\infty} \frac{\log_{q+1} \nu_f(r)}{\log r} = \mu.$$

By using similar proof of Lemma 3.5 in [17], we can easily extend it to the case $\rho_p(g) = \rho_p(f) = +\infty$.

Lemma 3.3. *Let $p \geqslant 1$ be an integer and let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire function satisfying $\mu_p(g) = \mu_p(f) \leqslant \rho_p(g) = \rho_p(f) \leqslant +\infty$, $i(d) < p$ or $i(d) = p$ and $\rho_p(d) = \beta < \mu_p(f)$. Let $\nu_g(r)$ be the central index of g. Then there exists a set E_2 of finite logarithmic measure such that the estimation*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)) \quad (j \in \mathbb{N})$$

holds for all $|z| = r \notin E_2$ and $|g(z)| = M(r,g)$.

Lemma 3.4. [18] *Let $p \geqslant 1$ be an integer. Suppose that $f(z)$ is a meromorphic function such that $i(f) = p$, $\rho_p(f) = \rho < +\infty$. Then, there exist entire functions $\pi_1(z)$, $\pi_2(z)$ and $D(z)$ such that*

$$f(z) = \frac{\pi_1(z)\, e^{D(z)}}{\pi_2(z)} \quad \text{and} \quad \rho_p(f) = \max\left\{\rho_p(\pi_1), \rho_p(\pi_2), \rho_p\left(e^{D(z)}\right)\right\}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$\exp\left\{-\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}\right\} \leqslant |f(z)| \leqslant \exp_p\left\{r^{\rho+\varepsilon}\right\} \quad (r \notin E_3),$$

where $E_3 \subset (1,+\infty)$ is a set of r of finite linear measure.

To avoid some problems caused by the exceptional set, we recall the following lemmas.

Lemma 3.5. [2] *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leqslant h(r)$ outside of an exceptional set E_4 of finite linear measure. Then for any $\lambda > 1$, there exists $r_0 > 0$ such that $g(r) \leqslant h(\lambda r)$ for all $r > r_0$.*

Lemma 3.6. [10] *Let $\varphi : [0,+\infty) \to \mathbb{R}$ and $\psi : [0,+\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin E_5 \cup [0,1]$, where $E_5 \subset (1,+\infty)$ is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \leqslant \psi(\alpha r)$ for all $r > r_1$.*

Lemma 3.7. *Assume that $k \geqslant 2$ and $A_0, A_1, \cdots, A_{k-1}$ ($A_0 \not\equiv 0$), F are meromorphic functions. Let $\rho = \max\{\rho_p(A_j)\ (j = 0,1,\cdots,k-1), \rho_p(F)\} < \infty$ and let f be a meromorphic solution of infinite iterated $p-$order of equation (1.2) with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. Then $\rho_{p+1}(f) \leqslant \rho$.*

Proof. We assume that f is a meromorphic solution of infinite iterated $p-$order $\rho_p(f) = \infty$ of equation (1.2). We can rewrite (1.2) as

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |A_0| + \sum_{j=1}^{k-1} |A_j| \left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right|. \tag{3.1}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions such that $\mu_p(g) = \mu_p(f) \leqslant \rho_p(g) = \rho_p(f) = +\infty$, $i(d) < p$ or $i(d) = p$ and $\rho_p(d) = \lambda_p(d) = \lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By Lemma 3.3, there exists a set $E_2 \subset (1,+\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$ at which $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \quad (j \geqslant 1). \tag{3.2}$$

By Lemma 3.4, for any given $\varepsilon \left(0 < \varepsilon < \frac{\mu_p(f) - \rho_p(d)}{2}\right)$, there exists a set $E_3 \subset (1,+\infty)$ with finite linear measure (and so with finite logarithmic measure) such that for all z satisfying $|z| = r \notin E_3$, we have

$$|A_j(z)| \leqslant \exp_p\{r^{\rho+\varepsilon}\} \quad (j = 0,\cdots,k-1), \quad |F(z)| \leqslant \exp_p\{r^{\rho+\varepsilon}\}. \tag{3.3}$$

It follows that for all z satisfying $|z| = r \notin E_3$ at which $|g(z)| = M(r,g)$, we have for any given $\varepsilon \left(0 < \varepsilon < \frac{\mu_p(f) - \rho_p(d)}{2}\right)$

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)F(z)}{g(z)}\right| \leqslant \frac{\exp_p\left\{r^{\rho_p(d)+\varepsilon}\right\} \exp_p\{r^{\rho+\varepsilon}\}}{\exp_p\left\{r^{\mu_p(f)-\varepsilon}\right\}} \leqslant \exp_p\{r^{\rho+\varepsilon}\}. \tag{3.4}$$

Substituting (3.2), (3.3) and (3.4) into (3.1), we obtain for all z satisfying $|z| = r \notin [0,1] \cup E_2 \cup E_3$ at which $|g(z)| = M(r,g)$,

$$\left|\frac{\nu_g(r)}{z}\right|^k |1 + o(1)| \leqslant \exp_p\{r^{\rho+\varepsilon}\}$$

$$+\sum_{j=1}^{k-1} \exp_p\left\{r^{\rho+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + \exp_p\left\{r^{\rho+\varepsilon}\right\}.$$

So, we get

$$\left|\nu_g(r)\right|^k |1+o(1)| \leqslant (k+1)\, r^k \exp_p\left\{r^{\rho+\varepsilon}\right\}\left|\nu_g(r)\right|^{k-1} |1+o(1)|. \tag{3.5}$$

Then, by Lemma 3.6 and Lemma 3.2, we obtain from (3.5) that $\rho_{p+1}(g) = \rho_{p+1}(f) \leqslant \rho+\varepsilon$. Since $\varepsilon \left(0 < \varepsilon < \frac{\mu_p(f)-\rho_p(d)}{2}\right)$ being arbitrary, then we get $\rho_{p+1}(f) \leqslant \rho$. ∎

Lemma 3.8. *Let $H \subset [0,+\infty)$ be a set with a positive upper density (or of infinite linear measure), and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ $(A_0 \not\equiv 0)$, $F(z)$ be meromorphic functions with finite iterated $p-$ order.*

If there exist positive constants $\sigma > 0$, $\alpha > 0$ such that $|A_0(z)| \geqslant \exp_p\{\alpha r^\sigma\}$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\left\{\rho_p\left(A_j\right)(j = 1, 2, \cdots, k-1), \rho_p(F)\right\} < \sigma$, then every meromorphic solution $f \not\equiv 0$ of equation (1.2) *is transcendental and satisfies $\rho_p(f) \geqslant \sigma$.*

Proof. Assume that $f \not\equiv 0$ is a meromorphic solution of (1.2) with $\rho_p(f) < \sigma$. It follows from (1.2) that

$$\frac{F}{f} - \frac{f^{(k)}}{f} - \sum_{j=1}^{k-1} A_j \frac{f^{(j)}}{f} = A_0. \tag{3.6}$$

Since $\rho_p\left(A_j\right) < \sigma$ $(j = 1, 2, \cdots, k-1)$, $\rho_p(F) < \sigma$ and $\rho_p(f) < \sigma$, then from (3.6) we obtain that the iterated $p-$order of A_0 is $\rho_1 = \rho_p(A_0) \leqslant \max\left\{\rho, \rho_p(f)\right\} < \sigma$. By Lemma 3.4, for any ε $(0 < \varepsilon < \sigma - \rho_1)$ there exists a set $E_3 \subset (1, +\infty)$ with a finite linear measure such that

$$|A_0(z)| \leqslant \exp_p\left\{r^{\rho_1+\varepsilon}\right\} \tag{3.7}$$

holds for all z satisfying $|z| = r \notin E_3$. From the hypotheses of Lemma 3.8, there exists a set H with $\overline{dens}H > 0$ (or $m(H) = \infty$), and there exist positive constants $\sigma > 0$, $\alpha > 0$ such that

$$|A_0(z)| \geqslant \exp_p\{\alpha r^\sigma\} \tag{3.8}$$

holds for all z satisfying $|z| = r \in H$, $r \to +\infty$. By (3.7) and (3.8), we conclude that for all z satisfying $|z| = r \in H \backslash E_3$, $r \to +\infty$, we have

$$\exp_p\{\alpha r^\sigma\} \leqslant \exp_p\left\{r^{\rho_1+\varepsilon}\right\}$$

and by ε $(0 < \varepsilon < \sigma - \rho_1)$ this is a contradiction as $r \to +\infty$. Consequently, any meromorphic solution $f \not\equiv 0$ of equation (1.2) is transcendental and satisfies $\rho_p(f) \geqslant \sigma$. ∎

Lemma 3.9. *Let $g(z)$ be a nonconstant entire function of finite iterated $p-$order. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, +\infty)$ with $\overline{dens}H_1 = 1$ such that*

$$M(r, g) \geqslant \exp_p \left\{ r^{\rho_p(g)-\varepsilon} \right\}$$

for all z satisfying $|z| = r \in H_1$.

Proof. When $p = 1$, the lemma is due to Kwon in [14]. Thus we assume that $p \geqslant 2$. Set $\alpha = \rho_p(g) - \varepsilon$ and $\beta = \rho_p(g) - \frac{\varepsilon}{2}$. Then there is a sequence $\{r_n\}$ of real numbers for which we have $(r_n)^{\frac{\varepsilon}{2}} \geqslant n^{\alpha}$ and

$$M(r_n, g) \geqslant \exp_p \left\{ r_n^{\beta} \right\}.$$

Therefore

$$M(r_n, g) \geqslant \exp_p \left\{ r_n^{\beta} \right\} = \exp_p \left\{ (r_n)^{\frac{\varepsilon}{2}} r_n^{\beta - \frac{\varepsilon}{2}} \right\} \geqslant \exp_p \{ (n r_n)^{\alpha} \} \tag{3.9}$$

for all $n \in \mathbb{N}$. By (3.9) for any $r \in [r_n, n r_n]$, we obtain

$$M(r, g) \geqslant M(r_n, g) \geqslant M\left(\frac{r}{n}, g\right) \geqslant \exp_p \{ r^{\alpha} \}.$$

Set $H_1 = \bigcup_{n=1}^{\infty} [r_n, n r_n]$. Now, we take a sequence $\{R_n\}$ such that $\frac{n r_n}{2} \leqslant R_n \leqslant n r_n$, then

$$\overline{dens}H_1 = \limsup_{r \longrightarrow +\infty} \frac{m(H_1 \cap [0, r])}{r} \geqslant \limsup_{n \longrightarrow +\infty} \frac{m([r_n, n r_n] \cap [0, R_n])}{R_n}$$

$$\geqslant \limsup_{n \longrightarrow +\infty} \frac{R_n - \frac{2R_n}{n}}{R_n} = \limsup_{n \longrightarrow +\infty} \left(1 - \frac{2}{n}\right) = 1.$$

By definition we have $0 \leqslant \overline{dens}H_1 \leqslant 1$. Hence $\overline{dens}H_1 = 1$. ∎

Lemma 3.10. ([4]) *Let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite iterated $p-$order meromorphic functions.*

If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of equation (1.2), then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.

4. PROOF OF THEOREM 2.1

Let $f \not\equiv 0$ be a meromorphic solution of (1.1). It follows from (1.1) that

$$|A_0| \leqslant \left| \frac{f^{(k)}}{f} \right| + \sum_{j=1}^{k-1} |A_j| \left| \frac{f^{(j)}}{f} \right|. \tag{4.1}$$

By Lemma 3.8, we know that f is transcendental. By using Lemma 3.1, there is a set $E_1 \subset (0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r\left[T(2r, f)\right]^{2k} \ (j = 1, 2, \cdots, k). \tag{4.2}$$

By Lemma 3.4, for any given ε $(0 < \varepsilon < \sigma - \rho)$ there exists a set $E_3 \subset (1, +\infty)$ with finite linear measure such that

$$\left|A_j(z)\right| \leqslant \exp_p\{r^{\rho+\varepsilon}\}, \ j = 1, 2, \cdots, k-1 \tag{4.3}$$

holds for all z satisfying $|z| = r \notin E_3$. Also, by the hypotheses of Theorem 2.1, there exists a set $H \subset [0, +\infty)$ with $m(H) = \infty$, such that for all z satisfying $|z| = r \in H$, $r \to +\infty$, we have

$$|A_0(z)| \geqslant \exp_p\{\alpha r^{\sigma}\}. \tag{4.4}$$

Hence it follows from (4.1), (4.2), (4.3) and (4.4) that for all z satisfying $|z| = r \in H \setminus (E_1 \cup E_3)$, $r \to +\infty$, we have

$$\begin{aligned}\exp_p\{\alpha r^{\sigma}\} &\leqslant r\left[T(2r, f)\right]^{2k} + \sum_{j=1}^{k-1}\exp_p\{r^{\rho+\varepsilon}\} r\left[T(2r, f)\right]^{2k} \\ &\leqslant kr\exp_p\{r^{\rho+\varepsilon}\}\left[T(2r, f)\right]^{2k}.\end{aligned} \tag{4.5}$$

By $0 < \varepsilon < \sigma - \rho$, it follows from Lemma 3.5 and (4.5) that

$$\mu_p(f) = \rho_p(f) = \infty \text{ and } \rho_{p+1}(f) \geqslant \sigma. \tag{4.6}$$

Furthermore, if $\lambda_p(1/f) < \infty$, then f is a meromorphic solution of (1.1) with $\rho_p(f) = \mu_p(f) = \infty$, $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ and by Remark 2.1, we have $\max\{\rho_p\left(A_j\right) : j = 0, \cdots, k-1\} = \rho_p(A_0) = \beta < \infty$. Thus, by Lemma 3.7, we get

$$\rho_{p+1}(f) \leqslant \rho_p(A_0). \tag{4.7}$$

By (4.6) and (4.7), we conclude that $\mu_p(f) = \rho_p(f) = \infty$ and $\sigma \leqslant \rho_{p+1}(f) \leqslant \rho_p(A_0)$.

5. PROOF OF THEOREM 2.2

Let f be a meromorphic solution of (1.2). Assume that $\rho_p(f) < \infty$. It follows from (1.2) that

$$|A_0| \leqslant \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\left|A_j\right|\left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right|. \tag{5.1}$$

By Lemma 3.8, we know that f is transcendental with $\rho_p(f) \geqslant \sigma$. By the hypothesis $\lambda_p\left(\frac{1}{f}\right) < \sigma$ and Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda_p(d) = \rho_p(d) = \lambda_p(1/f) < \sigma,$$

$\rho_p(f) = \rho_p(g) \geqslant \sigma$. By Lemma 3.9, for any given ε $\left(0 < \varepsilon < \rho_p(g)\right)$, there exists a set $H_1 \subset [0, +\infty)$ with $\overline{dens} H_1 = 1$ such that

$$M(r, g) \geqslant \exp_p\left\{r^{\rho_p(g)-\varepsilon}\right\} \tag{5.2}$$

holds for all z satisfying $|z| = r \in H_1$. We have

$$\rho = \max\{\rho_p\left(A_j\right) \ (j = 1, 2, \cdots, k-1), \rho_p(F)\} < \sigma.$$

Then by Lemma 3.4, we have by using (5.2) for any given ε $(0 < \varepsilon < \min\{\sigma - \rho, \frac{\rho_p(g)-\rho_p(d)}{2}\})$, there exists a set $E_3 \subset (1, +\infty)$ with finite linear measure such that for all z satisfying $|z| = r \in H_1 \backslash E_3$ at which $|g(z)| = M(r, g)$,

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)F(z)}{g(z)}\right| \leqslant \frac{\exp_p\left\{r^{\rho_p(d)+\varepsilon}\right\}\exp_p\left\{r^{\rho+\varepsilon}\right\}}{\exp_p\left\{r^{\rho_p(g)-\varepsilon}\right\}} \leqslant \exp_p\left\{r^{\rho+\varepsilon}\right\}. \tag{5.3}$$

By using similar arguments as in the proof of Theorem 2.1, for any given ε $\left(0 < \varepsilon < \min\left\{\sigma - \rho, \frac{\rho_p(g)-\rho_p(d)}{2}\right\}\right)$, there exists a set $H_2 = H \cap H_1 \subset [0, +\infty)$ with positive upper density such that for all z satisfying $|z| = r \in H_2 \backslash (E_1 \cup E_3)$, $r \to +\infty$, at which $|g(z)| = M(r, g)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r\left[T(2r, f)\right]^{2k} \ (j = 1, 2, \cdots, k), \tag{5.4}$$

$$\left|\frac{F(z)}{f(z)}\right| \leqslant \exp_p\left\{r^{\rho+\varepsilon}\right\}, \tag{5.5}$$

$$\left|A_j(z)\right| \leqslant \exp_p\left\{r^{\rho+\varepsilon}\right\} \ (j = 1, 2, \cdots, k-1), \tag{5.6}$$

$$|A_0(z)| \geqslant \exp_p\left\{\alpha r^{\sigma}\right\}. \tag{5.7}$$

Substituting (5.4), (5.5), (5.6) and (5.7) into (5.1), we obtain for all z satisfying $|z| = r \in H_2 \backslash (E_1 \cup E_3)$, $r \to +\infty$, at which $|g(z)| = M(r, g)$, that for any given $\varepsilon \left(0 < \varepsilon < \min\left\{\sigma - \rho, \frac{\rho_p(g)-\rho_p(d)}{2}\right\}\right)$

$$\begin{aligned}\exp_p\left\{\alpha r^{\sigma}\right\} &\leqslant r\left[T(2r, f)\right]^{2k} + \sum_{j=1}^{k-1} \exp_p\left\{r^{\rho+\varepsilon}\right\} r\left[T(2r, f)\right]^{2k} + \exp_p\left\{r^{\rho+\varepsilon}\right\} \\ &\leqslant (k+1)\, r\left[T(2r, f)\right]^{2k} \exp_p\left\{r^{\rho+\varepsilon}\right\}. \end{aligned} \tag{5.8}$$

Hence by (5.8), we have $\rho_p(f) = \infty$. This is a contradiction which means that the assumption of $\rho_p(f) < \infty$ is not true. Hence, we conclude that $\rho_p(f) = \infty$. Since $F \not\equiv 0$, then by Lemma 3.10, we obtain

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty \text{ and } \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f). \tag{5.9}$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \min\left\{\mu_p(f), \sigma\right\}$, then f is a meromorphic solution of (1.2) with $\rho_p(f) = \infty$, $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ and by Remark 2.1, we have

$$\max\{\rho_p\left(A_j\right)(j = 0, 1, \cdots, k-1), \rho_p(F)\} = \rho_p(A_0) = \beta < \infty.$$

Therefore, by Lemma 3.7, we get

$$\rho_{p+1}(f) \leqslant \rho_p(A_0). \tag{5.10}$$

By (5.9) and (5.10), we conclude that

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty \text{ and } \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leqslant \rho_p(A_0).$$

References

[1] M. Andasmas and B. Belaïdi, *On the growth and the zeros of solutions of higher order linear differential equations with meromorphic coefficients*, To appear in Publications de l'Institut Mathématique.

[2] S. Bank, *General theorem concerning the growth of solutions of first-order algebraic differential equations*, Compositio Math. 25 (1972), 61–70.

[3] B. Belaïdi, *Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions*, Electron. J. Qual. Theory Differ. Equ. 2002, no. 5, 1-8.

[4] B. Belaïdi, *Oscillation of fixed points of solutions of some linear differential equations*, Acta. Math. Univ. Comenianae 77 (2008), no. 2, 263–269.

[5] L. G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc. 101 (1987), no. 2, 317–322.

[6] T. B. Cao, Z. X. Chen, X. M. Zheng and J. Tu, *On the iterated order of meromorphic solutions of higher order linear differential equations*, Ann. Differential Equations 21 (2005), no. 2, 111–122.

[7] Z. X. Chen and C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J. 22 (1999), no. 2, 273–285.

[8] Y. Chen, *Estimates of the zeros and growths of meromorphic solutions of homogeneous and non-homogeneous second order linear differential equations,* Math. Appl. (Wuhan) 23(2010), no. 1, 18-26.

[9] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) 37 (1988), no. 1, 88–104.

[10] G. G. Gundersen*, Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), no. 1, 415–429.

[11] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[12] J. He, X. M. Zheng and H. Hu, *Iterated order of meromorphic solutions of certain higher order linear differential equations with meromorphic coefficients of finite iterated order*, Acta Univ. Apulensis Math. Inform. No. 33 (2013), 145-157.

[13] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. 22 (1998), no. 4, 385–405.

[14] K. H. Kwon, *On the growth of entire functions satisfying second order linear differential equations* , Bull. Korean Math. Soc. **33**(1996), no. 3, 487-496.

[15] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin-New York, 1993.

[16] X. Shen and H. Y. Xu, *The zeros and growth of solutions of higher order differential equations,* Fourth International Conference on Computational and Information Sciences (2012), 605-608.

[17] J. Tu and Z. X. Chen, *Growth of solutions of complex differential equations with meromorphic coefficients of finite iterated order*, Southeast Asian Bull. Math. 33 (2009), no. 1, 153–164.

[18] J. Tu and T. Long, *Oscillation of complex high order linear differential equations with coefficients of finite iterated order*, Electron. J. Qual. Theory Differ. Equ. 2009, No. 66, 1-13.

[19] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.