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ON THE (p,q)-ORDER OF SOLUTIONS OF SOME COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the growth of solutions of some complex linear differential equations and we obtain some results on the (p,q)-order of these solutions. The results presented in this paper mainly improve the corresponding results announced in the literatures.

Keywords. Entire function; Meromorphic function; Differential equation; Lacunary series; (p,q)-order.

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1. Introduction and main results

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory (see e.g. [8, 14, 20]). For $r \in [0,+\infty)$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r = \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. Furthermore, we define the linear measure of a set $E \subset [0,+\infty)$ by $m(E) = \int_E dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ by $m_l(F) = \int_F \frac{dt}{t}$. For the unity of notations, we present here the definition of $(p,q)-$order where p and q are integers with $p \geq q \geq 1$; see, e.g., [15, 16].

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Definition 1.1. The $(p,q)-$order of a meromorphic function $f(z)$ is defined by

$$\sigma_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of the function f. If f is an entire function, then

$$\sigma_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

where $M(r,f)$ is the maximum modulus of f in the circle $|z| = r$.

Definition 1.2. The lower $(p,q)-$order of a meromorphic function $f(z)$ is defined by

$$\mu_{(p.q)}(f) = \liminf_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

If $f(z)$ is an entire function, then

$$\mu_{(p.q)}(f) = \liminf_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Definition 1.3. The $(p,q)-$type of a meromorphic function $f(z)$ with $0 < \sigma_{(p,q)}(f) < +\infty$ is defined by

$$\tau_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

If $f(z)$ is an entire function, then

$$\tau_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

Definition 1.4. The (p,q)-exponent of convergence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_p N(r,\frac{1}{f})}{\log_q r}$$

and the $(p,q)-$exponent of convergence of distinct zeros of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q r},$$

where $N\left(r,\frac{1}{f}\right)$ $\left(\overline{N}\left(r,\frac{1}{f}\right)\right)$ is the integrated counting function of zeros (distinct zeros) of $f(z)$ in $\{z:|z|\leq r\}$. The lower $(p,q)-$exponent of convergence of zeros of a meromorphic function $f(z)$ is defined by

$$\underline{\lambda}_{(p.q)}(f)=\liminf_{r\to+\infty}\frac{\log_p N(r,\frac{1}{f})}{\log_q r}$$

and the lower $(p,q)-$exponent of convergence of distinct zeros of a meromorphic function $f(z)$ is defined by

$$\underline{\bar{\lambda}}_{(p.q)}(f)=\liminf_{r\to+\infty}\frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q r}.$$

The $(p,q)-$exponent of convergence of the sequence of poles of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}\left(\frac{1}{f}\right)=\limsup_{r\to+\infty}\frac{\log_p N(r,f)}{\log_q r}.$$

In the past years, many authors investigated the complex linear differential equations

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0 \tag{1.1}$$

and

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{1.2}$$

when $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ are entire functions and obtained some valuable results, (see e.g. [1], [9], [15−18], [21]). In 2013, Tu et al. investigated the growth of solutions of equation (1.2) when the dominant coefficient $A_d(z)$ $(0\leq d\leq k-1)$ is of maximal order and being Lacunary series.

Theorem A. ([17]) *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ be entire functions of finite iterated order satisfying*

$$\max\{\sigma_p(A_j)\ (j\neq d),\sigma_p(F)\}<\mu_p(A_d)=\sigma_p(A_d)=\sigma<\infty\ (0\leq d\leq k-1).$$

Suppose that $A_d=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap series

$$\frac{\lambda_n}{n}>(\log n)^{2+\eta}\ (\eta>0,n\in\mathbb{N}). \tag{1.3}$$

Then every transcendental solution $f(z)$ of (1.2) satisfies $\mu_{p+1}(f)=\sigma_{p+1}(f)=\sigma$. Furthermore if $F(z)\not\equiv 0$, then every transcendental solution $f(z)$ of (1.2) satisfies $\underline{\overline{\lambda}}_{p+1}(f)=\underline{\lambda}_{p+1}(f)=\bar{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma$.

Recently, Huang *et al.* [9] considered the equation (1.2) with different conditions on the coefficient $A_d(z)$ and obtained the following result.

Theorem B. ([9]) *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ *be entire functions. Suppose that there exists some* $d\in\{1,\cdots,k-1\}$ *such that* $\max\{\sigma(A_j),\sigma(F):j\neq d\}\leq\sigma(A_d)<\infty$, $\max\{\tau(A_j):\sigma(A_j)=\sigma(A_d),\tau(F)\}<\tau(A_d)$ *and that* $T(r,A_d)\sim\log M(r,A_d)$ *as* $r\to+\infty$ *outside a set of* r *of finite logarithmic measure. Then we have*
(i) *Every transcendental solution* f *of* (1.2) *satisfes* $\sigma_2(f)=\sigma(A_d)$, *and* (1.2) *may have polynomial solutions* f *of degree* $<d$.
(ii) *If* $F(z)\not\equiv 0$, *then every transcendental solution* f *of* (1.2) *satisfies* $\bar{\lambda}_2(f)=\lambda_2(f)=\sigma_2(f)=\sigma(A_d)$.
(iii) *If* $d=1$, *then every nonconstant solution* f *of* (1.2) *satisfies* $\sigma_2(f)=\sigma(A_1)$. *Furthermore, if* $F(z)\not\equiv 0$, *then every nonconstant solution* f *of* (1.2) *satisfies* $\bar{\lambda}_2(f)=\lambda_2(f)=\sigma_2(f)=\sigma(A_1)$.

As for the linear differential equations

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0 \tag{1.4}$$

and

$$A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{1.5}$$

where $k\geq 2$, $A_j(z)$ $(j=0,1,\cdots,k)$, $F(z)$ are entire functions with $A_0A_kF\not\equiv 0$, many authors investigated the properties of their solutions and obtained some interesting results, (see e.g. [3], [4], [7], [19]). It well-known that if $A_k(z)\equiv 1$, then all solutions of (1.4) and (1.5) are entire functions, but when $A_k(z)$ is a nonconstant entire function, then equation (1.4) or (1.5) can possess meromorphic solutions. For instance the equation

$$zf'''+4f''+\left(-1-\frac{1}{2}z^2-z\right)e^{-z}f'+\left(\left(1-\frac{1}{2}z^2+2z\right)e^{-2z}+ze^{-3z}\right)f=0$$

has a meromorphic solution $f(z)=\dfrac{1}{z^2}e^{e^{-z}}$ and the equation

$$z^3f'''-z^3f''-2z^2f'-(z^3+3z^2-6)f=(z^2-6)\sin z$$

has a meromorphic solution $f(z)=\dfrac{\cos z}{z}$. In 2015, Wu and Zheng have considered the equations (1.4) and (1.5), and obtained the following result when the coefficient $A_k(z)$ is of maximal order and Fabry gap series.

Theorem C. ([19]) *Suppose that $k \geq 2$, $A_j(z)$ $(j=0,1,\cdots,k)$ are entire functions satisfying $A_k(z)A_0(z) \not\equiv 0$ and $\sigma(A_j) < \sigma(A_k) < \infty$ $(j=0,1,\cdots,k-1)$. Suppose that $A_k(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ and the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap codition*

$$\frac{\lambda_n}{n} \to \infty \ (n \to \infty). \tag{1.6}$$

Then every rational solution $f(z)$ of (1.4) is a polynomial with $\deg f \leq k-1$ and every transcendental meromorphic solution $f(z)$, whose poles are of uniformly bounded multiplicities, of (1.4) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$, satisfies

$$\bar{\lambda}(f-\varphi) = \lambda(f-\varphi) = \sigma(f) = \infty, \ \bar{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \sigma_2(f) = \sigma(A_k),$$

where $\varphi(z)$ is a finite order meromorphic function and doesn't solve (1.4).

Now, these theorems leaves us with two questions : First, can we have the same properties as in Theorem B for the solutions of equation (1.2) when the coefficients are of $(p,q)-$order? and secondly, what about the growth of solutions of the equations (1.4) and (1.5) when we have the arbitrary coefficient $A_s(z)$ $(0 \leq s \leq k)$ instead of the coefficient $A_k(z)$? In this paper, we proceed this way and we obtain the following results.

Theorem 1.1. *Let $A_j(z)$ $(j=0,1,\cdots,k)$ with $A_k(z)A_0(z) \not\equiv 0$ be entire functions such that*

$$\max\{\sigma_{(p,q)}(A_j), \ j \neq s\} < \sigma_{(p,q)}(A_s) < \infty, \ (0 \leq s \leq k).$$

Suppose that $A_s(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ and the sequence of exponents $\{\lambda_n\}$ satisfies (1.3). Then every rational solution $f(z)$ of (1.4) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution $f(z)$ of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfies

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s).$$

Theorem 1.2. *Let $A_j(z)$ $(j=0,1,\cdots,k)$ with $A_k(z)A_0(z) \not\equiv 0$ be entire functions such that*

$$\max\{\sigma_{(p,q)}(A_j), j \neq s\} < \mu_{(p,q)}(A_s) = \sigma_{(p,q)}(A_s) = \sigma < \infty, \ (0 \leq s \leq k).$$

Suppose that $A_s(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ and the sequence of exponents $\{\lambda_n\}$ satisfies (1.3). Then every rational solution $f(z)$ of (1.4) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution $f(z)$ of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfies

$$\mu_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s) = \sigma.$$

Theorem 1.3. *Let $A_j(z)$ $(j=0,1,\cdots,k)$ be entire functions satisfying the hypotheses of Theorem 1.1 and $F(z)\not\equiv 0$ is an entire function.*

(i) *If $\sigma_{(p+1,q)}(F)<\sigma_{(p,q)}(A_s)$, then every transcendental meromorphic solution $f(z)$ of (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ satisfies*

$$\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s),$$

with at most one exceptional solution f_0 satisfying $\sigma_{(p+1,q)}(f_0)<\sigma_{(p,q)}(A_s)$.

(ii) *If $\sigma_{(p+1,q)}(F)>\sigma_{(p,q)}(A_s)$, then every transcendental meromorphic solution $f(z)$ of (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ satisfies*

$$\sigma_{(p+1,q)}(f)=\sigma_{(p+1,q)}(F).$$

Remark 1.1 The Theorems 1.1-1.3 had been proved in [3] for the case where $A_k(z)$ is the dominant coefficient with $(p,q)-$order for the equations (1.4) and (1.5) and in this paper, we gave similar results when the arbitrary coefficient $A_s(z)$ $(0\leq s\leq k)$ is the dominant one instead of $A_k(z)$.

For equation (1.2), we obtained the following result.

Theorem 1.4. *Let $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ be entire functions. Suppose that there exists some $s\in\{1,2,\cdots,k-1\}$ such that*

$$\max\{\sigma_{(p,q)}(A_j),\sigma_{(p,q)}(F),\ j\neq s\}\leq\sigma_{(p,q)}(A_s)=\sigma<\infty,$$

$$\max\{\tau_{(p,q)}(A_j):\sigma_{(p,q)}(A_j)=\sigma_{(p,q)}(A_s),\tau_{(p,q)}(F)\}<\tau_{(p,q)}(A_s)$$

and that $T(r,A_s)\sim\log M(r,A_s)$ as $r\to+\infty$ outside a set of r of finite logarithmic measure. Then

(i) *Every transcendental solution $f(z)$ of (1.2) satisfies $\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)$, and every non-transcendental solution $f(z)$ of (1.2) is a polynomial of degree $\deg(f)\leq s-1$.*

(ii) *If $F(z)\not\equiv 0$, then every transcendental solution $f(z)$ of (1.2) satisfies $\bar{\lambda}_{(p+1,q)}(f)=\lambda_{(p+1,q)}(f)=\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)$.*

(iii) *If $s=1$, then every nonconstant solution $f(z)$ of (1.2) satisfies $\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_1)$ and if $F(z)\not\equiv 0$, then every nonconstant solution $f(z)$ of (1.2) satisfies $\bar{\lambda}_{(p+1,q)}(f)=\lambda_{(p+1,q)}(f)=\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_1)$.*

2. Lemmas

Lemma 2.1. ([5]) *Let f be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then for any given constant and for any given $\varepsilon > 0$:*

(i) *There exist a set $E_1 \subset (1,+\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α such that for all z with $|z| = r \notin [0,1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\right)^{n-m} \quad (0 \leq m < n).$$

(ii) *There exist a set $H_1 \subset [0,2\pi)$ that has linear measure zero and a constant $B > 0$ depending only on α, for any $\theta \in [0,2\pi)\diagdown H_1$, there exists a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying* $\arg z = \theta$ *and $|z| = r > R_0$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)\log T(\alpha r,f)\right)^{n-m} \quad (0 \leq m < n).$$

By using the similar proof of Lemma 2.5 in [7], we easily obtain the following lemma when $\sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) = +\infty$.

Lemma 2.2. *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \leq \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) \leq +\infty$ and $\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu$. Then there exists a set $E_2 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_2$ and $|g(z)| = M(r,g)$ we have*

$$\left|\frac{f(z)}{f^{(k)}(z)}\right| \leq r^{2k},\ (k \in \mathbb{N}).$$

Lemma 2.3. ([13]) *Let $f(z) = \sum\limits_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function and the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition* (1.3). *Then for any given $\varepsilon > 0$,*

$$\log L(r,f) > (1-\varepsilon)\log M(r,f)$$

holds outside a set E_3 of finite logarithmic measure, where $M(r,f) = \sup\limits_{|z|=r} |f(z)|$, $L(r,f) = \inf\limits_{|z|=r} |f(z)|$.

Lemma 2.4. ([16]) *Let $f(z)$ be an entire function of $(p,q)-$order satisfying $0 < \sigma_{(p,q)}(f) = \sigma < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\sigma = \lim_{r\to+\infty, r\in E_4} \frac{\log_p T(r,f)}{\log_q r} = \lim_{r\to+\infty, r\in E_4} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

and

$$M(r,f) > \exp_{p+1}\{(\sigma-\varepsilon)\log_q r\}.$$

Lemma 2.5. ([6]) *Let $g:[0,+\infty)\to\mathbb{R}$ and $h:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that $g(r)\le h(r)$ for all $r\notin E_5\cup[0,1]$, where $E_5\subset(1,+\infty)$ is a set of finite logarithmic measure. Then for any $\alpha>1$, there exists an $r_0=r_0(\alpha)>0$ such that $g(r)\le h(\alpha r)$ for all $r>r_0$.*

By using the similar proof of Lemma 3.5 in [18], we easily obtain the following lemma when $\sigma_{(p,q)}(g)=\sigma_{(p,q)}(f)=+\infty$.

Lemma 2.6. *Let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu_{(p,q)}(g)=\mu_{(p,q)}(f)=\mu\le\sigma_{(p,q)}(g)=\sigma_{(p,q)}(f)\le+\infty$ and $\lambda_{(p,q)}(d)=\sigma_{(p,q)}(d)=\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu$. Then there exists a set $E_6\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin[0,1]\cup E_6$ and $|g(z)|=M(r,g)$ we have*

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n(1+o(1)),\quad (n\in\mathbb{N}),$$

where $\nu_g(r)$ is the central index of $g(z)$.

Lemma 2.7. *Let $f(z)$ be an entire function such that $\sigma_{(p,q)}(f)=\sigma<+\infty$. Then, there exist entire functions $\beta(z)$ and $D(z)$ such that*

$$f(z)=\beta(z)e^{D(z)},$$

$$\sigma_{(p,q)}(f)=\max\left\{\sigma_{(p,q)}(\beta),\sigma_{(p,q)}\left(e^{D(z)}\right)\right\}$$

and

$$\sigma_{(p,q)}(\beta)=\limsup_{r\to+\infty}\frac{\log_p N(r,\frac{1}{f})}{\log_q r}.$$

Moreover, for any given $\varepsilon>0$, we have

$$|\beta(z)|\ge\exp\left\{-\exp_p\left\{(\sigma_{(p,q)}(\beta)+\varepsilon)\log_q r\right\}\right\}\ \ (r\notin E_7),$$

where $E_7\subset(1,+\infty)$ is a set of r of finite linear measure.

Proof. By Theorem 10.2 in [11] and Theorem 2.2 in [12], we get that $f(z)$ can be represented by

$$f(z)=\beta(z)e^{D(z)},$$

with

$$\sigma_{(p,q)}(f)=\max\left\{\sigma_{(p,q)}(\beta),\sigma_{(p,q)}\left(e^{D(z)}\right)\right\}.$$

By using similar proof in Lemma 6.1 in [10], for any given $\varepsilon>0$, we obtain

$$|\beta(z)|\geq\exp\left\{-\exp_p\left\{(\sigma_{(p,q)}(\beta)+\varepsilon)\log_q r\right\}\right\}\ \ (r\notin E_7),$$

where $E_7\subset(1,+\infty)$ is a set of r of finite linear measure.

Lemma 2.8. *Let $f(z)$ be an entire function such that $\sigma_{(p,q)}(f)=\sigma<+\infty$. Then, there exists a set $E_8\subset(1,+\infty)$ of r of finite linear measure such that for any given $\varepsilon>0$, we have*

$$\exp\left\{-\exp_p\left\{(\sigma+\varepsilon)\log_q r\right\}\right\}\leq|f(z)|\leq\exp_{p+1}\left\{(\sigma+\varepsilon)\log_q r\right\}\ \ (r\notin E_8).$$

Proof. When $p=q=1$, the lemma is due to Chen [2]. Thus, we assume that $p\geq q>1$ or $p>q=1$. It is obvious that $|f(z)|\leq\exp_{p+1}\left\{(\sigma+\varepsilon)\log_q r\right\}$. By Lemma 2.7, there exist entire functions $\beta(z)$ and $D(z)$ such that

$$f(z)=\beta(z)e^{D(z)}\ \text{ and }\ \sigma_{(p,q)}(f)=\max\left\{\sigma_{(p,q)}(\beta),\sigma_{(p,q)}\left(e^{D(z)}\right)\right\}.$$

Since $\sigma_{(p-1,q)}(D)=\sigma_{(p,q)}\left(e^{D(z)}\right)\leq\sigma_{(p,q)}(f)$ and $\left|e^{D(z)}\right|\geq e^{-|D(z)|}$, for sufficiently large $|z|=r$, we have

$$\left|e^{D(z)}\right|\geq e^{-|D(z)|}\geq\exp\left\{-\exp_p\left\{(\sigma+\frac{\varepsilon}{2})\log_q r\right\}\right\}.$$

By Lemma 2.7 again, it follows that

$$\begin{aligned}|f(z)|&=|\beta(z)|\left|e^{D(z)}\right|\\&\geq\exp\left\{-\exp_p\left\{(\sigma_{(p,q)}(\beta)+\frac{\varepsilon}{2})\log_q r\right\}\right\}\exp\left\{-\exp_p\left\{(\sigma+\frac{\varepsilon}{2})\log_q r\right\}\right\}\\&\geq\exp\left\{-\exp_p\left\{(\sigma+\frac{\varepsilon}{2})\log_q r\right\}\right\}\exp\left\{-\exp_p\left\{(\sigma+\frac{\varepsilon}{2})\log_q r\right\}\right\}\\&=\exp\left\{-2\exp_p\left\{\sigma+\frac{\varepsilon}{2})\log_q r\right\}\right\}\geq\exp\left\{-\exp_p\left\{(\sigma+\varepsilon)\log_q r\right\}\right\},\end{aligned}$$

for $r\notin E_8$, where $E_8\subset(1,+\infty)$ is a set of r of finite linear measure. Thus, we complete the proof of Lemma 2.8.

Lemma 2.9. ([16]) *Let $f(z)$ be an entire function of $(p,q)-$order, and let $\nu_f(r)$ be a central index of $f(z)$. Then*

$$\sigma_{(p,q)}(f)=\limsup_{r\to+\infty}\frac{\log_p\nu_f(r)}{\log_q r}.$$

Lemma 2.10. *Let $f(z)$ be an entire function with $\sigma_{(p,q)}(f)=\sigma$, $0<\sigma<\infty$. Then for any given $\beta<\sigma$, there exists a set E_9 having infinite logarithmic measure such that for all $|z|=r\in E_9$, we have*

$$\log_{p+1} M(r,f)>\beta\log_q r,$$

where $M(r,f)=\sup\limits_{|z|=r}|f(z)|$.

Proof. By the definition of the $(p,q)-$order, for any given $\varepsilon>0$, there exists a sequence $\{r_n\}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n<r_{n+1}$ and

$$\lim_{n\to\infty}\frac{\log_{p+1}M(r_n,f)}{\log_q r_n}=\sigma.$$

Then, there exists a positive integer n_0 such that for all $n\geq n_0$ and for any given $\varepsilon>0$, we have

$$M(r_n,f)>\exp_{p+1}\{(\sigma-\varepsilon)\log_q r_n\}. \tag{2.1}$$

When $q\geq 1$, we have

$$\lim_{n\to\infty}\frac{\log_q\left(\frac{n}{n+1}\right)r}{\log_q r}=1.$$

Since $\beta<\sigma$, then we can choose sufficiently small $\varepsilon>0$ to satisfy $0<\varepsilon<\sigma-\beta$. Therefore, there exists a positive integer n_1 such that for all $n\geq n_1$, we have

$$\frac{\log_q\left(\frac{n}{n+1}\right)r}{\log_q r}>\frac{\beta}{\sigma-\varepsilon}. \tag{2.2}$$

Take $n_2=\max\left\{n_0,n_1\right\}$. Then, by (2.1) and (2.2) we get for $r\in[r_n,\left(1+\frac{1}{n}\right)r_n]$

$$\log_{p+1}M(r,f)\geq\log_{p+1}M(r_n,f)>(\sigma-\varepsilon)\log_q r_n$$

$$\geq(\sigma-\varepsilon)\log_q\left(\frac{n}{n+1}\right)r>\beta\log_q r.$$

Setting $E_9=\bigcup\limits_{n=n_2}^{\infty}[r_n,(1+\frac{1}{n})r_n]$, we have

$$m_l(E_9)=\sum_{n=n_2}^{\infty}\int_{r_n}^{(1+\frac{1}{n})r_n}\frac{dt}{t}=\sum_{n=n_2}^{\infty}\log(1+\frac{1}{n})=\infty.$$

Lemma 2.11. *Let $f(z)=\sum\limits_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ be an entire function with $\sigma_{(p,q)}(f)=\sigma$, $0<\sigma<\infty$. If the sequence of exponent $\{\lambda_n\}$ satisfies (1.3), then for any given $\beta<\sigma$, there exists a set E_{10} having* *infinite logarithmic measure such that for all* $|z| = r \in E_{10}$, *we have*

$$|f(z)| > \exp_{p+1}\{\beta \log_q r\}.$$

Proof. By Lemma 2.3, for any given $\varepsilon > 0$, there exists a set E_3 of finite logarithmic measure such that for all $r \notin E_3$, we have

$$\log L(r,f) > (1-\varepsilon)\log M(r,f).$$

For any given $\beta < \sigma$, we can choose $\delta > 0$ such that $\beta < \delta < \sigma$ and sufficiently small ε satisfying $0 < \varepsilon < \frac{\delta-\beta}{\delta}$. Then, by Lemma 2.10, there exists a set E_9 having infinite logarithmic measure such that for all $r \in E_9$, we have

$$|f(z)| > L(r,f) > [M(r,f)]^{1-\varepsilon} > \left(\exp_{p+1}\{\delta \log_q r\}\right)^{1-\varepsilon} > \exp_{p+1}\{\beta \log_q r\},$$

where $E_{10} = E_9 \diagdown E_3$ is a set with infinite logarithmic measure.

Lemma 2.12. *Let* $f(z)$ *be an entire function with* $\mu_{(p,q)}(f) = \mu < \infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_{11} \subset (1,+\infty)$ *having infinite logarithmic measure such that for all* $|z| = r \in E_{11}$, *we have*

$$\mu_{(p,q)}(f) = \lim_{r\to+\infty, r\in E_{11}} \frac{\log_{p+1} M(r,f)}{\log_q r}$$

and

$$M(r,f) < \exp_{p+1}\{(\mu+\varepsilon)\log_q r\}.$$

Proof. By the definition of the lower (p,q)-order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$, and

$$\lim_{r_n\to+\infty} \frac{\log_{p+1} M(r_n,f)}{\log_q r_n} = \mu_{(p,q)}(f).$$

Then for any given $\varepsilon > 0$, there exists an n_2 such that for $n \geq n_2$ and any $r \in [\frac{n}{n+1}r_n, r_n]$, we have

$$\frac{\log_{p+1} M(\frac{n}{n+1}r_n, f)}{\log_q r_n} \leq \frac{\log_{p+1} M(r,f)}{\log_q r} \leq \frac{\log_{p+1} M(r_n,f)}{\log_q \frac{n}{n+1}r_n}.$$

Letting $E_{11} = \bigcup_{n=n_2}^{\infty} [\frac{n}{n+1}r_n, r_n]$, then for any $r \in E_{11}$, we have

$$\lim_{r\to+\infty, r\in E_{11}} \frac{\log_{p+1} M(r,f)}{\log_q r} = \lim_{r_n\to+\infty} \frac{\log_{p+1} M(r_n,f)}{\log_q r_n} = \mu_{(p,q)}(f),$$

and

$$m_l(E_{11}) = \sum_{n=n_2}^{\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_2}^{\infty} \log(1+\frac{1}{n}) = \infty.$$

Lemma 2.13. ([15]) *If $f(z)$ is a meromorphic function, then $\sigma_{(p,q)}(f') = \sigma_{(p,q)}(f)$.*

Lemma 2.14. ([17]) *Let $f(z)$ be a transcendental entire function, and let $z_r = re^{i\theta_r}$ be a point satisfying $|f(z_r)| = M(r,f)$. Then, there exists a constant $\delta_r > 0$ such that for all z satisfying $|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\left| \frac{f(z)}{f^{(j)}(z)} \right| \leq 2r^j, \ (j \in \mathbb{N}).$$

Lemma 2.15. ([16]) *Let $f(z)$ be an entire function of $(p,q)-$order satisfying $0 < \sigma_{(p,q)}(f) = \sigma < \infty$ and $0 < \tau_{(p,q)}(f) = \tau < \infty$. Then for any given $\beta < \tau$, there exists a set $E_{13} \subset [1,+\infty)$ that has an infinite logarithmic measure such that for all $|z| = r \in E_{13}$, we have*

$$\log_p M(r,f) > \beta(\log_{q-1} r)^{\sigma} \quad (r \in E_{13}).$$

Lemma 2.16. *Let $f(z)$ be a transcendental entire function satisfying $0 < \sigma_{(p,q)}(f) = \sigma < \infty$, $0 < \tau_{(p,q)}(f) = \tau < \infty$ and $T(r,f) \sim \log M(r,f)$ as $r \to +\infty$ outside a set of r of finite logarithmic measure. Then for any $\beta < \tau$, there exists a set $E_{14} \subset (0,+\infty)$ having infinite logarithmic measure and a set $H_2 \subset [0,2\pi)$ that has linear measure zero such that for all z satisfying $|z| = r \in E_{14}$ and $\arg z = \theta \in [0,2\pi) \setminus H_2$, we have*

$$\left| f(re^{i\theta}) \right| > \exp_p\{\beta(\log_{q-1} r)^{\sigma}\}.$$

Proof. Since $m(r,f) \sim \log M(r,f)$ as $r \to +\infty$ and $r \notin F \subset (0,+\infty)$, where F is a set of r of finite logarithmic measure, by the definition of $m(r,f)$, we see that there exists a set $H_2 \subset [0,2\pi)$ with linear measure zero such that for all z satisfying $\arg z = \theta \in [0,2\pi) \setminus H_2$ and for any $\varepsilon > 0$, we have

$$\left| f(re^{i\theta}) \right| > [M(r,f)]^{1-\varepsilon}. \tag{2.3}$$

Otherwise, we find that there exists a set $H \subset [0,2\pi)$ with positive linear measure, i.e., $m(H) > 0$ such that, for all z satisfying $\arg z = \theta \in H$ and for any $\varepsilon > 0$, we have

$$\left| f(re^{i\theta}) \right| \leq [M(r,f)]^{1-\varepsilon}.$$

Then, for all $r \notin F$, we get

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left|f(re^{i\theta})\right| d\theta$$

$$= \frac{1}{2\pi}\int_H \ln^+ \left|f(re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{[0,2\pi)\setminus H} \ln^+ \left|f(re^{i\theta})\right| d\theta$$

$$\leq \frac{(1-\varepsilon)m(H)}{2\pi}\log M(r,f) + \frac{2\pi - m(H)}{2\pi}\log M(r,f)$$

$$= \frac{2\pi - \varepsilon m(H)}{2\pi}\log M(r,f). \tag{2.4}$$

Since $\varepsilon > 0$ and $m(H) > 0$, then (2.4) is a contradiction with $m(r,f) \sim \log M(r,f)$. For any $\beta < \tau$, we choose $\xi (> 0)$ satisfying $\beta < \xi < \tau$. By Lemma 2.15, there exists a set $E_{13} \subset [1,+\infty)$ that has an infinite logarithmic measure such that for all $|z| = r \in E_{13}$, we have

$$\log_p M(r,f) > \xi(\log_{q-1} r)^{\sigma}. \tag{2.5}$$

By (2.3) and (2.5), for any given ε $(0 < \varepsilon < 1 - \frac{\beta}{\xi})$ and for all $|z| = r \in E_{14} = E_{13} \setminus F$ and $\arg z = \theta \in [0,2\pi)\setminus H_2$, we have

$$\left|f(re^{i\theta})\right| > [M(r,f)]^{1-\varepsilon} > \left(\exp_p\{\xi(\log_{q-1} r)^{\sigma}\}\right)^{1-\varepsilon} > \exp_p\{\beta(\log_{q-1} r)^{\sigma}\}.$$

Thus, the proof of Lemma 2.16 is complete.

Lemma 2.17. ([15]) *Let $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ be meromorphic functions. If $f(z)$ is a meromorphic solution to* (1.2) *satisfying*

$$\max\{\sigma_{(p+1,q)}(F), \sigma_{(p+1,q)}(A_j) : j = 0,1,\cdots,k-1\} < \sigma_{(p+1,q)}(f),$$

then we have

$$\bar{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).$$

3. Proof of Theorem 1.1

Assume that $f(z)$ is a rational solution of (1.4). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. Since $\max\{\sigma_{(p,q)}(A_j), j \neq s\} < \sigma_{(p,q)}(A_s) < \infty$, then

$$\sigma_{(p,q)}(0) = \sigma_{(p,q)}(A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f)$$

$$= \sigma_{(p,q)}(A_s) > 0,$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$. By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1,+\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left(T(2r,f)\right)^{k+1}, \quad 1 \leq j \leq k. \tag{3.1}$$

Since $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, then by Hadamard's factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \leq \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f),$$

$$\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu.$$

Then by Lemma 2.2, there exists a set E_2 of finite logarithmic measure such that for all $|z| = r \notin E_2$ and $|g(z)| = M(r,g)$ and for r sufficiently large, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}, \quad (s \in \mathbb{N}). \tag{3.2}$$

Set $\alpha = \max\{\sigma_{(p,q)}(A_j) : j \neq s\} < \sigma_{(p,q)}(A_s) = \sigma < \infty$. Then, for any given ε $(0 < 2\varepsilon < \sigma - \alpha)$, we have

$$\left|A_j(z)\right| \leq \exp_{p+1}\{(\alpha + \varepsilon)\log_q r\}, \quad j \neq s. \tag{3.3}$$

By Lemma 2.3 and Lemma 2.4, there exists a set $E_{15} \subset (1,+\infty)$ of infinite logarithmic measure such that for all $|z| = r \in E_{15}$, we have

$$|A_s(z)| \geq L(r,A_s) > (M(r,A_s))^{1-\varepsilon} \geq (\exp_{p+1}\{(\sigma - \frac{\varepsilon}{2})\log_q r\})^{1-\varepsilon}$$

$$\geq \exp_{p+1}\{(\sigma - \varepsilon)\log_q r\}. \tag{3.4}$$

By (1.4), we have

$$|A_s(z)| \leq \left|\frac{f}{f^{(s)}}\right| \left[|A_k(z)|\left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f}\right|\right.$$

$$\left. + |A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)|\right]. \tag{3.5}$$

Hence, by substituting $(3.1)-(3.4)$ into (3.5), we obtain for all z satisfying $r\in E_{15}\diagdown(E_1\cup E_2\cup[0,1])$

$$\exp_{p+1}\{(\sigma-\varepsilon)\log_q r\}\le r^{2s}\exp_{p+1}\{(\alpha+\varepsilon)\log_q r\}kB\left(T(2r,f)\right)^{k+1}. \tag{3.6}$$

By (3.6) and Lemma 2.5, we have

$$\sigma_{(p+1,q)}(f)\ge\sigma_{(p,q)}(A_s).$$

Now, we prove that $\sigma_{(p+1,q)}(f)\le\sigma_{(p,q)}(A_s)$. We can rewrite (1.4) as

$$\begin{aligned}-A_k(z)\frac{f^{(k)}}{f}&=A_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+A_{s+1}(z)\frac{f^{(s+1)}}{f}\\&+A_s(z)\frac{f^{(s)}}{f}+A_{s-1}(z)\frac{f^{(s-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z).\end{aligned} \tag{3.7}$$

By Lemma 2.6, there exists a set $E_6\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin[0,1]\cup E_6$ and $|g(z)|=M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^j(1+o(1)),\quad (j=0,\cdots,k). \tag{3.8}$$

Since $\max\{\sigma_{(p,q)}(A_j),j\neq s\}<\sigma_{(p,q)}(A_s)<\infty$, then for sufficiently large r, we have

$$\left|A_j(z)\right|\le\exp_p\{(\sigma_{(p,q)}(A_s)+\varepsilon)\log_q r\},\ (j=0,\cdots,k). \tag{3.9}$$

By Lemma 2.8, there exists a set $E_8\subset(1,+\infty)$ of finite linear measure (and so of finite logarithmic measure) such that for all $|z|=r\notin E_8$, we have

$$\begin{aligned}|A_k(z)|&\ge\exp\left\{-\exp_p\{(\sigma_{(p,q)}(A_k)+\varepsilon)\log_q r\}\right\}\\&\ge\exp\left\{-\exp_p\{(\sigma_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\}.\end{aligned} \tag{3.10}$$

From (3.7) and (3.8), for all z satisfying $|z|=r\notin[0,1]\cup E_6$ and $|g(z)|=M(r,g)$, we have

$$\begin{aligned}-A_k(z)\left(\frac{\nu_g(r)}{z}\right)^k(1+o(1))&=A_{k-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{k-1}(1+o(1))\\+\cdots+A_{s+1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s+1}&(1+o(1))+A_s(z)\left(\frac{\nu_g(r)}{z}\right)^{s}(1+o(1))\\+A_{s-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s-1}(1+o(1))&+\cdots+A_1(z)\left(\frac{\nu_g(r)}{z}\right)(1+o(1))+A_0(z).\end{aligned}$$

It follows that

$$|A_k(z)|\left|\left(\frac{\nu_g(r)}{z}\right)^k\right||1+o(1)|\le|A_{k-1}(z)|\left|\left(\frac{\nu_g(r)}{z}\right)^{k-1}\right||1+o(1)|$$

$$+\cdots+|A_{s+1}(z)|\left|\left(\frac{\nu_g(r)}{z}\right)^{s+1}\right||1+o(1)|+|A_s(z)|\left|\left(\frac{\nu_g(r)}{z}\right)^{s}\right||1+o(1)|$$

$$+|A_{s-1}(z)|\left(\frac{\nu_g(r)}{z}\right)^{s-1}|1+o(1)|+\cdots+|A_1(z)|\left(\frac{\nu_g(r)}{z}\right)|1+o(1)|+|A_0(z)| \qquad (3.11)$$

and by (3.9) − (3.11) for all z satisfying $|z|=r\notin([0,1]\cup E_6\cup E_8)$ and $|g(z)|=M(r,g)$, we have

$$\exp\left\{-\exp_p\{(\sigma_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\}\left(\frac{\nu_g(r)}{r}\right)|1+o(1)|$$
$$\leq k\exp_{p+1}\{(\sigma_{(p,q)}(A_s)+\varepsilon)\log_q r\}\,|1+o(1)|\,.$$

So, we have

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_g(r)}{\log_q r}\leq\sigma_{(p,q)}(A_s)+\varepsilon. \qquad (3.12)$$

Since $\varepsilon>0$ is arbitrary, then by (3.12), Lemma 2.5 and Lemma 2.9, we have $\sigma_{(p+1,q)}(g)\leq\sigma_{(p,q)}(A_s)$, that is $\sigma_{(p+1,q)}(f)\leq\sigma_{(p,q)}(A_s)$. Therefore, we get $\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)$.

4. Proof of Theorem 1.2

Assume that $f(z)$ is a rational solution of (1.4). By the same reasoning as in the proof of Theorem 1.1, it is clear that $f(z)$ is a polynomial with $\deg f\leq s-1$. Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$. By Theorem 1.1, we have $\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)=\sigma$. Then, we only need to prove that $\mu_{(p+1,q)}(f)=\mu_{(p,q)}(A_s)=\sigma$. Since $\max\{\sigma_{(p,q)}(A_j)\ (j\neq s)\}<\sigma$, then there exist constants α_1,β_1 satisfying $\max\{\sigma_{(p,q)}(A_j)\ (j\neq s)\}<\alpha_1<\beta_1<\sigma$. Then

$$\left|A_j(z)\right|\leq\exp_{p+1}\{\alpha_1\log_q r\},\ j\neq s. \qquad (4.1)$$

Also, we have that $A_s(z)=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.3) and $\mu_{(p,q)}(A_s)=\sigma_{(p,q)}(A_s)=\sigma$. Then, by Lemma 2.11, there exists a set E_{10} having infinite logarithmic measure such that for all z satisfying

$$|z|=r\in E_{10},$$

we have

$$|A_s(z)|>\exp_{p+1}\{\beta_1\log_q r\}. \qquad (4.2)$$

Hence, by substituting (4.1), (4.2), (3.1), (3.2) into (3.5), for all z satisfying

$$|z|=r\in E_{10}\diagdown([0,1]\cup E_1\cup E_2),$$

we have

$$\exp_{p+1}\{\beta_1 \log_q r\} \le B \exp_{p+1}\{\alpha_1 \log_q r\} r^{2s} k\,[T(2r,f)]^{k+1}. \tag{4.3}$$

Since β_1 is arbitrarily close to σ, then by (4.3) and Lemma 2.5, we obtain

$$\mu_{(p+1,q)}(f) \ge \sigma = \mu_{(p,q)}(A_s).$$

On the other hand, by (1.4), we have

$$|A_k(z)|\left|\frac{f^{(k)}}{f}\right| \le |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f}\right| + |A_s(z)|\left|\frac{f^{(s)}}{f}\right|$$
$$+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)|. \tag{4.4}$$

By Lemma 2.12, for any given $\varepsilon > 0$, there exists a set $E_{11} \subset (1,+\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_{11}$, one has

$$\left|A_j(z)\right| \le \exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\},\ j = 0,\cdots,k. \tag{4.5}$$

By Lemma 2.8, there exists a set $E_8 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin E_8$, we have

$$|A_k(z)| \ge \exp\left\{-\exp_p\{(\sigma_{(p,q)}(A_k)+\varepsilon)\log_q r\}\right\}$$
$$\ge \exp\left\{-\exp_p\{(\sigma_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\} = \exp\left\{-\exp_p\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\}. \tag{4.6}$$

From (3.8), (4.4) – (4.6), for all z satisfying $|z| = r \in E_{11} \diagdown (E_6 \cup E_8)$ and $|g(z)| = M(r,g)$, we have

$$\exp\left\{-\exp_p\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\}\left(\frac{\nu_g(r)}{r}\right)^k |1+o(1)| \le$$
$$\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\left(\frac{\nu_g(r)}{r}\right)^{k-1} |1+o(1)|$$
$$+\cdots+\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\left(\frac{\nu_g(r)}{r}\right)^{s+1} |1+o(1)|$$
$$+\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\left(\frac{\nu_g(r)}{r}\right)^{s} |1+o(1)|$$
$$+\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\left(\frac{\nu_g(r)}{r}\right)^{s-1} |1+o(1)|$$
$$+\cdots+\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\left(\frac{\nu_g(r)}{r}\right) |1+o(1)|$$
$$+\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\},$$

that is, for all z satisfying $|z| = r \in E_{11} \setminus (E_6 \cup E_8)$ and $|g(z)| = M(r,g)$, we obtain

$$\exp\left\{-\exp_p\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}\right\}\left(\frac{\nu_g(r)}{r}\right)|1+o(1)|$$

$$\leq k\,|1+o(1)|\exp_{p+1}\{(\mu_{(p,q)}(A_s)+\varepsilon)\log_q r\}. \tag{4.7}$$

It follows that

$$\underline{\lim_{r\to+\infty}}\frac{\log_{p+1}\nu_g(r)}{\log_q r}\leq \mu_{(p,q)}(A_s)+\varepsilon. \tag{4.8}$$

Since $\varepsilon > 0$ is arbitrary, by (4.8), Lemma 2.5 and Lemma 2.9, we have

$$\mu_{(p+1,q)}(g)\leq \mu_{(p,q)}(A_s),$$

that is,

$$\mu_{(p+1,q)}(f)\leq \mu_{(p,q)}(A_s).$$

Therefore, we get

$$\mu_{(p+1,q)}(f)= \mu_{(p,q)}(A_s)=\sigma.$$

5. Proof of Theorem 1.3

(i) We assume that $f(z)$ is a transcendental meromorphic solution of (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, and $\{f_1, f_2, \cdots, f_k\}$ is a meromorphic solution base of the corresponding homogeneous equation (1.4) of (1.5). By Theorem 1.1, we get that

$$\sigma_{(p+1,q)}(f_j)=\sigma_{(p,q)}(A_s), (j=1,2,\cdots,k).$$

By the elementary theory of differential equations, all solutions of (1.5) can be represented in the form

$$f(z)=f_0(z)+B_1f_1(z)+B_2f_2(z)+\cdots+B_kf_k(z), \tag{5.1}$$

where $B_1,\cdots,B_k\in\mathbb{C}$ and the function f_0 has the form

$$f_0(z)=C_1(z)f_1(z)+C_2(z)f_2(z)+\cdots+C_k(z)f_k(z), \tag{5.2}$$

where $C_1(z),\cdots,C_k(z)$ are suitable meromorphic functions satisfying

$$C_j' = F.G_j(f_1,\cdots,f_k).\left[W(f_1,\cdots,f_k)\right]^{-1},\ j=1,2,\cdots,k, \tag{5.3}$$

where $G_j(f_1,\cdots,f_k)$ are differential polynomials in $f_1,\cdots,f_k$ and their derivatives with constant coefficients, and $W(f_1,\cdots,f_k)$ is the Wronskian of f_1, $\cdots$, f_k. Since the Wronskian $W(f_1,\cdots,f_k)$ is a differential polynomial in $f_1,\cdots,f_k$, it is easy to obtain

$$\sigma_{(p+1,q)}(W) \leq \max\{\sigma_{(p+1,q)}(f_j) : j=1,2,\cdots,k\} = \sigma_{(p,q)}(A_s). \tag{5.4}$$

Also, we have that $G_j(f_1,\cdots,f_k)$ are differential polynomials in $f_1,\cdots,f_k$ and their derivatives with constant coefficients. Then, we have

$$\sigma_{(p+1,q)}(G_j) \leq \max\{\sigma_{(p+1,q)}(f_j) : j=1,2,\cdots,k\} = \sigma_{(p,q)}(A_s),\ (j=1,2,\cdots,k). \tag{5.5}$$

By Lemma 2.13 and (5.5), for $j=1,\cdots,k$, we have

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}(C_j') \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_s)\} = \sigma_{(p,q)}(A_s). \tag{5.6}$$

Hence, from (5.1), (5.2) and (5.6), we obtain

$$\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j=1,2,\cdots,k\} = \sigma_{(p,q)}(A_s).$$

Now we assert that all meromorphic solutions f of equation (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$, with at most one exceptional solution f_0 with

$$\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_s).$$

In fact, if there exists another meromorphic solution f_1 of (1.5) satisfying $\sigma_{(p+1,q)}(f_1) < \sigma_{(p,q)}(A_s)$, then $f_0 - f_1$ is a nonzero meromorphic solution of (1.4) and satisfies $\sigma_{(p+1,q)}(f_0 - f_1) < \sigma_{(p,q)}(A_s)$. But by Theorem 1.1 we have any meromorphic solution f of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$. This is a contradiction. Therefore, we have that all meromorphic solutions f of equation (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$, with at most one exceptional solution f_0 with $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_s)$.

(ii) From (1.5), by a simple consideration of order, we get $\sigma_{(p+1,q)}(f) \geq \sigma_{(p+1,q)}(F)$. By Lemma 2.13 and $(5.3)-(5.5)$, for $j=1,\cdots,k$, we have

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}(C_j') \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_s)\} \leq \sigma_{(p+1,q)}(F). \tag{5.7}$$

By (5.1), (5.2) and (5.7), we have

$$\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j=1,2,\cdots,k\} \leq \sigma_{(p+1,q)}(F).$$

Therefore, we have $\sigma_{(p+1,q)}(f)=\sigma_{(p+1,q)}(F)$.

6. Proof of Theorem 1.4

(i) Suppose that $f(z)$ is a transcendental solution of (1.2). On one hand, by (1.2), we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|+|A_0(z)|+\left|\frac{F}{f}\right|. \tag{6.1}$$

By Wiman-Valiron theory $[11,\ \text{p. }187\text{-}199]$, there exists a set $E_{16}\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin[0,1]\cup E_{16}$ and $|f(z)|=M(r,f)>1$, we have

$$\frac{f^{(j)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^j(1+o(1)),\quad (j=0,\cdots,k). \tag{6.2}$$

By the definition of the $(p,q)-$order, for any given $\varepsilon>0$ and for sufficiently large r, we have

$$\left|A_j(z)\right|\leq\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\},\ j\neq s \tag{6.3}$$

and

$$|F(z)|\leq\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{6.4}$$

Since $|f(z)|=M(r,f)>1$, then for sufficiently large r we have

$$\left|\frac{F(z)}{f(z)}\right|=\frac{|F(z)|}{M(r,f)}\leq\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{6.5}$$

By substituting (6.2), (6.3) and (6.5) into (6.1), for sufficiently large $r\notin[0,1]\cup E_{16}$, we obtain

$$\left(\frac{\nu_f(r)}{r}\right)|1+o(1)|\leq(k+1)\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{6.6}$$

By (6.6), Lemma 2.5 and Lemma 2.9, we obtain $\sigma_{(p+1,q)}(f)\leq\sigma_{(p,q)}(A_s)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $\sigma_{(p+1,q)}(f)\leq\sigma_{(p,q)}(A_s)$. On the other hand, by (1.2), we obtain

$$|A_s(z)|\leq\left|\frac{f}{f^{(s)}}\right|\left[\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f}\right|\right.$$

$$\left.+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|+|A_0(z)|+\left|\frac{F}{f}\right|\right]. \tag{6.7}$$

For each sufficiently large circle $|z|=r$, we take $z_r=re^{i\theta_r}$ satisfying $|f(z_r)|=M(r,f)>1$. Then, by Lemma 2.14, there exists a constant $\delta_r>0$ and a set E_{12} such that for all z satisfying

$|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s. \tag{6.8}$$

By Lemma 2.1, there exist a set $H_1 \subset [0,2\pi)$ that has linear measure zero and a constant $B > 0$ such that for all z satisfying $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$ and for sufficiently large r, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left(T(2r,f)\right)^{k+1}, \ \ 1 \leq j \leq k. \tag{6.9}$$

We choose α_2, β_2 satisfying $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_s), \tau_{(p,q)}(F)\} < \alpha_2 < \beta_2 < \tau_{(p,q)}(A_s)$. Since $|f(z) - f(z_r)| < \varepsilon$ and $|f(z_r)| \to \infty$ as $r \to +\infty$, for all sufficiently large $|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|A_j(z)\right| \leq \exp_p\{\alpha_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_s)}\}, \ j \neq s \tag{6.10}$$

and

$$\left|\frac{F(z)}{f(z)}\right| \leq |F(z)| \leq \exp_p\{\alpha_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_s)}\}. \tag{6.11}$$

Since $T(r,A_s) \sim \log M(r,A_s)$ as $r \to +\infty$ $(r \notin E_{12})$, by Lemma 2.16, for any $\beta_2 < \tau_{(p,q)}(A_s)$, there exists a set $E_{14} \subset (0,+\infty)$ having infinite logarithmic measure and a set $H_2 \subset [0,2\pi)$ that has linear measure zero such that for all z satisfying $|z| = r \in E_{14}$ and $\arg z = \theta \in [0,2\pi)\setminus H_2$, we have

$$|A_s(z)| > \exp_p\{\beta_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_s)}\}. \tag{6.12}$$

Substituting (6.8)-(6.12) into (6.7), for all z satisfying $|z| = r \in E_{14}\setminus E_{12}$ and $\arg z = \theta \in [0,2\pi)\setminus(H_1 \cup H_2)$, we get

$$\exp_p\{\beta_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_s)}\} \leq 2r^s \exp_p\{\alpha_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_s)}\}(k+1)B\left(T(2r,f)\right)^{k+1}. \tag{6.13}$$

By (6.13) and Lemma 2.5, we obtain $\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_s)$. Thus, we have $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$.

Now, if $f(z)$ is a polynomial solution of (1.2) with $\deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. If

$$\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F), j \neq s\} < \sigma_{(p,q)}(A_s) < \infty,$$

then

$$\sigma_{(p,q)}(A_s) = \sigma_{(p,q)}(-A_s(z)f^{(s)}) = \sigma_{(p,q)}(f^{(k)} + A_{k-1}(z)f^{(k-1)}$$

$$+ \cdots + A_{s+1}(z)f^{(s+1)} + A_{s-1}(z)f^{(s-1)} + \cdots + A_1(z)f' + A_0(z)f - F(z))$$

$$\leq \max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F), j \neq s\} < \sigma_{(p,q)}(A_s),$$

which is a contradiction. If $\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F), j \neq s\} = \sigma_{(p,q)}(A_s) = \sigma$ and $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_s), \tau_{(p,q)}(F)\} < \tau_{(p,q)}(A_s)$, then we choose α_2, β_2 satisfying $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_s), \tau_{(p,q)}(F)\} < \alpha_2 < \beta_2 < \tau_{(p,q)}(A_s)$. By Lemma 2.15, there exists a set E_{13} having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_{13}$, we have

$$|A_s(z)| > \exp_p\{\beta_2(\log_{q-1} r)^\sigma\} \tag{6.14}$$

and for sufficiently large r

$$|F(z)| < \exp_p\{\alpha_2(\log_{q-1} r)^\sigma\}, \ \left|A_j(z)\right| < \exp_p\{\alpha_2(\log_{q-1} r)^\sigma\}, \ j \neq s. \tag{6.15}$$

Hence, from (6.7), (6.14) and (6.15), for all z satisfying $|z| = r \in E_{13}$, we have

$$\exp_p\{\beta_2(\log_{q-1} r)^\sigma\} \leq (k+1)r^M \exp_p\{\alpha_2(\log_{q-1} r)^\sigma\},$$

where M is a constant. This is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

(ii) If $F(z) \not\equiv 0$, then we find from Lemma 2.17 that every transcendental solution $f(z)$ of (1.2) satisfies $\bar{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$.

(iii) If $s = 1$ and $f(z)$ is a polynomial solution of (1.2), then by (ii), we get that $\deg f \leq s-1$. Thus $f(z)$ must be a constant. By (i) and (ii), every nonconstant solution $f(z)$ of (1.2) satisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$ and $\bar{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$ if $F(z) \not\equiv 0$.

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