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GROWTH OF LOGARITHMIC DIFFERENCES OF MEROMORPHIC FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In this paper, some properties about the behavior of growth of logarithmic differences of meromorphic functions are obtained, we prove also some relations between the exponent of convergence of meromorphic functions and the growth of their logarithmic differences. In addition, we give some applications in complex difference equations and uniqueness theory.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [10, 11, 17]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$, as $r \to +\infty$ except possibly a set of r of finite linear measure.

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Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points, where each common a-point of f and g has the same multiplicities. Throughout this paper, we denote by $\rho(f)$, $\lambda(f)$ $\big($resp. $\overline{\lambda}(f)\big)$ and $\lambda(\frac{1}{f})$ $\Big($resp. $\overline{\lambda}(\frac{1}{f})\Big)$ the order of f and the exponent of convergence of zeros (resp. distinct zeros) and poles $(resp. distinct poles)$ of f respectively, and by $\rho_2(f)$ the hyper-order of f (see [17]). We also need the following definition.

Definition 1.1 Let f be a nonconstant meromorphic function. We define difference operators as $\Delta_c f(z) = f(z+c) - f(z)$, $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$, where c is a nonzero complex number, $n \geq 2$ is a positive integer. If $c = 1$, we denote $\Delta_c f(z) = \Delta f(z)$.

The estimation of logarithmic derivatives play the key role in theory of differential equations. In his paper Gundersen [7] proved some interesting inequalities on the module of logarithmic derivatives of meromorphic functions. Recently Chiang and Feng [4, 5] established the Nevanlinna characteristic function of $f(z+\eta)$ in the complex plane, Laine and Yang [12] established the value distribution of difference polynomials, Halburd and Korhonen [9] established Nevanlinna theory for difference operators, Halburd and Korhonen [8] established the difference analogue of the lemma on the logarithmic derivative. Recently in [13], the authors have studied some properties about the behavior of growth of logarithmic derivatives of entire and meromorphic functions, and have obtained some relations between the zeros of entire functions and the growth of their logarithmic derivatives. In fact, they have proved.

Theorem A [13] *Let f be meromorphic function. If there exists an integer $k \geq 1$ such that $\rho\left(\frac{f^{(k)}}{f}\right) = \rho(f)$ and $\rho(f) > \rho_2(f)$, then*

$$\max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho(f).$$

Furthermore, if f is entire function, then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

Theorem B [13] *Let f be an entire function with finite number of zeros. Then for any integer $k \geq 1$*

$$\rho\left(\frac{f^{(k)}}{f}\right) = \rho_2(f).$$

In this paper, some properties about the behavior of growth of logarithmic differences of meromorphic functions are obtained, also we give some relations between the exponent of convergence of meromorphic functions and the growth of their logarithmic differences. In fact, we obtain the following results.

Theorem 1.1 *Let f be meromorphic function of finite order and let $n \in \mathbb{Z}\setminus\{0\}$. Then*

$$\max\left\{\rho\left(\frac{f(z+n)}{f(z)}\right), n \in \mathbb{Z}\setminus\{0\}\right\} = \rho\left(\frac{f(z+1)}{f(z)}\right). \tag{1.1}$$

Furthermore

$$\max_{n\in\mathbb{Z}\setminus\{0\}}\left\{\rho\left(\frac{f(z+n)}{f(z)}\right), \rho\left(\frac{f(z+n+1)}{f(z)}\right)\right\} = \rho\left(\frac{f(z+1)}{f(z)}\right). \tag{1.2}$$

Remark 1.1 In Theorem 1.1, the condition $n \in \mathbb{Z}\setminus\{0\}$ is necessary, for example the function $f(z) = \sin(2\pi z)$ satisfies

$$\rho\left(\frac{f(z+n)}{f(z)}\right) = 0, \text{ for all } n \in \mathbb{Z}\setminus\{0\}.$$

On the other hand, we have

$$\rho\left(\frac{f\left(z+\frac{1}{2\pi}\right)}{f(z)}\right) = 1.$$

Example 1.1 The function $f(z) = \sin\left(\frac{\pi}{2}z\right)$ satisfies $f(z+1) = \cos\left(\frac{\pi}{2}z\right)$ and $f(z+4) = f(z)$. So

$$\rho\left(\frac{f(z+1)}{f(z)}\right) = 1, \ \rho\left(\frac{f(z+4)}{f(z)}\right) = 0.$$

Theorem 1.2 *Let f be meromorphic function of finite order. If there exists a nonzero complex number c such that $\rho\left(\frac{f(z+c)}{f(z)}\right) = \rho(f)$, then*

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho(f). \tag{1.3}$$

Furthermore, if f is entire function, then

$$\lambda(f)=\rho(f).$$

Example 1.2 The function $f(z)=\frac{1}{\Gamma(z)}+1$ satisfies

$$\rho\left(\frac{f(z+1)}{f(z)}\right)=\rho\left(\frac{z\Gamma(z)+1}{z\Gamma(z)+z}\right)=\rho(f)=1$$

and $\lambda(f)=\lambda\left(\frac{1}{f}\right)=\rho(f)=1.$

Example 1.3 The functions $g(z)=\sin z$ and $h(z)=\frac{1}{e^z-1}$ satisfy

$$\rho\left(\frac{g(z+1)}{g(z)}\right)=\rho(g)=1,\ \rho\left(\frac{h(z+1)}{h(z)}\right)=\rho(h)=1,$$

where

$$0=\lambda\left(\frac{1}{g}\right)<\lambda(g)=\rho(g)=1$$

and

$$0=\lambda(h)<\lambda\left(\frac{1}{h}\right)=\rho(h)=1.$$

Corollary 1.1 *Let f be meromorphic function of finite order. If there exists $c\in\mathbb{C}\setminus\{0\}$ such that $\rho(\Delta_c f)<\rho(f)$, then*

$$\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}=\rho(f). \tag{1.4}$$

Theorem 1.3 *Let f be meromorphic function of finite order.*
(i) *If* $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\rho(f)-1$, *then for any* $c\in\mathbb{C}\setminus\{0\}$

$$\rho\left(\frac{f(z+c)}{f(z)}\right)=\rho(f)-1. \tag{1.5}$$

(ii) *If* $\rho(f)<1$, *then for any* $c\in\mathbb{C}\setminus\{0\}$

$$\rho\left(\frac{f(z+c)}{f(z)}\right)=\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}. \tag{1.6}$$

Remark 1.2 In Theorem 1.3 (i), the term $\rho(f)-1$ is sharp (i.e., we can not replace the condition $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\rho(f)-1$ by $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\rho(f)$). For example the function $f(z)=\cos z^{\frac{3}{2}}e^{z^2}$ satisfies $\lambda(f)=\frac{3}{2}>\rho(f)-1=1$, on the other hand

$$\frac{f(z+1)}{f(z)}=\frac{\cos(z+1)^{\frac{3}{2}}}{\cos z^{\frac{3}{2}}}e^{2z+1}.$$

That say

$$\rho\left(\frac{f(z+1)}{f(z)}\right)=\frac{3}{2}\neq\rho(f)-1=1.$$

Theorem 1.4 *Let f be meromorphic function of infinite order such that $\rho_2(f)<1$ and $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}<\infty$. Then for any $c\in\mathbb{C}\setminus\{0\}$*

$$\rho\left(\frac{f(z+c)}{f(z)}\right)=\rho(f)=\infty. \tag{1.7}$$

Example 1.4 The function $f(z)=\exp(\cos\sqrt{z})$ satisfies $\rho_2(f)=\frac{1}{2}<1$ and $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}=0$. Therefore, for any $c\in\mathbb{C}\setminus\{0\}$, $\frac{f(z+c)}{f(z)}$ is of infinite order.

2. Some Applications

In this section, we give simple proofs of some known results in complex difference equations and uniqueness theory.

Theorem 2.1 [14] *Let $a_0(z),a_1(z),\cdots,a_n(z)$, $F(z)$ $(\not\equiv 0)$ be finite order entire functions. If f is entire solution of the equation*

$$a_n(z)f(z+n)+a_{n-1}(z)f(z+n-1)+\cdots+a_1(z)f(z+1)$$

$$+a_0(z)f(z)=F(z) \tag{2.1}$$

with

$$\rho(f)>\max\{\rho(a_j)\ \ (j=0,\cdots,n),\rho(F)\}, \tag{2.2}$$

then $\lambda(f)=\rho(f)$.

*Proof***.** Dividing both sides of (2.1) by f, we obtain

$$\frac{F(z)}{f(z)}-a_0(z)=\sum_{i=1}^{n}a_i(z)\frac{f(z+i)}{f(z)}.$$

By using (2.2) and Theorem 1.1, we have

$$\rho(f)\leq\max\left\{\rho(a_i)\ ,\rho\left(\frac{f(z+i)}{f(z)}\right)\ \ (i=1,\cdots,n)\right\}=\rho\left(\frac{f(z+1)}{f(z)}\right).$$

Then $\rho(f)=\rho\left(\frac{f(z+1)}{f(z)}\right)$. Hence, by Theorem 1.2 we obtain $\lambda(f)=\rho(f)$.

Remark 2.1 We can obtain the same result of Theorem 2.1 for the equation

$$a_n(z) f(z+c_n) + a_{n-1}(z) f(z+c_{n-1}) + \cdots + a_1(z) f(z+c_1)$$
$$+a_0(z) f(z) = F(z),$$

where c_k $(k = 1, \cdots, n)$ are constants unequal to each other.

Theorem 2.2 *Let f be a transcendental entire function of finite order, c_i $(i = 0, ..., n)$ be constants, unequal to each other and let d_i $(i = 0, ...n)$ be the entire coefficients of the difference polynomial*

$$g(z) = d_n f(z+c_n) + d_{n-1} f(z+c_{n-1}) + \cdots + d_0 f(z+c_0) \tag{2.3}$$

such that

$$\max_{0 \le i \le n} \rho(d_i) < \rho(f). \tag{2.4}$$

If f has a Borel exceptional value a such that $g(z) - a\sum_{i=0}^{n} d_i \not\equiv 0$, then $\rho(g) = \rho(f)$.

*Proof***.** By (2.3) and (2.4), we have $\rho(g) \le \rho(f)$. We need to prove only $\rho(g) \ge \rho(f)$. We prove this by contraposition. Suppose that $\rho(g) < \rho(f)$. Then, dividing both sides of (2.3) by $f - a$ we have

$$\frac{g(z) - a\sum_{i=0}^{n} d_i}{f(z) - a} = \sum_{i=0}^{n} d_i \frac{f(z+c_i) - a}{f(z) - a}. \tag{2.5}$$

Set $F(z) = f(z) - a$ in (2.5), we get

$$\frac{g(z) - a\sum_{i=0}^{n} d_i}{F(z)} = \sum_{i=0}^{n} d_i \frac{F(z+c_i)}{F(z)}. \tag{2.6}$$

Since $g(z) - a\sum_{i=0}^{n} d_i \not\equiv 0$, then by (2.4) and (2.6)

$$\rho(f) = \rho(F) \le \max_{0 \le i \le n} \rho\left(\frac{F(z+c_i)}{F(z)}\right) \le \rho(F),$$

which means that there exist at least a constant $c \in \mathbb{C} \setminus \{0\}$ such that

$$\rho\left(\frac{F(z+c)}{F(z)}\right) = \rho(F).$$

Applying Theorem 1.2 to the equation above, we deduce that $\lambda(F) = \rho(F)$, so $\lambda(f-a) = \rho(f)$ (i.e., a is not a Borel exceptional value of f). Hence, $\rho(g) \geq \rho(f)$.

The Pielou logistic equation

$$y(z+1) = \frac{R(z)\,y(z)}{Q(z) + P(z)\,y(z)}, \tag{2.7}$$

where $P(z)$, $Q(z)$, $R(z)$ are nonzero polynomials, is an important difference equation because it is obtained by transform form the well-known Verhulst-Pearl equation (see [6], p. 99)

$$x'(t) = x(t)\,[a - bx(t)] \quad (a, b > 0),$$

which is the most popular continuous model of growth of a population. In [3], Chen obtained the following theorem.

Theorem C [3] *Let $P(z)$, $Q(z)$, $R(z)$ be polynomials with*

$$P(z)\,Q(z)\,R(z) \not\equiv 0,$$

and $y(z)$ be a finite order transcendental meromorphic solution of the equation (2.7). Then

$$\lambda\left(\frac{1}{y}\right) = \rho(y) \geq 1.$$

In this paper, we obtain the following result.

Theorem 2.3 *Let $P(z)$, $Q(z)$, $R(z)$ be meromorphic functions with*

$$P(z)\,Q(z)\,R(z) \not\equiv 0,$$

and $y(z)$ be a finite order meromorphic solution of the equation

$$y(z+c) = \frac{R(z)\,y(z)}{Q(z) + P(z)\,y(z)}, \quad (c \in \mathbb{C}\setminus\{0\}) \tag{2.8}$$

such that

$$\rho(y) > \max\{\rho(P), \rho(Q), \rho(R)\}. \tag{2.9}$$

Then

$$\max\left\{\lambda\left(\frac{1}{y}\right), \lambda(y)\right\} = \rho(y).$$

*Proof***.** Dividing both sides of equation (2.8) by $y(z)$, we obtain

$$\frac{y(z+c)}{y(z)} = \frac{R(z)}{Q(z) + P(z)\,y(z)}. \tag{2.10}$$

From (2.9) and (2.10), we get

$$\rho\left(\frac{y(z+c)}{y(z)}\right)=\rho(y).$$

So, by Theorem 1.2 we obtain $\max\left\{\lambda\left(\frac{1}{y}\right),\lambda(y)\right\}=\rho(y)$.

Recently, the difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded, which bring about a number of papers focusing on the uniqueness study of meromorphic functions sharing a small function with their difference operators. Furthermore, people obtained lots of results expressly for the meromorphic function whose order is less than 1 because if $\rho(f)<1$, then we have $g(z+\eta)=g(z)(1+o(1))$ as $z\to\infty$ (see [2]) possibly outside of a small set. For example, the authors in [16] obtained the following result.

Theorem D *Let f be a transcendental entire function such that $\rho(f)<1$. If f and $\Delta^n f$ share a finite value a* CM, *then*

$$\Delta^n f-a=c(f-a)$$

holds for some nonzero complex number c.

In [18], Zhang, Kang and Liao find that such probability $\Delta^n f-a=c(f-a)$ in the conclusion of Theorem D does not exist. That is to say if transcendental entire function f and $\Delta^n f$ share a finite value a CM, then $\rho(f)\geq 1$. In the following we give a simple proof for this result.

Theorem 2.4 *Let f be a transcendental entire function of finite order and let c be a nonzero complex number such that $f(z+c)\not\equiv f(z)$. If $f(z)$ and $f(z+c)$ shared a finite value a* CM, *then $\rho(f)\geq 1$. Furthermore, if*

$$f(z+c)-a=e^{P(z)}(f(z)-a),$$

where P is nonconstant polynomial and $\lambda(f-a)<\rho(f)-1$, then

$$\rho(f)=\deg P+1. \tag{2.11}$$

Proof. Suppose that $\rho(f)<1$. Since $f(z)$ and $f(z+c)$ shared a value a CM, then

$$\frac{f(z+c)-a}{f(z)-a}=e^{P(z)}, \tag{2.12}$$

which implies

$$1 > \rho(f) \geq \rho\left(\frac{f(z+c)-a}{f(z)-a}\right) = \rho\left(e^{P(z)}\right) = \deg P.$$

Hence $\deg P = 0$ and $e^{P(z)} = K$. We can rewrite (2.12) as

$$f(z+c) - Kf(z) = a(1-K). \tag{2.13}$$

It's clear that K can not be equal 1 because $f(z+c) \not\equiv f(z)$. Differentiating both sides of (2.13), we have

$$f'(z+c) - Kf'(z) = 0,$$

which implies

$$\Delta_c f'(z) + (1-K) f'(z) = 0.$$

By using Lemma 3.8 in Section 3 of this paper, we obtain the contradiction $\rho(f') \geq 1$. Hence $\rho(f) \geq 1$. Set $G(z) = f(z) - a$. Then, by using the hypothesis $\lambda(G) = \lambda(f-a) < \rho(f) - 1 = \rho(G) - 1$, Theorem 1.3 (i) and (2.12), we obtain

$$\rho(f) - 1 = \rho(G) - 1 = \rho\left(\frac{G(z+c)}{G(z)}\right) = \deg P,$$

and the proof of Theorem 2.4 is complete.

Remark 2.2 By the same reasoning, we can find the same conclusion of Theorem 2.4 if we replace $f(z+c)$ by $\Delta^n f$.

Remark 2.3 In fact, (2.11) was proved by Li, Yang and Yi in [15], with weaker condition $\lambda(f-a) < \rho(f)$ instead of $\lambda(f-a) < \rho(f) - 1$.

3. Some Lemmas

Lemma 3.1 ([4]) *Let f be a transcendental meromorphic function with finite order σ and η be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r),$$

$$\text{i.e., } T(r, f(z+\eta)) = T(r,f) + S(r,f). \tag{3.1}$$

Lemma 3.2 ([4]) *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let*

σ be the order of $f(z)$. Then for each $\varepsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma-1+\varepsilon}\right). \tag{3.2}$$

Lemma 3.3 ([4]) *Let f be a meromorphic function with exponent of convergence of poles $\lambda\left(\frac{1}{f}\right) = \lambda < +\infty$, $\eta \neq 0$ be fixed. Then for each $\varepsilon > 0$,*

$$\begin{aligned} N(r, f(z+\eta)) &= N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r), \\ \text{i.e., } N(r, f(z+\eta)) &= N(r,f) + S(r,f). \end{aligned}$$

Lemma 3.4 ([5]) *Let f be a meromorphic function of order $\rho(f) = \sigma < 1$, and let η be a fixed, non-zero number. Then for any $\varepsilon > 0$, there exists a set $E \subset (1,\infty)$ that depends on f and has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0,1]$*

$$\frac{\Delta_c f(z)}{f(z)} = c\frac{f'(z)}{f(z)} + O\left(r^{2\rho-2+\varepsilon}\right). \tag{3.3}$$

Lemma 3.5 ([3]) *Let f be a transcendental meromorphic function of order $\sigma(f) = \sigma < 1$, and let $g_1(z)$ and $g_2(z)$ $(\not\equiv 0)$ be polynomials, c_1, c_2 $(\neq c_1)$ be constants. Then*

$$h(z) = g_1(z) f(z+c_1) + g_2(z) f(z+c_2)$$

is transcendental.

Lemma 3.6 [10] *Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O(\log T(r,f) + \log r)$, possibly outside a set $E_1 \subset [0,\infty)$ of a finite linear measure. If f is a finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

To avoid some problems caused by the exceptional set we recall the following lemma.

Lemma 3.7 ([1]) *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure. Then for any $\lambda > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\lambda r)$ for all $r > r_0$.*

Lemma 3.8 ([5]) *Let $P_0(z), \cdots, P_n(z)$ be polynomials such that*

$$\max_{1\le j\le n}\{\deg P_j\} \le \deg P_0.$$

Let $f(z)$ be a meromorphic solution to the difference equation

$$P_n(z)\,\Delta_c^n f(z) + P_{n-1}\Delta_c^{n-1} f(z) + \cdots + P_0(z)\, f(z) = 0.$$

Then $\rho(f) \ge 1$.

4. Proofs of Theorems and Corollary

Proof of Theorem 1.1. Without loss of generality, we suppose that $n \ge 1$ is a positive integer. We can write

$$\frac{f(z+n)}{f(z)} = \frac{f(z+n)}{f(z+n-1)}\frac{f(z+n-1)}{f(z+n-2)}\cdots\frac{f(z+1)}{f(z)}, \tag{4.1}$$

which implies

$$\rho\left(\frac{f(z+n)}{f(z)}\right) \le \max\left\{\rho\left(\frac{f(z+i+1)}{f(z+i)}\right),\ i=0,\cdots,n-1\right\}. \tag{4.2}$$

Set $g(z) = \frac{f(z+1)}{f(z)}$. Then, by using Lemma 3.1 and Lemma 3.7, for $i \in \mathbb{N}$ we have $\rho(g(z+i)) = \rho(g(z))$. So

$$\rho\left(\frac{f(z+i+1)}{f(z+i)}\right) = \rho\left(\frac{f(z+1)}{f(z)}\right). \tag{4.3}$$

It follows that

$$\rho\left(\frac{f(z+n)}{f(z)}\right) \le \rho\left(\frac{f(z+1)}{f(z)}\right),\ \ (n\ge 1). \tag{4.4}$$

Hence

$$\max\left\{\rho\left(\frac{f(z+n)}{f(z)}\right),\ n\ge 1\right\} = \rho\left(\frac{f(z+1)}{f(z)}\right). \tag{4.5}$$

Suppose now, for all $n \ge 2$

$$\rho\left(\frac{f(z+n)}{f(z)}\right) < \rho\left(\frac{f(z+1)}{f(z)}\right). \tag{4.6}$$

Set $\varphi(z) = \frac{f(z+n)}{f(z-1)}$. Then, by Lemma 3.1 and Lemma 3.7, we have $\rho(\varphi(z+1)) = \rho(\varphi(z))$. So

$$\rho\left(\frac{f(z+n+1)}{f(z)}\right) = \rho\left(\frac{f(z+n)}{f(z-1)}\right) = \rho\left(\frac{f(z+n)}{f(z)}\frac{f(z)}{f(z-1)}\right),$$

and since

$$\rho\left(\frac{f(z)}{f(z-1)}\right)=\rho\left(\frac{f(z+1)}{f(z)}\right)>\rho\left(\frac{f(z+n)}{f(z)}\right)$$

then, we deduce

$$\rho\left(\frac{f(z+n+1)}{f(z)}\right)=\rho\left(\frac{f(z+1)}{f(z)}\right). \tag{4.7}$$

By $(4.5)-(4.7)$, we obtain

$$\begin{aligned}\max\left\{\rho\left(\frac{f(z+n)}{f(z)}\right),\ n\geq 1\right\} &= \max_{n\geq 1}\left\{\rho\left(\frac{f(z+n)}{f(z)}\right),\rho\left(\frac{f(z+n+1)}{f(z)}\right)\right\}\\ &=\rho\left(\frac{f(z+1)}{f(z)}\right).\end{aligned}$$

Proof of Theorem 1.2. Let f be a meromorphic function of order ρ and $c\in\mathbb{C}\setminus\{0\}$. Then, by using Lemmas 3.2-3.3, for any given $\varepsilon>0$, we have

$$\begin{aligned}T\left(r,\frac{f(z+c)}{f(z)}\right) &= m\left(r,\frac{f(z+c)}{f(z)}\right)+N\left(r,\frac{f(z+c)}{f(z)}\right)\\ &\leq O(r^{\rho-1+\varepsilon})+N\left(r,\frac{1}{f(z)}\right)+N(r,f(z+c))\\ &= N\left(r,\frac{1}{f}\right)+N(r,f)+S(r,f)\\ &\leq r^{\lambda(f)+\varepsilon}+r^{\lambda\left(\frac{1}{f}\right)+\varepsilon}+S(r,f)\\ &\leq 2r^{\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}+\varepsilon}+S(r,f).\end{aligned}$$

Then, by Lemma 3.7

$$\rho\left(\frac{f(z+c)}{f(z)}\right)\leq\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}\leq\rho(f)=\rho\left(\frac{f(z+c)}{f(z)}\right).$$

Hence

$$\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}=\rho(f).$$

Proof of Corollary 1.1. Since $\rho(\Delta_c f)<\rho(f)$, then

$$\rho\left(\frac{f(z+c)}{f(z)}\right)=\rho\left(\frac{\Delta_c f}{f}\right)=\rho(f).$$

By Theorem 1.2, we obtain $\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}=\rho(f)$.

Proof of Theorem 1.3. (i) By Hadamard factorization, f can be written as

$$f(z) = \Pi(z)\exp(P_n(z)), \tag{4.8}$$

where Π is the canonical product of zeros and poles of f. Since $\rho(\Pi) = \max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} < \rho(f) - 1$, then $\rho(f)$ is an integer and

$$\rho(f) = \rho(\exp(P_n(z))) = \deg P_n.$$

By the hypothesis of Theorem 1.3 (i), we have

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho(\Pi) < \rho(f) - 1 = \deg P_n - 1 = n - 1.$$

By the same method

$$f(z+c) = \Pi(z+c)\exp(P_n(z+c)),$$

then

$$\frac{f(z+c)}{f(z)} = \frac{\Pi(z+c)}{\Pi(z)}\exp(P_n(z+c) - P_n(z)). \tag{4.9}$$

On the other hand, we have

$$\begin{aligned} P_n(z+c) - P_n(z) &= \left(a_n(z+c)^n + a_{n-1}(z+c)^{n-1} + \cdots\right) \\ &\quad - \left(a_n z^n + a_{n-1}z^{n-1} + \cdots\right) \\ &= \left(a_n z^n + (nca_n + a_{n-1})z^{n-1} + \cdots\right) \\ &\quad - \left(a_n z^n + a_{n-1}z^{n-1} + \cdots\right) = nca_n z^{n-1} + \cdots . \end{aligned}$$

So

$$\deg(P_n(z+c) - P_n(z)) = n - 1. \tag{4.10}$$

By (4.9), (4.10) and since $\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} = \rho\left(\frac{\Pi(z+c)}{\Pi(z)}\right) < \rho(f) - 1$, then we have

$$\rho\left(\frac{f(z+c)}{f(z)}\right) = \deg(P_n(z+c) - P_n(z)) = n - 1.$$

(ii) Suppose that $\rho(f) < 1$. Then, by using Lemma 3.4, for any $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ that depends on f and that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$

$$\frac{\Delta_c f(z)}{f(z)} = c\frac{f'(z)}{f(z)} + O\left(r^{2\rho-2+\varepsilon}\right),$$

which implies

$$T\left(r,\frac{\Delta_c f(z)}{f(z)}\right)=T\left(r,\frac{f'(z)}{f(z)}\right)+S(r,f).$$

Then, by Lemma 3.7

$$\rho\left(\frac{f(z+c)}{f(z)}\right)=\rho\left(\frac{\Delta_c f(z)}{f(z)}\right)=\rho\left(\frac{f'}{f}\right). \tag{4.11}$$

On the other hand, by Lemma 3.6, we have

$$\begin{aligned}T\left(r,\frac{f'}{f}\right) &= m\left(r,\frac{f'}{f}\right)+N\left(r,\frac{f'}{f}\right)\\ &= O(\log r)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\\ &\le 2r^{\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}+\varepsilon}+O(\log r).\end{aligned}$$

So

$$\rho\left(\frac{f'}{f}\right)\le\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}. \tag{4.12}$$

By the same method, we have

$$\overline{N}(r,f)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)=N\left(r,\frac{f'}{f}\right)\le T\left(r,\frac{f'}{f}\right)$$

and

$$\overline{N}\left(r,\frac{1}{f}\right)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)=N\left(r,\frac{f'}{f}\right)\le T\left(r,\frac{f'}{f}\right),$$

which implies

$$\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}\le\rho\left(\frac{f'}{f}\right). \tag{4.13}$$

From $(4.11)-(4.13)$, we get

$$\max\left\{\overline{\lambda}(f),\overline{\lambda}\left(\frac{1}{f}\right)\right\}=\rho\left(\frac{f(z+c)}{f(z)}\right).$$

Proof of Theorem 1.4. By the same reasoning of Theorem 1.3

$$\frac{f(z+c)}{f(z)}=\frac{\Pi(z+c)}{\Pi(z)}\exp\left(g(z+c)-g(z)\right), \tag{4.14}$$

where g is transcendental function of order less than one. By using Lemma 3.5, $\Delta_c g(z) = g(z+c) - g(z)$ is transcendental. Hence

$$\rho\left(\frac{f(z+c)}{f(z)}\right) = \rho(f) = \infty.$$

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