# On the spectral boundary value problems and boundary approximate controllability of linear systems

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# ON THE SPECTRAL BOUNDARY VALUE PROBLEMS AND BOUNDARY APPROXIMATE CONTROLLABILITY OF LINEAR SYSTEMS

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Abstract. The main subject of this paper is the study of a general spectral boundary value problems with right invertible (resp. left invertible) operators and corresponding initial boundary operators. The obtained results are used to describe the approximate boundary controllability of linear systems in abstract operator-theoretic setting.

## 1. Introduction

Spectral theory of boundary value problems in abstract operator-theoretic setting has received a lot of attention in the recent past, in particular, were applied to extend the spectral study for symmetric and self-adjoint elliptic differential operators on bounded and unbounded domains. The classic results known in this context are the ones of J. von Neumann, H. Weyl, D. Hilbert, K. Friedrichs, M. Kreĭn and those of many other authors. We refer the reader to the recent contributions [1, 2, 3, 4, 7, 9, 13, 14] and the references therein.

In [9], the author has developed a study of a general class of linear equations with right invertible operators and corresponding initial, boundary and mixed boundary value problems. Moreover, he has also investigated controllability of linear systems with right invertible operators and with generalized almost invertible operators. Recently, V. Ryzhov [14] has considered a general spectral boundary value problems, in Hilbert spaces setting. He used the left inverse of an operator to reformulate Poincaré, Hilbert and Riemman problems for harmonic and analytic functions in abstract setting.

This paper is a continuation and refinement of the research treatment of boundary value problems from the point of view of the Banach space operators theory in terms of general initial boundary operators. Note that this treatment yields certain useful properties and new techniques for studies of many problems in the literature. We treat here general abstract boundary value problems with generalized boundary conditions in the case when the first member is right or left invertible and the corresponding spectral parameter is in the Browder resolvent set. As an example, it turns out that it allows to interpret boundary value problems when the "boundary" does not exist a priori and is constructed artificially as

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a certain perturbation of the original problem. Problems of this type frequently arise in the case of singular perturbations of differential operators. For operators in Hilbert spaces often so-called boundary triplets are used in the context of abstract boundary value problems and the analysis becomes particularly challenging when the boundary conditions depend on the spectral parameter in linear or nonlinear way as well ($\lambda$-dependent boundary conditions), see, for example, [2, 7].

The paper consists of two parts. In the first part, we develop the spectral boundary value problems from the perspective of general theory of left and right invertible linear operators in Banach spaces. An abstract form of spectral boundary value problems with generalized boundary conditions is introduced and results on their solvability complemented by representation formulas of solutions are obtained. The question of existence of solutions of boundary value problems with singularities defined by a given boundary condition is also studied. This question is addressed on the basis of a version of Browder's resolvent formula derived from the obtained representations of solutions. In the second part of the paper, we develop a theoretical framework for the concepts of controllability. Recall that, in infinite dimensional spaces, exact controllability is not always realized. To overcome these restrictions, H. V. Thi in [16] (see also [9]) defined the so-called $F_1$-controllability, in the sens of: A system is approximate controllable if any state can be transfered to a neighborhood of other state by an admissible control. In this work we consider a new concepts of approximate controllability called $\Gamma_1$-controllability, in view to generalizes the work of [16] and cover a large class of linear control systems, in particular, those with boundary conditions. This controllability refers to the boundary approximate controllability, in the sense that any boundary state can be transfered to a neighborhood of other boundary state by an admissible control. The necessary and sufficient conditions for a linear system to be boundary approximately reachable, and boundary exactly controllable are also given.

The paper is organized as follows. In Section 2 we give some preliminary results of the theory of right and left invertible operators. In particular, we define the so-called initial boundary operators corresponding to a right (resp. left) invertible operator and we show that this notion generalize those of initial operators introduced by D. Przeworska-Rolewicz in [12]. In Section 3, we illustrate the general solutions of a general abstract boundary value problems defined by ordered pairs $(D, A)$ of linear operators acting in Banach spaces with $D$ is a right (resp. a left) invertible operator. The section ends with a brief discussion on the solution of some boundary value problems with singularities. In Section 4, based on the results obtained in the last section and the concept of boundary controllability and boundary reachability, we give necessary and sufficient conditions for an abstract control linear system to be boundary approximately reachable, boundary exactly controllable and boundary approximately controllable. Finally, by a typical example, we show that the concept and results of the boundary approximate reachability are completely coincide with the approximate reachability of the evolution linear control systems in infinite dimensional spaces.

## 2. Preliminaries

Let $X$ be a complex Banach space. Let a linear operator $A$ defined in a linear subset $\mathcal{D}(A)$ of $X$, called the domain of $A$, and mapping $\mathcal{D}(A)$ into $X$. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are respectively the range and the null space of $A$ and $A^*$ is the adjoint of $A$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded operators on $X$. An operator $A$ is closed if its graph is a closed subspace of $X \times X$. The spectrum and resolvent of a closed linear operator $A$ on $X$ are respectively denoted by $\sigma(A)$ and $\rho(A)$.

An operator $D : \mathcal{D}(D) \subseteq X \longrightarrow X$, is said to be right invertible if there exists an operator $R$ such that

$$\mathcal{R}(R) \subset \mathcal{D}(D) \quad \text{and} \quad DR = I_X.$$

Where $I_X$ is the identity operator on $X$. In this case $R$ is called a right inverse of $D$. By $\mathbf{R}_D$ we denote the set of all right inverses of $D$. If $R \in \mathbf{R}_D$ is a given right inverse of $D$, the family $\mathbf{R}_D$ is characterized by

$$\mathbf{R}_D = \{R + (I_X - RD)S : S \in \mathcal{L}(X)\}.$$

The theory of right invertible operators started with the works of D. Przeworska-Rolewicz [11], and then developed through many mathematicians (see [9, 16]). An operator $D : \mathcal{D}(D) \subseteq X \longrightarrow X$, is said to be left invertible if there exists an operator $L$ such that

$$\mathcal{R}(D) \subset \mathcal{D}(L) \quad \text{and} \quad LD = I_X.$$

In this case $L$ is called a left inverse of $D$. By $\mathbf{L}_D$ we denote the set of all left inverses of $D$. If $L \in \mathbf{L}_D$ is a given left inverse of $D$, the family $\mathbf{L}_D$ is characterized by

$$\mathbf{L}_D = \{L + T(I_X - DL) : T \in \mathcal{L}(X)\}.$$

**Proposition 2.1.** (i) *If $D$ is a right invertible operator then for every $R \in \boldsymbol{R}_D$*

$$\mathcal{D}(D) = \mathcal{R}(R) \oplus \mathcal{N}(D).$$

(i) *If $D$ is a left invertible operator then for every $L \in \boldsymbol{L}_D$*

$$\mathcal{D}(L) = \mathcal{R}(D) \oplus \mathcal{N}(L).$$

Now, let $E$ another complex Banach space, called boundary space.

**Definition 2.2.** An operator $\Gamma : X \longrightarrow E$ is said to be an initial boundary operator for a right invertible operator $D : \mathcal{D}(D) \subseteq X \longrightarrow X$ corresponding to a right inverse $R$ of $D$, if

- $\mathcal{N}(\Gamma) = \mathcal{R}(R)$.
- There exists an operator $\Pi : E \longrightarrow X$ for which $\mathcal{R}(\Pi) = \mathcal{N}(D)$ and $\Gamma\Pi = I_E$.

The set of all initial boundary operators for $D$ will be denoted by $\mathcal{I}_D$. If $\Gamma \in \mathcal{I}_D$, then by Proposition 2.1 and Definition 2.2, we have

$$\mathcal{D}(D) = \mathcal{R}(R) \oplus \mathcal{R}(\Pi). \tag{2.1}$$

In [12] D. Przeworska-Rolewicz introduced the class of initial operators. Recall that an operator $F : X \to X$ is said to be an initial operator for $D$ corresponding to $R \in \mathbf{R}_D$ if $F^2 = F$, $FX = \mathcal{N}(D)$ and $FR = 0$ on $\mathcal{D}(R)$. Let us remark that if $\Gamma$ and $\Pi$ are as in the Definition 2.2 and if we put $F = \Pi\Gamma$, then $F$ is an initial operator for $D$. By the same way we define the set of initial boundary operators for a left invertible operator as follows.

**Definition 2.3.** An operator $\Lambda : X \longrightarrow E$ is said to be an initial boundary operator for a left invertible operator $D$ corresponding to a left inverse $L$, if

- $\mathcal{N}(\Lambda) = \mathcal{R}(D)$.
- There exists an operator $\Theta : E \longrightarrow X$ for which $\mathcal{R}(\Theta) = \mathcal{N}(L)$ and $\Lambda\Theta = I_E$.

The set of all initial boundary operators for $D$ will be denoted by $\mathcal{K}_D$. If $\Lambda \in \mathcal{K}_D$, then by Proposition 2.1 and Definition 2.3, we have

$$\mathcal{D}(L) = \mathcal{R}(D) \oplus \mathcal{R}(\Theta). \tag{2.2}$$

Let us remark that all the definitions and the results of this section are also valid in the algebraic setting, i.e it can suppose that $X$ and $E$ are also linear spaces over the same field.

## 3. Spectral boundary value problem

3.1. **Regular spectral boundary value problem.** Let $X, E$ be a complex Banach spaces. Suppose that $D : \mathcal{D}(D) \subset X \longrightarrow X$, with $\dim \mathcal{N}(D) \neq 0$, be right invertible with a right inverse $R$, $\Gamma$ be a boundary operator of $D$ corresponding to $R \in \mathbf{R}_D$, and $A$ be a linear operator such that $\mathcal{D}(D) \subset \mathcal{D}(A)$. We consider the following spectral boundary value problem for the ordered pairs $(D, A)$ for unknown $x \in \mathcal{D}(D)$:

$$\begin{cases} Dx = \lambda Ax + f \\ \\ \Gamma x = \varphi \end{cases} \tag{3.1}$$

where $f \in X$, $\varphi \in E$ and $\lambda \in \mathbb{C}$ is spectral parameter. We state and prove the following key mathematical result.

**Theorem 3.1.** *Let $A, B$ be two linear operators on $X$ such that $\mathcal{R}(A) \subset \mathcal{D}(B)$ and $\mathcal{R}(B) \subset \mathcal{D}(A)$, then*

*$I - \lambda AB$ is invertible if and only if $I - \lambda BA$ is invertible for all $\lambda \neq 0$.*

*In this case, we have*

$$(I - \lambda BA)^{-1} = I + \lambda B(I - \lambda AB)^{-1}A, \tag{3.2}$$

*and*

$$(I - \lambda AB)^{-1} = I + \lambda A(I - \lambda BA)^{-1}B. \tag{3.3}$$

*Proof.* Let $\lambda \neq 0$. Assuming $I - \lambda AB$ is invertible with inverse $C$, then

$$(I - \lambda AB)C = C(I - \lambda AB) = I \Longleftrightarrow C - \lambda ABC = C - \lambda CAB = I.$$

Hence

$$\lambda CAB = C - I.$$

Otherwise,

$$\begin{aligned}(I+\lambda BCA)(I-\lambda BA) &= I-\lambda BA+\lambda BCA-\lambda^2 BCABA\\ &= I-\lambda BA+\lambda BCA-\lambda B(C-I)A\\ &= I\end{aligned}$$

By the same calculation and the fact that $\lambda ABC = C - I$, we obtain $(I - \lambda BA)(I+\lambda BCA) = I$. Then $(I-\lambda BA)$ is invertible and

$$(I-\lambda BA)^{-1} = (I+\lambda BCA) = I+\lambda B(I-\lambda AB)^{-1}A.$$

The converse holds by interchanging $A$ and $B$. $\square$

As in our case $\mathcal{R}(A) \subset \mathcal{D}(R)$ and $\mathcal{R}(R) \subset \mathcal{D}(A)$ , then by virtue of the Theorem 3.1, we obtain

**Lemma 3.2.** *Let $R \in \boldsymbol{R}_D$. If $\lambda^{-1} \in \rho(RA)$ then*

$$\mathcal{N}(D-\lambda A) = \mathcal{R}((I-\lambda RA)^{-1}\Pi). \tag{3.4}$$

*Proof.* Let $x \in \mathcal{D}(D)$ then, by (2.1), there exist $f \in X$ and $\varphi \in E$ such that $x = Rf + \Pi\varphi$. Hence

$$\begin{aligned}(D-\lambda A)x &= (D-\lambda A)(Rf+\Pi\varphi)\\ &= f-\lambda ARf-\lambda A\Pi\varphi\\ &= (I-\lambda AR)f-\lambda A\Pi\varphi.\end{aligned}$$

Now, for $\lambda^{-1} \in \rho(RA)$ and $x \in \mathcal{N}(D-\lambda A)$, we obtain $f = \lambda(I-\lambda AR)^{-1}A\Pi\varphi$. Thus

$$\begin{aligned}x &= \lambda R(I-\lambda AR)^{-1}A\Pi\varphi+\Pi\varphi\\ &= [\lambda R(I-\lambda AR)^{-1}A+I]\Pi\varphi\\ &= (I-\lambda RA)^{-1}\Pi\varphi.\end{aligned}$$

This implies that $x \in \mathcal{R}((I-\lambda RA)^{-1}\Pi)$.
To prove the inverse; put $z = (I-\lambda RA)^{-1}\Pi\varphi$, for some $\varphi \in E$. By using (3.2) we have

$$\begin{aligned}(D-\lambda A)z &= (D-\lambda A)(I-\lambda RA)^{-1}\Pi\varphi\\ &= (D-\lambda A)[I+\lambda R(I-\lambda AR)^{-1}A]\Pi\varphi\\ &= [D-\lambda A+\lambda(I-\lambda AR)(I-\lambda AR)^{-1}A]\Pi\varphi\\ &= (D-\lambda A+\lambda A)\Pi\varphi\\ &= 0.\end{aligned}$$

Therefore, formula (3.4) holds. $\square$

**Theorem 3.3.** *Let $R \in \boldsymbol{R}_D$. If $\lambda^{-1} \in \rho(RA)$, then the problem* (3.1) *is uniquely solvable for any $f \in X$, $\varphi \in E$ with the solution*

$$x_\lambda^{f,\varphi} = (I-\lambda RA)^{-1}Rf+(I-\lambda RA)^{-1}\Pi\varphi. \tag{3.5}$$

*Proof.* Let $\lambda^{-1} \in \rho(RA), f \in X$ and $\varphi \in E$. Due to Lemma 3.2 we have

$$\begin{aligned}(D-\lambda A)x_\lambda^{f,\varphi} &= (D-\lambda A)(I-\lambda RA)^{-1}Rf + (D-\lambda A)(I-\lambda RA)^{-1}\Pi\varphi \\ &= (D-\lambda A)(I-\lambda AR)^{-1}Rf \\ &= (D-\lambda A)[I+\lambda R(I-\lambda AR)^{-1}A]Rf \\ &= f.\end{aligned}$$

and

$$\begin{aligned}\Gamma x_\lambda^{f,\varphi} &= \Gamma(I-\lambda AR)^{-1}(Rf+\Pi\varphi) \\ &= \Gamma[I+\lambda R(I-\lambda AR)^{-1}A](Rf+\Pi\varphi) \\ &= \Gamma\Pi\varphi \\ &= \varphi.\end{aligned}$$

Let us prove the uniqueness. Namely, if $x_1, x_2 \in \mathcal{D}(D)$ are two solutions of the problem (3.1), then for their difference $x_0 = x_1 - x_2 = Rf_0 + \Pi\varphi_0$ with some $f_0 \in X, \varphi_0 \in E$ we have $(D-\lambda A)x_0 = 0$ and $\Gamma x_0 = 0$, i.e. $\Gamma_0(Rf_0 + \Pi\varphi_0) = 0$. Since $\mathcal{N}(\Gamma) = \mathcal{R}(R)$ and $\Gamma\Pi = I_E$, we infer that $\varphi_0 = 0$. On the other hand, $0 = (D-\lambda A)x_0 = (D-\lambda A)Rf_0 = (I-\lambda AR)f_0$, by assumption $\lambda^{-1} \in \rho(RA) = \rho(AR)$, thus $f_0 = 0$. □

A version of Lemma 3.2 and Theorem 3.3 in the case of left invertible operators is given as follows:

**Theorem 3.4.** *If $D$ is a left inverse of $R$ and $\lambda^{-1} \in \rho(RA)$, then*

$$\mathcal{N}(D-\lambda A) = \mathcal{R}((I-\lambda RA)^{-1}\Pi),$$

*and the problem* (3.1) *is uniquely solvable for any $f \in X$, $\varphi \in E$ with the solution*

$$x_\lambda^{f,\varphi} = (I-\lambda RA)^{-1}Rf + (I-\lambda RA)^{-1}\Pi\varphi.$$

*Remark* 3.5. Theorem 3.4 generalize [14, Lemma 1.1] and [14, Theorem 1.2] respectively, by taking $A = I$.

**Example 3.6** (The abstract Cauchy problem)**.** Let $\mathcal{X} = C([0,a], X)$ the space of all continuous functions over $[0,a]$ to a Banach space $X$. We consider the following abstract Cauchy problem:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t), \quad 0 < t \leq a, \\ x(0) = x_0 \end{cases} \tag{3.6}$$

where $A : D(A) \longrightarrow X$ is the generator of a $C_0$-semigroupe $T(t)$ on $X$ and $x_0 \in X$. We denote by $D = \frac{d}{dt}x(t)$ and $R = \int_0^t x(\tau)d\tau$. The operator $D$ is right invertible and $R$ is a right inverse of $D$ on $\mathcal{X}$. Now, if we denote by

$$Cx(t) = \int_0^t T(t-\tau)x(\tau)d\tau,$$

then

$$(I+CA)(I-RA)x(t) = (I-RA)(I+CA)x(t) = x(t), \quad \text{for all } t \in [0,a].$$

This means that the operator $I - RA$ is invertible and its inverse is given by

$$(I - RA)^{-1}x(t) = x(t) + A\int_0^t T(t-\tau)x(\tau)d\tau.$$

We define the initial operator by $\Gamma_0 x = x(0)$, for $x(.) \in \mathcal{X}$. Now, We can rewrite the abstract Cauchy problem as follows:

$$\begin{cases} Dx = Ax(t) + f(t) \\ \Gamma_0 x = x_0 \end{cases}$$

Since the operator $I - RA$ is invertible, this problem is well-posed and has the unique solution

$$x(t) = (I - RA)^{-1}(Rf + x_0)(t) = \int_0^t T(t-\tau)x(\tau)d\tau + T(t)x_0, \quad \text{for all } t \in [0, a].$$

This is exactly the solution given by the classical Cauchy theory, see [10].

**Example 3.7** (The Darboux problem for hyperbolic equations)**.** We consider the following hyperbolic differential equation

$$\frac{\partial^2 x(t,s)}{\partial t \partial s} = a(t,s)\frac{\partial x(t,s)}{\partial t} + b(t,s)\frac{\partial x(t,s)}{\partial s} + c(t,s)x(t,s) + y(t,s), \tag{3.7}$$

with initial conditions

$$\begin{cases} x(t, s_1) = \varphi(t), & \text{for all } t \in I_t = [t_1, t_2], \\ x(t_1, s) = \psi(s), & \text{for all } t \in I_s = [s_1, s_2] \end{cases} \tag{3.8}$$

where $a, b, c, y \in C(I_t \times I_s)$, $\varphi \in C(I_t)$ and $\psi \in C(I_s)$ with $\varphi(t_1) = \psi(s_1) = 0$. We put,

- $X = C^2(I_t \times I_s)$,
- $E = \{(f, g) : (f, g) \in C(I_t) \times C(I_s) \text{ such that } f(t_1) = g(s_1) = 0\}$,
- $Dx(t,s) = \frac{\partial^2 x(t,s)}{\partial t \partial s}$,
- $Rx(t,s) = \int_{t_1}^t \int_{s_1}^s x(\tau, \sigma)d\tau d\sigma$,
- $Ax(t,s) = a(t,s)\frac{\partial x(t,s)}{\partial t} + b(t,s)\frac{\partial x(t,s)}{\partial s} + c(t,s)x(t,s)$,
- $\Gamma : X \longrightarrow E$, $\Gamma x(t,s) = (x(t, s_1), x(t_1, s))$,
- $\Pi : E \longrightarrow X$, $\Pi(f(t), g(s)) = f(t) + g(s)$.

With these notations, the problem (3.7)-(3.8) take the form

$$\begin{cases} Dx = Ax + y, \\ \Gamma x(t,s) = (\varphi, \psi) \end{cases}$$

Since the operator $I - RA$ is invertible, the problem (3.7)-(3.8) is well-posed and has the unique solution

$$x(t,s) = (I - RA)^{-1}(Ry + \Pi(\varphi, \psi))(t,s) = (I - RA)^{-1}(Ry(t,s) + \varphi(t) + \psi(s)).$$

3.2. **Singular boundary value problem.** Let $T$ be a closed operator on $X$ and $\lambda$ an isolated point of the spectrum of $T$. Form a contour

$$\Gamma_\lambda = \{\xi \in \mathbb{C} :| \lambda - \xi |= \varepsilon\}),$$

with a bounded region inside $\Gamma_\lambda$ intersecting the spectrum of $T$ only at the point $\Gamma_\lambda$. We define the Riesz projection of $T$ associated to the contour $\Gamma_\lambda$ by

$$P_\lambda = \frac{1}{2\pi i} \oint_{\Gamma_\lambda} (T - \mu)^{-1}\, d\mu, \tag{3.9}$$

The discrete spectrum of $T$, denoted $\sigma_d(T)$, is just the set of isolated points $\lambda \in \mathbb{C}$ of the spectrum such that the corresponding Riesz projectors $P_\lambda$ are finite dimensional. The points of $\sigma_d(T)$ being poles of finite rank, i.e., around each of these points there is a punctured disk in which the resolvent has a Laurent expansion whose singular part has only finitely many nonzero terms, the coefficients in these being of finite rank. It follows that $\sigma_{eb}(T) = \sigma(T) \setminus \sigma_d(T)$ is an important part of $\sigma(T)$ called the Browder essential spectrum of $T$ (see [5]). Denotes by $\rho_B(T) := \mathbb{C} \setminus \sigma_{eb}(T)$ the Browder resolvent set of $T$. $\rho_B(T)$ is the largest open set on which the resolvent is finitely meromorphic.

For $\lambda \in \rho_B(T)$, let $P_\lambda$ be the corresponding finite rank Riesz projector. From the fact that $D(T)$ is $P_\lambda-$invariant, we may define the operator

$$T_\lambda = (\lambda - T)(I - P_\lambda) + P_\lambda$$

with domain $D(T)$. With respect to the decomposition $X = \mathcal{N}(P_\lambda) \oplus \mathcal{R}(P_\lambda)$ we can write:

$$T_\lambda = (\lambda - T\mid_{N(P_\lambda)}) \oplus I$$

Since $\sigma(T_\lambda) = \sigma((\lambda - T)(I - P_\lambda)) = \sigma(\lambda - T) \setminus \{0\}$, $T_\lambda$ has a bounded inverse which we denote by $R_B(\lambda, T)$ and called the Browder resolvent operator, i.e.,

$$R_B(\lambda, T) = ((\lambda - T)\mid_{N(P_\lambda)})^{-1}(I - P_\lambda) + P_\lambda.$$

Clearly $R_B(\lambda, T) = (\lambda - T)^{-1}$, for $\lambda \in \rho(T)$ and $R_B(\lambda, T)$ may be viewed as an extension of the usual resolvent from $\rho(T)$ to $\rho_B(T)$ and retains many of its important properties. For example, because $P_\lambda T_\lambda = P_\lambda$ on $D(T)$ and $T_\lambda P_\lambda = P_\lambda$ on $X$ it follows that $P_\lambda R_B(\lambda, T) = P_\lambda = R_B(\lambda, T)P_\lambda$. Now, we assume $A = I - P_{\lambda^{-1}}$ where $P_{\lambda^{-1}}$ is the Riesz projector of $R$ and $\lambda^{-1} \in \rho_B(R)$, where $R \in \mathbf{R}_D$, then the problem (3.1) becomes

$$\begin{cases} (D - \lambda I)x + \lambda P_{\lambda^{-1}}x = f \\ \Gamma x = \varphi \end{cases} \tag{3.10}$$

where $f \in X$, $\varphi \in E$. By noting that $f = (I - P_{\lambda^{-1}})f + P_{\lambda^{-1}}f$ and by proceeding as in the proofs of Lemma 3.2 and Theorem 3.3, we obtain the following results.

**Theorem 3.8.** *Let $R \in \boldsymbol{R}_D$. If $\lambda^{-1} \in \rho_B(R)$, then*

$$\mathcal{N}(D - \lambda I + \lambda P_{\lambda^{-1}}) = \mathcal{R}(R_B(\lambda^{-1}, R)\Pi),$$

*and then the problem (3.10) is uniquely solvable for any $f \in X$, $\varphi \in E$ with the solution*

$$x_\lambda^{f,\varphi} = R_B(\lambda^{-1}, R)(Rf + \Pi\varphi).$$

Similarly, in the case of left invertible operators, we have

**Theorem 3.9.** *If $D$ is a left inverse of $R$ and $\lambda^{-1} \in \rho_B(R)$, then*

$$\mathcal{N}(D - \lambda I + \lambda P_{\lambda^{-1}}) = \mathcal{R}(R_B(\lambda^{-1}, R)\Pi),$$

*and then the problem* (3.10) *is uniquely solvable for any $f \in X$, $\varphi \in E$ with the solution*

$$x_\lambda^{f,\varphi} = R_B(\lambda^{-1}, R)(Rf + \Pi\varphi).$$

## 4. The boundary approximate controllability of linear systems

Let $D$, $A$ and $\Gamma$ as in the problem (3.1). Furthermore, let $U$ another Banach space and $B$ is bounded operator from $U$ to $X$. Consider the following abstract control linear system:

$$\begin{cases} Dx = Ax + Bu \\ \Gamma x = \varphi_0 \end{cases} \tag{4.1}$$

The spaces $X$ and $U$ are called the space of states and space of controls, respectively. The element $\varphi_0 \in E$ is said be an initial boundary state. According to Theorem 3.3, if the operator $I - RA$ (or $I - AR$) is invertible, then for every fixed pair $(\varphi_0, u) \in E \times U$, the linear system (4.1) is well-posed and has a unique solution, which is given by:

$$x(\varphi_0, u) = \mathcal{E}_A^D(RBu + \Pi\varphi_0), \qquad \text{where } \mathcal{E}_A^D = (I - RA)^{-1}. \tag{4.2}$$

$x(\varphi_0, u)$ is called output corresponding to the input $u$. We denote by

$$\Re(\varphi_0) = \cup_{u \in U} x(\varphi_0, u) \tag{4.3}$$

the set of all solutions of (4.1) for arbitrary fixed initial boundary state $\varphi_0 \in E$. This set is called the reachable set from the initial boundary state $\varphi_0$ by means of control $u \in U$.

**Definition 4.1.** (1) A state $x \in X$ is called approximately reachable by the initial boundary state $\varphi_0 \in E$ if for every $\epsilon > 0$ there exists a control $u \in U$ such that

$$\|x - x(\varphi_0, u)\|_X < \epsilon.$$

(2) The linear system (4.1) is said to be approximately reachable from the initial boundary state $\varphi_0 \in E$ if

$$\overline{\Re(\varphi_0)} = X.$$

Where $\overline{M}$ denote the closure of a subspace $M$ of $X$.

**Definition 4.2.** Let $\Gamma_1$ be a bounded operator from $X$ to $E$.

(i) A state $x_1 \in X$, such that $\Gamma_1 x_1 = \varphi_1$, for some $\varphi_1 \in E$, is said to be $\Gamma_1$-reachable (resp. $\Gamma_1$-approximately reachable) by the initial boundary state $\varphi_0 \in E$ if there exists a control $u \in U$ such that

$$\Gamma_1 x(\varphi_0, u) = \varphi_1, \quad (\text{ resp. } \|\Gamma_1 x(\varphi_0, u) - \varphi_1\|_E < \epsilon \text{ for all } \epsilon > 0).$$

The state $x_1$ is called a final state.

(ii) The linear system (4.1) is said to be $\Gamma_1$-controllable (resp. $\Gamma_1$-approximately controllable) if for every initial boundary state $\varphi_0 \in E$,

$$\Gamma_1(\Re(\varphi_0)) = E, \quad (\text{ resp. } \overline{\Gamma_1(\Re(\varphi_0))} = E).$$

(iii) The linear system (4.1) is said to be $\Gamma_1$-controllable (resp. $\Gamma_1$-approximately controllable) to zero if

$$0 \in \Gamma_1(\Re(\varphi_0)), \quad (\text{ resp. } 0 \in \overline{\Gamma_1(\Re(\varphi_0))}),$$

for every initial boundary state $\varphi_0 \in E$.

**Theorem 4.3.** *Let $R \in \boldsymbol{R}_D$ and $\Gamma_1$ be a bounded operator from $X$ to $E$. The linear system* (4.1) *is $\Gamma_1$-controllable if and only if the operator $\Gamma_1 \mathcal{E}_A^D RB$ is surjective.*

*Proof.* Suppose that the system (4.1) is $\Gamma_1$-controllable, then we have

$$\Gamma_1(\Re(\varphi_0)) = \Gamma_1 \mathcal{E}_A^D RBU + \Pi\varphi_0 = E,$$

for every $\varphi_0 \in E$. Let $\varphi_1 \in E$, there exists a control $u \in U$ such that

$$\Gamma_1 \mathcal{E}_A^D RBu + \Pi\varphi_0 = \varphi_1$$

and

$$\Gamma_1 \mathcal{E}_A^D RBu = \varphi_1 - \Gamma_1 \mathcal{E}_A^D \Pi\varphi_0$$

The arbitrariness of $\varphi_0, \varphi_1 \in E$ implies that $\Gamma_1 \mathcal{E}_A^D RB$ is surjective. Conversely, for all $\varphi_1 \in E$ there exists a control $u \in U$ such that

$$\Gamma_1 \mathcal{E}_A^D RBu = \varphi_1.$$

This means that the system (4.1) is $\Gamma_1$-reachable by the initial boundary state $\varphi_0 \in E$, hence $\Gamma_1(\Re(0)) = E$, by the linearity of the reachable set we obtain the $\Gamma_1$-controllability of the system (4.1). □

**Theorem 4.4.** *Let $R \in \boldsymbol{R}_D$ and $\Gamma_1$ be a bounded operator from $X$ to $E$. Then the linear system* (4.1) *is $\Gamma_1$-approximately reachable from zero if and only if the operator $B^* R^* (\mathcal{E}_A^D)^* \Gamma_1^*$ is injective.*

*Proof.* Suppose that the system (4.1) is $\Gamma_1$-approximately reachable from the boundary state zero, we have

$$\overline{\Gamma_1(\Re(0))} = \overline{\Gamma_1 \mathcal{E}_A^D RBU} = E.$$

This is equivalent to the injectivity of $B^* R^* (\mathcal{E}_A^D)^* \Gamma_1^*$. □

**Lemma 4.5.** *Let $R \in \boldsymbol{R}_D$ and $\Gamma_1$ be a bounded operator from $X$ to $E$. If the linear system* (4.1) *is $\Gamma_1$-approximately controllable to zero and $\Gamma_1 \mathcal{E}_A^D \Pi$ is surjective, then every final state $x_1$ is $\Gamma_1$-approximately reachable by the initial boundary zero.*

*Proof.* Let $x_1 \in X$ such that $x_1 = \Gamma_1 \varphi_1$ for some $\varphi_1 \in E$. By assumption, $0 \in \overline{\Gamma_1(\Re(\varphi_0))}$, for every initial boundary state $\varphi_0 \in E$. Therefore, for every $\varphi_0 \in E$ and $\epsilon > 0$, there exists a control $u_0 \in U$ such that

$$\left\| \Gamma_1 \mathcal{E}_A^D (RBu_0 + \Pi\varphi_0) \right\|_E < \epsilon. \tag{4.4}$$

The surjectivity of $\Gamma_1\mathcal{E}_A^D\Pi$ implies that, for any $\varphi_1 \in E$, there exists $\varphi_2 \in E$ such that $\Gamma_1\mathcal{E}_A^D\Pi\varphi_2 = -\varphi_1$. This equality and (4.4) together imply that for every $\varphi_1 \in E$ and $\epsilon > 0$, there exists a control $u_1 \in U$ such that

$$\left\|\Gamma_1\mathcal{E}_A^D RBu_1 - \varphi_1\right\|_E < \epsilon.$$

This proves that every finale state $x_1$ is $\Gamma_1$-approximately reachable by the initial boundary zero. □

**Theorem 4.6.** *Suppose that all assumptions of Lemma 4.5 are satisfied. Then the linear system* (4.1) *is $\Gamma_1$-approximately controllable.*

*Proof.* According to assumptions of Lemma 4.5, for every initial boundary state $\varphi_0 \in E$ and $\epsilon > 0$, there exists a control $u_0 \in U$ such that

$$\left\|\Gamma_1\mathcal{E}_A^D(RBu_0 + \Pi\varphi_0)\right\|_E < \frac{\epsilon}{2}. \tag{4.5}$$

By the result of Lemma 4.5, for every $\varphi_1 \in E$ and $\epsilon > 0$, there exists a control $u_1 \in U$ such that

$$\left\|\Gamma_1\mathcal{E}_A^D RBu_1 - \varphi_1\right\|_E < \frac{\epsilon}{2}. \tag{4.6}$$

From (4.5) and (4.6), it follows that for every $\varphi_0, \varphi_1 \in E$ and $\epsilon > 0$, there exists a control $u = u_0 + u_1 \in U$ such that

$$\begin{aligned}\left\|\Gamma_1\mathcal{E}_A^D(RBu + \Pi\varphi_0) - \varphi_1\right\|_E &\le \left\|\Gamma_1\mathcal{E}_A^D(RBu_0 + \Pi\varphi_0)\right\|_E - \left\|\Gamma_1\mathcal{E}_A^D RBu_1 - \varphi_1\right\|_E \\ &< \epsilon.\end{aligned}$$

The arbitrariness of $\varphi_0, \varphi_1 \in E$ and $\epsilon > 0$ implies that $\overline{\Gamma_1(\Re(\varphi_0))} = E$. □

**Theorem 4.7.** *Suppose that $U$ is a separable Hilbert space. Let $R \in \boldsymbol{R}_D$ and $\Gamma_1$ be a bounded operator from $X$ to $E$. Then the linear system* (4.1) *is $\Gamma_1$-controllable to zero if and only if there exists a real number $\alpha > 0$ such that*

$$\left\|B^*R^*(\mathcal{E}_A^D)^*\Gamma_1^*\psi\right\| \ge \alpha\left\|(\Pi^*(\mathcal{E}_A^D)^*\Gamma_1^*)\psi\right\|, \quad \text{for every } \psi \in E^*. \tag{4.7}$$

*Where $E^*$ denotes the dual space of $E$.*

*Proof.* Suppose that the linear system (4.1) is $\Gamma_1$- controllable, we have

$$0 \in \Gamma_1(\Re(\varphi_0)) \quad \text{for every } \varphi_0 \in E.$$

Therefore, for arbitrary $\varphi_0 \in E$, there exists $u \in U$ such that $\Gamma_1\mathcal{E}_A^D RBu = -\Gamma_1\mathcal{E}_A^D\Pi\varphi_0$. It implies that

$$\mathcal{R}(\Gamma_1\mathcal{E}_A^D\Pi) \subset \mathcal{R}(\Gamma_1\mathcal{E}_A^D RB).$$

Now, by [17, Theorem 2.2, pp. 208], we obtain (4.7). The converse is obvious. □

**Example 4.8.** Let $X = C(\Omega)$ be the space of all continuous functions over $\Omega = [0, a] \times [0, a]$, $a > 0$, $E = C([0, a])$ and $U = C(\mathbb{R})$. We consider the following control system

$$\frac{\partial x}{\partial t}(t, s) = \lambda x(t, s) + u(t), \tag{4.8}$$

with an initial condition

$$x(0, s) = \varphi(s). \tag{4.9}$$

Write $D = \frac{\partial}{\partial t}x(t,s)$, $Rx(t,s) = \int_0^t x(\tau,s)d\tau$. We have

$$\mathcal{D}(D) = \{x \in X : x(.,s) \in C^1([0,a]) \text{ for every fixed } s \in [0,a]\} \text{ and } \mathcal{D}(R) = X.$$

In addition, The operator $D$ is right invertible and $R$ is a right inverse of $D$. An initial boundary operator $\Gamma$ for $D$ corresponding to $R$ is defined by

$$\Gamma x(t,s) = x(0,s) - \varphi(s), \quad \text{for all } t,s \in [0,a].$$

We can define the operator $\Pi$ by $\Pi f(s) = f(s) + \varphi(s)$, for all $f \in C([0,a])$. Therefore, with $A = \lambda I$, $B = I$ and $\varphi_0 = 0$ the problem (4.8)-(4.9) can be written in the form (4.1).

Now, if we denote by

$$Cx(t,s) = \int_0^t e^{\lambda(t-\tau)} x(\tau,s)d\tau,$$

then

$$(I+\lambda C)(I-\lambda R)x(t,s) = (I-\lambda R)(I+\lambda C)x(t,s) = x(t,s), \quad \text{for all } t,s \in [0,a].$$

This means that the operator $I - \lambda R$ is invertible and its inverse is given by

$$\mathcal{E}_A^D x(t,s) = x(t,s) + \lambda \int_0^t e^{\lambda(t-\tau)} x(\tau,s)d\tau.$$

Hence, by formula (4.2), for every $u$, the solution of (4.8)-(4.9) is given by

$$x(t,s) = \mathcal{E}_A^D(RBu + \Pi\varphi_0)(t,s) = e^{\lambda t}\varphi(s) + \int_0^t e^{\lambda(t-\tau)} u(\tau)d\tau.$$

Now, if $\Gamma_1$ is a bounded operator from $C(\Omega)$ to $C([0,a])$, then by Theorem 4.3 and Theorem 4.4 respectively, the linear system (4.8)-(4.9) is

- $\Gamma_1$-controllable if and only if the operator $\Gamma_1\mathcal{E}_A^D RB$ is surjective.
- $\Gamma_1$-approximately reachable from zero if and only if the operator $B^*R^*(\mathcal{E}_A^D)^*\Gamma_1^*$ is injective.

For example, let $\Gamma_1$ defined by $\Gamma_1 x(t,s) = x(t_1,s) = \varphi_1(s)$, for fixed $t_1 \in ]0,a]$. it is easy to check $\Gamma_1\mathcal{E}_A^D\Pi\varphi_0 = e^{\lambda t_1}\varphi = T(t_1)\varphi$ for every $\varphi \in C([0,a])$, where $T(t)$ is a semi group of continuous linear operators generated by $A$. The injectivity of $B^*R^*(\mathcal{E}_A^D)^*\Gamma_1^*$ is equivalent to

$$B^*T(t_1)^*\psi = 0 \Longrightarrow \psi = 0.$$

Note that this condition is necessary and sufficient for the linear system in infinite dimensional space to be approximately reachable from zero. This example shows that in the case where $D$ is a differential operator, the concept and results of $\Gamma_1$-approximately reachable are completely coincide with the approximate reachability of the linear control systems in infinite dimensional space (see [17]).

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