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SOME PROPERTIES OF COMBINATION OF SOLUTIONS TO SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS OF $[P,Q]$-ORDER IN THE UNIT DISC

BENHARRAT BELAÏDI[1] AND ZINELÂABIDINE LATREUCH

ABSTRACT. In this paper, we consider some properties on the growth and oscillation of combination of solutions of the linear differential equation

$$f'' + A(z) f' + B(z) f = 0,$$

with analytic coefficients $A(z)$ and $B(z)$ with $[p,q]$ −order in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

1. Introduction and preliminaries

In the year 2000, Heittokangas firstly investigated the growth and oscillation theory of complex differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1.1}$$

where $A_0(z), \cdots, A_{k-1}(z)$ are analytic functions in the unit disc (see, [15]). It is well-known that all solutions of (1.1) are analytic functions (see, [15]). After him many authors (see, [4], [5], [8], [9], [10], [11], [12], [13], [16], [22]) have investigated the complex differential equation (1.1) and the second-order differential equations

$$f'' + A(z) f' + B(z) f = 0, \tag{1.2}$$

$$f'' + A(z) f = 0 \tag{1.3}$$

with analytic and meromorphic coefficients in the unit disc Δ. In ([17], [18]), Juneja and his co-authors investigated some properties of entire functions of $[p,q]$ −order, and obtained some results concerning their growth. Later, Liu, Tu and Shi; Xu, Tu and Xuan; Li and Cao; Belaïdi; Latreuch and Belaïdi applied the concepts of entire (meromorphic) functions in the complex plane and analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ of $[p,q]$ −order to investigate the complex differential equation (1.1) (see [6], [7], [22], [23], [24], [26]). In this paper, we will use this concept to study the growth and the oscillation of the combination of two linearly independent solutions f_1 and f_2 of equation (1.2) in the unit disc.

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, see ([14], [15], [19], [20], [25]).

In the following, we will give similar definitions as in ([17], [18]) for analytic and meromorphic functions of $[p,q]$-order, $[p,q]$-type and $[p,q]$-exponent of convergence of the zero-sequence in the unit disc.

Definition 1.1. ([6],[22]) *Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Δ, the $[p,q]$-order of $f(z)$ is defined by*

$$\rho_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Remark 1.1. *It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq +\infty$ ($0 \leq \rho_{M,[p,q]}(f) \leq +\infty$), for any $p \geq q \geq 1$. By Definition 1.1, we have that $\rho_{[1,1]} =$*

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$\rho(f)$ $(\rho_{M,[1,1]} = \rho_M(f))$ *and* $\rho_{[2,1]} = \rho_2(f)$ $(\rho_{M,[2,1]} = \rho_{M,2}(f))$.

For the relationship between $\rho_{[p,q]}(f)$ and $\rho_{M,[p,q]}(f)$ we have the following double inequality.

Proposition 1.1. *([6]) Let $p \geq q \geq 1$ be integers, and let f be an analytic function in Δ of $[p,q]$-order.*
(i) If $p = q \geq 1$, then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f) + 1.$$

(ii) If $p > q \geq 1$, then

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

Definition 1.2. *([22]) Let $p \geq q \geq 1$ be integers. The $[p,q]$-type of a meromorphic function $f(z)$ in Δ of $[p,q]$-order ρ $(0 < \rho < +\infty\ )$ is defined by*

$$\tau_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p-1} T(r,f)}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\rho}}.$$

Definition 1.3. *([22]) Let $p \geq q \geq 1$ be integers. The $[p,q]$-exponent of convergence of the zero-sequence of $f(z)$ in Δ is defined by*

$$\lambda_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_p N\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ in Δ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$.

The study of the properties of linearly independent solutions of complex differential equations is an old problem. In ([2], [3]), Bank and Laine obtained some results about the product $E = f_1 f_2$ of two linearly independent solutions f_1 and f_2 of (1.3) in the complex plane. In [21], the authors have investigated the relations between the polynomial of solutions of (1.2) and small functions in the complex plane. They showed that $w = d_1 f_1 + d_2 f_2$ keeps the same properties of growth and oscillation of f_j $(j = 1,2)$, where f_1 and f_2 are two linearly independent solutions of (1.2) and obtained the following results.

Theorem 1.1. *([21]) Let $A(z)$ and $B(z)$ be entire functions of finite order such that $\rho(A) < \rho(B)$ and $\tau(A) < \tau(B) < +\infty$ if $\rho(B) = \rho(A) > 0$. Let $d_j(z)$ $(j = 1,2)$ be entire functions that are not all vanishing identically such that $\max\{\rho(d_1), \rho(d_2)\} < \rho(B)$. If f_1 and f_2 are two nontrivial linearly independent solutions of (1.2), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(w) = \rho(f_1) = \rho(f_2) = +\infty$$

and

$$\rho_2(w) = \rho(B).$$

In the same paper, the authors studied also the zeros of the difference between the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ and entire functions of finite order.

The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers. Section 3 is for some lemmas and basic theorems. The other sections are for the proofs of our main results.

2. Main results

A natural question arises: What can be said about similar situations in the unit disc Δ for equation (1.2) in the terms of $[p,q]$ $-$order? Before we state our results, we define h and $\psi(z)$ by

$$h = \begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix},$$

where

$$H_1 = d_1,\ H_2 = 0,\ H_3 = d_2,\ H_4 = 0,\ H_5 = d_1',\ H_6 = d_1,$$
$$H_7 = d_2',\ H_8 = d_2,\ H_9 = d_1'' - d_1 B,$$
$$H_{10} = 2d_1' - d_1 A,\ H_{11} = d_2'' - d_2 B,$$
$$H_{12} = 2d_2' - d_2 A,\ H_{13} = d_1^{(3)} - 3d_1' B + d_1 AB - d_1 B',$$
$$H_{14} = 3d_1'' - 2d_1' A - d_1 B + d_1 A^2 - d_1 A',$$
$$H_{15} = d_2^{(3)} - 3d_2' B + d_2 AB - d_2 B',$$
$$H_{16} = 3d_2'' - 2d_2' A - d_2 B + d_2 A^2 - d_2 A'$$

and

$$\psi(z) = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi,$$

where $\varphi \not\equiv 0$, d_j $(j = 1,2)$ are analytic functions of finite $[p,q]$-order in Δ and

$$\phi_2 = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) A - 3d_1 d_2 d_2'' + 3d_2^2 d_1''}{h}, \tag{2.1}$$

$$\phi_1 = \frac{6d_2\left(d_1' d_2'' - d_2' d_1''\right) + 2d_2\left(d_1 d_2' - d_2 d_1'\right) B}{h}$$
$$+ \frac{2d_2\left(d_1 d_2' - d_2 d_1'\right) A' + 3d_2\left(d_2 d_1'' - d_1 d_2''\right) A}{h}, \tag{2.2}$$

$$\begin{aligned}\phi_0 = & \frac{1}{h}[(d_1 d_2' d_2'' - 3d_2 d_2' d_1'' + 2d_2 d_1' d_2'')\, A \\ & + \left(4d_1 (d_2')^2 + 3d_2^2 d_1'' - 3d_1 d_2 d_2'' - 4d_2 d_1' d_2'\right) B \\ & +2\left(d_2 d_1' d_2' - d_1 (d_2')^2\right) A' \\ & +2\left(d_1 d_2 d_2' - d_2^2 d_1'\right) B' + 6(d_2')^2 d_1'' \\ & -2d_1 d_2' d_2''' + 2d_2 d_1' d_2''' \\ & -3d_2 d_1'' d_2'' - 6d_1' d_2' d_2'' + 3d_1 (d_2'')^2]. \qquad (2.3)\end{aligned}$$

Theorem 2.1. *Let $p \geq q \geq 1$ be integers, and let $A(z)$ and $B(z)$ be analytic functions in Δ of finite $[p,q]$-order such that $\rho_{[p,q]}(A) < \rho_{[p,q]}(B)$ and $0 < \tau_{[p,q]}(A) < \tau_{[p,q]}(B) < +\infty$ if $\rho_{[p,q]}(B) = \rho_{[p,q]}(A) > 0$. Let $d_j(z)$ $(j = 1,2)$ be analytic functions that are not all vanishing identically such that $\max\{\rho_{[p,q]}(d_1), \rho_{[p,q]}(d_2)\} < \rho_{[p,q]}(B)$. If f_1 and f_2 are two nontrivial linearly independent solutions of (1.2), then the polynomial of solutions*

$$w = d_1 f_1 + d_2 f_2 \qquad (2.4)$$

satisfies

$$\rho_{[p,q]}(w) = \rho_{[p,q]}(f_1) = \rho_{[p,q]}(f_2) = +\infty$$

and

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(w) \leq \alpha_M,$$

where and in the following $\alpha_M = \max\{\rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B)\}$. Furthermore, if $p > q \geq 1$, then

$$\rho_{[p+1,q]}(w) = \rho_{[p,q]}(B).$$

Example 2.1. *([16]) For $\beta > 0$, the functions $f_1(z) = \exp(\exp((1-z)^{-\beta}))$ and $f_2(z) = \exp((1-z)^{-\beta})\exp(\exp((1-z)^{-\beta}))$ are linearly independent solutions of (1.2) satisfying*

$$\rho_{[1,1]}(f_1) = \rho_{[1,1]}(f_2) = +\infty$$

and

$$\rho_{[2,1]}(f_1) = \rho_{[2,1]}(f_2) = \beta,$$

where

$$A(z) = -\frac{2\beta \exp((1-z)^{-\beta})}{(1-z)^{\beta+1}} - \frac{\beta}{(1-z)^{\beta+1}} - \frac{1+\beta}{1-z}$$

and

$$B(z) = \frac{\beta^2 \exp(2(1-z)^{-\beta})}{(1-z)^{2\beta+2}}.$$

It is clear that $\rho_{[1,1]}(A) = \rho_{[1,1]}(B)$ and $\tau_{[1,1]}(A) < \tau_{[1,1]}(B)$. Then, by Theorem 2.1 for any two analytic functions $d_i(z)$ $(i = 1,2)$ of finite order $\rho_{[1,1]}(d_i) < +\infty$ $(i = 1,2)$ that are not all vanishing identically such that $\max\{\rho_{[1,1]}(d_1), \rho_{[1,1]}(d_2)\} < \rho_{[1,1]}(B)$, the combination $w = d_1 f_1 + d_2 f_2$ is of infinite order $\rho_{[1,1]}(w) = +\infty$ and $\rho_{[2,1]}(w) = \beta$.

From Theorem 2.1, we can obtain the following result.

Corollary 2.1. *Let $p \geq q \geq 1$ be integers, and let $f_i(z)$ $(i = 1,2)$ be two nontrivial linearly independent solutions of (1.2), where $A(z)$ and $B(z) \not\equiv 0$ are analytic functions of finite $[p,q]-$order in Δ such that $\rho_{[p,q]}(A) < \rho_{[p,q]}(B)$ or $\rho_{[p,q]}(A) = \rho_{[p,q]}(B) > 0$ and $0 < \tau_{[p,q]}(A) < \tau_{[p,q]}(B) < +\infty$, and let $d_j(z)$ $(j = 1,2,3)$ be analytic functions in Δ satisfying*

$$\max\{\rho_{[p,q]}(d_j) : j = 1,2,3\} < \rho_{[p,q]}(B)$$

and

$$d_2(z) f_2 + d_1(z) f_1 = d_3(z).$$

Then $d_j(z) \equiv 0$ $(j = 1,2,3)$.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, let $\varphi(z) \not\equiv 0$ be an analytic function in Δ with finite $[p,q]-$order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent solutions of (1.2), then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

(2.5)
$$\overline{\lambda}_{[p,q]}(w-\varphi) = \lambda_{[p,q]}(w-\varphi) = \rho_{[p,q]}(w) = +\infty$$

and

$$\rho_{[p,q]}(B) \leq \overline{\lambda}_{[p+1,q]}(w-\varphi) =$$

$$\lambda_{[p+1,q]}(w-\varphi) = \rho_{[p+1,q]}(w) \leq \alpha_M. \qquad (2.6)$$

Furthermore, if $p > q \geq 1$, then

$$\overline{\lambda}_{[p+1,q]}(w-\varphi) = \lambda_{[p+1,q]}(w-\varphi)$$

$$= \rho_{[p+1,q]}(w) = \rho_{[p,q]}(B). \qquad (2.7)$$

Theorem 2.3. *Let $p \geq q \geq 1$ be integers, and let $A(z)$ and $B(z)$ be analytic functions in Δ of finite $[p,q]-$order such that $\rho_{[p,q]}(A) < \rho_{[p,q]}(B)$. Let $d_j(z), b_j(z)$ $(j = 1,2)$ be finite $[p,q]-$order analytic functions in Δ such that $d_1(z) b_2(z) - d_2(z) b_1(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent solutions of (1.2), then*

$$\rho_{[p,q]}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = +\infty$$

and

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) \leq \alpha_M.$$

Furthermore, if $p > q \geq 1$, then

$$\rho_{[p+1,q]}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \rho_{[p,q]}(B).$$

3. Auxiliary lemmas

Lemma 3.1. *([14], [15], [25]) Let f be a meromorphic function in the unit disc and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r,f) = O\left(\log^{+} T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < +\infty$.

Lemma 3.2. *([1], [15]) Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ holds outside of an exceptional set $E_2 \subset [0,1)$ for which $\int_{E_2} \frac{dr}{1-r} < +\infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \le h(s(r))$ for all $r \in [0,1)$.*

Lemma 3.3. *([6]) Let $p \ge q \ge 1$ be integers. If $A_0(z), \cdots, A_{k-1}(z)$ are analytic functions of $[p,q]-$order in the unit disc Δ, then every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f)$$
$$\le \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\}.$$

Lemma 3.4. *([22]) Let $p \ge q \ge 1$ be integers. Let A_j $(j = 0, \cdots, k-1)$, $F \not\equiv 0$ be analytic functions in Δ, and let $f(z)$ be a solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F$$

satisfying

$$\max\left\{\rho_{[p,q]}(A_j) \quad (j = 0, \cdots, k-1), \rho_{[p,q]}(F)\right\}$$
$$< \rho_{[p,q]}(f) = \rho \le +\infty.$$

Then we have

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$$

and

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Lemma 3.5. *([22]) Let $p \ge q \ge 1$ be integers, and let f and g be non-constant meromorphic functions of $[p,q]$-order in Δ. Then we have*

$$\rho_{[p,q]}(f+g) \le \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}$$

and

$$\rho_{[p,q]}(fg) \le \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then we obtain

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f).$$

Lemma 3.6. *([22]) Let $p \ge q \ge 1$ be integers, and let f and g be meromorphic functions of $[p,q]$-order in Δ such that $0 < \rho_{[p,q]}(f), \rho_{[p,q]}(g) < +\infty$ and $0 < \tau_{[p,q]}(f), \tau_{[p,q]}(g) < +\infty$. Then, we have*
(i) If $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then

$$\tau_{[p,q]}(f+g) = \tau_{[p,q]}(fg) = \tau_{[p,q]}(f).$$

(ii) If $\rho_{[p,q]}(f) = \rho_{[p,q]}(g)$ and $\tau_{[p,q]}(f) \ne \tau_{[p,q]}(g)$, then

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f) = \rho_{[p,q]}(g).$$

Lemma 3.7. *([22]) Let $p \ge q \ge 1$ be integers, and let $A_j(z)$ $(j = 0, \cdots, k-1)$ be analytic functions in Δ satisfying*

$$\max\left\{\rho_{[p,q]}(A_j) : j = 1, \cdots, k-1\right\} < \rho_{[p,q]}(A_0).$$

If $f \not\equiv 0$ is a solution of (1.1), then $\rho_{[p,q]}(f) = +\infty$ and

$$\rho_{[p,q]}(A_0) \le \rho_{[p+1,q]}(f)$$
$$\le \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, \cdots, k-1\right\}.$$

Furthermore, if $p > q \ge 1$, then

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Lemma 3.8. *Let $p \ge q \ge 1$ be integers, and let $A(z)$ and $B(z)$ be analytic functions in Δ of finite $[p,q]-$order such $\rho_{[p,q]}(A) < \rho_{[p,q]}(B)$. If f_1 and f_2 are two nontrivial linearly independent solutions of (1.2), then $\frac{f_1}{f_2}$ is of infinite $[p,q]$-order and*

$$\rho_{[p,q]}(B) \le \rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) \le \alpha_M.$$

Furthermore, if $p > q \ge 1$, then

$$\rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) = \rho_{[p,q]}(B).$$

Proof. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2). Since $\rho_{[p,q]}(B) > \rho_{[p,q]}(A)$, then by Lemma 3.7

$$\rho_{[p,q]}(f_1) = \rho_{[p,q]}(f_2) = +\infty, \ \rho_{[p,q]}(B) \le$$

$$\rho_{[p+1,q]}(f_1) = \rho_{[p+1,q]}(f_2) \le \alpha_M. \tag{3.1}$$

Furthermore, if $p > q \ge 1$, then

$$\rho_{[p+1,q]}(f_1) = \rho_{[p+1,q]}(f_2) = \rho_{[p,q]}(B).$$

On the other hand

$$\left(\frac{f_1}{f_2}\right)' = -\frac{W(f_1,f_2)}{f_2^2}, \tag{3.2}$$

where $W(f_1,f_2) = f_1 f_2' - f_2 f_1'$ is the Wronskian of f_1 and f_2. By using (1.2) we obtain that

$$W'(f_1,f_2) = -A(z) W(f_1,f_2),$$

which implies that

$$W(f_1,f_2) = K \exp(-\int A(z)dz), \tag{3.3}$$

where $\int A(z)dz$ is the primitive of $A(z)$ and $K \in \mathbb{C}\setminus\{0\}$. By (3.2) and (3.3) we have

$$\left(\frac{f_1}{f_2}\right)' = -K\frac{\exp(-\int A(z)dz)}{f_2^2}. \tag{3.4}$$

Since $\rho_{[p,q]}(f_2) = +\infty$, $\rho_{[p+1,q]}(f_2) \geq \rho_{[p,q]}(B) > \rho_{[p,q]}(A)$ if $p \geq q \geq 1$ and $\rho_{[p+1,q]}(f_2) = \rho_{[p,q]}(B) > \rho_{[p,q]}(A)$ if $p > q \geq 1$, then by using (3.1) and Lemma 3.5 we obtain from (3.4)

$$\rho_{[p,q]}\left(\frac{f_1}{f_2}\right) = \rho_{[p,q]}(f_2) = +\infty,$$

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) = \rho_{[p+1,q]}(f_2) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$\rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) = \rho_{[p+1,q]}(f_2) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. □

Lemma 3.9. *([6]) Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in the unit disc Δ such that $\rho_{[p,q]}(f) = \rho < +\infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_3 \subset [0,1)$ with $\int_{E_3}\frac{dr}{1-r} < +\infty$.

Lemma 3.10. *Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Δ with $[p,q]$ - order $0 < \rho_{[p,q]}(f) = \rho < +\infty$ and $[p,q]$ - type $0 < \tau_{[p,q]}(f) = \tau < +\infty$. Then for any given $\beta < \tau$, there exists a subset E_4 of $[0,1)$ that has an infinite logarithmic measure $\int_{E_4}\frac{dr}{1-r} = +\infty$ such that $\log_{p-1} T(r,f) > \beta\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\rho}$ holds for all $r \in E_4$.*

Proof. By the definitions of $[p,q]$ - order and $[p,q]$ - type, there exists an increasing sequence $\{r_m\}_{m=1}^{+\infty} \subset [0,1)$ $(r_m \to 1^-)$ satisfying $\frac{1}{m} + \left(1-\frac{1}{m}\right)r_m < r_{m+1}$ and

$$\lim_{m\to+\infty}\frac{\log_{p-1}T(r_m,f)}{\left(\log_{q-1}\left(\frac{1}{1-r_m}\right)\right)^{\rho}} = \tau.$$

Then there exists a positive integer m_0 such that for all $m \geq m_0$ and for any given $0 < \varepsilon < \tau - \beta$, we have

$$\log_{p-1}T(r_m,f) > (\tau-\varepsilon)\left(\log_{q-1}\left(\frac{1}{1-r_m}\right)\right)^{\rho}. \tag{3.5}$$

For any given $\beta < \tau - \varepsilon$, there exists a positive integer m_1 such that for all $m \geq m_1$ we have

$$\left[\frac{\log_{q-1}\left(1-\frac{1}{m}\right)\left(\frac{1}{1-r}\right)}{\log_{q-1}\left(\frac{1}{1-r}\right)}\right]^{\rho} > \frac{\beta}{\tau-\varepsilon}. \tag{3.6}$$

Take $m \geq m_2 = \max\{m_0, m_1\}$. By (3.5) and (3.6), for any $r \in [r_m, \frac{1}{m} + \left(1-\frac{1}{m}\right)r_m]$, we have

$$\begin{aligned}\log_{p-1}T(r,f) &\geq \log_{p-1}T(r_m,f)\\ &> (\tau-\varepsilon)\left(\log_{q-1}\left(\frac{1}{1-r_m}\right)\right)^{\rho}\\ &\geq (\tau-\varepsilon)\left(\log_{q-1}\left(1-\frac{1}{m}\right)\left(\frac{1}{1-r}\right)\right)^{\rho}\\ &> \beta\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho}.\end{aligned}$$

Set $E_4 = \bigcup_{m=m_2}^{+\infty}\left[r_m, \frac{1}{m} + \left(1-\frac{1}{m}\right)r_m\right]$, then there holds

$$\begin{aligned}m_l E_4 &= \sum_{m=m_2}^{+\infty}\int_{r_m}^{\frac{1}{m}+\left(1-\frac{1}{m}\right)r_m}\frac{dt}{1-t}\\ &= \sum_{m=m_2}^{+\infty}\log\frac{m}{m-1} = +\infty.\end{aligned}$$

□

Lemma 3.11. *Let $p \geq q \geq 1$ be integers, and let $A(z)$ and $B(z)$ be analytic functions in Δ of finite $[p,q]$ −order such $\rho_{[p,q]}(A) < \rho_{[p,q]}(B)$ and $0 < \tau_{[p,q]}(A) < \tau_{[p,q]}(B) < +\infty$ if $\rho_{[p,q]}(B) = \rho_{[p,q]}(A) > 0$. If $f \not\equiv 0$ is a solution of (1.2), then*

$$\rho_{[p,q]}(f) = +\infty\ ,\ \rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(f) \leq \alpha_M$$

and

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$.

Proof. If $\rho_{[p,q]}(B) > \rho_{[p,q]}(A)$, then the result can be deduced by Lemma 3.7. We prove only the case when $\rho_{[p,q]}(B) = \rho_{[p,q]}(A) = \rho$ and $\tau_{[p,q]}(B) > \tau_{[p,q]}(A) > 0$. Since $f \not\equiv 0$, then by (1.2)

$$B = -\left(\frac{f''}{f} + A\frac{f'}{f}\right).$$

Suppose that f is of finite $[p,q]$ −order $\rho_{[p,q]}(f) = \mu < +\infty$. Then by Lemma 3.9

$$T(r,B) \leq T(r,A) + O\left(\exp_{p-1}\left\{(\mu+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_3 \subset [0,1)$ with $\int_{E_3}\frac{dr}{1-r} < +\infty$, which implies by using Lemma 3.2 the contradiction

$$\tau_{[p,q]}(B) \leq \tau_{[p,q]}(A).$$

Hence $\rho_{[p,q]}(f) = +\infty$. By Lemma 3.3, we have

$$\begin{aligned}\rho_{[p+1,q]}(f) &= \rho_{M,[p+1,q]}(f)\\ &\leq \max\{\rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B)\}.\end{aligned}$$

On the other hand, since $\rho_{[p,q]}(f) = +\infty$, then by Lemma 3.1

$$T(r,B) \leq T(r,A) + O\left(\log^{+}T(r,f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.7}$$

holds for all r outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < +\infty$. By $\tau_{[p,q]}(B) > \tau_{[p,q]}(A) > 0$, we choose α_0, α_1 satisfying $\tau_{[p,q]}(B) > \alpha_0 > \alpha_1 > \tau_{[p,q]}(A)$ such that for $r \to 1^-$, we have

$$T(r,A) \leq \exp_{p-1}\left\{\alpha_1\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho}\right\}. \tag{3.8}$$

By Lemma 3.10, there exists a subset $E_4 \subset [0,1)$ of infinite logarithmic measure such that

$$T(r,B) > \exp_{p-1}\left\{\alpha_0\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho}\right\}. \tag{3.9}$$

By (3.7)-(3.9) we obtain for all $r \in E_4 \backslash E_1$

$$\exp_{p-1}\left\{\alpha_0\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho}\right\}$$
$$\leq \exp_{p-1}\left\{\alpha_1\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho}\right\}$$
$$+O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right). \tag{3.10}$$

By using (3.10) and Lemma 3.2, we obtain

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(f).$$

Hence

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(f) \leq \max\left\{\rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B)\right\}$$

if $p \geq q \geq 1$ and $\rho_{[p+1,q]}(f) = \rho_{[p,q]}(B)$ if $p > q \geq 1$. □

4. Proof of Theorem 2.1

Proof. In the case when $d_1(z) \equiv 0$ or $d_2(z) \equiv 0$, then the conclusions of Theorem 2.1 are trivial. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2) and $d_j(z) \not\equiv 0$ $(j=1,2)$. Then by Lemma 3.11, we have

$$\rho_{[p,q]}(f) = +\infty, \ \rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(f) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$\rho_{[p+1,q]}(f) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. Suppose that $d_1 = cd_2$, where c is a complex number. Then, we obtain

$$w = d_1 f_1 + d_2 f_2 = c d_2 f_1 + d_2 f_2 = (c f_1 + f_2) d_2.$$

Since $f = cf_1 + f_2$ is a solution of (1.2) and $\rho_{[p,q]}(d_2) < \rho_{[p,q]}(B)$, then we have

$$\rho_{[p,q]}(w) = \rho_{[p,q]}(cf_1 + f_2) = +\infty,$$

$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(cf_1 + f_2) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(cf_1 + f_2) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. Suppose now that $d_1 \not\equiv cd_2$ where c is a complex number. Differentiating both sides of (2.4), we obtain

$$w' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2'. \tag{4.1}$$

Differentiating both sides of (4.1), we obtain

$$w'' = d_1'' f_1 + 2d_1' f_1' + d_1 f_1'' + d_2'' f_2 + 2d_2' f_2' + d_2 f_2''. \tag{4.2}$$

Substituting $f_j'' = -A(z) f_j' - B(z) f_j$ $(j=1,2)$ into equation (4.2), we have

$$w'' = (d_1'' - d_1 B) f_1 + (2d_1' - d_1 A) f_1'$$
$$+ (d_2'' - d_2 B) f_2 + (2d_2' - d_2 A) f_2'. \tag{4.3}$$

Differentiating both sides of (4.3) and by substituting $f_j'' = -A(z) f_j' - B(z) f_j$ $(j=1,2)$, we obtain

$$w''' = \left(d_1^{(3)} - 3d_1' B + d_1(AB - B')\right) f_1$$
$$+ \left(3d_1'' - 2d_1' A + d_1\left(A^2 - A' - B\right)\right) f_1'$$
$$+ \left(d_2^{(3)} - 3d_2' B + d_2(AB - B')\right) f_2$$
$$+ \left(3d_2'' - 2d_2' A + d_2\left(A^2 - A' - B\right)\right) f_2'. \tag{4.4}$$

By (2.4) and (4.1)-(4.4) we have

$$\begin{cases} w = d_1 f_1 + d_2 f_2, \\ w' = d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2', \\ w'' = (d_1'' - d_1 B) f_1 + (2d_1' - d_1 A) f_1' \\ \quad + (d_2'' - d_2 B) f_2 + (2d_2' - d_2 A) f_2', \\ w''' = \left(d_1^{(3)} - 3d_1' B + d_1(AB - B')\right) f_1 \\ \quad + \left(3d_1'' - 2d_1' A + d_1\left(A^2 - A' - B\right)\right) f_1' \\ \quad + \left(d_2^{(3)} - 3d_2' B + d_2(AB - B')\right) f_2 \\ \quad + \left(3d_2'' - 2d_2' A + d_2\left(A^2 - A' - B\right)\right) f_2'. \end{cases}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations we obtain

$$h = \begin{vmatrix} H_1 & H_2 & H_3 & H_4 \\ H_5 & H_6 & H_7 & H_8 \\ H_9 & H_{10} & H_{11} & H_{12} \\ H_{13} & H_{14} & H_{15} & H_{16} \end{vmatrix}$$
$$= 2\left(d_1 d_2' - d_2 d_1'\right)^2 B$$
$$+ \left(d_2^2 d_1' d_1'' + d_1^2 d_2' d_2'' - d_1 d_2 d_1' d_2'' - d_1 d_2 d_2' d_1''\right) A$$
$$-2\left(d_1 d_2' - d_2 d_1'\right)^2 A' + 2d_1 d_2 d_1' d_2''' + 2d_1 d_2 d_2' d_1'''$$
$$-6d_1 d_2 d_1'' d_2'' - 6d_1 d_1' d_2' d_2'' - 6d_2 d_1' d_2' d_1''$$
$$+6d_1 (d_2')^2 d_1'' + 6d_2 (d_1')^2 d_2'' - 2d_2^2 d_1' d_1'''$$
$$-2d_1^2 d_2' d_2''' + 3d_1^2 (d_2'')^2 + 3d_2^2 (d_1'')^2.$$

It is clear that $\left(d_1 d_2' - d_2 d_1'\right)^2 \not\equiv 0$ because $d_1 \neq cd_2$. Since

$$\max\{\rho_{[p,q]}(d_1), \rho_{[p,q]}(d_2)\} < \rho_{[p,q]}(B)$$

and $(d_1 d_2' - d_2 d_1')^2 \not\equiv 0$, then by using Lemma 3.6 we can deduce that $\rho_{[p,q]}(h) = \rho_{[p,q]}(B) > 0$. Hence $h \not\equiv 0$. By Cramer's method we have

$$f_1 = \frac{\begin{vmatrix} w & H_2 & H_3 & H_4 \\ w' & H_6 & H_7 & H_8 \\ w'' & H_{10} & H_{10} & H_{12} \\ w^{(3)} & H_{14} & H_{15} & H_{16} \end{vmatrix}}{h}$$

$$(4.5) \quad = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w^{(3)} + \phi_2 w'' + \phi_1 w' + \phi_0 w,$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions in Δ of finite $[p, q]$-order which are defined in (2.1)-(2.3). By (4.5) and Lemma 3.5, we have $\rho_{[p,q]}(f_1) \leq \rho_{[p,q]}(w)$ $(\rho_{[p+1,q]}(f_1) \leq \rho_{[p+1,q]}(w))$ and by (2.4) we have $\rho_{[p,q]}(w) \leq \rho_{[p,q]}(f_1)$ $(\rho_{[p+1,q]}(w) \leq \rho_{[p+1,q]}(f_1))$. Thus $\rho_{[p,q]}(w) = \rho_{[p,q]}(f_1)$ and $\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(f_1)$. □

5. Proof of Corollary 2.1

Proof. We suppose there exists $j = 1, 2, 3$ such that $d_j(z) \not\equiv 0$ and we obtain a contradiction. If $d_1(z) \not\equiv 0$ or $d_2(z) \not\equiv 0$, then by Theorem 2.1 we have $\rho_{[p,q]}(d_1 f_1 + d_2 f_2) = +\infty = \rho_{[p,q]}(d_3) < \rho_{[p,q]}(B)$ which is a contradiction. Now if $d_1(z) \equiv 0$, $d_2(z) \equiv 0$ and $d_3(z) \not\equiv 0$ we obtain also a contradiction. Hence $d_j(z) \equiv 0$ $(j = 1, 2, 3)$. □

6. Proof of Theorem 2.2

Proof. By Theorem 2.1, we have

$$\rho_{[p,q]}(w) = +\infty, \ \rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(w) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$\rho_{[p+1,q]}(w) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. Set $g(z) = d_1 f_1 + d_2 f_2 - \varphi$. Since $\rho_{[p,q]}(\varphi) < +\infty$, then by Lemma 3.5 we have $\rho_{[p,q]}(g) = \rho_{[p,q]}(w) = +\infty, \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(w)$. In order to prove $\overline{\lambda}_{[p,q]}(w - \varphi) = \lambda_{[p,q]}(w - \varphi) = +\infty$ and $\overline{\lambda}_{[p+1,q]}(w - \varphi) = \lambda_{[p+1,q]}(w - \varphi) = \rho_{[p+1,q]}(w)$, we need to prove only $\overline{\lambda}_{[p,q]}(g) = \lambda_{[p,q]}(g) = +\infty$ and $\overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) = \rho_{[p+1,q]}(w)$. By $w = g + \varphi$ we get from (4.5)

$$(6.1) \quad f_1 = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(3)} + \phi_2 g'' + \phi_1 g' + \phi_0 g + \psi,$$

where

$$\psi = 2\frac{\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} \varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi.$$

Substituting (6.1) into equation (1.2), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} g^{(5)} + \sum_{j=0}^{4} \beta_j g^{(j)}$$
$$= -\left(\psi'' + A(z)\psi' + B(z)\psi\right) = F(z),$$

where β_j $(j = 0, \cdots, 4)$ are meromorphic functions of finite $[p, q]$-order in Δ. Since $\psi \not\equiv 0$ and $\rho_{[p,q]}(\psi) < +\infty$, it follows that ψ is not a solution of (1.2), which implies that $F(z) \not\equiv 0$. Then, by applying Lemma 3.4 we obtain (2.5), (2.6) and (2.7). □

7. Proof of Theorem 2.3

Proof. Suppose that f_1 and f_2 are two nontrivial linearly independent solutions of (1.2). Then by Lemma 3.8, we have

$$(7.1) \quad \rho_{[p,q]}\left(\frac{f_1}{f_2}\right) = +\infty, \ \rho_{[p,q]}(B) \leq \rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$(7.2) \quad \rho_{[p+1,q]}\left(\frac{f_1}{f_2}\right) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. Set $g = \frac{f_1}{f_2}$. Then

$$w(z) = \frac{d_1(z) f_1(z) + d_2(z) f_2(z)}{b_1(z) f_1(z) + b_1(z) f_2(z)}$$
$$= \frac{d_1(z) g(z) + d_2(z)}{b_1(z) g(z) + b_2(z)}.$$

By Lemma 3.5, it follows that

$$\rho_{[p+1,q]}(w)$$
$$\leq \max\{\rho_{[p+1,q]}(d_j), \rho_{[p+1,q]}(b_j)\,(j = 1, 2), \rho_{[p+1,q]}(g)\}$$
$$(7.3) \quad = \rho_{[p+1,q]}(g).$$

On the other hand

$$g(z) = -\frac{b_2(z) w(z) - d_2(z)}{b_1(z) w(z) - d_1(z)},$$

which implies by Lemma 3.5 that $\rho_{[p,q]}(w) \geq \rho_{[p,q]}(g) = +\infty$ and

$$\rho_{[p+1,q]}(g)$$
$$\leq \max\{\rho_{[p+1,q]}(d_j), \rho_{[p+1,q]}(b_j)\,(j = 1, 2), \rho_{[p+1,q]}(w)\}$$
$$(7.4) \quad = \rho_{[p+1,q]}(w).$$

By using (7.1)-(7.4), we obtain

$$\rho_{[p,q]}(w) = \rho_{[p,q]}(g) = +\infty,$$
$$\rho_{[p,q]}(B) \leq \rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g) \leq \alpha_M$$

if $p \geq q \geq 1$ and

$$\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g) = \rho_{[p,q]}(B)$$

if $p > q \geq 1$. □

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