Growth of Solutions of Complex Differential Equations in a Sector of the Unit Disc

Benharrat BELAÏDI
Department of Mathematics,
Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB), B. P. 227
Mostaganem-Algeria
benharrat.belaidi@univ-mosta.dz

Abstract. In this paper, we deal with the growth of solutions of homogeneous linear complex differential equation by using the concept of lower $[p,q]$-order and lower $[p,q]$-type in a sector of the unit disc instead of the whole unit disc, and we obtain similar results as in the case of the unit disc.

AMS (2010) : 34M10, 30D35.
Key words : Complex differential equation, analytic function, $[p,q]$-order, lower $[p,q]$-order, lower $[p,q]$-type, sector.

1 Introduction and main results

Throughout this paper, we shall assume that readers are familiar with the fundamental results and the standard notations of Nevanlinna's theory in the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, see [5, 6, 7, 9, 15, 23].

Consider for $k \geq 2$ the complex differential equation

$$f^{(k)}(z) + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1.1}$$

where coefficients A_j $(j = 0, 1, \ldots, k-1)$ are analytic functions in the unit disc Δ. It is well-known that every solution of (1.1) is analytic in Δ, and there are exactly k linearly independent solutions of equation (1.1) (see e.g. [7]). The theory of complex differential equations in the unit disc has been developed since 1980's, see [13]. In the year 2000, Heittokangas in [7] firstly investigated the growth and oscillation theory of equation (1.1) when the coefficients A_j $(j = 0, 1, \ldots, k-1)$ are analytic functions in the unit disc

Δ by introducing the definition of the function spaces. His results also gave some important tools for further investigations on the theory of meromorphic solutions of equations (1.1). In 1994, S. J. Wu [17, 18] used the Nevanlinna theory in an angle to study the order of growth of solutions of the second-order linear differential equation in an angular region. Later Xu and Yi [22], N. Wu [19] , N. Wu and Y. Z. Li [20] , Zhang [24] generalized some results of [17, 18] to the case of linear higher order differential equations in angular domains by using the concepts of iterated $p-$order and the spread relation. Recently, Wu in [21] developed a new investigation related to linear differential equations with analytic coefficients in a sector of the unit disc

$$\Omega_{\alpha,\beta} = \left\{z \in \mathbb{C} : \alpha < \arg z < \beta,\ \ |z| < 1\right\},$$

and obtained some results about the order of growth of solutions of the differential equation

$$A_k(z)f^{(k)}(z) + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1.2}$$

where coefficients A_j $(j = 0, 1, \ldots, k)$ are analytic functions in the sector $\Omega_{\alpha,\beta}$. After that, Long in [11, 12] , Zemirni and Belaidi in [25] obtained different results concerning the growth of solutions of (1.1) and (1.2) by using the concepts of iterated p-order and $[p,q]$-order in the sector $\Omega_{\alpha,\beta}$. In this paper, we continue to investigate this new problem and study the growth of solutions of equation (1.1) when the coefficients A_j $(j = 0, 1, \ldots, k-1)$ are analytic functions of $[p,q]$-order in the sector $\Omega_{\alpha,\beta}$. Before stating our main results, we give some notations and basic definitions of meromorphic functions in the unit disc Δ and in a sector $\Omega_{\alpha,\beta}$ of the unit disc. The order of a meromorphic function f in Δ is defined by

$$\rho\left(f\right) = \limsup_{r\to1^-}\frac{\log T\left(r,f\right)}{\log\frac{1}{1-r}},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. If f is analytic function in Δ, then

$$\rho_M\left(f\right) = \limsup_{r\to1^-}\frac{\log\log M\left(r,f\right)}{\log\frac{1}{1-r}},$$

where $M(r, f) = \max\limits_{\substack{|z|=r\\ z\in\Delta}}|f(z)|$ is the maximum modulus function.

Remark 1.1 The following two statements hold [15, p. 205].
(a) If f is an analytic function in Δ, then

$$\rho(f) \leq \rho_M(f) \leq \rho(f) + 1$$

(b) There exist analytic functions f in Δ which satisfy $\rho_M(f) \neq \rho(f)$. For example, let $\mu > 1$ be a constant, and set

$$h(z) = \exp\left\{(1-z)^{-\mu}\right\},$$

where we choose the principal branch of the logarithm. Then $\rho(h) = \mu - 1$ and $\rho_M(h) = \mu$, see [4].

In contrast, the possibility that occurs in (b) cannot occur in the whole plane $\mathbb{C}$, because if $\rho(f)$ and $\rho_M(f)$ denote the order of an entire function f in the plane $\mathbb{C}$ (defined by the Nevanlinna characteristic and the maximum modulus, respectively), then it is well-know that

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \rho_M(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

The meromorphic functions in the unit disc can be divided into the following three classes:
(1) bounded type if $T(r,f) = O(1)$ as $r \to 1^-$;
(2) rational or non-admissible type if $T(r,f) = O\left(\frac{1}{1-r}\right)$ and f does not belong to (1);
(3) admissible in Δ if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = \infty.$$

Definition 1.1 [2, 3] Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in Δ, the $[p,q]$-order of f is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}},$$

where $\log_1^+ r := \log^+ r = \max(0, \log r)$, $\log_{p+1}^+ r := \log^+\left(\log_p^+ r\right)$, $p \in \mathbb{N}$. For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}.$$

It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq +\infty$. If f is non-admissible, then $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.1, $\rho_{[1,1]}(f) = \rho(f)$ is the order of f in Δ, $\rho_{[2,1]}(f) = \rho_2(f)$ is the hyper-order of f in Δ and $\rho_{[p,1]}(f) = \rho_p(f)$ is the p-iterated order of f in Δ.

Proposition 1.1 [2] Let $p \geq q \geq 1$ be integers, and let f be an analytic function in Δ of $[p,q]$-order. The following two statements hold :
(i) If $p = q$, then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f) + 1.$$

(ii) If $p > q$, then

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

Proposition 1.2 [8] Let $p \geq q \geq 1$ be integers, and let f be an analytic function in Δ of $[p,q]$-order. The following two statements hold :
(i) If $p = q$, then

$$\mu_{[p,q]}(f) \leq \mu_{M,[p,q]}(f) \leq \mu_{[p,q]}(f) + 1.$$

(ii) If $p > q$, then

$$\mu_{[p,q]}(f) = \mu_{M,[p,q]}(f).$$

In what follows, we give some notations and definitions of a meromorphic function in a sector in unit disc. Throughout this paper, Ω usually denotes the sector $\Omega_{\alpha,\beta}$ $(0 \leq \alpha < \beta \leq 2\pi)$ of the unit disc, and for any given $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$, Ω_ε denotes the sector

$$\Omega_{\alpha,\beta,\varepsilon} = \{z \in \mathbb{C} : \alpha + \varepsilon < \arg z < \beta - \varepsilon, \ \ |z| < 1\}.$$

In [21], Wu has used the Ahlfors-Shimizu characteristic function to measure the order of growth of a meromorphic function f in Ω. We recall the definition of the Ahlfors-Shimizu characteristic function, see [5, 6]. Let f be a meromorphic function in Ω, set

$$\Omega(r) = \Omega \cap \{z \in \mathbb{C} : 0 < |z| < r < 1\}$$
$$= \{z \in \mathbb{C} : \alpha < \arg z < \beta, \ 0 < |z| < r < 1\}.$$

Then, the Ahlfors-Shimizu characteristic function is defined by

$$T_0(r, \Omega, f) = \int_0^r \frac{S(t, \Omega, f)}{t} dt,$$

where

$$S(r,\Omega,f)=\frac{1}{\pi}\iint\limits_{\Omega(r)}\left(\frac{|f'(z)|}{1+|f(z)|^{2}}\right)^{2}d\sigma,\ z=re^{i\theta},\ d\sigma=rdrd\theta.$$

It follows by Hayman [6], Goldberg and Ostrovskii [5] that

$$T_0(r,\mathbb{C},f)=T(r,f)+O(1),\ 0<r<1.$$

The meromorphic functions in a sector Ω of the unit disc can be divided into the following three classes:
(1) bounded type if $T_0(r,\Omega,f)=O(1)$ as $r\to 1^-$;
(2) rational or non-admissible type if $T_0(r,\Omega,f)=O\left(\frac{1}{1-r}\right)$ and f does not belong to (1);
(3) admissible in Ω if

$$\limsup_{r\to 1^-}\frac{T_0(r,\Omega,f)}{\log\frac{1}{1-r}}=\infty.$$

Now, we introduce the concept of $[p,q]$-order and $[p,q]$-type of meromorphic functions in a sector Ω.

Definition 1.2 [12, 25] Let $p\geq q\geq 1$ be integers. Let f be a meromorphic function in Ω, the $[p,q]$-order of f is defined by

$$\rho_{[p,q],\Omega}(f)=\limsup_{r\to 1^-}\frac{\log_p^+ T_0(r,\Omega,f)}{\log_q\frac{1}{1-r}}.$$

It is clear that $0\leq\rho_{[p,q],\Omega}(f)\leq+\infty$. If f is non-admissible in Ω, then $\rho_{[p,q],\Omega}(f)=0$. By Definition 1.2, $\rho_{[1,1],\Omega}(f)=\rho_{\Omega}(f)$ is the order of f in Ω, see [21], $\rho_{[p,1],\Omega}(f)=\rho_{p,\Omega}(f)$ is the iterated p-order of f in Ω, see [11, 24].

Definition 1.3 [25] Let $p\geq q\geq 1$ be integers and f be a meromorphic function in Ω with $[p,q]$-order $0<\rho_{[p,q],\Omega}(f)<+\infty$. Then, the $[p,q]$-type of f is defined by

$$\tau_{[p,q],\Omega}(f)=\limsup_{r\to 1^-}\frac{\log_{p-1}^+ T_0(r,\Omega,f)}{\left(\log_{q-1}\frac{1}{1-r}\right)^{\rho_{[p,q],\Omega}(f)}}.$$

Now, we introduce the concept of lower $[p,q]$-order and lower $[p,q]$-type of a meromorphic function in a sector Ω.

Definition 1.4 Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in Ω, the lower $[p,q]$-order of f is defined by

$$\mu_{[p,q],\Omega}(f) = \liminf_{r \to 1^-} \frac{\log_p^+ T_0(r, \Omega, f)}{\log_q \frac{1}{1-r}}.$$

It is clear that $0 \leq \mu_{[p,q],\Omega}(f) \leq +\infty$. If f is non-admissible in Ω, then $\mu_{[p,q],\Omega}(f) = 0$. By Definition 1.4, $\mu_{[1,1],\Omega}(f) = \mu_\Omega(f)$ is the lower order of f in Ω and $\mu_{[p,1],\Omega}(f) = \mu_{p,\Omega}(f)$ is the lower iterated p-order of f in Ω.

Definition 1.5 Let $p \geq q \geq 1$ be integers and f be a meromorphic function in Ω with lower $[p,q]$-order $0 < \mu_{[p,q],\Omega}(f) < +\infty$. Then, the lower $[p,q]$-type of f is defined by

$$\underline{\tau}_{[p,q],\Omega}(f) = \liminf_{r \to 1^-} \frac{\log_{p-1}^+ T_0(r, \Omega, f)}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\mu_{[p,q],\Omega}(f)}}.$$

Several authors [2, 3, 8, 10, 16] have investigated the growth of solutions of the equation (1.1) by using the concepts of $[p,q]$-order in the unit disc Δ. Recently in [25], Zemirni and Belaïdi have investigated the growth of solutions of equation (1.1) in a sector of the unit disc with analytic coefficients of finite $[p,q]$-order, and have obtained the following results.

Theorem A [25] *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Ω. If*

$$\max_{1 \leq j \leq k-1} \left\{\rho_{[p,q],\Omega}(A_j)\right\} < \rho_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) satisfies $\rho_{[p,q],\Omega}(f) = +\infty$ and

$$\rho_{[p,q],\Omega_\varepsilon}(A_0) \leq \rho_{[p+1,q],\Omega}(f), \; \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\rho_{[p,q],\Omega_\varepsilon}(A_0) \leq \rho_{[p+1,q],\Omega}(f), \; \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p,q],\Omega}(A_0).$$

Theorem B [25] *Let $p \geq q \geq 1$ be integers and $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in Ω. Suppose that*

$$\max_{1 \leq j \leq k-1} \left\{\rho_{[p,q],\Omega}(A_j)\right\} \leq \rho_{[p,q],\Omega_\varepsilon}(A_0) = \rho \;\; (0 < \rho < +\infty).$$

and

$$\max_{1\leq j\leq k-1}\{\tau_{[p,q],\Omega}(A_j):\rho_{[p,q],\Omega}(A_j)=\rho_{[p,q],\Omega_\varepsilon}(A_0)\}$$
$$<\tau_{[p,q],\Omega_\varepsilon}(A_0)=\tau\ (0<\tau<+\infty).$$

Then every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f)=+\infty$ *and*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\leq\rho_{[p+1,q],\Omega}(f),\ \rho_{[p+1,q],\Omega_\varepsilon}(f)\leq\rho_{[p,q],\Omega}(A_0)+1.$$

Furthermore, if $p>q$, *then*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\leq\rho_{[p+1,q],\Omega}(f),\quad \rho_{[p+1,q],\Omega_\varepsilon}(f)\leq\rho_{[p,q],\Omega}(A_0).$$

Thus, the following questions arise naturally: (i) Whether the results similar to Theorem A can be obtained in Ω if $A_0(z)$ to dominate other coefficients in the sense of lower $[p,q]$-order?

(ii) If we use the lower $[p,q]$-type of $A_0(z)$ to dominate other coefficients, what can be said about $\mu_{[p+1,q],\Omega}(f)$ similar to Theorem B? In this paper, we give some answers to the above questions. We mainly obtain the following results.

Theorem 1.1 *Let* $p\geq q\geq 1$ *be integers and* $\varepsilon\in\left(0,\frac{\beta-\alpha}{2}\right)$. *Let* $A_0(z),A_1(z)$, $\ldots$, $A_{k-1}(z)$ *be analytic functions in* Ω. *If*

$$\max_{1\leq j\leq k-1}\left\{\rho_{[p,q],\Omega}(A_j)\right\}<\mu_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f)=\mu_{[p,q],\Omega}(f)=+\infty$,

$$\mu_{[p,q],\Omega_\varepsilon}(A_0)\leq\mu_{[p+1,q],\Omega}(f)\leq\rho_{[p+1,q],\Omega}(f)$$

and

$$\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0)+1.$$

Furthermore, if $p>q$, *then*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0)\leq\mu_{[p+1,q],\Omega}(f)\leq\rho_{[p+1,q],\Omega}(f)$$

and

$$\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0).$$

Remark 1.2 The Theorem 1.1 is similar to Theorem 2.2 (i) in [16] in the unit disc Δ.

Corollary 1.1 *Let* $p \geq q \geq 1$ *be integers and* $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ *be analytic functions in* Ω. *If*

$$\max_{1\leq j\leq k-1}\left\{\rho_{[p,q],\Omega}(A_j)\right\} < \mu_{[p,q],\Omega_\varepsilon}(A_0) = \rho_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f) = \mu_{[p,q],\Omega}(f) = +\infty$ *and*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, *then*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f)$$

and

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Theorem 1.2 *Let* $p \geq q \geq 1$ *be integers and* $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$. *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ *be analytic functions in* Ω *such that* $0 < \mu = \mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \rho_{[p,q],\Omega_\varepsilon}(A_0) < +\infty$. *Suppose that*

$$\max_{1\leq j\leq k-1}\left\{\rho_{[p,q],\Omega}(A_j)\right\} \leq \mu_{[p,q],\Omega_\varepsilon}(A_0)$$

and

$$\max_{1\leq j\leq k-1}\{\tau_{[p,q],\Omega}(A_j) : \rho_{[p,q],\Omega}(A_j) = \mu_{[p,q],\Omega_\varepsilon}(A_0)\} < \underline{\tau}_{[p,q],\Omega_\varepsilon}(A_0) < +\infty.$$

Then every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f) = \mu_{[p,q],\Omega}(f) = +\infty$ *and*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, *then*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f)$$

and

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Remark 1.3 The Theorem 1.2 is similar to Theorem 2.1 in [8] in the unit disc Δ.

2 Auxiliary lemmas

Lemma 2.1 [14] *Let*

$$u(z)=\frac{\left(ze^{-i\theta_0}\right)^{\pi/\delta}+2\left(ze^{-i\theta_0}\right)^{\pi/(2\delta)}-1}{\left(ze^{-i\theta_0}\right)^{\pi/\delta}-2\left(ze^{-i\theta_0}\right)^{\pi/(2\delta)}-1}, \tag{2.1}$$

where $0\leq\theta_0=\frac{\alpha+\beta}{2}<2\pi$, $0<\delta=\frac{\beta-\alpha}{2}<\pi$. *Then* $u(z)$ *is a conformal map of angular domain* Ω, $(0<\beta-\alpha<2\pi)$ *onto the unit disc* Δ. *Moreover, for any positive number* ε *satisfying* $0<\varepsilon<\delta$, *the transformation* (2.1) *satisfies*

$$u\left(\left\{z:\frac{1}{2}<|z|<r\right\}\cap\{z:|\arg z-\theta_0|<\delta-\varepsilon\}\right)$$

$$\subset\left\{u:|u|<1-\frac{\varepsilon}{2^{\frac{\pi}{2\delta}+1}\delta}(1-r)\right\},$$

$$u^{-1}\left(\{u:|u|<\varrho\}\right)\subset\left(\left\{z:|z|<1-\frac{\delta}{8\pi}(1-\varrho)\right\}\cap\{z:|\arg z-\theta_0|<\delta\}\right),$$

where $\varrho<1$ *is a constant. The inverse transformation of* (2.1) *is*

$$z(u)=e^{i\theta_0}\left(\frac{-(1+u)+\sqrt{2(1+u^2)}}{1-u}\right)^{\frac{2\delta}{\pi}}. \tag{2.2}$$

Lemma 2.2 [21] *Let* f *be a meromorphic function in* Ω, *where* $0<\beta-\alpha<2\pi$. *For any given* $\varepsilon\in\left(0,\frac{\beta-\alpha}{2}\right)$, *set* $\delta=\frac{\beta-\alpha}{2}$ *and* $b=\frac{\varepsilon}{2^{\pi/(2\delta)+1}\delta}$. *Then the following statements hold*

$$T_0\left(\varrho,\mathbb{C},f\left(z\left(u\right)\right)\right)\leq\frac{16\pi}{\delta}T_0\left(1-\frac{\delta}{8\pi}\left(1-\varrho\right),\Omega,f(z)\right)+O(1), \tag{2.3}$$

$$T_0\left(r,\Omega_\varepsilon,f(z)\right)\leq\frac{2}{b}T_0\left(1-b\left(1-r\right),\mathbb{C},f\left(z(u)\right)\right)+O(1), \tag{2.4}$$

where $z(u)$ *is the inverse transformation of* (2.1).

Remark 2.1 *By applying the formula* $T\left(r,f\right)=T_0\left(r,\mathbb{C},f\right)+O(1)\ (0<r<1)$, *Lemma 2.2, the definition of* $[p,q]$*-order and lower* $[p,q]$*-order, we immediately obtain that*

$$\rho_{[p,q],\Omega_\varepsilon}\left(f\left(z\right)\right)\leq\rho_{[p,q]}\left(f\left(z\left(u\right)\right)\right)\leq\rho_{[p,q],\Omega}\left(f\left(z\right)\right)$$

and

$$\mu_{[p,q],\Omega_\varepsilon}\left(f\left(z\right)\right) \leq \mu_{[p,q]}\left(f\left(z\left(u\right)\right)\right) \leq \mu_{[p,q],\Omega}\left(f\left(z\right)\right).$$

Lemma 2.3 [21] *Let f be a meromorphic function in Ω, where $0 < \beta - \alpha < 2\pi$ and $z(u)$ be the inverse transformation of (2.1). Set $F(u) = f\left(z\left(u\right)\right)$, $\psi\left(u\right) = f^{(\ell)}\left(z\left(u\right)\right)$, then*

$$\psi\left(u\right) = \sum_{j=1}^{\ell} \alpha_j F^{(j)}(u), \tag{2.5}$$

where the coefficients α_j are polynomials (with numerical coefficients) in the variables $V(u)\left(= \frac{1}{z'(u)}\right), V'(u), V''(u), \ldots$. Moreover, we have

$$T\left(\varrho, \alpha_j\right) = O\left(\log \frac{1}{1-\varrho}\right), \quad j = 1, 2, \ldots, \ell. \tag{2.6}$$

For the convenience of the readers, we give the statement and the proof of Lemma 2.4 [25, Lemma 3.4] with more precisions.

Lemma 2.4 *Suppose $f \not\equiv 0$ is a solution of (1.1) in Ω. Then $F(u) = f\left(z\left(u\right)\right)$ is a solution of*

$$F^{(k)}(u) + B_{k-1}(u)F^{(k-1)}(u) + \cdots + B_0(u)F(u) = 0 \tag{2.7}$$

in Δ, where

$$B_0(u) = \frac{1}{\alpha_k} A_0\left(z\left(u\right)\right) \tag{2.8}$$

and for $j = 1, 2, \ldots, k-1$

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k} \sum_{n=j}^{k-1} A_n\left(z\left(u\right)\right). \tag{2.9}$$

Consequently,

$$T\left(\varrho, B_0\right) \leq T\left(r, A_0\left(z\left(u\right)\right)\right) + O\left(\log \frac{1}{1-\varrho}\right) \tag{2.10}$$

and

$$T\left(\varrho, B_j\right) \leq \sum_{n=j}^{k-1} T\left(r, A_n\left(z\left(u\right)\right)\right) + O\left(\log \frac{1}{1-\varrho}\right). \tag{2.11}$$

Proof. Suppose that $f \not\equiv 0$ is a solution of (1.1) in the sector Ω. By using Lemma 2.3, we have

$$f^{(k)}(z(u)) + \sum_{n=1}^{k-1} A_n(z(u)) f^{(n)}(z(u)) + A_0(z(u)) f(z(u))$$

$$= \sum_{j=1}^{k} \alpha_j F^{(j)}(u) + \sum_{n=1}^{k-1} A_n(z(u)) \sum_{j=1}^{n} \alpha_j F^{(j)}(u) + A_0(z(u)) f(z(u))$$

$$= \sum_{j=1}^{k} \alpha_j F^{(j)}(u) + \sum_{j=1}^{k-1} \left(\alpha_j \sum_{n=j}^{k-1} A_n(z(u)) \right) F^{(j)}(u) + A_0(z(u)) f(z(u))$$

$$= \alpha_k F^{(k)}(u) + \sum_{j=1}^{k-1} \left(\alpha_j \sum_{n=j}^{k-1} A_n(z(u)) + \alpha_j \right) F^{(j)}(u) + A_0(z(u)) F(u).$$

It follows that $F(u) = f(z(u))$ is a solution of

$$F^{(k)}(u) + B_{k-1}(u)F^{(k-1)}(u) + \cdots + B_0(u)F(u) = 0,$$

where $B_0(u) = \dfrac{1}{\alpha_k} A_0(z(u))$ and

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k} \sum_{n=j}^{k-1} A_n(z(u)),\ j = 1, 2, \ldots, k-1.$$

By the proof of Lemma 2.3, we can get that [21, p. 63]

$$\alpha_k = V^k(u) = \left(\frac{1}{z'(u)} \right)^k$$

$$= \left(\frac{\omega}{e^{i\theta_0}} \left(\frac{1-u}{-(1+u) + \sqrt{2(1+u^2)}} \right)^{\frac{1}{\omega}-1} \frac{(1-u)^2 \sqrt{1+u^2}}{\sqrt{2}(1+u) - 2\sqrt{1+u^2}} \right)^k,$$

which is analytic in Δ, where $\theta_0 = \frac{\alpha+\beta}{2}$ and $\omega = \frac{\pi}{\beta-\alpha}$. Since $\alpha_k = V^k(u) \neq 0$ in Δ, then $B_0(u) = \dfrac{1}{\alpha_k} A_0(z(u))$ and

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k} \sum_{n=j}^{k-1} A_n(z(u)),\ j = 1, 2, \ldots, k-1$$

are also analytic in Δ. Because

$$T\left(\varrho,\alpha_j\right)=O\left(\log\frac{1}{1-\varrho}\right),\quad j=1,2,\ldots,k,$$

it follows from this and the properties of Nevanlinna's characteristic function that

$$\begin{aligned}T\left(\varrho,B_0\right)&\leq T\left(\varrho,\frac{1}{\alpha_k}\right)+T\left(\varrho,A_0\left(z\left(u\right)\right)\right)\\&=T\left(\varrho,\alpha_k\right)+T\left(\varrho,A_0\left(z\left(u\right)\right)\right)+O\left(1\right)\\&=T\left(\varrho,A_0\left(z\left(u\right)\right)\right)+O\left(\log\frac{1}{1-\varrho}\right),\end{aligned}$$

and for $j=1,2,\ldots,k-1$

$$T\left(\varrho,B_j\right)\leq T\left(\varrho,\frac{\alpha_j}{\alpha_k}\right)+\sum_{n=j}^{k-1}T\left(\varrho,A_n\left(z\left(u\right)\right)\right)+O\left(1\right)$$

$$\leq T\left(\varrho,\alpha_j\right)+T\left(\varrho,\frac{1}{\alpha_k}\right)+\sum_{n=j}^{k-1}T\left(\varrho,A_n\left(z\left(u\right)\right)\right)+O\left(1\right)$$

$$=T\left(\varrho,\alpha_j\right)+T\left(\varrho,\alpha_k\right)+\sum_{n=j}^{k-1}T\left(\varrho,A_n\left(z\left(u\right)\right)\right)+O\left(1\right)$$

$$=\sum_{n=j}^{k-1}T\left(\varrho,A_n\left(z\left(u\right)\right)\right)+O\left(\log\frac{1}{1-\varrho}\right).$$

Lemma 2.5 [16] *Let $p\geq q\geq 1$ be integers. If $B_0(u),B_1(u),...,B_{k-1}(u)$ are analytic functions of $[p,\ q]$-order in the unit disc Δ, then every solution $F\not\equiv 0$ of (2.7) satisfies*

$$\mu_{[p+1,q]}\left(F\right)=\mu_{M,[p+1,q]}\left(F\right)\leq\max_{1\leq j\leq k-1}\left\{\mu_{M,[p,q]}\left(B_0\right),\rho_{M,[p,q]}\left(B_j\right)\right\}.$$

Lemma 2.6 *Let $p\geq q\geq 1$ be integers. If $A_0(z),...,A_{k-1}(z)$ are analytic functions of $[p,\ q]$-order in sector Ω satisfying $\max\limits_{1\leq j\leq k-1}\left\{\rho_{[p,q],\Omega}\left(A_j\right)\right\}<\mu_{[p,q],\Omega_\varepsilon}\left(A_0\right)$, then for any given $\varepsilon\in\left(0,\frac{\beta-\alpha}{2}\right)$, every solution $f\not\equiv 0$ of (1.1) satisfies*

$$\mu_{[p+1,q],\Omega_\varepsilon}\left(f\right)\leq\mu_{[p,q],\Omega}\left(A_0\right)+1.$$

Furthermore, if $p > q$ *then*

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Proof. Let $f \not\equiv 0$ be a solution of equation (1.1). Then by Lemma 2.4, $F(u) = f(z(u))$ is a solution of equation (2.7) and by using Remark 2.1, Proposition 1.1, Proposition 1.2 and Lemma 2.5, we obtain

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p+1,q]}(F) = \mu_{M,[p+1,q]}(F)$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{M,[p,q]}(B_0), \rho_{M,[p,q]}(B_j)\right\}$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{[p,q]}(B_0), \rho_{[p,q]}(B_j)\right\} + 1$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{[p,q],\Omega}(A_0), \rho_{[p,q],\Omega}(A_j)\right\} + 1$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{[p,q],\Omega}(A_0), \mu_{[p,q],\Omega_\varepsilon}(A_0)\right\} + 1 = \mu_{[p,q],\Omega}(A_0) + 1.$$

If $p > q$, we obtain

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p+1,q]}(F) = \mu_{M,[p+1,q]}(F)$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{M,[p,q]}(B_0), \rho_{M,[p,q]}(B_j)\right\}$$
$$= \max_{1\leq j\leq k-1}\left\{\mu_{[p,q]}(B_0), \rho_{[p,q]}(B_j)\right\}$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{[p,q],\Omega}(A_0), \rho_{[p,q],\Omega}(A_j)\right\}$$
$$\leq \max_{1\leq j\leq k-1}\left\{\mu_{[p,q],\Omega}(A_0), \mu_{[p,q],\Omega_\varepsilon}(A_0)\right\} = \mu_{[p,q],\Omega}(A_0).$$

Lemma 2.7 [7, 15] *Let* f *be a meromorphic function in the unit disc* Δ *and let* $k \in \mathbb{N}$. *Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$, *possibly outside a set* $F \subset [0, 1)$ *with* $\int_E \frac{dr}{1-r} < \infty$.

Lemma 2.8 [1, 7] *Let* $g : (0, 1) \to \mathbb{R}$ *and* $h : (0, 1) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *holds outside of an exceptional* *set* $E \subset [0,1)$ *for which* $\int_E \frac{dr}{1-r} < \infty$*. Then there exists a constant* $d \in (0,1)$ *such that if* $s(r) = 1 - d(1-r)$*, then* $g(r) \le h(s(r))$ *for all* $r \in [0,1)$*.*

Lemma 2.9 [25] *Let* $p \ge q \ge 1$ *be integers. If* $A_0(z), ..., A_{k-1}(z)$ *are analytic functions of [p, q]-order in sector* Ω *satisfying* $\max_{0\le j\le k-1} \left\{\rho_{[p,q],\Omega}(A_j)\right\} \le \eta$*, then for any given* $\varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right)$*, every solution* $f \not\equiv 0$ *of* (1.1) *satisfies*

$$\rho_{[p+1,q],\Omega_\varepsilon}(f) \le \eta + 1.$$

Furthermore, if $p > q$ *then*

$$\rho_{[p+1,q],\Omega_\varepsilon}(f) \le \eta.$$

3 Proofs of the Theorems

Proof of Theorem 1.1. Suppose that $f \not\equiv 0$ is a solution of (1.1) in the sector Ω. From Lemma 2.4, the function $F(u) = f(z((u))$ is a solution of (2.7), where $z(u)$ is defined by (2.2). Then, by Lemma 2.2 and the properties of characteristic function of Nevanlinna, we have

$$T(\varrho, B_0(u)) = T\left(\varrho, \frac{1}{\alpha_k} A_0(z(u))\right) \ge T(\varrho, A_0(z(u))) - T(\varrho, \alpha_k)$$
$$= T_0(\varrho, \mathbb{C}, A_0(z(u))) + O(1) - T(\varrho, \alpha_k)$$
$$\ge \frac{b}{2} T_0\left(1 - \frac{1-\varrho}{b}, \Omega_\varepsilon, A_0(z)\right) + O(1) - T(\varrho, \alpha_k). \qquad (3.1)$$

By (2.3), (2.11) and the formula $T(r,f) = T_0(r, \mathbb{C}, f) + O(1)$ $(0 < r < 1)$, for $j = 1, 2, \ldots, k-1$ we have

$$T(\varrho, B_j(u)) \le \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O\left(\log \frac{1}{1-\varrho}\right)$$
$$= \sum_{n=j}^{k-1} T_0(\varrho, \mathbb{C}, A_n(z(u))) + O(1) + O\left(\log \frac{1}{1-\varrho}\right)$$
$$\le \frac{16\pi}{\delta} \sum_{n=j}^{k-1} T_0\left(1 - \frac{\delta}{8\pi}(1-\varrho), \Omega, A_n(z)\right) + O\left(\log \frac{1}{1-\varrho}\right). \qquad (3.2)$$

Set

$$\eta=\max_{1\le j\le k-1}\left\{\rho_{[p,q],\Omega}\left(A_j\right)\right\}<\mu_{[p,q],\Omega_\varepsilon}\left(A_0\right)=\mu.$$

Then, for any given $\epsilon\,(0<2\epsilon<\mu-\eta)$ and $r\to1^-$, we have for $j=1,2,\ldots,k-1$

$$T_0\left(r,\Omega,A_j(z)\right)\le\exp_p\left\{(\eta+\epsilon)\log_q\frac{1}{1-r}\right\}. \tag{3.3}$$

By the definition of lower $[p,q]$ order

$$T_0\left(r,\Omega_\varepsilon,A_0(z)\right)>\exp_p\left\{(\mu-\epsilon)\log_q\frac{1}{1-r}\right\}. \tag{3.4}$$

Now, as $|u|=\varrho\to1^-$, it follows from (3.1), (3.2), (3.3) and (3.4) that

$$\begin{aligned}T\left(\varrho,B_0\right)&\ge\frac{b}{2}T_0\left(1-\frac{1-\varrho}{b},\Omega_\varepsilon,A_0(z)\right)+O(1)-T\left(\varrho,\alpha_k\right)\\&\ge\frac{b}{2}\exp_p\left((\mu-\epsilon)\log_q\left(\frac{b}{1-\varrho}\right)\right)+O(1)-T\left(\varrho,\alpha_k\right)\\&=O\left(\exp_p\left((\mu-\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)\right)-T\left(\varrho,\alpha_k\right)\end{aligned} \tag{3.5}$$

and for $j=1,2,\ldots,k-1$

$$\begin{aligned}T\left(\varrho,B_j\right)&\le\frac{16\pi}{\delta}(k-j)\exp_p\left((\eta+\epsilon)\log_q\left(\frac{8\pi}{\delta\left(1-\varrho\right)}\right)\right)+O\left(\log\frac{1}{1-\varrho}\right)\\&=O\left(\exp_p\left((\eta+\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)+\log\frac{1}{1-\varrho}\right).\end{aligned} \tag{3.6}$$

By (2.7), we can write

$$\begin{aligned}T\left(\varrho,B_0\right)=m\left(\varrho,B_0\right)&\le\sum_{j=1}^{k-1}m\left(\varrho,B_j\right)+\sum_{j=1}^{k}m\left(\varrho,\frac{F^{(j)}}{F}\right)+O(1)\\&=\sum_{j=1}^{k-1}T\left(\varrho,B_j\right)+\sum_{j=1}^{k}m\left(\varrho,\frac{F^{(j)}}{F}\right)+O(1).\end{aligned} \tag{3.7}$$

It follows by (3.5), (3.6), (3.7) and Lemma 2.7 that

$$O\left(\exp_p\left((\mu-\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)\right)\le O\left(\exp_p\left((\eta+\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)+\log\frac{1}{1-\varrho}\right)$$

$$+T(\varrho,\alpha_k)+O\left(\log^+ T(\varrho,F)+\log\frac{1}{1-\varrho}\right) \tag{3.8}$$

holds for all u satisfying $|u|=\varrho\notin E$ as $\varrho\to 1^-$ and $E\subset(0,1)$ is a set with $\int_E \frac{d\varrho}{1-\varrho}<+\infty$. By using Lemma 2.8 and (3.8), for all u satisfying $|u|=\varrho$ as $\varrho\to 1^-$, we obtain

$$\exp_p\left((\mu-\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)\leq O\left(\exp_p\left((\eta+\epsilon)\log_q\left(\frac{1}{1-\varrho}\right)\right)\right)$$
$$+O\left(\log\frac{1}{d(1-\varrho)}\right)+O\left(\log^+ T(1-d(1-\varrho),F)\right). \tag{3.9}$$

Thus, from (3.9) we get $\sigma_{[p,q]}(F)=\mu_{[p+1,q]}(F)=+\infty$ and $\sigma_{[p+1,q]}(F)\geq\mu_{[p+1,q]}(F)\geq\mu$. Then, by Remark 2.1, we get that

$$\rho_{[p,q],\Omega}(f(z))=\mu_{[p,q]}(f(z))=+\infty \text{ and } \rho_{[p+1,q],\Omega}(f(z))\geq\mu_{[p+1,q],\Omega}(f(z))\geq\mu.$$

On the other hand, by Lemma 2.6 we have $\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0)+1$, and if $p>q$, we have $\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0)$.

Proof of Corollary 1.1. By using Theorem 1.1 and Lemma 2.9, we easily obtain Corollary 1.1.

Proof of Theorem 1.2. Suppose that $f\not\equiv 0$ is a solution of (1.1) in the sector Ω. From Lemma 2.4, the function $F(u)=f(z((u))$ is a solution of (2.7), where $z(u)$ is defined by (2.2). If $\rho_{[p,q],\Omega}(A_j)<\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$ for all $j=1,\cdots,k-1$, then Theorem 1.2 reduces to Theorem 1.1. Thus, we assume that at least one of A_j $(j=1,\cdots,k-1)$ satisfies $\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$. So, there exists a set $I\subseteq\{1,\cdots,k-1\}$ such that for $j\in I$ we have $\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$ and

$$\tau_1=\max_{j\in I}\{\tau_{[p,q],\Omega}(A_j):\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)\}<\underline{\tau}_{[p,q],\Omega_\varepsilon}(A_0)=\tau<+\infty.$$

and for $j\in\{1,\cdots,k-1\}\backslash I$, we have $b=\max_{j\in\{1,\cdots,k-1\}\backslash I}\{\rho_{[p,q],\Omega}(A_j)\}<\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$. Then that for any given ϵ $(0<2\epsilon<\min\{\mu-b,\tau-\tau_1\})$ and for $r\to 1^-$, we have for $j\in\{1,\cdots,k-1\}\backslash I$

$$T_0(r,\Omega,A_j(z))\leq\exp_p\left\{(b+\epsilon)\log_q\frac{1}{1-r}\right\}\leq\exp_p\left\{(\mu-\epsilon)\log_q\frac{1}{1-r}\right\} \tag{3.10}$$

and for $j \in I$, we get

$$T_0\left(r, \Omega, A_j(z)\right) \leq \exp_{p-1}\left\{(\tau_1+\epsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\mu}\right\}. \tag{3.11}$$

By the definition of lower $[p, q]$ order, we have for $r \to 1^-$

$$T_0\left(r, \Omega_\varepsilon, A_0(z)\right) > \exp_{p-1}\left\{(\tau-\epsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\mu}\right\}. \tag{3.12}$$

Then, by (3.1) and (3.12) as $|u| = \varrho \to 1^-$

$$\begin{aligned} T\left(\varrho, B_0(u)\right) &= T\left(\varrho, \frac{1}{\alpha_k} A_0\left(z\left(u\right)\right)\right) \\ &\geq \frac{b}{2} T_0\left(1-\frac{1-\varrho}{b}, \Omega_\varepsilon, A_0(z)\right) + O(1) - T\left(\varrho, \alpha_k\right) \\ &\geq \frac{b}{2}\exp_{p-1}\left\{(\tau-\epsilon)\left(\log_{q-1}\frac{b}{1-\varrho}\right)^{\mu}\right\} + O(1) - T\left(\varrho, \alpha_k\right) \\ &= O\left(\exp_{p-1}\left\{(\tau-\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\mu}\right\}\right) - T\left(\varrho, \alpha_k\right). \end{aligned} \tag{3.13}$$

Also, by (3.2), (3.10) and (3.11) for $j = 1, 2, \ldots, k-1$

$$\begin{aligned} T\left(\varrho, B_j\right) &\leq \frac{16\pi}{\delta}\sum_{n=j}^{k-1} T_0\left(1-\frac{\delta}{8\pi}\left(1-\varrho\right), \Omega, A_n\left(z\right)\right) + O\left(\log\frac{1}{1-\varrho}\right) \\ &\leq O\left(\exp_p\left\{(\mu-\epsilon)\log_q\frac{8\pi}{\delta\left(1-\varrho\right)}\right\}\right) \\ &+O\left(\exp_{p-1}\left\{(\tau_1+\epsilon)\left(\log_{q-1}\frac{8\pi}{\delta\left(1-\varrho\right)}\right)^{\mu}\right\}\right) + O\left(\log\frac{1}{1-\varrho}\right) \\ &= O\left(\exp_{p-1}\left\{(\tau_1+\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\mu}\right\} + \log\frac{1}{1-\varrho}\right). \end{aligned} \tag{3.14}$$

It follows by (3.7), (3.13), (3.14) and Lemma 2.7 that

$$O\left(\exp_{p-1}\left\{(\tau-\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\mu}\right\}\right) \leq O\left(\exp_{p-1}\left\{(\tau_1+\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\mu}\right\}\right)$$

$$+T(\varrho,\alpha_k)+O\left(\log^+T(\varrho,F)+\log\frac{1}{1-\varrho}\right) \tag{3.15}$$

holds for all u satisfying $|u|=\varrho\notin E$ as $\varrho\to 1^-$, where $E\subset(0,1)$ is a set with $\int_E\frac{d\varrho}{1-\varrho}<+\infty$. By using Lemma 2.8 and (3.15), for all u satisfying $|u|=\varrho\to 1^-$, we obtain

$$\exp_{p-1}\left\{(\tau-\epsilon)\left(\log_{q-1}\frac{1}{1-\varrho}\right)^{\mu}\right\}\leq O\left(\exp_{p-1}\left\{(\tau_1+\epsilon)\left(\log_{q-1}\frac{1}{d(1-\varrho)}\right)^{\mu}\right\}\right)$$

$$+O\left(\log\frac{1}{d(1-\varrho)}\right)+O\left(\log^+T(1-d(1-\varrho),F)\right). \tag{3.16}$$

Thus, from (3.16) we get $\rho_{[p,q]}(F)=\mu_{[p,q]}(F)=+\infty$ and $\rho_{[p+1,q]}(F)\geq\mu_{[p+1,q]}(F)\geq\mu$. Then, by Remark 2.1, we get that

$$\rho_{[p,q],\Omega}(f(z))=\mu_{[p,q]}(f(z))=+\infty \text{ and } \rho_{[p+1,q],\Omega}(f(z))\geq\mu_{[p+1,q],\Omega}(f(z))\geq\mu.$$

On the other hand, by Lemma 2.6 we have $\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0)+1$, and if $p>q$, we have $\mu_{[p+1,q],\Omega_\varepsilon}(f)\leq\mu_{[p,q],\Omega}(A_0)$.

Reference

[1] S. Bank, *General theorem concerning the growth of solutions of first-order algebraic differential equations*, Compositio Math. 25 (1972), 61–70.
[2] B. Belaïdi, *Growth of solutions to linear equations with analytic coefficients of [p, q]-order in the unit disc*, Electron. J. Differential Equations 2011, No. 156, 1-11.
[3] B. Belaïdi, *On the [p,q]-order of analytic solutions of linear differential equations in the unit disc*, Novi Sad J. Math. 42 (2012), no. 1, 117–129.
[4] I. Chyzhykov, G. Gundersen, J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. (3) 86 (2003), no. 3, 735–754.
[5] A.A. Goldberg and I.V. Ostrovskii, *Value distribution of meromorphic functions*, Translations of Mathematical Monographs, 236. American Mathematical Society, Providence, RI, 2008.
[6] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.
[7] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. No. 122 (2000), 54 pp.

[8] H. Hu and X. M. Zheng, *Growth of solutions of linear differential equations with analytic coefficients of* $[p,q]$*-order in the unit disc*, Electron. J. Differential Equations 2014, No. 204, 12 pp.
[9] I. Laine, *Complex differential equations*, Handbook of Differential Equations : Ordinary Differential Equations, Vol. IV, 269–363, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.
[10] Z. Latreuch and B. Belaïdi, *Linear differential equations with analytic coefficients of* $[p,q]$*-order in the unit disc*, Sarajevo J. Math. 9(21) (2013), no. 1, 71–84.
[11] J. Long, *Growth of solutions of higher order complex linear differential equations in an angular domain of unit disc*, J. Math. Study 48 (2015), no. 3, 306–314.
[12] J. Long, *On* $[p,\ q]$*-order of solutions of higher-order complex linear differential equations in an angular domain of unit disc*, J. Math. Study 50 (2017), no. 1, 91–100.
[13] Ch. Pommerenke, *On the mean growth of the solutions of complex linear differential equations in the disk*, Complex Variables Theory Appl. 1 (1982/83), no. 1, 23–38.
[14] D. C. Sun and J. R. Yu, *On the distribution of random Dirichlet series* (II), Chin. Ann. Math. 11(B) (1990), 33–44.
[15] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, (1975), reprint of the 1959 edition.
[16] J. Tu and H. X. Huang, *Complex oscillation of linear differential equations with analytic coefficients of* $[p,q]$*-order in the unit disc*, Comput. Methods Funct. Theory 15 (2015), no. 2, 225–246.
[17] S. J. Wu, *Estimates for the logarithmic derivative of a meromorphic function in an angle, and their application*, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 235–240, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.
[18] S. J. Wu, *On the growth of solutions of second order linear differential equations in an angle*, Complex Variables Theory Appl. 24 (1994), no. 3-4, 241–248.
[19] N. Wu, *Growth of solutions to linear complex differential equations in an angular region*, Electron. J. Differential Equations 2013, No. 183, 8 pp.
[20] N. Wu and Y. Z. Li, *On the growth of solutions of higher order linear differential equations*, New Zealand J. Math. 42 (2012), 27–35.
[21] N. Wu, *On the growth order of solutions of linear differential equations in a sector of the unit disk*, Results Math. 65 (2014), no. 1-2, 57–66.

[22] J. F. Xu and H. X. Yi, *Solutions of higher order linear differential equations in an angle*, Appl. Math. Lett. 22 (2009), no. 4, 484–489.
[23] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.
[24] G. Zhang, *Value distributions of solutions to complex linear differential equations in angular domains*, Open Math. 15 (2017), no. 1, 884–894.
[25] M. A. Zemirni and B. Belaïdi, $[p,q]$*-order of solutions of complex differential equations in a sector of the unit disc*, An. Univ. Craiova Ser. Mat. Inform. 45 (2018), no. 1, 37–49.