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Symmetry and uniqueness of positive solutions for a Neumann boundary value problem

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Abstract

This work deals with the existence and symmetry of positive solutions for a Neumann boundary value problem. It is a generalization of the work of Pedro J. Torres. The main result is the uniqueness of positive solutions, which is proved by an analytical method, for a given interval of the positive parameter q.

Keywords: Krasnoselskii theorem; Neumann conditions; Green's function; Symmetry; Uniqueness and positivity of solutions

1. Introduction

In this work we are concerned with the existence, symmetry and uniqueness of positive solutions for the following problem:

$$(P_p)\begin{cases} Lu \equiv -u'' + q^2 u = |u|^p f(x), & x \in]a, b[\\ u'(a) = 0 = u'(b), \end{cases}$$

where $p > 1$, $q > 0$, $0 \le a < b \le \pi$, and f is a continuous positive symmetric function on $[a, b]$.

For $f(x) = 1 + \sin x$ and $[a, b] = [0, \pi]$, Mays and Norbury [2] have considered the problem (P_2) arising in fluid dynamics. They have proved numerically the existence of positive solutions if $q^2 \in]0, 10[$. Torres [4] has confirmed analytically the results of [2] by using a fixed-point Theorem for Krasnoselskii operators [1]; he also proved the symmetry of the solutions. We remark that the analytical proof of uniqueness of positive solutions for (P_2) remains an open problem. It is strongly suggested numerically [2] on a given range of values of the parameter q.

In this work we generalize the work of [4] by considering the term $|u|^p f(x)$ instead of $u^2(1 + \sin x)$ and we give a uniqueness result. The work is organized as follows. In Section 2 we give an existence result, Section 3 is concerned with the properties of the solutions and the last one contains a uniqueness result.

In this work we use the following notation. $\|u\|_0 = \sup\{|u(x)|,\ x \in [a, b]\}$, and $\|u\|_\gamma = \left(\int_a^b |u(x)|^\gamma \mathrm{d}x\right)^{\frac{1}{\gamma}}$ for $\gamma \ge 1$.

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The proof of the existence of positive solutions is based on the following theorems.

Theorem 1 ([1]). *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous, where P is a cone of E, and such that one of the following conditions is satisfied:*

$$(1)\ \|Ax\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1 \quad \text{and} \quad \|Ax\| \ge \|x\|, \quad \forall x \in P \cap \partial\Omega_2;$$
$$(2)\ \|Ax\| \ge \|x\|, \quad \forall x \in P \cap \partial\Omega_1 \quad \text{and} \quad \|Ax\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_2.$$

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2 ([3]). *Let $C(K, \mathbb{R})$ be the space of continuous functions on the compact set $K \subset \mathbb{R}^n$. Then a subset $S \subset C(K, \mathbb{R})$ is precompact if and only if the functions of S are uniformly bounded and equicontinuous.*

2. Existence result

Theorem 3. *Assume that f is a positive continuous function on $[a, b]$. Then the problem (P_p) has at least one positive solution for any positive q and any $p > 1$.*

Proof. As was observed in [2], the Green's function $G(x, y)$ of the operator L, with the Neumann conditions, is a positive and continuous function on $[a, b] \times [a, b]$. Thus the problem (P_p) can be written as the fixed-point problem

$$u(x) = \int_a^b G(x, y)\, |u(y)|^p f(y)\, dy \equiv Au(x).$$

Define

$$m = \min\{G(x, y);\, (x, y) \in [a, b] \times [a, b]\}, \qquad M = \max\{G(x, y);\, (x, y) \in [a, b] \times [a, b]\},$$
$$\alpha = \min\{f(x);\, a \le x \le b\}, \qquad \beta = \max\{f(x);\, a \le x \le b\}, \qquad l = b - a;$$

then m, α and l are positive.

Now consider the Banach space $E = C([a, b])$ endowed with the norm $\|.\|_0$, and define the cone

$$P = \left\{u \in E : \min_{a \le x \le b} u(x) \ge \frac{m}{M}\|u\|_0\right\}.$$

We start by proving that $AP \subset P$.

For any given $u \in P$, we have

$$\begin{aligned} Au(x) &\ge m \int_a^b u^p(y)\, f(y)\, dy \\ &\ge \frac{m}{M} \int_a^b G(s, y)\, u^p(y)\, f(y)\, dy \\ &= \frac{m}{M} Au(s), \quad \text{for all } x, s \in [a, b], \end{aligned}$$

so

$$\min_{a \le x \le b} Au(x) \ge \frac{m}{M}\|Au\|_0,$$

and then $Au \in P$.

Now let us prove that $A : P \to P$ is completely continuous. For any fixed $u_0 \in P$, and any $u \in P$, by the mean-value theorem, we obtain

$$|Au(x) - Au_0(x)| \le p \int_a^b |u(y) - u_0(y)|\, G(x, y)\, f(y)\, (v(y))^{p-1}\, dy, \quad \forall x \in [a, b],$$

where the real number $v(y)$ is between $u(y)$ and $u_0(y)$.

Thus

$$\|Au - Au_0\|_0 \leq p\beta M \|u - u_0\|_0 \int_a^b (v(y))^{p-1} \, dy,$$

which proves that the operator A is continuous on P.

Let $(u_n)_n$ be a bounded sequence in P, that is,

$$\exists C > 0, \quad \|u_n\|_0 \leq C, \quad \forall n \in \mathbb{N}.$$

Let us prove that the set $S := \{Au_n, \ n \in \mathbb{N}\}$ is precompact.

First we verify that the functions Au_n are uniformly bounded. For any $x \in [a, b]$, and any $n \in \mathbb{N}$, we have

$$Au_n(x) = \int_a^b G(x, y) u_n^p(y) f(y) \, dy \leq \beta l M C^p,$$

that is $\|Au_n\|_0 \leq \beta l M C^p$, $\forall n \in \mathbb{N}$.

Now we prove that the functions Au_n are equicontinuous.

For any x_1 and x_2 fixed in $[a, b]$,

$$|Au_n(x_1) - Au_n(x_2)| = \left| \int_a^b (G(x_1, y) - G(x_2, y)) u_n^p(y) f(y) \, dy \right|,$$

and we remark that for any y fixed in $[a, b]$, the function $x \longmapsto G(x, y)$ is uniformly continuous in $[a, b]$, i.e.

$$\forall \varepsilon > 0, \quad \exists \delta(\varepsilon, y) > 0 : |x_1 - x_2| < \delta \Rightarrow |G(x_1, y) - G(x_2, y)| < \varepsilon$$

and since $y \in [a, b]$ which is a compact set, there exists $\delta(\varepsilon) > 0$ independent of y, such that for any given $\varepsilon > 0$,

$$\forall x_1, \ x_2 \in [a, b] : |x_1 - x_2| < \delta(\varepsilon) \Rightarrow |G(x_1, y) - G(x_2, y)| < \varepsilon$$

and then

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : |x_1 - x_2| < \delta \Rightarrow |Au_n(x_1) - Au_n(x_2)| < \beta l C^p \varepsilon, \quad \forall n \in \mathbb{N}$$

which confirms that the functions Au_n are equicontinuous, and in consequence of Theorem 2 the set S is precompact, and so the operator A is completely continuous.

In order to apply Theorem 1, we consider the open balls

$$\Omega_1 = \{u \in E, \ \|u\|_0 < r_1\} \quad \text{and} \quad \Omega_2 = \{u \in E, \ \|u\|_0 < r_2\}$$

where

$$r_1 = (\beta l M)^{-\frac{1}{p-1}} \quad \text{and} \quad r_2 = \left(\frac{M^p}{\alpha l m^{p+1}} \right)^{\frac{1}{p-1}}.$$

Clearly $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ because $r_1 < r_2$.

Now, if $u \in P \cap \partial\Omega_1$, we get

$$\|Au\|_0 \leq \beta l M \|u\|_0^p = \|u\|_0$$

and if $u \in P \cap \partial\Omega_2$,

$$\begin{aligned} \|Au\|_0 &\geq \alpha m \int_a^b u^p(y) \, dy \\ &\geq \alpha m l \left(\frac{m}{M} \|u\|_0 \right)^p = \frac{\alpha l m^{p+1}}{M^p} \|u\|_0^p = \|u\|_0, \end{aligned}$$

then the operator A satisfies condition (1) of Theorem 1.

Therefore it has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and so the problem (P_p) has at least a positive solution for any positive q. $\square$

Remark that the solution u of our problem satisfies the inequalities

$$m\left(\beta l M^{p}\right)^{-\frac{1}{p-1}} \leq u(x) \leq \left(\alpha l m^{p+1} M^{-p}\right)^{-\frac{1}{p-1}}, \quad \forall x \in [a,b].$$

3. Uniform upper bound and symmetry of the solutions

At the beginning of this section, we deduce the uniform upper bound for every positive solution of the problem (P_p).

Theorem 4. *Assume that f is a positive continuous function on $[a,b]$. Then there exists a constant $C_q := \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}}\left(1+(ql)^2\right)$, such that any positive solution of the problem (P_p) verifies*

$$u(x) < C_q, \quad \forall x \in [a,b].$$

Proof. Let u be a positive solution of the problem. Integrating the equation of (P_p), we obtain

$$q^2\|u\|_1 = \int_a^b u^p(y) f(y)\,\mathrm{d}y \geq \alpha \|u\|_p^p,$$

and using the Hölder inequality, we can write

$$\|u\|_1 = \int_a^b u(x)\,\mathrm{d}x \leq \left[\int_a^b \mathrm{d}x\right]^{\frac{p-1}{p}} \left[\int_a^b u^p(x)\,\mathrm{d}x\right]^{\frac{1}{p}} = l^{\frac{p-1}{p}} \|u\|_p,$$

and then

$$\|u\|_p \leq \left[\frac{q^2}{\alpha}\right]^{\frac{1}{p-1}} l^{\frac{1}{p}} \quad \text{and} \quad \|u\|_1 \leq \left[\frac{q^2}{\alpha}\right]^{\frac{1}{p-1}} l.$$

Moreover, for any $x \in]a,b[$,

$$\begin{aligned} u'(x) &= \int_a^x u''(s)\,\mathrm{d}s = \int_a^x \left(q^2 u(s) - u^p(s) f(s)\right)\mathrm{d}s \\ &< q^2\|u\|_1 \leq q^2 \left[\frac{q^2}{\alpha}\right]^{\frac{1}{p-1}} l, \\ -u'(x) &= \int_x^b u''(s)\,\mathrm{d}s = \int_x^b \left(q^2 u(s) - u^p(s) f(s)\right)\mathrm{d}s \\ &< q^2\|u\|_1 \leq q^2 \left[\frac{q^2}{\alpha}\right]^{\frac{1}{p-1}} l, \end{aligned}$$

so

$$\|u'\|_0 < q^2 \left[\frac{q^2}{\alpha}\right]^{\frac{1}{p-1}} l.$$

On the other hand

$$u'(a) = u'(b) \Rightarrow \exists x_0 \in]a,b[: u''(x_0) = 0,$$

and then from the equation,

$$u^{p-1}(x_0) = \frac{q^2}{f(x_0)},$$

which gives us

$$\left(\frac{q^2}{\beta}\right)^{\frac{1}{p-1}} \leq u\left(x_0\right) \leq \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}}.$$

We can deduce the constant C_q:

$$u(x) = u\left(x_0\right) + \int_{x_0}^{x} u'(s)\,\mathrm{d}s < \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}} + (ql)^2 \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}}.$$

Then

$$C_q \equiv \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}} \left(1 + (ql)^2\right). \quad \square$$

The constant C_q will be used to prove the symmetry of the positive solutions.

Theorem 5. *Assume that f is a positive, symmetric and continuous function on $[a, b]$ and the positive parameter q satisfies the following inequality:*

$$p\beta\frac{q^2}{\alpha}\left(1 + q^2 l^2\right)^{p-1} < 1 + q^2; \tag{3.1}$$

then any positive solution of the problem $\left(P_p\right)$ is symmetric.

Proof. We follow along the lines of [4]. Let u_1 be a positive solution; then u_2 such that $u_2(x) = u_1(a + b - x)$ is also a solution, because f is symmetric.

Let us prove that $u_1 \equiv u_2$. Define $z = u_1 - u_2$; then z is a solution of the problem

$$\begin{cases} z'' + g(x)\,z = 0, \\ z'(a) = 0 = z'(b), \end{cases} \tag{3.2}$$

where $g(x) = pf(x)(w(x))^{p-1} - q^2$, and the real number $w(x)$ is between $u_1(x)$ and $u_2(x)$ and such that

$$u_1^p(x) - u_2^p(x) = p\,(w(x))^{p-1}\left(u_1(x) - u_2(x)\right).$$

Using the fact that u_1 and u_2 are strictly less than C_q and the condition (3.1), we verify that

$$g(x) < 1, \quad \forall x \in [a, b]. \tag{3.3}$$

Our purpose is to prove that $z \equiv 0$. Suppose that z is not a trivial solution and let us change to polar coordinates:

$$z = r\cos\theta, \qquad z' = -r\sin\theta, \quad r > 0,\ 0 \leq \theta < 2\pi.$$

By deriving z and z', we get

$$z' = r'\cos\theta - r\theta'\sin\theta = -r\sin\theta,$$

and

$$z'' = -r'\sin\theta - r\theta'\cos\theta = -g(x)\,r\cos\theta.$$

From these equations, we obtain

$$\theta' = g(x)\cos^2\theta + \sin^2\theta. \tag{3.4}$$

Integrating (3.4) in the interval $[a, x]$, $a < x \leq b$, and using (3.3) we get

$$\theta(x) - \theta(a) = \int_a^x g(s)\cos^2\theta(s) + \sin^2\theta(s)\,\mathrm{d}s < \int_a^x \mathrm{d}s = x - a. \tag{3.5}$$

Now remark that

$$z(x) = -z(a + b - x),$$

and therefore

$$z\left(\frac{a+b}{2}\right)=0.$$

By using the Sturm comparison theorem with the equation

$$z''+\left(\frac{\pi}{l}\right)^2 z=0,$$

which admits the solution

$$z_0(x)=\sin\frac{\pi}{l}(x-a),$$

we deduce that $\frac{a+b}{2}$ is the unique zero of z in the interval $[a,b]$.

The solution z is supposed not identically zero and $z(a)=-z(b)$; then

$$z(a)\,z(b)<0.$$

Assume that $z(a)>0$; then from $z'(a)=0$, we get

$$\theta(a)=0.$$

On the other hand,

$$z\left(\frac{a+b}{2}\right)=0,$$

and so,

$$\theta\left(\frac{a+b}{2}\right)=\frac{\pi}{2},\quad \text{or}\quad \theta\left(\frac{a+b}{2}\right)=\frac{3\pi}{2}.$$

Now, using (3.5) we get

$$\pi<b-a,\quad \text{or}\quad 3\pi<b-a;$$

this is a contradiction. Then $z\equiv 0$, and therefore

$$u_1(x)=u_1(a+b-x),\quad \forall x\in[a,b].\quad \square$$

4. Uniqueness result for the positive solution

Let λ_1 be the first positive eigenvalue of the following problem with Neumann boundary conditions:

$$\begin{cases}-u''=\lambda u, & x\in\,]a,b[\\ u'(a)=u'(b)=0.\end{cases}$$

Theorem 6. *Under the hypothesis for the function f, and if the positive parameter q satisfies the relation*

$$\lambda_1+q^2-p\frac{\beta}{\alpha}q^2\left(1+q^2l^2\right)^{p-1}>0, \tag{4.1}$$

then the problem (P_p) admits a unique positive solution.

Proof. Let u_1 and u_2 be two positive solutions of the problem (P_p). Then, if we put $v=u_1-u_2$, we get the following problem:

$$\begin{aligned}&-v''+q^2v=\left(u_1^p(x)-u_2^p(x)\right)f(x), \quad x\in\,]a,b[,\\ &v'(a)=v'(b)=0.\end{aligned} \tag{4.2}$$

Now by the mean-value theorem, there exists a real number $w(x)$ between $u_1(x)$ and $u_2(x)$ such that

$$u_1^p(x)-u_2^p(x)=pw^{p-1}(x)\,(u_1(x)-u_2(x)).$$

Then the problem (4.2) becomes

$$\begin{aligned} &-v'' + \left(q^2 - pw^{p-1}(x) f(x)\right) v = 0, \quad x \in]a, b[\\ &v'(a) = v'(b) = 0. \end{aligned} \tag{4.3}$$

Note that the function $x \longmapsto w(x)$ is continuous in $[a, b]$. We can define it by

$$w^{p-1}(x) = \frac{u_1^p(x) - u_2^p(x)}{p(u_1(x) - u_2(x))}, \quad \text{if } u_1(x) \neq u_2(x),$$
$$w(x) = u_1(x), \quad \text{if } u_1(x) = u_2(x).$$

Return to the last problem and put $h(x) = q^2 - pw^{p-1}(x) f(x)$; then

$$q^2 - p\beta C_q^{p-1} \leq h(x) \leq q^2, \quad \forall x \in [a, b], \tag{4.4}$$

where $C_q = \left(\frac{q^2}{\alpha}\right)^{\frac{1}{p-1}} \left(1 + q^2 l^2\right)$.

Multiplying the equation of the problem (4.3) by v and integrating in the interval $[a, b]$, we obtain

$$\int_a^b \left(v'(x)\right)^2 dx + \int_a^b h(x) v^2(x) dx = 0.$$

Now using the characterization of λ_1, we know, if $I =]a, b[$, that

$$\lambda_1 = \inf\left\{\int_I \left(v'(x)\right)^2 dx : v \in H^1(I),\ v'(a) = 0 \text{ and } \int_I v^2 dx = 1\right\}.$$

In fact $\lambda_1 = \frac{\pi^2}{l^2}$, and it is attained by the function v_1:

$$v_1(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{\pi}{l}(x - a)\right).$$

From the characterization of λ_1, we have

$$\lambda_1 \int_I v^2 dx \leq \int_I \left(v'\right)^2 dx,$$

and then

$$\int_I (\lambda_1 + h(x)) v^2 dx \leq \int_I \left(v'\right)^2 dx + \int_I h(x) v^2 dx = 0.$$

Hence $v(x) = 0$, $\forall x \in I$, i.e. $u_1 \equiv u_2$ if $\lambda_1 + h(x) > 0$, $\forall x \in I$, but this is satisfied from (4.1) and (4.4). □

Application

For the particular case [4]:

$$p = 2, \qquad f(x) = 1 + \sin x, \quad \text{and} \quad (a, b) = (0, \pi),$$

and then

$$\alpha = 1, \qquad \beta = 2, \qquad l = \pi, \quad \text{and} \quad \lambda_1 = 1.$$

By Theorem 6, this problem admits a unique positive solution if

$$4\pi^2 q^4 + 3q^2 - 1 < 0,$$

which means if

$$q \in]0, 0,354446\ldots[.$$

And this is the same range of values of the parameter q for which, the solution is symmetric.

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