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NONHOMOGENEOUS ELLIPTIC EQUATIONS WITH DECAYING CYLINDRICAL POTENTIAL AND C R – I TICAL EXPONENT

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ABSTRACT . We prove the existence and multiplicity of solutions for a nonho - mogeneous elliptic equation involving decaying cylindrical potential and criti - cal exponent .

1 . INTRODUCTION

In this article , we consider the problem

$$-\mathrm{div}(|\, y\,|^{-2a}\ \nabla u)-\mu\,|\, y\,|^{-2(a+1)}\ u=uh_{\in}^{|y|^{-2_*b}|u|^{2_*-2}u}{}_{\mathcal{D}_0^{1,2}},+\lambda g\quad \mathrm{in}\mathbb{R}^N,\quad y\neq 0 \tag{1.1}$$

where each point in $\mathbb{R}^N$ is written as a pair $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}, k$ and N are integers such that $N\geq 3$ and k belongs to $\{1,...,N\}; -\infty<a<(k-2)/2; a\leq b<a+1;$

$2_*=2N/(N-2+2(b-a)); -\infty<\mu<\bar{\mu}a,k:=((k-2(a+1))/2)^2; g\in\mathcal{H}'_\mu\cap C(\mathbb{R}^N);$

h is a bounded positive function on $\mathbb{R}^k$ and λ is real parameter . Here $\mathcal{H}'_\mu$ is the dual of $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ and $\mathcal{D}_0^{1,2}$ will be defined later .

Some results are already available for (1 . 1) in the case $k=N$; see for example [1 0 , 1 1] and the references therein . Wang and Zhou [1 0] proved that there exist at least two solutions for (1 . 1) with $a=0$, $0<\mu\leq\bar{\mu}0,N=$ $((N-2)/2)^2$ and $h\equiv$ 1, under certain conditions on g. Bouchekif and Matallah [2] showed the existence of two solutions of (1 . 1) under certain conditions on functions g and h, when $0<\mu\leq\bar{\mu}0,N,\lambda\in(0,\Lambda_*), -\infty<a<(N-2)/2$ and $a\leq b<a+1$, with Λ_* a positive constant .

Concerning existence results in the case $k<N$, we cite [6 , 7] and the references therein . Musina [7] considered (1 . 1) with $-a/2$ instead of a and $\lambda=0$, also (1 . 1) with $a=0,b=0,\lambda=0$, with $h\equiv 1$ and $a\neq 2-k$. She established the existence of a ground state solution when $2<k\leq N$ and $0<\mu<\bar{\mu}a,k=((k-2+a)/2)^2$ for (1 . 1) with $-a/2$ instead of a and $\lambda=0$. She also showed that (1 . 1) with $a=0$, $b=0,\lambda=0$ does not admit ground state solutions . Badiale et al [1] studied (1 . 1) with $a=0,b=0,\lambda=0$ and $h\equiv 1$. They proved the existence of at least a nonzero nonnegative weak solution u, satisfying $u(y,z)=u(|\,y\,|,z)$ when $2\leq k<N$ and

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$\mu < 0$. Bouchekif and El Mokhtar [3] proved that (1 . 1) with $a = 0, b = 0$ admits two distinct solutions when $2 < k \leq N$, $b = N - p(N-2)/2$ with $p \in (2, 2^*]$, $\mu < \bar{\mu}0, k$, and $\lambda \in (0, \Lambda_*)$ where Λ_* is a positive constant . Terracini [9] proved that there are no positive solutions of (1 . 1) with $b = 0, \lambda = 0$ when $a \neq 0, h \equiv 1$ and $\mu < 0$. The regular problem corresponding to $a = b = \mu = 0$ and $h \equiv 1$ has been considered on a regular bounded domain Ω by Tarantello [8] . She proved that for g in $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition , the problem considered admits two distinct solutions .

Before formulating our results , we give some definitions and notation . We denote by $\mathcal{D}_0^{1,2} = \mathcal{D}_0^{1,2}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ and $\mathcal{H}_\mu = \mathcal{H}_\mu(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$, the closure of $C_0^\infty(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ with respect to the norms

$$\| u \|_a, 0 = (\int_{\mathbb{R}^N} | y |^{-2a} | \nabla u |^2 \, dx) 1/2$$

and

$$\| u \| a, \mu = (\int_{\mathbb{R}^N} (| y |^{-2a} | \nabla u |^2 - \mu | y |^{-2(a+1)} | u |^2) dx) 1/2,$$

respectively , with $\mu < \bar{\mu}a, k = ((k - 2(a+1))/2)^2$ for $k \neq 2(a+1)$.

From the Hardy - Sobolev - Maz ' ya inequality , it is easy to see that the norm $\| u \|_{a,\mu}$ is equivalent to $\| u \|_a, 0$.

Since our approach is variational , we define the functional $I_{a,b,\lambda,\mu}$ on $\mathcal{H}_\mu$ by

$$I(u) := I_{a,b,\lambda,\mu}(u) := (1/2) \| u \|_{a,\mu}^2 - (1/2_*) \int_{\mathbb{R}^N} h \, | y |^{-2_* b} | u |^{2_*} \, dx - \lambda \int_{\mathbb{R}^N} g u dx.$$

We say that $u \in \mathcal{H}_\mu$ is a weak solution of (1 . 1) if it satisfies

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} (| y |^{-2a} \nabla u \nabla v - \mu | y |^{-2(a+1)} uv - h | y |^{-2_* b} | u |^{2_* - 2} uv - \lambda g v) dx$$
$$= 0, \quad \text{for} v \in \mathcal{H}_\mu.$$

Here $\langle \cdot, \cdot \rangle$ denotes the product in the duality $\mathcal{H}'_\mu, \mathcal{H}_\mu$.

Throughout this work , we consider the following assumptions :

(G) There exist $\nu_0 > 0$ and $\delta_0 > 0$ such that $g(x) \geq \nu_0$, for all x in $B(0, 2\delta_0)$;

$$\text{(H)} \quad \lim_{|y| \to 0} h(y) = \lim_{|y| \to \infty} h(y) = h_0 > 0, h(y) \geq h_0, y \in \mathbb{R}^k.$$

Here , $B(a, r)$ denotes the ball centered at a with radius r.

Under some conditions on the coefficients of (1 . 1) , we split $\mathcal{N}$ in two disj oint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$, thus we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$.

Remark 1 . 1 . Note that all solutions of (1 . 1) are nontrivial .

We shall state our main results .

Theorem 1 . 2 . *Assume that* $3 \leq k \leq N, -1 < a < (k-2)/2, 0 \leq \mu < \bar{\mu}a, k$, *and* (G) *holds , then there exists* $\Lambda_1 > 0$ *such that the* (1 . 1) *has at least one nontrivial s o lutio n on* $\mathcal{H}_\mu$ *for al l* $\lambda \in (0, \Lambda_1)$. **Theorem 1 . 3 .** *In addition to the assumptions of the Theorem 1 . 2 , if* (H) *ho lds ,*
then there exists $\Lambda_2 > 0$ *such that* (1 . 1) *has at least two nontrivial s o lutions on* $\mathcal{H}_\mu$
for al l $\lambda \in (0, \Lambda_2)$.

This article is organized as follows . In Section 2 , we give some preliminaries . Section 3 and 4 are devoted to the proofs of Theorems 1 . 2 and 1 . 3 .

2 . Preliminaries

We list here a few integral inequalities . The first one that we need is the Hardy inequality with cylindrical weights [7] . It states that

$$\bar{\mu}a, k \int_{\mathbb{R}^N} \mid y \mid^{-2(a+1)} v^2 dx \leq \int_{\mathbb{R}^N} \mid y \mid^{-2a} \mid \nabla v \mid^2 dx, \quad \text{forall} v \in \mathcal{H}_\mu,$$

The starting point for studying (1 . 1) is the Hardy - Sobolev - Maz ' ya inequality that
is particular to the cylindrical case $k < N$ and that was proved by Maz ' ya in [6] . It states that there exists positive constant $C_{a,2_*}$ such that

$$C_{a,2_*}(\int_{\mathbb{R}^N} \mid y \mid^{-2_* b} \mid v \mid^{2_*} dx) 2/2_* \leq \int_{\mathbb{R}^N} (\mid y \mid^{-2a} \mid \nabla v \mid^2 - \mu \mid y \mid^{-2(a+1)} v^2) dx,$$
$$\text{forany} v \in C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}).$$

Proposition 2 . 1 ([6]) **.** *The value*

$$S_{\mu,2_*} = S_{\mu,2_*}(k, 2_*) := v \inf_{\mathcal{H} \in \mu \setminus} \{0\} \frac{\int_{\mathbb{R}^N} (\mid y \mid^{-2a} \mid \nabla v \mid^2 - \mu \mid y \mid^{-2(a+1)} v^2) dx}{(\int_{\mathbb{R}^N} \mid y \mid^{-2} *_{|v|}^{b} {}^2 * dx)^{2/2*}} \tag{2.1}$$

is a chieved on $\mathcal{H}_\mu$*, for* $2 \leq k < N$ *and* $\mu \leq \bar{\mu}a, k$.

Definition 2 . 2 . Let $c \in \mathbb{R}, E$ be a Banach space and $I \in C^1(E, \mathbb{R})$.

(i) $(u_n)_n$ is a Palais - Smale sequence at level $c($ in short $(PS)_c)$ in E for I if $I(u_n) = c + o_n(1)$ and $I'(u_n) = o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence .

2 . 1 . **Nehari manifold .** It is well known that I is of class C^1 in $\mathcal{H}_\mu$ and the so - lutions of (1 . 1) are the critical points of I which is not bounded below on $\mathcal{H}_\mu$. Consider the Nehari manifold

$$\mathcal{N} = \{u \in \mathcal{H}_\mu \setminus \{0\} : \langle I'(u), u \rangle = 0\},$$

Thus , $u \in \mathcal{N}$ if and only if

$$\parallel u \parallel_{a,\mu}^2 - \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u \mid^{2_*} dx - \lambda \int_{\mathbb{R}^N} g u dx = 0. \tag{2.2}$$

Note that $\mathcal{N}$ contains every nontrivial solution of (1 . 1) . Moreover , we have the following results .

Lemma 2 . 3 . *The functional* I *is coercive and bounded from below on* $\mathcal{N}$*. Proof .* If $u \in \mathcal{N}$, then by ((2 . 2) and the H ö lder inequality , we deduce that

$$\begin{aligned} I(u) &= ((2_* - 2)/2_* 2) \parallel u \parallel_{a,\mu}^2 - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} g u dx \\ &\geq ((2_* - 2)/2_* 2) \parallel u \parallel 2_a, \mu - \lambda(1 - (1/2_*)) \parallel u \parallel a, \mu \parallel g \parallel \mathcal{H}'_\mu \\ &\geq -\lambda^2 C_0, \end{aligned} \tag{2.3}$$

where

$$C_0 := C_0(\parallel g \parallel \mathcal{H}'_\mu{}^{)} = [(2_* - 1)^2 / 2_* 2(2_* - 2)] \parallel g \parallel_{\mathcal{H}'_\mu}^2 > 0.$$

Thus , I is coercive and bounded from below on $\mathcal{N}$. □

 Define

$$\Psi_\lambda(u) = \langle I'(u), u\rangle.$$

Then , for $u \in \mathcal{N}$,

$$\langle \Psi'_\lambda(u), u\rangle = 2 \parallel u \parallel 2_a, \mu - 2_* \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u \mid^{2_*} dx - \lambda \int_{\mathbb{R}^N} gudx$$
$$= \parallel u \parallel 2_a, \mu - (2_* - 1) \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u \mid^{2_*} dx \qquad (2.4)$$
$$= \lambda(2_* - 1) \int_{\mathbb{R}^N} gudx - (2_* - 2) \parallel u \parallel^2_{a,\mu}.$$

Now , we split $\mathcal{N}$ in three parts :

$$\mathcal{N}^+ = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u\rangle > 0\}, \quad \mathcal{N}^0 = \{u \in \mathcal{N} \langle \Psi'_\lambda(u), u\rangle = 0\},$$
$$\mathcal{N}^- = \{u \in \mathcal{N} : \langle \Psi'_\lambda(u), u\rangle < 0\}$$

We have the following results . **Lemma 2 . 4 .** *Suppose that the re exists a local minimizer* u_0 *for* I *on* $\mathcal{N}$ *and* $u_0 element - slash$

$$Then, I'(u_0) = 0 in \mathcal{H}'_\mu. \qquad \mathcal{N}^0.$$

Proof . If u_0 is a lo cal minimizer for I on $\mathcal{N}$, then there exists $\theta \in \mathbb{R}$ such that

$$\langle I'(u_0), \varphi\rangle = \theta \langle \Psi'_\lambda(u_0), \varphi\rangle$$
$$\text{forany} \varphi \in \mathcal{H}_\mu.$$

If $\theta = 0$, then the lemma is proved . If not , taking $\varphi \equiv u_0$ and using the assumption $u_0 \in \mathcal{N}$, we deduce

$$0 = \langle I'(u_0), u_0\rangle = \theta\langle \Psi'_\lambda(u_0), u_0\rangle.$$

Thus

$$\langle \Psi'_\lambda(u_0), u_0\rangle = 0,$$

which contradicts that $u_0 element - slash \mathcal{N}^0$. □ Let

$$\Lambda_1 := (2_* - 2)(2_* - 1)^{-(2_* - 1)/(2_* - 2)} [(h_0)^{-1} S_{\mu, 2_*}]^{2_*/2(2_* - 2)} \parallel g \parallel \mathcal{H'}^{\mu}_{-1}. \qquad (2.5)$$

Lemma 2 . 5 . *We have* $\mathcal{N}^0 = \varnothing$ *for al l* $\lambda \in (0, \Lambda_1)$. *Proof .* Let us reason by contradiction . Suppose $\mathcal{N}^0 \neq \varnothing$ for some $\lambda \in (0, \Lambda_1)$. Then ,
by (2 . 4) and for $u \in \mathcal{N}^0$, we have

$$\parallel u \parallel 2_a, \mu =^= \binom{2_* -}{\lambda \ ((2_*} 1^)_- \mathbb{R}^{N (2_*}_{\int_{1)/}} \ {}^{y|^- 2_* b|u|^{2_*}}_{h| \ - 2)) \int_{\mathbb{R}^N}} dx_{gudx}. \qquad (2.6)$$

Moreover , by (G) , the H ö lder inequality and the Sobolev embedding theorem , we obtain

$$[^((h_0)^{-1} S_{\mu, 2_*})^{2_*/2} / (2_* - 1)] 1/(2_* - 2) \leq \parallel u \parallel_{a,\mu} \leq [\lambda((2_* - 1) \parallel g \parallel \mathcal{H}'_\mu{}^{/(2} * - 2))]. \qquad (2.7)$$

This implies that $\lambda \geq \Lambda_1$, which is a contradiction to $\lambda \in (0, \Lambda_1)$. □

 Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ for $\lambda \in (0, \Lambda_1)$. Define

$$c := \inf_{u\in\mathcal{N}} I(u), \quad c^+ := \inf_{\in_u \mathcal{N}+} I(u), \quad c^- := \inf_{\in_u \mathcal{N}-} I(u).$$

We need also the following Lemma . **Lemma 2 . 6 .** *(i) If* $\lambda \in (0, \Lambda_1)$, *th en* $c \le c^+ < 0$.

(ii) If $\lambda \in (0, (1/2)\Lambda_1)$, *th en* $c^- > C_1$, *where*

$$C_1 = C_1(\lambda, S_{\mu,2_*} \parallel g \parallel \mathcal{H}'_\mu) = ((2_* - 2)/2_*2)(2_* - 1)^{2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} \\ -\lambda(1 - (1/2_*))(2_* - 1)^{2/(2_*-2)} \parallel g \parallel \mathcal{H}'_\mu.$$

Proof . (i) Let $u \in \mathcal{N}^+$. By (2 . 4) ,

$$[1/(2_* - 1)] \parallel u \parallel 2_a, \mu > \int_{\mathbb{R}^N} h \mid y \mid^{-2_*b} \mid u \mid^{2_*} dx$$

and so

$$I(u) = (-1/2) \parallel u \parallel^2_{a,\mu} + (1 - (1/2_*)) \int_{\mathbb{R}^N} h \mid y \mid^{-2_*b} \mid u \mid^{2_*} dx \\ < [(-1/2) + (1 - (1/2_*))(1/(2_* - 1))] \parallel u \parallel^2_{a,\mu} \\ = -((2_* - 2)/2_*2) \parallel u \parallel^2_{a,\mu};$$

we conclude that $c \le c^+ < 0$. (ii) Let $u \in \mathcal{N}^-$. By (2 . 4) ,

$$[1/(2_* - 1)] \parallel u \parallel^2_{a,\mu} < \int_{\mathbb{R}^N} h \mid y \mid^{-2_*b} \mid u \mid^{2_*} dx.$$

Moreover , by Sobolev embedding theorem , we have

$$\int_{\mathbb{R}^N} h \mid y \mid^{-2_*b} \mid u \mid^{2_*} dx \le (S_{\mu,2_*})^{-2_*/2} \parallel u \parallel^{2_*}_{a,\mu}.$$

This implies

$$\parallel u \parallel a, \mu > [(2_* - 1)]^{-1/(2_*-2)}(S_{\mu,2_*})^{2_*/2(2_*-2)}, \quad \text{forall} u \in \mathcal{N}^-.$$

By (2 . 3) ,

$$I(u) \ge ((2_* - 2)/2_*2) \parallel u \parallel^2_{a,\mu} - \lambda(1 - (1/2_*)) \parallel u \parallel_{a,\mu} \parallel g \parallel \mathcal{H}'_\mu.$$

Thus , for all $\lambda \in (0, (1/2)\Lambda_1)$, we have $I(u) \ge C_1$. ☐ For each $u \in \mathcal{H}_\mu$, we write

$$t_m := t_{\max}(u) = [\frac{\parallel u \parallel a, \mu}{(2_* - 1) \int_{\mathbb{R}^N} h \mid y \mid^{-2} *^b_{|u|}{}^2 * dx}]^{1/(2_*-2)} > 0.$$

Lemma 2 . 7 . *Let* $\lambda \in (0, \Lambda_1)$. *For each* $u \in \mathcal{H}_\mu$, *one has the following :*

(i) If $\int_{\mathbb{R}^N} g(x)udx \le 0$, *then there exists a unique* $t^- > t_m$ *such that* $t^-u \in \mathcal{N}^-$ *and*

$$I(t^-u) = \sup_{t \ge 0} I(tu).$$

(ii) If $\int_{\mathbb{R}^N} g(x)udx > 0$, *th en there exist unique* t^+ *and* t^- *such that* $0 < t^+ <$

$$t_m < t^-, t^+u \in \mathcal{N}^+, t^-u \in \mathcal{N}-,$$
$$I(t^+u) = \inf_{\leq_0 t \leq t_m} I(tu) and I(t^-u) = \sup_{t \geq 0} I(tu).$$

The proof of the above lemma follows from a proof in [5] , with minor modifica - tions .

3 . Proof of Theorem 1 . 2

For the proof we need the following results .

Proposition 3 . 1 ([5]) **.** *(i) If* $\lambda \in (0, \Lambda_1)$, *then there exists a minimizing s equence* $(u_n)_n$ *in* $\mathcal{N}$ *such that*

$$I(u_n) = c + o_n(1), \quad I'(u_n) = o_n(1) \quad in \mathcal{H}'_{\mu}, \tag{3.1}$$

where $o_n(1)$ *te nds to* 0 *as* n *t ends to* ∞.

(ii) if $\lambda \in (0, (1/2)\Lambda_1)$, *then there exists a minimizing s equence* $(u_n)_n$ *in* $\mathcal{N}-$ *such that*

$$I(u_n) = c^- + o_n(1), \quad I'(u_n) = o_n(1) \quad in \mathcal{H}'_{\mu}.$$

Now , t aking as a starting point the work of Tarantello [8] , we establish the existence of a lo cal minimum for I on $\mathcal{N}^+$. **Proposition 3 . 2 .** *If* $\lambda \in (0, \Lambda_1)$, *then* I *has a minimizer* $u_1 \in \mathcal{N}^+$ *and it satisfies*

$$\text{(i)} \quad I(u_1) = c = c^+ < 0,$$

(i i) u_1 *is a s o lution of* (1 . 1) . *Proo* $f - period$ (i) By Lemma 2.3, I is coercive and bounded below on $\mathcal{N}$. We can assume that there exists $u_1 \in \mathcal{H}_\mu$ such that

$$\begin{gathered} u_n \rightharpoonup u_1 \quad \text{weaklyin} \mathcal{H}_\mu, \\ u_n \rightharpoonup u_1 \quad \text{weaklyin} L^2 * (\mathbb{R}^N, |\, y \,|^{-2_* b}), \\ u_n \to u_1 \quad \text{a . e in } \mathbb{R}^N. \end{gathered} \tag{3.2}$$

Thus , by (3 . 1) and (3.2), u_1 is a weak solution of (1 . 1) since $c < 0$ and $I(0) = 0$. Now , we show that u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Suppose otherwise . Then $\| u_1 \| a, \mu < \liminf_{n\to\infty} \| u_n \|_{a,\mu}$ and we obtain

$$\begin{aligned} c \le I(u_1) = ((2_* - 2)/2_* 2) \| u_1 \| 2_a, \mu - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} g u1 dx \\ < \lim_{n\to\infty} \inf I(u_n) = c. \end{aligned}$$

We have a contradiction . Therefore , u_n converges to u_1 strongly in $\mathcal{H}_\mu$. Moreover , we have $u_1 \in \mathcal{N}^+$. If not , then by Lemma 2 . 7 , there are two numbers t_0^+ and t_0^-, uniquely defined so that $t_0^+ u_1 \in \mathcal{N}^+$ and $t_0^- u_1 \in \mathcal{N}^-$. In particular , we have

$$t_0^+ < t_0^- = 1.\text{Since}$$
$$\frac{d}{dt} I(tu_1)|_t = t_0^+ = 0, \quad \frac{d^2}{dt^2} I(tu_1)|t = t_0^+ > 0,$$

there exists $t_0^+ < t^- \le t_0^-$ such that $I(t_0^+ u_1) < I(t^- u_1)$. By Lemma 2 . 7 ,

$$I(t_0^+ u_1) < I(t^- u_1) < I(t_0^- u_1) = I(u_1),$$

which is a contradiction . □

4 . Proof of Theorem 1 . 3

In this section , we est ablish the existence of a second solution of (1 . 1) . For this ,
we require the following Lemmas , with C_0 is given in (2 . 3) . **Lemma 4 . 1 .** *Assume that* (G) *ho lds and le t* $(u_n)_n \subset \mathcal{H}_\mu$ *be a* $(PS)_c$ *s equence for* I *for s ome* $c \in \mathbb{R}$ *with* $u_n \rightharpoonup u$ *in* $\mathcal{H}_\mu$. *Then* , $I'(u) = 0$ *and*

$$I(u) \geq -C_0\lambda^2.$$

Proo f − period It is easy to prove that $I'(u) = 0$, which implies that $\langle I'(u), u\rangle = 0$, and

$$\int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u \mid^{2_*} dx = \| u \|^2_{a,\mu} - \lambda \int_{\mathbb{R}^N} gudx.$$

Therefore ,

$$I(u) = ((2_* - 2)/2_* 2) \| u \|^2_{a,\mu} - \lambda(1 - (1/2_*)) \int_{\mathbb{R}^N} gudx.$$

Using (2 . 3) , we obtain

$$I(u) \geq -C_0\lambda^2.$$

□

Lemma 4 . 2 . *Assume that* (G) *holds and for any* $(PS)_c$ *s equence with* c *is a real number such that* $c < c^*\lambda$. *Then , there exists a subsequence which converges s trongly .*

$$Herec^*\lambda := ((2_* - 2)/2_* 2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} - C_0\lambda^2.$$

Proof . Using standard arguments , we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$. Thus , there exist a subsequence of $(u_n)_n$ which we still denote by $(u_n)_n$ and $u \in \mathcal{H}_\mu$ such that

$$u_n \rightharpoonup u \quad \text{weaklyin} \mathcal{H}_\mu,$$
$$u_n \rightharpoonup u \quad \text{weaklyin} L^2 * (\mathbb{R}^N, \mid y \mid^{-2_* b}).$$

$u_n \to u$ a . e in $\mathbb{R}^N$. Then , u is a weak solution of (1 . 1) . Let $v_n = u_n - u$, then by Br é zis - Lieb [4] , we
obtain

$$\| v_n \|^2_{a,\mu} = \| u_n \| 2_a, \mu - \| u \|^2_{a,\mu} + o_n(1) \tag{4.1}$$

and

$$\int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid v_n \mid^{2_*} dx = \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u_n \mid^{2_*} dx - \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid u \mid^{2_*} dx + o_n(1). \tag{4.2}$$

On the other hand , by using the assumption (H) , we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x) \mid y \mid^{-2_* b} \mid v_n \mid^{2_*} dx = h_0 \lim_{n\to\infty} \int_{\mathbb{R}^N} \mid y \mid^{-2_* b} \mid v_n \mid^{2_*} dx. \tag{4.3}$$

Since $I(u_n) = c + o_n(1), I'(u_n) = o_n(1)$ and by (4 . 1) , (4 . 2) , and (4 . 3) we deduce that

$$(1/2) \parallel v_n \parallel \mu_{a_,^2}^{\parallel v_n} - \parallel_{a,\mu}^{(_2 1/2 \ *)\int_{\mathbb{R}^N}} - \int_{\mathbb{R}^N} h^{h|y|^{-2_* b}|v_n|^{2_*}}_{|y|^{-2_* b}|v_n|^{2_*}} dx^{dx} =^{=} o_{n(1).}^{c - I(u)} + o_n(1), \tag{4.4}$$

Hence , we may assume that

$$\parallel v_n \parallel 2_a, \mu \to l, \quad \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid v_n \mid^{2_*} dx \to l. \tag{4.5}$$

 Sobolev inequality gives $\| v_n \|^2_{a,\mu} \geq (S_{\mu,2_*}) \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid v_n \mid^{2_*} dx.$ Combining this inequality with (4 . 5) , we obtain

$$l \geq S_{\mu,2_*}(l^{-1}h_0)^{-2/2_*}.$$

Either $l = 0$ or $l \geq (h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}$. Suppose that

$$l \geq (h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}.$$

Then , from (4 . 4) , (4 . 5) and Lemma 4 . 1 , we obtain

$$c \geq ((2_* - 2)/2_*2)l + I(u) \geq c^*\lambda,$$

which is a contradiction . Therefore , $l = 0$ and we conclude that u_n converges to u strongly in $\mathcal{H}_\mu$. □

Lemma 4 . 3 . *Assume that (G) and (H) ho ld . Then , there exist* $v \in \mathcal{H}_\mu$ *and* $\Lambda_* > 0$ *such that for* $\lambda \in (0, \Lambda_*)$, *one has*

$$\sup_{t \geq 0} I(tv) < c^*\lambda.$$

In particular , $c^- < c^*\lambda$ *for al l* $\lambda \in (0, \Lambda_*)$. *Proo f − period* Let $\varphi\varepsilon$ be such that

$$\varphi\varepsilon(x) = \{\omega_\varepsilon(x -_{\omega_\varepsilon(x)}^{-\omega_\varepsilon(x)} x_0) \quad \text{if}_{\text{if}}^{\text{if}} g(x_0^{g(x)}{}_{g(x)}) \leq^{>} 0 \text{for} x_0 \in \mathbb{R}^N {}^{0\text{forall}x \quad \in \mathbb{R}^N}_{\geq 0\text{forall}x \in \mathbb{R}^N}$$

where ω_ε satisfies (2 . 1) . Then , we claim that there exists $\varepsilon_0 > 0$ such that

$$\lambda \int_{\mathbb{R}^N} g(x)\varphi\varepsilon(x)dx > 0 \quad \text{forany} \varepsilon \in (0, \varepsilon_0). \tag{4.6}$$

In fact , if $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, (4.6) obviously holds . If there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, then by the continuity of $g(x)$, there exists $\eta > 0$ such that $g(x) > 0$ for all $x \in B(x_0, \eta)$. Then by the definition of $\omega_\varepsilon(x - x_0)$, it is easy to see that there exists an ε_0 small enough such that

$$\lambda \int_{\mathbb{R}^N} g(x)\omega_\varepsilon(x - x_0)dx > 0, \quad \text{forany} \varepsilon \in (0, \varepsilon_0).$$

Now , we consider the functions

$$f(t) = I(t\varphi\varepsilon), \quad \tilde{f}(t) = (t^2/2) \| \varphi\varepsilon \|^2_{a,\mu} -(t^2 * /2_*) \int_{\mathbb{R}^N} h \mid y \mid^{-2_* b} \mid \varphi\varepsilon \mid^{2_*} dx.$$

Then , for all $\lambda \in (0, \Lambda_1)$,

$$f(0) = 0 < c^*\lambda.$$

By the continuity of f, there exists $t_0 > 0$ small enough such that

$$f(t) < c^*\lambda, \quad \text{forall} t \in (0, t_0).$$

On the other hand ,

$$\max_{t \geq 0} \tilde{f}(t) = ((2_* - 2)/2_*2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)}.$$

Then , we obtain

$$\sup_{t\geq 0} I(t\varphi\varepsilon) < ((2_* - 2)/2_*2)(h_0)^{-2/(2_*-2)}(S_{\mu,2_*})^{2_*/(2_*-2)} - \lambda t_0 \int_{\mathbb{R}^N} g\varphi\varepsilon dx.$$

 Now , taking $\lambda > 0$ such that

$$-\lambda t_0 \int_{\mathbb{R}^N} g\varphi\varepsilon dx < -C_0\lambda^2,$$

and by (4 . 6) , we obtain

$$0 < \lambda < (t_0/C_0)(\int_{\mathbb{R}^N} g\varphi\varepsilon), \quad \text{for} \varepsilon << \varepsilon_0.$$

Set

$$\Lambda_* = \min\{\Lambda_1, \quad (t_0/C_0)(\int_{\mathbb{R}^N} g\varphi\varepsilon)\}.$$

We deduce that

$$\sup_{t \geq 0} I(t\varphi\varepsilon) < c_\lambda, \quad \text{for all } \lambda \in (0, \Lambda_*). \quad (4.7)$$

Now , we prove that

$$c^- < c^*\lambda, \quad \text{forall} \lambda \in (0, \Lambda_*).$$

By (G) and the existence of w_n satisfying (2 . 1) , we have

$$\lambda \int_{\mathbb{R}^N} gwndx > 0.$$

Combining this with Lemma 2 . 7 and from the definition of c^- and (4 . 7) , we obtain that there exists $t_n > 0$ such that $t_n w_n \in \mathcal{N}^-$ and for all $\lambda \in (0, \Lambda_*)$,

$$c^- \leq I(t_n w_n) \leq \sup_{t \geq 0} I(tw_n) < c^*\lambda.$$

□

Now we establish the existence of a lo cal minimum of I on $\mathcal{N}-$. **Proposition 4 . 4 .** *There exists* $\Lambda_2 > 0$ *such that for* $\lambda \in (0, \Lambda_2)$, *th e functional* I *has a minimizer* u_2 *in* $\mathcal{N}-$ *and satisfies*

(i) $I(u_2) = c^-$,

(i i) u_2 *is a s o lution of* (1 . 1) *in* $\mathcal{H}_\mu$,

where $\Lambda_2 = \min\ \{(1/2)\Lambda_1, \Lambda_*\}$ *with* Λ_1 *defined as in* (2 . 5) *and* Λ_* *defined as in the proof of Lemma 4 . 3 .*

Proof . By Proposition 3 . 1 (ii) , there exists a $(PS)_{c^-}$ sequence for $I, (u_n)_n$ in $\mathcal{N}^-$ for all $\lambda \in (0, (1/2)\Lambda_1)$. From Lemmas 4 . 2 , 4 . 3 and 2 . 6 (i i) , for $\lambda \in (0, \Lambda_*)$, I satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then , we get that $(u_n)_n$ is bounded in $\mathcal{H}_\mu$.

Therefore , there exist a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and $u_2 \in \mathcal{N}-$ such that u_n converges to u_2 strongly in $\mathcal{H}_\mu$ and $I(u_2) = c^-$ for all $\lambda \in (0, \Lambda_2)$. Finally , by using the same arguments as in the proof of the Proposition 3 . 2 , for all $\lambda \in (0, \Lambda_1)$, we have that u_2 is a solution of (1.1). □

Now , we complete the proof of Theorem 1 . 3 . By Propositions 3 . 2 and 4 . 4 , we obtain that (1 . 1) has two solutions u_1 and u_2 such that $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \varnothing$, this implies that u_1 and u_2 are distinct .

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