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Growth and Oscillation of Polynomial of Linearly Independent Meromorphic Solutions of Second Order Linear Differential Equations in the Unit Disc

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Abstract. In this paper, we deal with the growth and oscillation of $w = d_1 f_1 + d_2 f_2$, where d_1, d_2 are meromorphic functions of finite iterated $p-$order that are not all vanishing identically and f_1, f_2 are two linearly independent meromorphic solutions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ satisfying $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, of the linear differential equation

$$f'' + A(z) f = 0,$$

where $A(z)$ is admissible meromorphic function of finite iterated $p-$order in Δ.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z : |z| < 1\}$ (see [12, 13, 15, 16, 18, 19]). We need to give some definitions. Firstly, let us give the definition about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (see [10, 11]).

Definition 1. Let f be a meromorphic function in Δ, and

$$D(f) := \limsup_{r \to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = b.$$

If $b < \infty$, we say that f is of finite b degree (or is non-admissible). If $b = \infty$, we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r, f)$.

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Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$ (see [3, 14, 15]). Let us define inductively, for $r \in [0,1)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r$, $\log_0 r = r$, $\exp_{-1} r = \log_1 r$, $\log_{-1} r = \exp_1 r$.

Definition 2 ([4])**.** The iterated $p-$order of a meromorphic function f in Δ is defined as

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log \frac{1}{1-r}}, \quad (p \geq 1),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+(\log_p^+ x)$.

Definition 3 ([4])**.** The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined as

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\left\{p \in \mathbb{N} : \rho_p(f) < \infty\right\}, & \text{if } f \text{ is admissible,} \\ & \text{and } \rho_p(f) < \infty \text{ for some } p \in \mathbb{N}, \\ +\infty, & \text{if } \rho_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 4 ([8])**.** Let f be a meromorphic function in Δ. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $N(r, \frac{1}{f})$ is the counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $\overline{N}(r, \frac{1}{f})$ is the counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$.

Definition 5 ([8])**.** The growth index of the convergence exponent of the sequence of zeros of a meromorphic $f(z)$ in Δ is defined as

$$i_\lambda(f) = \begin{cases} 0, & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min\left\{p \in \mathbb{N} : \lambda_p\left(f\right) < \infty\right\}, & \text{if some } p \in \mathbb{N} \text{ with } \lambda_p(f) < \infty \text{ exists,} \\ +\infty, & \text{if } \lambda_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Remark 1. Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Definition 6 ([8]). For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f in Δ is defined as

$$\delta(a,f) = \liminf_{r\to 1^-} \frac{m\left(r, \frac{1}{f-a}\right)}{T\left(r,f\right)} = 1 - \limsup_{r\to 1^-} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}$$

provided that f has unbounded characteristic.

We consider the linear differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

and the polynomial of solutions

$$w = d_1 f_1 + d_2 f_2, \tag{1.2}$$

where $A(z)$ and $d_j(z)$, $(j = 1,2)$, are finite iterated $p-$order meromorphic functions in Δ. The growth and oscillation theory of complex differential equation (1.1) in the complex plane were firstly investigated by Bank and Laine in 1982-1983 (see [1,2]). After their many authors (see [5,7,8,9,14,15,17]) have investigated the complex differential equation (1.1) in the unit disc Δ and in the complex plane. Recently in [17], the authors have investigated the relations between the polynomial of solutions of (1.1) and small functions in the complex plane. They showed that $w = d_1 f_1 + d_2 f_2$ keeps the same properties of the growth and oscillation of f_j, $(j = 1,2)$, where f_1 and f_2 are two linearly independent solutions of (1.1) and obtained the following results.

Theorem A ([17]). *Let $A(z)$ be a transcendental entire function of finite order. Let $d_j(z)$, $(j = 1,2)$, be finite order entire functions that are not all vanishing identically such that* $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. *If f_1 and f_2 are two linearly independent solutions of* (1.1)*, then the polynomial of solutions* (1.2) *satisfies*

$$\rho(w) = \rho(f_j) = \infty, \quad (j = 1,2)$$

and

$$\rho_2(w) = \rho_2(f_j) = \rho(A), \quad (j = 1,2).$$

Theorem B ([17]). *Under the hypotheses of Theorem A, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that* $\psi(z) = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} +$

$\phi_2\varphi'' + \phi_1\varphi' + \phi_0\varphi \not\equiv 0$, *where*

$$\begin{aligned}
\phi_2 &= \frac{3d_2^2 d_1'' - 3d_1 d_2 d_2''}{h}, \\
\phi_1 &= \frac{2d_1 d_2 d_2' A + 6d_2 d_1' d_2'' - 6d_2 d_2' d_1'' - 2d_2^2 d_1' A}{h}, \\
\phi_0 &= \frac{2d_2 d_1' d_2''' - 2d_1 d_2' d_2''' - 3d_1 d_2 d_2'' A - 3d_2 d_1'' d_2'' + 2d_1 d_2 d_2' A'}{h} \\
&\quad - \frac{4d_2 d_1' d_2' A - 6d_1' d_2' d_2'' + 3d_1 \left(d_2''\right)^2 + 4d_1 \left(d_2'\right)^2 A + 3d_2^2 d_1'' A}{h} \\
&\quad + \frac{6\left(d_2'\right)^2 d_1'' - 2d_2^2 d_1' A'}{h}.
\end{aligned}$$

If f_1 and f_2 are two linearly independent solutions of (1.1)*, then the polynomial of solutions* (1.2) *satisfies*

$$\overline{\lambda}(w-\varphi) = \lambda(w-\varphi) = \rho(f_j) = \infty, \quad (j=1,2)$$

and

$$\overline{\lambda}_2(w-\varphi) = \lambda_2(w-\varphi) = \rho_2(f_j) = \rho(A), \quad (j=1,2).$$

The question which is arises: Can we obtain similar results of Theorems A-B in the unit disc Δ? Thus it is interesting to consider the growth and complex oscillation of the polynomial of solutions of equation (1.1) for the case where $A(z)$ is a meromorphic function in the unit disc Δ in the terms of the idea of iterated order. Before we state our results we define h and ψ by

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix},$$

$$\psi(z) = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2\varphi'' + \phi_1\varphi' + \phi_0\varphi,$$

where $\varphi \not\equiv 0$ is a meromorphic of finite iterated $p-$order in the unit disc Δ and

$$\phi_2 = \frac{3d_2^2 d_1'' - 3d_1 d_2 d_2''}{h}, \tag{1.3}$$

$$\phi_1 = \frac{2d_1 d_2 d_2' A + 6d_2 d_1' d_2'' - 6d_2 d_2' d_1'' - 2d_2^2 d_1' A}{h}, \tag{1.4}$$

$$
\begin{aligned}
\phi_0 = {} & \frac{2d_2d_1'd_2''' - 2d_1d_2'd_2''' - 3d_1d_2d_2''A - 3d_2d_1''d_2'' + 2d_1d_2d_2'A'}{h} \\
& - \frac{4d_2d_1'd_2'A - 6d_1'd_2'd_2'' + 3d_1(d_2'')^2 + 4d_1(d_2')^2A + 3d_2^2d_1''A}{h} \\
& + \frac{6(d_2')^2d_1'' - 2d_2^2d_1'A'}{h}.
\end{aligned} \tag{1.5}
$$

Theorem 1. *Let $A(z)$ be an admissible meromorphic function in Δ such that $i(A) = p$, $(1 \leq p < \infty)$, and $\delta(\infty, A) = \delta > 0$. Let $d_j(z)$, $(j = 1, 2)$, be finite iterated $p-$order meromorphic functions in Δ that are not all vanishing identically such that $\max\{\rho_p(d_1), \rho_p(d_2)\} < \rho_p(A)$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of (1.1) such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, then the polynomial of solutions $w = d_1f_1 + d_2f_2$ satisfies $i(w) = p + 1$,*

$$\rho_p(w) = \rho_p(f_j) = \infty, \quad (j = 1, 2)$$

and

$$\rho_{p+1}(w) = \rho_{p+1}(f_j) = \rho_p(A), \quad (j = 1, 2)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(w) = \rho_{p+1}(f_j) \leq \rho_p(A) + 1, \quad (j = 1, 2)$$

if $p = 1$.

From Theorem 1, we can obtain the following result.

Corollary 1. *Let $f_j(z)$, $(j = 1, 2)$, be two nontrivial linearly independent meromorphic solutions of (1.1) such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, where $A(z)$ is admissible meromorphic function in Δ such that $i(A) = p$, $(1 \leq p < \infty)$, and $\delta(\infty, A) = \delta > 0$, and let $d_j(z)$, $(j = 1, 2, 3)$ be meromorphic functions in Δ satisfying*

$$\max\{\rho_p(d_j) : j = 1, 2, 3\} < \rho_p(A)$$

and

$$d_1(z)f_1 + d_2(z)f_2 = d_3(z).$$

Then $d_j(z) \equiv 0$, $(j = 1, 2, 3)$.

Proof. We suppose there exists $j = 1, 2, 3$ such that $d_j(z) \not\equiv 0$ and we obtain a contradiction. If $d_1(z) \not\equiv 0$ or $d_2(z) \not\equiv 0$, then by Theorem 1 we have $\rho_p(d_1f_1 + d_2f_2) = \infty = \rho_p(d_3) < \rho_p(A)$ which is a contradiction. Now if $d_1(z) \equiv 0$, $d_2(z) \equiv 0$ and $d_3(z) \not\equiv 0$ we obtain also a contradiction. Hence $d_j(z) \equiv 0$, $(j = 1, 2, 3)$. □

Theorem 2. *Under the assumptions of Theorem 1, let $\varphi(z) \not\equiv 0$ be a meromorphic function in Δ with finite iterated $p-$order such that $\psi(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of* (1.1) *such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, then the polynomial of solutions $w = d_1 f_1 + d_2 f_2$ satisfies*

$$\overline{\lambda}_p(w - \varphi) = \lambda_p(w - \varphi) = \rho_p(w) = \infty \tag{1.6}$$

and

$$\overline{\lambda}_{p+1}(w - \varphi) = \lambda_{p+1}(w - \varphi) = \rho_{p+1}(w) = \rho_p(A)$$

if $p > 1$, while

$$\rho_p(A) \leq \overline{\lambda}_{p+1}(w - \varphi) = \lambda_{p+1}(w - \varphi) = \rho_{p+1}(w) \leq \rho_p(A) + 1 \tag{1.7}$$

if $p = 1$.

Theorem 3. *Let $A(z)$ be an admissible meromorphic function in Δ such that $i(A) = p$, $(1 \leq p < \infty)$, and $\delta(\infty, A) = \delta > 0$. Let $d_j(z), b_j(z)$, $(j = 1, 2)$, be finite iterated $p-$order meromorphic functions in Δ such that $d_1(z)b_2(z) - d_2(z)b_1(z) \not\equiv 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of* (1.1) *such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, then*

$$i\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = p + 1,$$

$$\rho_p\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \infty$$

and

$$\rho_{p+1}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) = \rho_p(A)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}\left(\frac{d_1 f_1 + d_2 f_2}{b_1 f_1 + b_2 f_2}\right) \leq \rho_p(A) + 1$$

if $p = 1$.

2. Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 1 ([4])**.** *If f and g are meromorphic functions in Δ, $p \geq 1$ is an integer, then we have*

(i) $\rho_p(f) = \rho_p(1/f)$, $\rho_p(a.f) = \rho_p(f)$, $(a \in \mathbb{C} - \{0\})$;
(ii) $\rho_p(f) = \rho_p(f')$;
(iii) $\max\{\rho_p(f + g), \rho_p(fg)\} \leq \max\{\rho_p(f), \rho_p(g)\}$;
(iv) *if $\rho_p(f) < \rho_p(g)$, then $\rho_p(f + g) = \rho_p(g)$, $\rho_p(fg) = \rho_p(g)$.*

Lemma 2 ([8]). *Let $A(z)$ be an admissible meromorphic function in Δ such that $i(A) = p$, $(1 \leq p < \infty)$, and $\delta(\infty, A) = \delta > 0$, and let f be a nonzero meromorphic solution of* (1.1). *If $\delta(\infty, f) > 0$, then $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A)$ if $p > 1$, while*

$$\rho_p(A) \leq \rho_{p+1}(f) \leq \rho_p(A) + 1$$

if $p = 1$.

Lemma 3. *Let $A(z)$ be an admissible meromorphic function in Δ such that $i(A) = p$, $(1 \leq p < \infty)$, and $\delta(\infty, A) > 0$. If f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of* (1.1) *such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, then $\frac{f_1}{f_2}$ satisfies $i\left(\frac{f_1}{f_2}\right) = p + 1$ and $\rho_{p+1}\left(\frac{f_1}{f_2}\right) = \rho_p(A)$ if $p > 1$, while*

$$\rho_p(A) \leq \rho_{p+1}\left(\frac{f_1}{f_2}\right) \leq \rho_p(A) + 1$$

if $p = 1$.

Proof. Suppose that f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of (1.1) such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$. Then by Lemma 2, we have $i(f_j) = p + 1$, $\rho_p(f_j) = \infty$, $(j = 1, 2)$, and

$$\rho_{p+1}(f_j) = \rho_p(A), \quad (j = 1, 2) \tag{2.1}$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(f_j) \leq \rho_p(A) + 1, \quad (j = 1, 2) \tag{2.2}$$

if $p = 1$. On the other hand, we have (see, [15])

$$\left(\frac{f_1}{f_2}\right)' = -\frac{W(f_1, f_2)}{f_2^2} = -\frac{c}{f_2^2}, \tag{2.3}$$

where $W(f_1, f_2) = f_1 f_2' - f_2 f_1' = c \neq 0$ is the Wronskian of f_1 and f_2. By Lemma 1, (2.1), (2.2) and (2.3) we obtain that $i\left(\frac{f_1}{f_2}\right) = p+1$, $\rho_p\left(\frac{f_1}{f_2}\right) = \infty$ and

$$\rho_{p+1}\left(\frac{f_1}{f_2}\right) = \rho_p(A)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}\left(\frac{f_1}{f_2}\right) \leq \rho_p(A) + 1$$

if $p = 1$. □

Lemma 4 ([6]). *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions in Δ, and let f be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z)$$

such that

$$\max\{\rho_p(A_j) \;\; (j = 0, 1, \ldots, k-1), \rho_p(F)\} < \rho_p(f) \leq +\infty.$$

Then

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$$

and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).$$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. In the case when $d_1(z) \equiv 0$ or $d_2(z) \equiv 0$, then the conclusions of Theorem 1 are trivial. Suppose that f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of (1.1) such that $\delta(\infty, f_j) > 0$, $(j = 1, 2)$, and $d_j(z) \not\equiv 0$, $(j = 1, 2)$. Then by Lemma 2, we have $i(f_j) = p + 1$, $\rho_p(f_j) = \infty$, $(j = 1, 2)$, and

$$\rho_{p+1}(f_j) = \rho_p(A), \quad (j = 1, 2)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(f_j) \leq \rho_p(A) + 1, \quad (j = 1, 2)$$

if $p = 1$. Suppose that $d_1 = cd_2$, where c is a complex number. Then, by (1.2) we obtain

$$w = cd_2f_1 + d_2f_2 = (cf_1 + f_2)\,d_2.$$

Since $f = cf_1 + f_2$ is a solution of (1.1) and $\rho_p(d_2) < \rho_p(A)$, then we have

$$\rho_p(w) = \rho_p(cf_1 + f_2) = \infty$$

and

$$\rho_{p+1}(w) = \rho_{p+1}(cf_1 + f_2) = \rho_p(A)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(w) = \rho_{p+1}(cf_1 + f_2) \leq \rho_p(A) + 1$$

if $p = 1$. Suppose now that $d_1 \not\equiv cd_2$ where c is a complex number. Differentiating both sides of (1.2), we obtain

$$w' = d_1'f_1 + d_1f_1' + d_2'f_2 + d_2f_2'. \tag{3.1}$$

Differentiating both sides of (3.1), we obtain

$$w'' = d_1''f_1 + 2d_1'f_1' + d_1f_1'' + d_2''f_2 + 2d_2'f_2' + d_2f_2''. \tag{3.2}$$

Substituting $f_j'' = -Af_j$, $(j = 1, 2)$, into equation (3.2), we have

$$w'' = \left(d_1'' - d_1A\right)f_1 + 2d_1'f_1' + \left(d_2' - d_2A\right)f_2 + 2d_2'f_2'. \tag{3.3}$$

Differentiating both sides of (3.3) and by substituting $f_j'' = -Af_j$, $(j = 1, 2)$, we obtain

$$\begin{aligned} w''' = {} & \left(d_1''' - 3d_1'A - d_1A'\right)f_1 + \left(d_1'' - d_1A + 2d_1''\right)f_1' \\ & + \left(d_2''' - 3d_2'A - d_2A'\right)f_2 + \left(d_2'' - d_2A + 2d_2''\right)f_2'. \end{aligned} \tag{3.4}$$

By (1.2), (3.1), (3.3) and (3.4) we have

$$\begin{aligned} w &= d_1 f_1 + d_2 f_2, \\ w' &= d_1' f_1 + d_1 f_1' + d_2' f_2 + d_2 f_2', \\ w'' &= \left(d_1'' - d_1 A\right) f_1 + 2d_1' f_1' + \left(d_2'' - d_2 A\right) f_2 + 2d_2' f_2', \\ w''' &= \left(d_1''' - 3d_1' A - d_1 A'\right) f_1 + \left(d_1'' - d_1 A + 2d_1''\right) f_1' \\ &\quad + \left(d_2''' - 3d_2' A - d_2 A'\right) f_2 + \left(d_2'' - d_2 A + 2d_2''\right) f_2'. \end{aligned} \tag{3.5}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations we obtain

(3.6)

$$\begin{aligned} h &= \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 A & 2d_2' \\ d_1''' - 3d_1' A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix} \\ &= \left(4d_1^2 (d_2')^2 + 4d_2^2 (d_1')^2 - 8d_1 d_2 d_1' d_2'\right) A + 2d_1 d_2 d_1' d_2''' + 2d_1 d_2 d_2' d_1''' - 6d_1 d_2 d_1'' d_2'' \\ &\quad - 6d_1 d_1' d_2' d_2'' - 6d_2 d_1' d_2' d_1'' + 6d_1 (d_2')^2 d_1'' + 6d_2 (d_1')^2 d_2'' - 2d_2^2 d_1' d_1''' \\ &\quad - 2d_1^2 d_2' d_2''' + 3d_1^2 (d_2'')^2 + 3d_2^2 (d_1'')^2. \end{aligned}$$

To show that $4d_1^2 (d_2')^2 + 4d_2^2 (d_1')^2 - 8d_1 d_2 d_1' d_2' \not\equiv 0$, we suppose that

$$d_1^2 (d_2')^2 + d_2^2 (d_1')^2 - 2d_1 d_2 d_1' d_2' = 0. \tag{3.7}$$

Dividing both sides of (3.7) by $(d_1 d_2)^2$, we obtain

$$\left(\frac{d_2'}{d_2}\right)^2 + \left(\frac{d_1'}{d_1}\right)^2 - 2\frac{d_1'}{d_1}\frac{d_2'}{d_2} = 0 \tag{3.8}$$

equivalent to

$$\left(\frac{d_1'}{d_1} - \frac{d_2'}{d_2}\right)^2 = 0, \tag{3.9}$$

which implies that $d_1 = c d_2$ where c is a complex number which is a contradiction. Since $\max\{\rho_p(d_1), \rho_p(d_2)\} < \rho_p(A)$ and $4d_1^2 (d_2')^2 + 4d_2^2 (d_1')^2 - 8d_1 d_2 d_1' d_2' \not\equiv 0$, then by Lemma 1 we can deduce from (3.6) that $\rho_p(h) = \rho_p(A) > 0$. Hence $h \not\equiv 0$. By Cramer's method we have

(3.10)

$$\begin{aligned} f_1 &= \frac{\begin{vmatrix} w & 0 & d_2 & 0 \\ w' & d_1 & d_2' & d_2 \\ w'' & 2d_1' & d_2'' - d_2 A & 2d_2' \\ w''' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2' A - d_2 A' & d_2'' - d_2 A + 2d_2'' \end{vmatrix}}{h} \\ &= \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w, \end{aligned}$$

where ϕ_j, $(j=0,1,2)$, are meromorphic functions in Δ of finite iterated $p-$order which are defined in (1.3)–(1.5). Suppose now $\rho_p(w)<\infty$, then by (3.10) we obtain $\rho_p(f_1)<\infty$ which is a contradiction. Hence $\rho_p(w)=\infty$. By (1.2) we have $\rho_{p+1}(w)\leq\rho_{p+1}(f_1)$. Suppose that $\rho_{p+1}(w)<\rho_{p+1}(f_1)$, then by (3.10) we obtain $\rho_{p+1}(f_1)\leq\rho_{p+1}(w)$ which is a contradiction. Hence $\rho_{p+1}(w)=\rho_{p+1}(f_1)$. □

Proof of Theorem 2. By Theorem 1 we have $\rho_p(w)=\infty$ and $\rho_{p+1}(w)=\rho_p(A)$ if $p>1$, while

$$\rho_p(A)\leq\rho_{p+1}(w)\leq\rho_p(A)+1$$

if $p=1$. Set $g(z)=d_1f_1+d_2f_2-\varphi$. Since $\rho_p(\varphi)<\infty$, then we have $\rho_p(g)=\rho_p(w)=\infty$ and $\rho_{p+1}(g)=\rho_{p+1}(w)$. In order to prove $\overline{\lambda}_p(w-\varphi)=\lambda_p(w-\varphi)=\rho_p(w)=\infty$, $\overline{\lambda}_{p+1}(w-\varphi)=\lambda_{p+1}(w-\varphi)=\rho_{p+1}(w)$ we need to prove only $\overline{\lambda}_p(g)=\lambda_p(g)=\rho_p(w)=\infty$, $\overline{\lambda}_{p+1}(g)=\lambda_{p+1}(g)=\rho_{p+1}(w)$. By $w=g+\varphi$ we get from (3.10)

$$f_1=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}g^{(3)}+\phi_2g''+\phi_1g'+\phi_0g+\psi, \tag{3.11}$$

where $\psi=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\varphi^{(3)}+\phi_2\varphi''+\phi_1\varphi'+\phi_0\varphi$. Substituting (3.11) into equation (1.1), we obtain

$$\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}g^{(5)}+\sum_{j=0}^{4}\beta_jg^{(j)}=-\left(\psi''+A\psi\right)=B,$$

where β_j, $(j=0,\ldots,4)$ are meromorphic functions in Δ of finite iterated $p-$order. Since $\psi\not\equiv0$ and $\rho_p(\psi)<\infty$, it follows that ψ is not a solution of (1.1), which implies that $B\not\equiv0$. Then, by applying Lemma 4 we obtain (1.6) and (1.7). □

Proof of Theorem 3. Suppose that f_1 and f_2 are two nontrivial linearly independent meromorphic solutions of (1.1) such that $\delta(\infty,f_j)>0$, $(j=1,2)$. Then by Lemma 3, we have $i\left(\frac{f_1}{f_2}\right)=p+1$, $\rho_p\left(\frac{f_1}{f_2}\right)=\infty$ and $\rho_{p+1}\left(\frac{f_1}{f_2}\right)=\rho_p(A)$ if $p>1$, while

$$\rho_p(A)\leq\rho_{p+1}\left(\frac{f_1}{f_2}\right)\leq\rho_p(A)+1$$

if $p=1$. Set $g=\frac{f_1}{f_2}$. Then

$$w(z)=\frac{d_1(z)f_1(z)+d_2(z)f_2(z)}{b_1(z)f_1(z)+b_1(z)f_2(z)}=\frac{d_1(z)g(z)+d_2(z)}{b_1(z)g(z)+b_2(z)}. \tag{3.12}$$

It follows that $i(w)\leq p+1$ and

$$\rho_{p+1}(w)\leq\max\{\rho_{p+1}(d_j),\rho_{p+1}(b_j)\,(j=1,2),\rho_{p+1}(g)\}=\rho_{p+1}(g). \tag{3.13}$$

On the other hand, we have

$$g(z) = -\frac{b_2(z)\,w(z) - d_2(z)}{b_1(z)\,w(z) - d_1(z)},$$

which implies that $i(w) \geq p+1$ and

$$\begin{aligned} &\rho_p(w) \geq \rho_p(g) = \infty, \\ &\rho_{p+1}(g) \leq \max\Big\{\rho_{p+1}(d_j), \rho_{p+1}(b_j)\ \ (j=1,2), \rho_{p+1}(w)\Big\} \\ &\qquad = \rho_{p+1}(w). \end{aligned} \tag{3.14}$$

By using (3.13) and (3.14), we obtain $i(w) = p+1$ and

$$\rho_p(w) = \rho_p(g) = \infty,$$
$$\rho_{p+1}(w) = \rho_{p+1}(g) = \rho_p(A)$$

if $p > 1$, while

$$\rho_p(A) \leq \rho_{p+1}(w) = \rho_{p+1}(g) \leq \rho_p(A) + 1$$

if $p = 1$. □

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