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A CLASS OF GENERALIZED INTEGRAL OPERATORS

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Abstract . In this paper , we introduce a class of generalized integral operators that includes Fourier integral operators . We establish some conditions on these operators such that they do not have bounded extension on $L^2(\mathbb{R}n)$. This permit us in particular to construct a class of Fourier integral operators with bounded symbols in $S^0_{1,1}(\mathbb{R}n \times \mathbb{R}n)$ and in $T_{0<\rho<1}S^0_{\rho,1}(\mathbb{R}n \times \mathbb{R}n)$ which cannot be extended to bounded operators in $L^2(\mathbb{R}^n)$.

1 . Introduction

The integral operators of type

$$A\varphi(x) = \int e^{iS(x,\theta)} a(x,\theta)\mathcal{F}\varphi(\theta)d\theta \tag{1.1}$$

appear naturally for solving the hyperbolic partial differential equations and ex - pressing the $C^\infty-$ solution of the associate Cauchy problem ' s (see e . g . [1 0 , 1 1]) .

If we write formally the expression of the Fourier transform $\mathcal{F}\varphi(\theta)$ in (1 . 1) , we obtain the following Fourier integral operators , so - called canonical transformations ,

$$A\varphi(x) = \int e^{i(S(x,\theta)-y\theta)} a(x,y,\theta)\varphi(y)dyd\theta \tag{1.2}$$

in which appear two $C^\infty-$ functions , the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ and the amplitude a called the symbol of the operator A. In the particular case where $S(x,\theta) = x\theta$, one recovers the notion of pseudodifferential operators (see e . g [6 , 1 5]) .

Since 1 970 , many of Mathematicians have been interested to these type of op - erators : Duistermaat [3] , H ö rmander [6 , 7] Kumano - Go [8] , and Fuj iwara [2] . We mention also the works of Hasanov [4] , and the recent results of Messirdi Senous - saoui [1 2] and Aiboudi - Messirdi - Senoussaoui [1] .

In this paper we study a general class of integral operators including the class of Fourier integral operators , specially we are interested in their continuity on $L^2(\mathbb{R}^n)$.

The continuity of the operator A on $L^2(\mathbb{R}^n)$ is guaranteed if the weight of the symbol a is bounded , if this weight tends to zero then A is compact on $L^2(\mathbb{R}^n)$(see eg . [1 2]) .

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If the symbol a is only bounded the associated Fourier integral operator A is not necessary bounded on $L^2(\mathbb{R}^n)$. Indeed , in 1 998 Hasanov [4] constructed an example of unbounded Fourier integral operators on $L^2(\mathbb{R})$.

Aiboudi - Messirdi - Senoussaoui [1] constructed recently in a class of Fourier inte - gral operators with bounded symbols in the H ö rmander class $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ that cannot be extended to be a bounded operator in $L^2(\mathbb{R}^n), n \geq 1$.

These results of unboundedness was obtained by using the properties of the operators

$$B\varphi(x) = \int_{\mathbb{R}n} k(z)\varphi((b(x)z + a(x))dz \tag{1.3}$$

on $L^2(\mathbb{R}^n), n \geq 1$, where $k(z) \in S(\mathbb{R}^n)$(the space of $C^\infty -$ functions on $\mathbb{R}^n$, whose derivatives decrease faster than any power of $\mid x \mid$ as $\mid x \mid \to +\infty), a(x)$ and $b(x)$ are real - valued , measurable functions on $\mathbb{R}^n$. Operators of type (1 . 3) was considered by Hasanov [4] and a slightly different way by Aiboudi Messirdi Senoussaoui [1] .

We also give in this paper a generalization of these results since we consider a class of integral operators which is general than thus of type (1 . 3) :

$$C\varphi(x) = \int_{\mathbb{R}n} K(x,z)\varphi(F(x,z))dz \tag{1.4}$$

where $K(x,z)$ and $F(x,z)$ are real - valued , measurable functions on $\mathbb{R}^{2n}$. The gener - alized integral operator C includes Hilbert , Mellin and the Fourier - Bros - Iagolnitzer transforms which they has been used by many authors and for many purposes , in particular respectively by H ö rmander [5] for the analysis of linear partial differential operators , Robert [1 3] about the functional calculus of pseudodiffrential operators , Sj ö strand [1 4] in the area of microlocal and semiclassical analysis and Stein [1 5] for the study of singular integral operators .

The operators C appears also in the study of the width of the quantum reso - nances (see e . g . [9]) .

We shall discuss in the second section bounded extension problems for the class of operators type C. We give some technical conditions on the functions $K(x,z)$ and $F(x,z)$ such that C do not admit a bounded extension on $L^2(\mathbb{R}^n)$. We also indicate a connection between transformations C and Fourier integral operators .

In the third section , we construct an example of Fourier integral with bounded symbols belongs respectively to $S^0_{1,1}(\mathbb{R}^n)$, (the case $n = 1$ is given in [4] and gen - eralized for $n \geq 2$ in [1]) , and $\cap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on $L^2(\mathbb{R}^n), n \geq 2$. In the case of the H ö rmander symbolic class $S^0_{1,1}(\mathbb{R}^n)$ our constructions are direct and technical .

2 . Unboundedness of the generalized integral operators

In this section we construct a class of operators C that cannot be extended to be a bounded operator in $L^2(\mathbb{R}^n), n \geq 1$. We have first an easy boundedness criterion of the operator C.

 Proposition 2 . 1 . *Let* $F(x,.) \in C^1(\mathbb{R}^n)$, *and* $K(x,.) \in L^2(\mathbb{R}^n)$ *for al l* $x \in R^n$.
Suppose that th ere exits a function $g(x)$ *such that*

$$g(x) > 0, \quad \forall x \in \mathbb{R}^n$$

$$|\det(\frac{\partial F(x,z)}{\partial z})| \geq g(x), \quad \forall x, z \in \mathbb{R}^n$$

$$\| K(x,.) \| L^2(\mathbb{R}n)/\sqrt{g(x)} \in L^2(\mathbb{R}^n)$$

then C *is a bounded operator on* $L^2(\mathbb{R}^n)$. *Proof .* Using H ö lder inequality and the change of variable $y = F(x,z)$, it ' s inverse is denoted $z = G(x,y)$, we obtain for all $\varphi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\| C\varphi \| 2L_2(\mathbb{R}n) &= \int_{\mathbb{R}n} | \int_{\mathbb{R}n} K(x,z)\varphi(F(x,z))dz|^2 dx \\
&\leq \int_{\mathbb{R}n} [\int_{\mathbb{R}n} | K(x,z)\varphi(F(x,z)) | dz]2_{dx} \\
&\leq \int_{\mathbb{R}n} [\| K(x,.) \| 2L_2(\mathbb{R}n) \int_{\mathbb{R}n} | \varphi(F(x,z)) |^2 dz]dx \\
&= \int_{\mathbb{R}n} [\| K(x,.) \|^2_{L\ 2(\mathbb{R}n)} \int_{\mathbb{R}n} | \varphi(y) |^2 |\det(\frac{\partial F(x,z)}{\partial z})_{(z=}G(x,y))|^{-1} dy]dx \\
&\leq \| \varphi \|^2_{L\ 2(\mathbb{R}n)} \int_{\mathbb{R}n} \frac{\| K(x,.) \|^2_{L^2(\mathbb{R}n)}}{g(x)} dx
\end{aligned}$$

(2 . 1) hence C is bounded operator on $L^2(\mathbb{R}^n)$ with $\| C \| \leq M = \| \frac{\|K(x,.)\|L^2(\mathbb{R}^n)}{\sqrt{g(x)}} \| L^2(\mathbb{R}n)$.

□

Now we give the main result of this paper . We proof that under some conditions the operator C do not admit a bounded extension on $L^2(\mathbb{R}^n)$. **Theorem 2 . 2 .** *Let* $\delta \in]0,1[$ *and th e operator* C *defined by* (1 . 4) *on* $L^2(\mathbb{R}^n)$ *for*

$$x = (x_1, ..., x_n) \in]0, \delta[^n such that :$$

(H 1) *For* $\varepsilon > 0$ *and for al l* $x \in \mathbb{R}^n$

$$\{z \in \mathbb{R}^n :| F(x,z) |\leq \varepsilon\} = \prod_{i=1}^{n} [a_i^-(x,\varepsilon), i_a^+(x,\varepsilon)]$$

where $i_a^{\pm}(x,t)$ *are real - measurable functions on* $\mathbb{R}^n \times]0,+\infty[$ *satisfying 1 - for any* $p \in \mathbb{N}^*$ *and* $i \in \{1,...,n\}$,

$$\lim_{i_x \to 0^+} i_a^{\pm}(px, x_i) = \pm\infty$$

2 - for any $\lambda \in]0,1[$, $i \in \{1,...,n\}$ *and* $p \in \mathbb{N}^*$, *the functions* $i_a^+(px,\lambda)$ *and* $a_i^-(px,\lambda)$ *are respectively decreasing and in creasing with respect to* x

$in]0,\delta[^n$

.

(H 2) *There exists a constant* $R > 0$ *such that for any* $r \geq R$ *and for al l* $x \in]0,\delta[^n$

$$|\int_{[-r,r]^n} K(x,z)dz| \geq \delta$$

Then the operator C *cannot be extended to a bounded operator on* $L^2(\mathbb{R}^n)$.

 Proof . Let us define the generalized sequence of functions

$$\varphi\varepsilon(x) = \{^{1,}_{0}, \quad \text{otherwise}^{\text{if} x\in[-\varepsilon,}\varepsilon]^{n} \tag{2.2}$$

then $\varphi\varepsilon \in L^2(\mathbb{R}^n)$ for all $\varepsilon > 0$ and we have

$$C\varphi\varepsilon(x) = \int_{Q^n_{i=1}} [a_i^-(x,\varepsilon), i_a^+(x,\varepsilon)]K(x,z)dz$$

Consequently ,

$$C\varphi\varepsilon_j(x) = integraldisplay - Qni = 1^{[a^-_i(}x,\varepsilon_j), i_a^+(x,\varepsilon_j)]K(x,z)dz \quad (2.3)$$
$$\text{where} \varepsilon_j \geq 0 \text{and} \lim_{j\to+\infty} \varepsilon_j = 0.$$

By condition 1 of the the assumption $(H1)$, for any $p \in \mathbb{N}^*$ there exists a number $\varepsilon_p \geq 0$ such that

$$i_a^+(p\Lambda_p, \varepsilon_p) \geq R \tag{2.4}$$

and

$$a_i^-(p\Lambda_p, \varepsilon_p) \leq -R \quad (2.5)$$
$$\text{for} \Lambda_p = (\varepsilon_p, \varepsilon_p, ...\varepsilon_p), p\varepsilon_p \leq \delta < 1 \text{and} i \in \{1, ..., n\}.$$

It follows from (2 . 4) , (2 . 5) and condition 2 of the assumption (H 1) that for $x \in]0, p\varepsilon_p]^n$ and $i \in \{1, ..., n\}$ we have

$$i_a^+(x, \varepsilon_p) \geq i_a^+(p\Lambda_p, \varepsilon_p) \geq R, \tag{2.6}$$
$$a_i^-(x, \varepsilon_p) \leq a_i^-(p\Lambda_p, \varepsilon_p) \leq -R \tag{2.7}$$

Finally using (H 2) , (2 . 3) , (2 . 6) and (2 . 7) we deduce

$$\| C\varphi\varepsilon_p \|^2_{L\ 2(\mathbb{R}n)} \geq \int_{]0,p\varepsilon_p]^n} | C\varphi\varepsilon p^{(x)|^2 dx} \geq \delta^2 p^n \varepsilon_p^n \tag{2.8}$$

If we consider that C has a bounded extension to $L^2(\mathbb{R}^n)$, then by virtue of (2 . 1) we obtain for $\varphi = \varphi\varepsilon_p \in L^2(\mathbb{R}^n)$

$$\delta^2 p^n \varepsilon_p^n \leq \| C\varphi\varepsilon_p \|^2_{L\ 2(\mathbb{R}n)} \leq M^2 \varepsilon_p^n$$

and for any $p \in \mathbb{N}^*$

$$p^n \leq \frac{M^2}{\delta^2}$$

This is a contradiction . Consequently A cannot be a bounded operator in $L^2(\mathbb{R}^n)$.

□

Remark 2 . 3 . (1) If in particular $K(x,z) = K(z)$ is independent on x and $F(x,z) = b(x) \circ z + a(x)$, where $K(z)$ is a real - valued measurable function , $b(x), a(x) \in \mathbb{R}^n$ are measurable functions on $\mathbb{R}^n$, we obtain the so - called generalized Hilbert transforms introduced in [4]

(2) The operator C is an Fourier integral operator for an appropriate choice of the functions $K(x,z)$ and $F(x,z)$.

$$C\varphi(x) = \int_{\mathbb{R}n} K(x,z)\varphi(F(x,z))dz$$

$$= \int_{\mathbb{R}n}\int_{\mathbb{R}n} e^{iz.\xi}\mathcal{F}K(x,\xi)\varphi(F(x,z))d\xi dz,$$

 where $\mathcal{F}K(x,\xi)$ is the Fourier transform of the partial function $z \to K(x,z)$. Set - ting $y = F(x,z)$ and $z = G(x,y)$, we have

$$C\varphi(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{iG(x,y).\xi}\mathcal{F}K(x,\xi)\varphi(y) \mid \det(\frac{\partial G}{\partial y}) \mid d\xi dy$$

which is a Fourier integral operator with the phase function $\phi(x,y,\xi) = G(x,y).\xi$ and the symbol $p(x,y,\xi) = \mathcal{F}K(x,\xi) \mid \det\,(\frac{\partial G}{\partial y}) \mid$ if K and G are infinitely regular with respect to x, y and ξ.

3 . A class of unbounded Fourier integral operators on $L^2(\mathbb{R}^n)$

It follows from theorem 2 . 2 that with an appropriate choice of $K(x,z)$ and $F(x,z)$ we can construct a class of Fourier integral operators which cannot be extended as bounded operators on $L^2(\mathbb{R}^n)$.

An example of unbounded fourier integral operator with a symbol in $S^0_{1,1}(\mathbb{R}\times\mathbb{R})$ and $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n\times\mathbb{R}^n)$ was given respectively in [4] and [1] , where if $\rho \in \mathbb{R}$,

$$S^0_{\rho,1}(\mathbb{R}^n\times\mathbb{R}^n) = \{p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n \exists C_{\alpha,\beta} > 0; |\partial_x^\alpha \partial_\theta^\beta p(x,\theta)| \le C_{\alpha,\beta}\lambda^{-\rho|\beta|+|\alpha|}(\theta) \tag{3.1}$$

3 . 1 . A class with symbols in $S^0_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$. Hre , we generalize the example given by Hasanov on $\mathbb{R}$ to high dimensions . Namely , in the same spirit of [8] . we have easily if we get $K(z) \in \mathcal{S}(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n,\mathbb{R})$.

Proposition 3 . 1 . *If $K(z) \in \mathcal{S}(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n,\mathbb{R})$, then for all $\alpha,\beta \in \mathbb{N}^n$ there exists $C_{\alpha\beta} > 0$ such that*

$$\mid \partial_x^\alpha\partial_\xi^\beta K(b(x)\xi) \mid \le C_{\alpha\beta}(1+ \mid \xi \mid)^{|\alpha|-|\beta|} \tag{3.2}$$

for al l $(x,\xi) \in [-1,1]^n \times R^n$.

Proof . It suffices to use the fact that $K \in \mathcal{S}(\mathbb{R}^n)$ and β is bounded on $[-1,1]^n$. □

Let also $a = (a_1, a_2, ..., a_n) \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ such that a, b, K satisfy (H 1) and (H 2) , with

$$b(x) > 0$$

$$i_a^\pm(x,t) = \frac{\pm t + a_i(x)}{b(x)}, \quad t > 0, x \in \mathbb{R}^n \tag{3.3}$$

Then , for $q(x,\xi) = K(b(x)\xi)$ defined on $[-1,1]^n \times \mathbb{R}^n$, we have

$$\mid \partial_x^\alpha\partial_\xi^\beta q(x,\xi) \mid \le C_{\alpha\beta}(1+ \mid \xi)^{|\alpha|-|\beta|}$$

on $[-1,1]^n \times \mathbb{R}^n, \alpha,\beta \in \mathbb{N}^n, C_{\alpha\beta}$ being constants .

Thus , $q \in S^0_{1,1}([-1,1]^n \times \mathbb{R}^n)$, in particular $q(x,\xi)$ is a well bounded symbol . Take a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with supp $\eta \subset [-1,1]^n$ and $\eta(x) = 1$ for $x \in [-\delta,\delta]^n, \delta < 1$. It is now obvious to see that the function $p(x,\xi) = \eta(x)q(x,\xi) \in S^0_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.

Now the Fourier integral operator defined by

$$C\varphi(x) = \int_{\mathbb{R}^2n} e^{-i(a(x).\xi+y.\xi)}p(x,\xi)\varphi(\xi)dyd\xi$$

$$= \int_{\mathbb{R}^2n} e^{-i(a(x).\xi+y.\xi)}\eta(x)K(b(x)\xi)\varphi(\xi)dyd\xi$$

 is of the type (1 . 4) . Indeed , for $s = b(x)\xi$ and $x \in]0, \delta]^n$

$$C\varphi(x) = \int_{\mathbb{R}^2 n} e^{-i\frac{(a(x)+t).s}{\beta(x)}} K(s) \frac{1}{bn(x)} \varphi(y) dy ds$$

Finally , if we pose $\frac{a(x)+y}{b(x)} = z$, we have

$$C\varphi(x) = \int \mathcal{F}K(z)\varphi(b(x)z - a(x))dz$$

By theorem 2 . 2 , we conclude that the operator C cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$.

3 . 2 . **A class with symbols in** $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$. We describe in this sec - tion the results of Aiboudi - Messirdi - Senoussaoui [1] , they constructed a class of unbounded Fourier integral operators with a separated variables phase function and a symbol in the H ö rmander class $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$.

Precisely , let $K \in S(\mathbb{R})$ with $K(t) = 1$ on $[-\delta, \delta]$ and $b(t)$ is continuous function on [0 , 1] such that

$$\begin{aligned} b(t) \in C^\infty(]0,1]), \quad b(0) = 0, \quad b'(t) > 0 \text{in}]0,1] \\ \mid b^{(k)}(t) \mid \leq \frac{C_k}{t^k} \text{in}]0,1], k \in \mathbb{N}^*, C_k > 0 \end{aligned} \tag{3.4}$$

$\chi(x), \psi(\xi) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ homogeneous of degree 1 . Thus the function

$$q(x,\xi) = e^{-i\chi(x)\psi(\xi)} \prod_{j=1}^{n} K(b(\mid x \mid) \mid x \mid \xi_j), \quad \xi = (\xi 1, ..., \xi_n) \tag{3.5}$$

belongs to $C^\infty([-1,1]^n \times \mathbb{R}^n)$ and satisfies , as in the proposition 3 . 1 , the following estimates

Proposition 3 . 2 . *For all* α, β *in* $\mathbb{N}^n$,

$$\begin{aligned} \mid \partial_x^\alpha \partial_\xi^\beta q(x,\xi) \mid \leq C_{\alpha\beta} \frac{(1+|\xi|)^{|\alpha|-|\beta|}}{b((1+|\xi|)^{-1})^{|\beta|}} \quad (3.6) \\ on [-1,1]^n \times \mathbb{R}^n where C_{\alpha\beta} > 0. \end{aligned}$$

Now if $\phi(x)$ is a $C_0^\infty(\mathbb{R})-$ function such that

$$\begin{aligned} \phi(s) = 1 \quad \text{on} [-\delta, \delta], \delta < 1 \\ \text{supp}\phi \subset [-1,1] \end{aligned}$$

define the global C^∞ symbol on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\begin{aligned} p(x,\xi) = e^{-i\chi(x)\psi(\xi)} \prod_{j=1}^{n} \phi(x_j) K(b(\mid x \mid) \mid x \mid \xi_j) \\ x = (x_1, ... x_n), \quad \xi = (\xi 1, ..., \xi_n). \end{aligned} \tag{3.7}$$

Then $p(x,\xi) \in \cap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and the corresponding Fourier integral oper - ator is

$$C\varphi(x) = \int_{\mathbb{R}n} e^{i\chi(x)\psi(\xi)} p(x,\xi)\mathcal{F}\varphi(\xi)d\xi$$

$$= \prod_{n}^{j=1} \phi(x_j) \int_{\mathbb{R}n} K(b(\mid x \mid) \mid x \mid \xi_j)\mathcal{F}\varphi(\xi)d\xi \tag{3.8}$$

 By using an adequate change of variable in the integral (3 . 8) , we have

$$C\varphi(x)=\int_{\mathbb{R}n}\varphi(b(\mid x\mid)\mid x\mid z)\prod_{n}^{j=1}\mathcal{F}K(z_j)d\xi,\quad z=(z_1,...,z_n) \tag{3.9}$$

which is of the form C in theorem 2 . 2 where the functions $F(x,z) \quad = b(\mid x\mid)\mid x\mid z$ and $K(x,z)=\prod_{j=1}^{n}\mathcal{F}K(z_j)$ satisfy (H 1) and (H 2) . Consequently , the operator C cannot be continuously extended on $L^2(\mathbb{R}^n)$.

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