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ON THE VALUE DISTRIBUTION THEORY OF DIFFERENTIAL POLYNOMIALS IN THE UNIT DISC

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Abstract. In this paper, we investigate the relationship between small functions and non-homogeneous differential polynomials $g_k = d_k(z) f^{(k)} + \cdots + d_1(z) f' + d_0(z) f + b(z)$, where $d_0(z), d_1(z), \cdots, d_k(z)$ and $b(z)$ are finite $[p,q]-$order meromorphic functions in the unit disc Δ and $k \geq 2$ is an integer, which are not all equal to zero generated by the complex higher order non-homogeneous linear differential equation $f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F$, for $(k \geq 2)$, where $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ are finite $[p,q]-$order meromorphic functions in unit disc Δ.

Keywords. Non-homogeneous linear differential equations; Differential polynomials; Analytic solutions; Meromorphic solutions; Unit disc.

1. Introduction

The study on value distribution of differential polynomials generated by solutions of a given complex differential equation in the case of complex plane seems to have been started by Bank [2]. Many authors have investigated the growth and oscillation of solutions of complex linear differential equations in $\mathbb{C}$, see [2, 6, 7, 19, 20, 22]. In the unit disc, there already exist many results $[3, 4, 5, 8, 9, 10, 11, 12, 15, 16, 23, 24]$ but the study is more difficult than that in the

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complex plane. In [11] Fenton-Strumia obtained some results of Wiman-Valiron type for power series in the unit disc, and Fenton-Rossi [12] obtained an asymptotic equality of Wiman-Valiron type for the derivatives of analytic functions in the unit disc and applied to ODEs with analytic coefficients.

Throughout this paper, we shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z : |z| < 1\}$ (see $[14], [15], [20], [21], [25]$).

Firstly, we will recall some notations about the finite iterated order and the growth index to classify generally meromorphic functions of fast growth in Δ as those in $\mathbb{C}$ (see $[7], [19], [20]$). Let us define inductively, for $r \in (0, +\infty)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, +\infty)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\exp_{-1} r := \log_1 r$, $\log_{-1} r := \exp_1 r$.

Definition 1.1. $[8]$ The iterated $p-$order of a meromorphic function f in Δ is defined by

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r, f)}{\log \frac{1}{1-r}} \ (p \geq 1 \text{ is an integer}),$$

where $T(r, f)$ is the characteristic function of Nevanlinna of f, and $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+\left(\log_p^+ x\right)$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order (see $[15, 21]$). For an analytic function f in Δ we also define

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log \frac{1}{1-r}} \ (p \geq 1 \text{ is an integer}),$$

in which $M(r, f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function.

Remark 1.1. It follows by M. Tsuji in $[25]$ that if f is an analytic function in Δ, then $\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1$. However, it follows by Proposition 2.2.2 in $[20]$, that we have $\rho_{M,p}(f) = \rho_p(f)$, for $p \geq 2$.

Definition 1.2. $[9]$ Let f be a meromorphic function. Then the iterated $p-$convergence exponent of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}} \ (p \geq 1 \text{ is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z\in\mathbb{C}:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of zeros and for $p=2$ hyper-exponent of convergence of the sequence of zeros. Similarly, the iterated $p-$convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f)=\limsup_{r\to 1^-}\frac{\log_p^+\overline{N}\left(r,\frac{1}{f}\right)}{\log\frac{1}{1-r}}\ (p\geq 1\text{ is an integer}),$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z\in\mathbb{C}:|z|\leq r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyper-exponent of convergence of the sequence of distinct zeros.

In the following, we will give similar definitions as in [17, 18] for analytic and meromorphic functions of $[p,q]$-order, $[p,q]$-type and $[p,q]$-exponent of convergence of the zero-sequence in the unit disc.

Definition 1.3. [4] Let $p\geq q\geq 1$ be integers, and let f be a meromorphic function in Δ, the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p^+T(r,f)}{\log_q\frac{1}{1-r}}.$$

For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p+1}^+M(r,f)}{\log_q\frac{1}{1-r}}.$$

Remark 1.2. It is easy to see that $0\leq\rho_{[p,q]}(f)\leq+\infty$ ($0\leq\rho_{M,[p,q]}(f)\leq+\infty$), for any $p\geq q\geq 1$. By Definition 1.3, we have that $\rho_{[p,1]}=\rho_p(f)$ ($\rho_{M,[p,1]}=\rho_{M,p}(f)$).

In [24], Tu and Huang extended Proposition 1.1 in [4] with more details, as follows.

Proposition 1.1. [24] *Let f be an analytic function of $[p,q]$-order in Δ. Then the following five statements hold:*

(i) *If $p=q=1$, then $\rho(f)\leq\rho_M(f)\leq\rho(f)+1$.*

(ii) *If $p=q\geq 2$ and $\rho_{[p,q]}(f)<1$, then $\rho_{[p,q]}(f)\leq\rho_{M,[p,q]}(f)\leq 1$.*

(iii) *If $p=q\geq 2$ and $\rho_{[p,q]}(f)\geq 1$, or $p>q\geq 1$, then $\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f)$.*

(iv) *If* $p \geq 1$ *and* $\rho_{[p,p+1]}(f) > 1$, *then* $D(f) = \limsup\limits_{r \to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = \infty$; *if* $\rho_{[p,p+1]}(f) < 1$, *then* $D(f) = 0$.

(v) *If* $p \geq 1$ *and* $\rho_{M,[p,p+1]}(f) > 1$, *then* $D_M(f) = \limsup\limits_{r \to 1^-} \frac{\log^+ M(r,f)}{\log \frac{1}{1-r}} = \infty$; *if* $\rho_{M,[p,p+1]}(f) < 1$, *then* $D_M(f) = 0$.

Definition 1.4. [23] Let $p \geq q \geq 1$ be integers. The $[p,q]$-type of a meromorphic function $f(z)$ in Δ of $[p,q]$-order $\rho_{[p,q]}(f)$ $\left(0 < \rho_{[p,q]}(f) < +\infty\right)$ is defined by

$$\tau_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_{p-1}^+ T(r,f)}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\rho_{[p,q]}(f)}}.$$

For an analytic function $f(z)$ in Δ, the $[p,q]$-type about maximum modulus of f of $[p,q]$-order $\rho_{M,[p,q]}(f)$ $\left(0 < \rho_{M,[p,q]}(f) < +\infty\right)$ is defined by

$$\tau_{M,[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ M(r,f)}{\left(\log_{q-1} \frac{1}{1-r}\right)^{\rho_{M,[p,q]}(f)}}.$$

By Definition 1.4, we have that $\tau_{[p,1]} = \tau_p(f)$ $(\tau_{M,[p,1]} = \tau_{M,p}(f))$ and $\tau_{[1,1]} = \tau(f)$ $(\tau_{M,[1,1]} = \tau_M(f))$.

Definition 1.5. [23] Let $p \geq q \geq 1$ be integers. The $[p,q]$-exponent of convergence of the zero-sequence of $f(z)$ in Δ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}}.$$

Consider the complex nonhomogeneous linear differential equation

$$f'' + A_1(z) f' + A_0(z) f = F, \tag{1.1}$$

where $A_0(z), A_1(z)$ and F are analytic functions in Δ of finite order. In $[22]$, Laine and Rieppo considered value distribution theory of differential polynomials generated by solutions of linear differential equations in the complex plane. It is well-known that all solutions of equation (1.1) are analytic functions in Δ and that there are exactly two linearly independent solutions of (1.1), see [15]. In $[10]$, El Farissi, Belaïdi and Latreuch studied the complex oscillation of differential polynomial generated by solutions of second order linear differential equation (1.1) with analytic coefficients in Δ and obtained the following results. Set

$$\begin{aligned}\alpha_0 &= d_0 - d_2A_0,\ \beta_0 = d_2A_0A_1 - (d_2A_0)' - d_1A_0 + d_0',\\ \alpha_1 &= d_1 - d_2A_1,\ \beta_1 = d_2A_1^2 - (d_2A_1)' - d_1A_1 - d_2A_0 + d_0 + d_1',\end{aligned}$$

$$h = \alpha_1\beta_0 - \alpha_0\beta_1 \tag{1.2}$$

and

$$\psi(z) = \frac{\alpha_1\left(\varphi' - (d_2F)' - \alpha_1F\right) - \beta_1(\varphi - d_2F)}{h}.$$

Theorem A. [10] *Let $A_1(z), A_0(z) \not\equiv 0$ and F be analytic functions in Δ of finite order. Let $d_0(z)$, $d_1(z)$, $d_2(z)$ be analytic functions in Δ that are not all equal to zero with $\rho\left(d_j\right) < \infty$ $(j = 0,1,2)$ such that $h \not\equiv 0$, where h is defined by (1.2). If f is an infinite order solution of (1.1) with $\rho_2(f) = \rho$, then the differential polynomial $g_f = d_2f'' + d_1f' + d_0f$ satisfies $\rho\left(g_f\right) = \rho(f) = \infty$ and $\rho_2\left(g_f\right) = \rho_2(f) = \rho$.*

Theorem B. [10] *Let $A_1(z)$, $A_0(z) \not\equiv 0$ and F be analytic functions in Δ of finite order. Let $d_0(z)$, $d_1(z)$, $d_2(z)$ be analytic functions in Δ which are not all equal to zero with $\rho\left(d_j\right) < \infty$ $(j = 0,1,2)$ such that $h \not\equiv 0$, and let $\varphi(z) \not\equiv 0$ be an analytic function in Δ of finite order such that $\psi(z)$ is not a solution of (1.1). If f is an infinite order solution of (1.1) with $\rho_2(f) = \rho$, then the differential polynomial $g_f = d_2f'' + d_1f' + d_0f$ satisfies*

$$\overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \rho(f) = \infty,$$

$$\overline{\lambda}_2\left(g_f - \varphi\right) = \lambda_2\left(g_f - \varphi\right) = \rho_2(f) = \rho.$$

In 2011, Belaïdi studied the growth, the oscillation and the relation between small functions and some class of differential polynomials generated by non-homogeneous second order linear differential equation (1.1) when the solutions are of finite iterated $p-$order (see [3]). Set

$$\beta_0 = d_0' - d_1A_0,\ \beta_1 = d_1' + d_0 - d_1A_1,$$

$$h = d_1\beta_0 - d_0\beta_1 \tag{1.3}$$

and

$$\eta(z)=\frac{d_1(\varphi'-b'-d_1F)-\beta_1(\varphi-b)}{h}.$$

Theorem C. [3] *Let $A_1(z)$, $A_0(z)\not\equiv 0$, F be analytic functions of finite iterated $p-$order in Δ. Let $d_0(z)$, $d_1(z)$, $b(z)$ be analytic functions of finite iterated $p-$ order in Δ such that at least one of d_0, d_1 does not vanish identically and that $h\not\equiv 0$, where h is defined by (1.3). If f is a finite iterated $p-$order solution of (1.1) such that*

$$\max\left\{\rho_p(A_j),\ \rho_p(d_j)\ (j=0,1),\ \rho_p(b),\ \rho_p(F)\right\}<\rho_p(f), \tag{1.4}$$

then the differential polynomial $g_f=d_1f'+d_0f+b$ satisfies $\rho_p(g_f)=\rho_p(f)=\rho$.

Theorem D. [3] *Assume that the assumptions of Theorem C hold, and let $\varphi(z)$ be an analytic function in Δ with $\rho_p(\varphi)<\rho_p(f)$ such that $\eta(z)$ is not a solution of (1.1). If f is a finite iterated $p-$order solution of (1.1) such that (1.4) holds, then the differential polynomial $g_f=d_1f'+d_0f+b$ satisfies $\overline{\lambda}(g_f-\varphi)=\lambda(g_f-\varphi)=\rho_p(f)$.*

The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers. Section 3 is for some lemmas and basic theorems. The last section is for the proofs of our main results.

2. Main results

In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by meromorphic solutions of non-homogeneous linear differential equations in the unit disc. The main purpose of this paper is to study the controllability of solutions of the non-homogeneous higher order linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F,\ k\geq 2 \tag{2.1}$$

and the differential polynomial

$$g_k=d_k(z)f^{(k)}+d_{k-1}(z)f^{(k-1)}+\cdots+d_0(z)f+b(z), \tag{2.2}$$

where $A_i(z)\ (i=0,1,\cdots,k-1)$, $F(z)$ and $d_0(z),d_1(z),\cdots,d_k(z)$, $b(z)$ are meromorphic functions in Δ of finite $[p,q]-$order.

There exists a natural question: How about the growth and oscillation of the differential polynomial (2.2) with meromorphic coefficients of finite $[p,q]$-order generated by solutions of equation (2.1) in the unit disc?

The main purpose of this paper is to consider the above question. Before we state our results, we define the sequence of meromorphic functions $\alpha_{i,j}, \beta_j, (i=0,\cdots,k-1; j=0,\cdots,k-1)$ in Δ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \text{ for all } i = 1,\cdots,k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, \text{ for } i = 0, \end{cases} \tag{2.3}$$

$$\alpha_{i,0} = d_i - d_k A_i, \text{ for } i = 0,\cdots,k-1 \tag{2.4}$$

and

$$\beta_j = \begin{cases} \beta'_{j-1} + \alpha_{k-1,j-1} F \text{ for all } j = 1,\cdots,k-1, \\ d_k F + b \text{ for } j = 0, \end{cases} \tag{2.5}$$

we define also h_k by

$$h_k = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ \alpha_{0,k-1} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}$$

and $\psi_k(z)$ by

$$\psi_k(z) = \frac{\begin{vmatrix} \varphi - \beta_0 & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ \varphi' - \beta_1 & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ \varphi^{(k-1)} - \beta_{k-1} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h_k(z)},$$

where $h_k \not\equiv 0$ and $\alpha_{i,j}, \beta_j \ (i=0,..,k-1; j=0,\cdots,k-1)$ are defined in $(2.3)-(2.5)$, and $\varphi(z)$ is a meromorphic function in Δ with $\rho_{[p,q]}(\varphi) < \infty$.

The main results of this paper state as follows.

Theorem 2.1. *Let $A_i(z)$ $(i=0,1,\cdots,k-1)$, $F(z)$ be meromorphic functions in Δ of finite $[p,q]-$order. Let $d_j(z)$ $(j=0,1,\cdots,k)$, $b(z)$ be finite $[p,q]-$order meromorphic functions in Δ that are not all vanishing identically such that $h_k\not\equiv 0$. If $f(z)$ is an infinite $[p,q]-$order meromorphic solution in Δ of (2.1) with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial (2.2) satisfies*

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=\infty$$

and

$$\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.1) in Δ such that

$$\rho_{[p,q]}(f)>\max\left\{\max_{0\leq i\leq k-1}\rho_{[p,q]}(A_i),\max_{0\leq j\leq k-1}\rho_{[p,q]}\left(d_j\right),\rho_{[p,q]}(F),\ \rho_{[p,q]}(b)\right\}, \tag{2.6}$$

then

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f).$$

Remark 2.1. In Theorem 2.1, if we do not have the condition $h_k\not\equiv 0$, then the conclusions of Theorem 2.1 cannot hold. For example, if we take $d_i=d_kA_i$ $(i=0,\cdots,k-1)$, then $h_k\equiv 0$. It follows that $g_k\equiv d_kF+b$ and $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(d_kF+b)$. So, if $f(z)$ is an infinite $[p,q]-$order meromorphic solution of (2.1), then $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(d_kF+b)<\rho_{[p,q]}(f)=\infty$, and if f is a finite $[p,q]-$order meromorphic solution of (2.1) such that (2.6) holds, then

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(d_kF+b)\leq\max\left\{\rho_{[p,q]}(d_k),\rho_{[p,q]}(F),\ \rho_{[p,q]}(b)\right\}<\rho_{[p,q]}(f).$$

Theorem 2.2. *Under the hypotheses of Theorem 2.1, let $\varphi(z)$ be a meromorphic function in Δ with finite $[p,q]-$order such that $\psi_k(z)$ is not a solution of (2.1). If $f(z)$ is an infinite $[p,q]-$order meromorphic solution in Δ of (2.1) with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial (2.2) satisfies*

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.1) in Δ such that

$$\rho_{[p,q]}(f) > \max\left\{\max_{0\leq i\leq k-1}\rho_{[p,q]}(A_i), \max_{0\leq j\leq k-1}\rho_{[p,q]}\left(d_j\right), \rho_{[p,q]}(F),\ \rho_{[p,q]}(b), \rho_{[p,q]}(\varphi)\right\}, \tag{2.7}$$

then

$$\overline{\lambda}_{[p,q]}(g_k-\varphi) = \lambda_{[p,q]}(g_k-\varphi) = \rho_{[p,q]}(f).$$

Remark 2.2. Obviously, Theorems 2.1 and 2.2 are a generalization of Theorems A, B, C and D.

Remark 2.3. By setting $b(z)\equiv 0$, $\varphi(z)\equiv 0$ and $k=2$ in Theorem 2.2 we obtain Theorem 1.1 in [13].

We consider now the differential equation

$$f''+A_1(z)f+A_0(z)f=0, \tag{2.8}$$

where $A_0(z), A_1(z)$ are finite $[p,q]-$order analytic functions in the unit disc Δ. In the following we will give sufficient conditions on A_0, A_1 which satisfied the results of Theorem 2.1 and Theorem 2.2 without the conditions " $h_k\not\equiv 0$ " and " $\psi_k(z)$ is not a solution of (2.8) " where $k=2$.

Corollary 2.1. *Let $p\geq q\geq 1$ be integers, and let $A_0(z)$, $A_1(z)$ be analytic functions in Δ. Assume that $\rho_{[p,q]}(A_1) < \rho_{[p,q]}(A_0) = \rho$ $(0<\rho<+\infty)$ and $\tau_{[p,q]}(A_0)=\tau$ $(0<\tau<+\infty)$. Let $d_0(z)$, $d_1(z)$, $d_2(z)$ and $b(z)$ be analytic functions in Δ that are not all vanishing identically such that*

$$\max\left\{\rho_{[p,q]}(b),\ \rho_{[p,q]}\left(d_j\right) : j=0,1,2\right\} < \rho_{[p,q]}(A_0).$$

If $f(z)\not\equiv 0$ is a solution of the differential equation (2.8), then the differential polynomial $g_2=d_2f''+d_1f'+d_0f+b$ satisfies $\rho_{[p,q]}(g_2)=\rho_{[p,q]}(f)=\infty$ and

$$\rho_{[p,q]}(A_0)\leq\rho_{[p+1,q]}(g_2)=\rho_{[p+1,q]}(f)\leq\max\left\{\rho_{M,[p,q]}(A_i)\ \ (i=0,1)\right\}.$$

Furthermore, if $p>q$ then

$$\rho_{[p+1,q]}(g_2)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0).$$

Corollary 2.2. *Let $p \geq q \geq 1$ be integers, and let $A_0(z)$, $A_1(z)$ be analytic functions in Δ. Assume that $\rho_{[p,q]}(A_1) < \rho_{[p,q]}(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\tau_{[p,q]}(A_0) = \tau$ $(0 < \tau < +\infty)$. Let $d_0(z)$, $d_1(z)$, $d_2(z)$ and $b(z)$ be analytic functions in Δ that are not all vanishing identically such that*

$$\max\left\{\rho_{[p,q]}(b),\ \rho_{[p,q]}(d_j) : j = 0,1,2\right\} < \rho_{[p,q]}(A_1).$$

Let $\varphi(z)$ be an analytic function in Δ of finite $[p,q]-$order such that $\varphi - b \not\equiv 0$. If $f(z) \not\equiv 0$ is a solution of the differential equation (2.8), then the differential polynomial $g_2 = d_2 f'' + d_1 f' + d_0 f + b$ satisfies $\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f) = \infty$ and

$$\begin{aligned}\rho_{[p,q]}(A_0) &\leq \overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(g_2) \\ &= \rho_{[p+1,q]}(f) \leq \max\left\{\rho_{M,[p,q]}(A_0),\ \rho_{M,[p,q]}(A_1)\right\}.\end{aligned}$$

Furthermore, if $p > q$ then

$$\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Remark 2.4 By setting $b(z) \equiv 0$ in Corollaries 2.1 and 2.2 we obtain Theorems 1.3 and 1.4 in [23].

3. Auxiliary lemmas

Lemma 3.1. [4] *Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in the unit disc Δ such that $\rho_{[p,q]}(f) = \rho < \infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho + \varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$.

Lemma 3.2. $[1, 15]$ *Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_2 \subset [0,1)$ for which $\int_{E_2} \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

By using similar proof of Lemma 3.5 in [16], we easily obtain the following lemma when $\rho_{[p,q]}(f)=+\infty$.

Lemma 3.3. *Let $p\geq q\geq 1$ be integers. Let $A_i(z)$ $(i=0,\cdots,k-1)$, $F\not\equiv 0$ be meromorphic functions in Δ, and let $f(z)$ be a solution of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F$$

satisfying $\max\left\{\rho_{[p,q]}(A_i)\ \ (i=0,\cdots,k-1),\rho_{[p,q]}(F)\right\}<\rho_{[p,q]}(f)=\rho\leq+\infty$. *Then we have*

$$\overline{\lambda}_{[p,q]}(f)=\lambda_{[p,q]}(f)=\rho_{[p,q]}(f)$$

and

$$\overline{\lambda}_{[p+1,q]}(f)=\lambda_{[p+1,q]}(f)=\rho_{[p+1,q]}(f).$$

Lemma 3.4. [23] *Let $p\geq q\geq 1$ be integers, and let $A_i(z)$ $(i=0,\cdots,k-1)$ be analytic functions in Δ satisfying*

$$\max\left\{\rho_{[p,q]}(A_i)\ \ (i=1,\cdots,k-1)\right\}<\rho_{[p,q]}(A_0).$$

If $f(z)\not\equiv 0$ is a solution of the differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0,$$

then $\rho_{[p,q]}(f)=+\infty$ and

$$\rho_{[p,q]}(A_0)\leq\rho_{[p+1,q]}(f)\leq\max\left\{\rho_{M,[p,q]}(A_i)\ \ (i=0,\cdots,k-1)\right\}.$$

Furthermore, if $p>q$ then

$$\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0).$$

Lemma 3.5. [5] *Let $p\geq q\geq 1$ be integers, and let f be a meromorphic function of $[p,q]-$order in Δ. Then $\rho_{[p,q]}(f')=\rho_{[p,q]}(f)$.*

Lemma 3.6. [5] *Let $p\geq q\geq 1$ be integers, and let f and g be non-constant meromorphic functions of $[p,q]$-order in Δ. Then we have*

$$\rho_{[p,q]}(f+g)\leq\max\left\{\rho_{[p,q]}(f),\rho_{[p,q]}(g)\right\}$$

and

$$\rho_{[p,q]}(fg) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, *then we obtain*

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f).$$

Lemma 3.7. [23] *Let* $p \geq q \geq 1$ *be integers, and let* f *and* g *be meromorphic functions of* $[p,q]$*-order in* Δ *such that* $0 < \rho_{[p,q]}(f), \rho_{[p,q]}(g) < \infty$ *and* $0 < \tau_{[p,q]}(f), \tau_{[p,q]}(g) < \infty$. *We have*
(i) *If* $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, *then*

$$\tau_{[p,q]}(f+g) = \tau_{[p,q]}(fg) = \tau_{[p,q]}(f).$$

(ii) *If* $\rho_{[p,q]}(f) = \rho_{[p,q]}(g)$ *and* $\tau_{[p,q]}(f) \neq \tau_{[p,q]}(g)$, *then*

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f) = \rho_{[p,q]}(g).$$

Lemma 3.8. *Assume that* $f(z)$ *is a solution of equation* (2.1). *Then the differential polynomial* g_k *defined in* (2.2) *satisfies the system of equations*

$$\begin{cases} g_k - \beta_0 = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_k' - \beta_1 = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_k'' - \beta_2 = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_k^{(k-1)} - \beta_{k-1} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}, \end{cases}$$

where

$$\begin{aligned} \alpha_{i,j} &= \begin{cases} \alpha_{i,j-1}' + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,j-1}' - A_0 \alpha_{k-1,j-1}, \text{ for } i = 0, \end{cases} \\ \alpha_{i,0} &= d_i - d_k A_i, \text{ for } i = 0, \cdots, k-1 \end{aligned}$$

and

$$\beta_j = \begin{cases} \beta_{j-1}' + \alpha_{k-1,j-1} F, \text{ for all } j = 1, 2, \cdots, k-1, \\ d_k F + b, \text{ for } j = 0. \end{cases}$$

Proof. Suppose that f is a solution of (2.1). We can rewrite (2.1) as

$$f^{(k)} = F - \sum_{i=0}^{k-1} A_i f^{(i)}, \tag{3.1}$$

which implies

$$g_k = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f + b = \sum_{i=0}^{k-1} \left(d_i - d_k A_i\right) f^{(i)} + d_k F + b. \tag{3.2}$$

We can rewrite (3.2) as

$$g_k - \beta_0 = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.3}$$

where $\alpha_{i,0}$ are defined in (2.4) and $\beta_0 = d_k F + b$. Differentiating both sides of equation (3.3) and replacing $f^{(k)}$ with $f^{(k)} = F - \sum\limits_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$\begin{aligned} g_k' - \beta_0' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} + \alpha_{k-1,0} F \\ &= \left(\alpha_{0,0}' - \alpha_{k-1,0} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i\right) f^{(i)} + \alpha_{k-1,0} F. \end{aligned} \tag{3.4}$$

We can rewrite (3.4) as

$$g_k' - \beta_1 = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0} - A_i \alpha_{k-1,0}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha_{0,0}' - A_0 \alpha_{k-1,0}, \text{ for } i = 0 \end{cases} \tag{3.6}$$

and

$$\beta_1 = \beta_0' + \alpha_{k-1,0} F.$$

Differentiating both sides of equation (3.5) and replacing $f^{(k)}$ with $f^{(k)} = F - \sum\limits_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$\begin{aligned} g_k'' - \beta_1' &= \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)} \\ &= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)} \end{aligned}$$

$$= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)} + \alpha_{k-1,1} F$$

$$= \left(\alpha'_{0,1} - \alpha_{k-1,1} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha'_{i,1} + \alpha_{i-1,1} - A_i \alpha_{k-1,1}\right) f^{(i)} + \alpha_{k-1,1} F, \tag{3.7}$$

which implies that

$$g''_k - \beta_2 = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha'_{i,1} + \alpha_{i-1,1} - A_i \alpha_{k-1,1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,1} - A_0 \alpha_{k-1,1}, \text{ for } i = 0 \end{cases} \tag{3.9}$$

and

$$\beta_2 = \beta'_1 + \alpha_{k-1,1} F.$$

By using the same method as above we can easily deduce that

$$g_k^{(j)} - \beta_j = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \; j = 0, 1, \cdots, k-1, \tag{3.10}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, \text{ for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, \text{ for } i = 0, \end{cases} \tag{3.11}$$

$$\alpha_{i,0} = d_i - d_k A_i, \text{ for all } i = 0, 1, \cdots, k-1$$

and

$$\beta_j = \begin{cases} \beta'_{j-1} + \alpha_{k-1,j-1} F, \;\; \text{for all } j = 1, 2, \cdots, k-1, \\ d_k F + b, \;\; \text{for } j = 0. \end{cases} \tag{3.12}$$

By $(3.3) - (3.12)$ we obtain the system of equations

$$\begin{cases} g_k - \beta_0 = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g'_k - \beta_1 = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g''_k - \beta_2 = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_k^{(k-1)} - \beta_{k-1} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{cases} \tag{3.13}$$

This completes the proof of Lemma 3.8.

4. Proof of the theorems and corollaries

Proof of Theorem 2.1. Suppose that $f(z)$ is an infinite $[p,q]-$order meromorphic solution of (2.1) with $\rho_{[p+1,q]}(f)=\rho$. By Lemma 3.8, g_k satisfies the system of equations

$$\begin{cases} g_k-\beta_0=\alpha_{0,0}f+\alpha_{1,0}f'+\cdots+\alpha_{k-1,0}f^{(k-1)}, \\ g_k'-\beta_1=\alpha_{0,1}f+\alpha_{1,1}f'+\cdots+\alpha_{k-1,1}f^{(k-1)}, \\ g_k''-\beta_2=\alpha_{0,2}f+\alpha_{1,2}f'+\cdots+\alpha_{k-1,2}f^{(k-1)}, \\ \cdots \\ g_k^{(k-1)}-\beta_{k-1}=\alpha_{0,k-1}f+\alpha_{1,k-1}f'+\cdots+\alpha_{k-1,k-1}f^{(k-1)}, \end{cases} \tag{4.1}$$

where

$$\alpha_{i,j}=\begin{cases} \alpha_{i,j-1}'+\alpha_{i-1,j-1}-A_i\alpha_{k-1,j-1}, \text{ for all } i=1,\cdots,k-1, \\ \alpha_{0,j-1}'-A_0\alpha_{k-1,j-1}, \text{ for } i=0, \end{cases} \tag{4.2}$$

$$\alpha_{i,0}=d_i-d_kA_i, \text{ for all } i=0,1,\cdots,k-1 \tag{4.3}$$

and

$$\beta_j=\begin{cases} \beta_{j-1}'+\alpha_{k-1,j-1}F, \ \text{ for all } j=1,2,\cdots,k-1, \\ d_kF+b, \ \text{ for } j=0. \end{cases} \tag{4.4}$$

By Cramer's rule, and since $h_k\not\equiv 0$, then we have

$$f=\frac{\begin{vmatrix} g_k-\beta_0 & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g_k'-\beta_1 & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_k^{(k-1)}-\beta_{k-1} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h_k}.$$

It follows that

$$\begin{aligned} f&=C_0(g_k-\beta_0)+C_1\left(g_k'-\beta_1\right)+\cdots+C_{k-1}\left(g_k^{(k-1)}-\beta_{k-1}\right) \\ &=C_0g_k+C_1g_k'+\cdots+C_{k-1}g_k^{(k-1)}-\sum_{k=0}^{k-1}C_j\beta_j, \end{aligned} \tag{4.5}$$

where C_j are finite $[p,q]-$order meromorphic functions in Δ depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (4.2), (4.3) and β_j are defined in (4.4).

If $\rho_{[p,q]}(g_k)<+\infty$, then by (4.5) we obtain $\rho_{[p,q]}(f)<+\infty$, which is a contradiction. Hence $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=+\infty$.

Now, we prove that $\rho_{[p+1,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho$. By (2.2), we get $\rho_{[p+1,q]}(g_k) \leq \rho_{[p+1,q]}(f)$ and by (4.5) we have $\rho_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(g_k)$. This yield $\rho_{[p+1,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho$.

Furthermore, if $f(z)$ is a finite $[p,q]-$order meromorphic solution of equation (2.1) such that

$$\rho_{[p,q]}(f) > \max\left\{\max_{0\leq i\leq k-1}\rho_{[p,q]}(A_i), \max_{0\leq j\leq k-1}\rho_{[p,q]}(d_j), \rho_{[p,q]}(F), \ \rho_{[p,q]}(b)\right\}, \tag{4.6}$$

then

$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(\alpha_{i,j}), \rho_{[p,q]}(C_j\beta_j) : i = 0,\cdots,k-1; j = 0,\cdots,k-1\right\}. \tag{4.7}$$

By (2.2) and (4.6) we have $\rho_{[p,q]}(g_k) \leq \rho_{[p,q]}(f)$. Now, we prove $\rho_{[p,q]}(g_f) = \rho_{[p,q]}(f)$. If $\rho_{[p,q]}(g_k) < \rho_{[p,q]}(f)$, then by (4.5) and (4.7) we get

$$\rho_{[p,q]}(f) \leq \max\left\{\rho_{[p,q]}(C_j\beta_j) \ \ (j = 0,\cdots,k-1), \rho_{[p,q]}(g_k)\right\} < \rho_{[p,q]}(f)$$

which is a contradiction. Hence $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f)$.

Remark 4.1. From (4.5), it follows that the condition $h_k \not\equiv 0$ is equivalent to the condition $g_k - \beta_0, g_k' - \beta_1, \cdots, g_k^{(k-1)} - \beta_{k-1}$ are linearly independent over the field of meromorphic functions of finite $[p,q]-$order. As it was noted in the paper by Laine and Rieppo [22], one may assume that $d_k \equiv 0$. Note that the linear dependence of $g_k - \beta_0, g_k' - \beta_1, \cdots, g_k^{(k-1)} - \beta_{k-1}$ implies that f satisfies a linear differential equation of order smaller than k with appropriate coefficients, and vise versa (e.g. Theorem 2.3 in the paper of Laine and Rieppo [22]).

Proof of Theorem 2.2. Suppose that $f(z)$ is an infinite $[p,q]-$order meromorphic solution of equation (2.1) with $\rho_{[p+1,q]}(f) = \rho$. Set $w(z) = g_k - \varphi$. Since $\rho_{[p,q]}(\varphi) < \infty$, then by Lemma 3.6 and Theorem 2.1 we have $\rho_{[p,q]}(w) = \rho_{[p,q]}(g_k) = \infty$ and $\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g_k) = \rho$. To prove $\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \infty$ and $\overline{\lambda}_{[p+1,q]}(g_k - \varphi) = \lambda_{[p+1,q]}(g_k - \varphi) = \rho$ we need to prove $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \infty$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho$. By $g_k = w + \varphi$, and using (4.5), we get

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi_k(z), \tag{4.8}$$

where

$$\psi_k(z) = C_0(\varphi - \beta_0) + C_1(\varphi' - \beta_1) + \cdots + C_{k-1}\left(\varphi^{(k-1)} - \beta_{k-1}\right).$$

Substituting (4.8) into (2.1), we obtain

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_j w^{(j)}=F-\left(\psi_k^{(k)}+A_{k-1}(z)\,\psi_k^{(k-1)}+\cdots+A_0(z)\,\psi_k\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions of finite $[p,q]$-order. Since $\psi_k(z)$ is not a solution of (2.1), it follows that $H\not\equiv 0$. Then by Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho$, i. e.,

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho.$$

Suppose that $f(z)$ is a finite $[p,q]-$order meromorphic solution of equation (2.1) such that (2.7) holds. Set $w(z)=g_k-\varphi$. Since $\rho_{[p,q]}(\varphi)<\rho_{[p,q]}(f)$, then by Lemma 3.6 and Theorem 2.1 we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)$. To prove $\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)$ we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_j w^{(j)}=F-\left(\psi_k^{(k)}+A_{k-1}(z)\,\psi_k^{(k-1)}+\cdots+A_0(z)\,\psi_k\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions in Δ with $[p,q]-$order such that $\rho_{[p,q]}\left(\phi_j\right)$ $<\rho_{[p,q]}(f)$ $(j=0,\cdots,2k-2)$ and

$$\psi_k(z)=C_0(\varphi-\beta_0)+C_1\left(\varphi'-\beta_1\right)+\cdots+C_{k-1}\left(\varphi^{(k-1)}-\beta_{k-1}\right),$$

$$\rho_{[p,q]}(H)<\rho_{[p,q]}(f).$$

Since $\psi_k(z)$ is not a solution of (2.1), it follows that $H\not\equiv 0$. Then by Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$, i. e., $\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)$.

Proof of Corollary 2.1. Suppose that $f\not\equiv 0$ is a solution of (2.8). Then by Lemma 3.4, we have $\rho_{[p,q]}(f)=\infty$ and

$$\rho_{[p,q]}(A_0)\le\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)\le\max\left\{\rho_{M,[p,q]}(A_i)\ \ (i=0,1)\right\}.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

On the other hand, we have

$$g_2 = d_2 f'' + d_1 f' + d_0 f + b. \tag{4.9}$$

It follows by Lemma 3.8 that

$$\begin{cases} g_2 - \beta_0 = \alpha_{0,0} f + \alpha_{1,0} f', \\ g_2' - \beta_1 = \alpha_{0,1} f + \alpha_{1,1} f'. \end{cases} \tag{4.10}$$

By (2.4), we obtain

$$\alpha_{i,0} = \begin{cases} d_1 - d_2 A_1, \text{ for } i = 1, \\ d_0 - d_2 A_0, \text{ for } i = 0. \end{cases} \tag{4.11}$$

Now, by (2.3), we get

$$\alpha_{i,1} = \begin{cases} \alpha_{1,0}' + \alpha_{0,0} - A_1 \alpha_{1,0}, \text{ for } i = 1, \\ \alpha_{0,0}' - A_0 \alpha_{1,0}, \text{ for } i = 0, \end{cases}$$

and by (2.5) we get

$$\beta_0 = d_2 F + b = b,\ \beta_1 = \beta_0' + \alpha_{1,0} F = b' \ (F \equiv 0).$$

Hence

$$\begin{cases} \alpha_{0,1} = d_2 A_0 A_1 - (d_2 A_0)' - d_1 A_0 + d_0', \\ \alpha_{1,1} = d_2 A_1^2 - (d_2 A_1)' - d_1 A_1 - d_2 A_0 + d_0 + d_1' \end{cases} \tag{4.12}$$

and

$$\begin{aligned} h_2 = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} \\ \alpha_{0,1} & \alpha_{1,1} \end{vmatrix} &= d_2^2 A_0^2 + d_0 d_2 A_1^2 - \left(-d_2' d_1 + d_1' d_2 + 2 d_0 d_2 - d_1^2\right) A_0 \\ &- \left(d_2' d_0 - d_2 d_0' + d_0 d_1\right) A_1 - d_1 d_2 A_1 A_0 + d_1 d_2 A_0' - d_0 d_2 A_1' \\ &- d_2^2 A_0' A_1 + d_2^2 A_0 A_1' - d_0' d_1 + d_0 d_1' + d_0^2. \end{aligned} \tag{4.13}$$

First we suppose that $d_2 \not\equiv 0$. By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemmas 3.6-3.7 we have $\rho_{[p,q]}(h) = \rho_{[p,q]}(A_0) > 0$. Hence $h \not\equiv 0$. Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Then, by using a similar reasoning as above we get $h_2 \not\equiv 0$. By $h_2 \not\equiv 0$ and (4.10), we obtain

$$f = \frac{\alpha_{1,1}(g_2 - \beta_0) - \alpha_{1,0}(g_2' - \beta_1)}{h_2} = \frac{\alpha_{1,1}(g_2 - b) - \alpha_{1,0}(g_2' - b')}{h_2}. \tag{4.14}$$

By (4.9), Lemma 3.5 and Lemma 3.6, we have $\rho_{[p,q]}(g_2) \leq \rho_{[p,q]}(f)$ $(\rho_{[p+1,q]}(g_2) \leq \rho_{[p+1,q]}(f))$ and by (4.14) we have $\rho_{[p,q]}(f) \leq \rho_{[p,q]}(g_2)$ $(\rho_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(g_2))$. Hence

$$\rho_{[p,q]}(g_2) = \rho_{[p,q]}(f) \quad (\rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f)).$$

Proof of Corollary 2.2. Set $w(z) = d_2 f'' + d_1 f' + d_0 f + b - \varphi$. Then, by $\rho_{[p,q]}(\varphi) < \infty$, we have $\rho_{[p,q]}(w) = \rho_{[p,q]}(g_2) = \rho_{[p,q]}(f)$ and $\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f)$. In order to prove $\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f)$, we need to prove only $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \rho_{[p,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho_{[p+1,q]}(f)$. Using $g_2 = w + \varphi$, we get from (4.14)

$$f = \frac{-\alpha_{1,0} w' + \alpha_{1,1} w}{h_2} + \psi_2, \tag{4.15}$$

where

$$\psi_2(z) = \frac{\alpha_{1,1}(\varphi - b) - \alpha_{1,0}(\varphi' - b')}{h_2}. \tag{4.16}$$

Substituting (4.15) into equation (2.8), we obtain

$$\frac{-\alpha_{1,0}}{h_2} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\psi_2'' + A_1(z)\psi_2' + A_0(z)\psi_2\right) = G, \tag{4.17}$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions in Δ with $\rho_{[p,q]}(\phi_j) < \infty$ $(j = 0,1,2)$. First, we prove that $\psi_2 \not\equiv 0$. Suppose that $\psi_2 \equiv 0$. By $\varphi - b \not\equiv 0$ and (4.16) we obtain

$$\alpha_{1,1} = \alpha_{1,0} \frac{\varphi' - b'}{\varphi - b}. \tag{4.18}$$

Since $\rho_{[p,q]}(\varphi - b) \leq \max\{\rho_{[p,q]}(\varphi), \rho_{[p,q]}(b)\} = \alpha < \infty$, then it follows that by using Lemma 3.1, we have

$$m(r, \alpha_{1,1}) \leq m(r, \alpha_{1,0}) + O\left(\exp_{p-1}\left\{(\alpha + \varepsilon)\left(\log_q \frac{1}{1-r}\right)\right\}\right),$$

holds for all r outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$, that is,

$$\begin{aligned} m(r, d_2 A_1^2 - (d_2 A_1)' - d_1 A_1 - d_2 A_0 + d_0 + d_1') &\leq m(r, d_1 - d_2 A_1) \\ &+ O\left(\exp_{p-1}\left\{(\alpha + \varepsilon)\left(\log_q \frac{1}{1-r}\right)\right\}\right), \quad r \notin E_1. \end{aligned} \tag{4.19}$$

(i) If $d_2 \not\equiv 0$, then by Lemma 3.2 and (4.19) we obtain

$$\rho_{[p,q]}(A_0) \leq \rho_{[p,q]}(A_1),$$

this is a contradiction.

(ii) If $d_2 \equiv 0$ and $d_1 \not\equiv 0$, by Lemma 3.2 and (4.19) we obtain

$$\rho_{[p,q]}(A_1) \leq \rho_{[p,q]}(d_1),$$

this is a contradiction.

(iii) If $d_2 = d_1 \equiv 0$ and $d_0 \not\equiv 0$, then we have by (4.18)

$$d_0 = 0 \times \frac{\varphi' - b'}{\varphi - b} \equiv 0,$$

which is a contradiction. It is clear now that $\psi_2 \not\equiv 0$ cannot be a solution of (2.8) because $\rho_{[p,q]}(\psi_2) < \infty$. Hence $G \not\equiv 0$. By Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \infty$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho_{[p+1,q]}(f)$, i.e., $\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi)$$
$$= \rho_{[p+1,q]}(f) \leq \max\left\{\rho_{M,[p,q]}(A_i) : \ i = 0, 1\right\}.$$

Furthermore, if $p > q$, we have $\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A_0)$. This completes the proof.

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