# RELATIONSHIP BETWEEN THE ESSENTIAL QUASI-FREDHOLM SPECTRUM AND THE CLOSED-RANGE SPECTRUM



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# RELATIONSHIP BETWEEN THE ESSENTIAL QUASI-FREDHOLM SPECTRUM AND THE CLOSED-RANGE SPECTRUM

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Abstract. J. P. Labrousse [13] studied and characterized in the case of Hilbert spaces, a relation of the essential quasi-Fredholm spectrum and the spectrum defined by Goldberg in [9] by $\sigma_{ec}(A) = \{\lambda \in \mathbb{C} \ ; \ R(\lambda I - A) \text{ is not closed}\}$. In this paper, we investigate this relation in the case of Banach spaces. A similar relation is given, if we consider a B-Fredholm spectrum instead of the essential quasi-Fredholm spectrum.

## 1. Introduction

Let $X$ be a Banach space and $\mathcal{L}(X)$ the set of all bounded linear operators from $X$ into $X$. For $A \in \mathcal{L}(X)$ we denote by $R(A)$ its range, $N(A)$ its null space and $\sigma(A)$ its spectrum. Let $I$ denotes the identity operator in $X$. An operator $A \in \mathcal{L}(X)$ is said to be semi-regular if $R(A)$ is closed and $N(A^n) \subseteq R(A)$, for all $n \geq 0$ (see [19]). $A$ is called a Kato type operator if we can write $A = A_1 \oplus A_0$ where $A_0$ is a nilpotent operator and $A_1$ is a semi-regular one. In 1958, T. Kato proved that a closed semi-Fredholm operator is of Kato type, J.P Labrousse [12] studied and characterized a new class of operators named quasi-Fredholm operators (see Definition 2.3), in the case of Hilbert spaces, and he proved that this class coincides with the set of Kato type operators. In the case of Banach spaces the Kato type operator is also quasi-Fredholm, the inverse is not true ( see remark of Theorem 3.2.2 in [12]). The study of such a class of operators gives a new definition of essential spectrum called the essential quasi-Fredholm spectrum which is the set of all complex $\lambda$ such that $\lambda I - A$ is not quasi-Fredholm operator. The main question motivated by J.P Labrousse [13], in the Hilbert spaces, is the relationship between the essential quasi-Fredholm spectrum and the closed-range spectrum, a subset of the spectrum containing all the complex numbers $\lambda$ such that $R(\lambda I - A)$ is not closed, noted $\sigma_{ec}(A)$ (see [9]), he proved that the symmetric difference between them is at most countable. This results is examined in Banach space in [4] for the Kato spectrum, which is the set of all complex $\lambda$ such that $\lambda I - A$ is not of Kato type operator, of course in Hilbert space a Kato type equivalent to a quasi-Fredholm one. The main results of this paper, is to prove, in the Banach space, that the symmetric difference between the essential quasi-Fredholm spectrum and the closed-range spectrum is at most contable, ours results generalize those of J. P. Labrousse from Hilbert spaces operators to the Banach space operators. In particular, we also give a same results if instead of the essential quasi-Fredholm spectrum, we consider some B-Fredholm spectra introduced by M. Berkani, [5, 6, 7].

Precisely, we establish in this paper the following results:
In section 2, we give some preliminary results which our investigation will be need.

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In section 3, we extended in the Theorem 3.3 to the Banach spaces, the result concerning the symmetric difference between the essential quasi-Fredholm spectrum and the closed range spectrum of an operator $A$ proved by J.P Labrousse [13] in the case of Hilbert spaces. We proves that a similar statements of Theorem 3.3 hold if, instead of the essential quasi-Fredholm spectrum, we consider the B-Fredholm spectra.
Finally, in section 4 we apply the previously obtained results to study the symmetric difference between the two spectrums for the class of operators which satisfy a polynomial growth condition.

## 2. Preliminary Results

In this section, we collect some technical results which we will need use in the sequel. Let $A$ be a bounded linear operator acting in a complex Banach space $X$.

**Definition 2.1.** (see [19]) Let $A \in \mathcal{L}(X)$, $A$ is said to be semi-regular if $R(A)$ is closed and $N(A^n) \subseteq R(A)$, for all $n \in \mathbb{N}$.

**Definition 2.2.** An operator $A \in \mathcal{L}(X)$, is said to be of Kato type of order $d$, if there exists $d \in \mathbb{N}$ and a pair of closed subspaces $(M, N)$ of $X$ such that :

(1) $X = M \oplus N$.
(2) $A(M) \subset M$ and $A_{/M}$ is semi-regular.
(3) $A(N) \subset N$ and $(A_{/N})^d = 0$ (i.e $A_{/N}$ is nilpotent).

An operator $A$ is said to be of Kato type if $A$ is of Kato type of order $d$, for some $d \in \mathbb{N}$.

Clearly, every semi-regular operator is of Kato type with $M = X$ and $N = \{0\}$ and a nilpotent operator has a decomposition with $M = \{0\}$ and $N = X$.

An operator is said to be *essentially semi-regular* if it admits a decomposition $(M, N)$ such that $N$ is finite-dimensional vector space. Note that if $A$ is essentially semi-regular then $A_{/N}$ is nilpotent and $A$ is of Kato type. For every operator $A \in \mathcal{L}(X)$, let us define the Kato spectrum, the semi-regular spectrum and the essentially semi-regular spectrum as follows respectively:
$\sigma_k(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not of Kato type}\}$,
$\sigma_{se}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not semi-regular}\}$,
$\sigma_{es}(A) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular}\}$.
The Kato spectrum is not necessarily non-empty. For example, each nilpotent operator has empty Kato spectrum. The following results shows that the Kato spectrum of a bounded operator $A$ is a closed subset of the spectrum $\sigma(A)$ of $A$. The next theorem is due to P. Aiena and E. Rosas:

**Theorem 2.1.** [1, Theorem 1.44.] *Suppose that $A \in \mathcal{L}(X)$, is of Kato type. Then there exists an open disc $\mathbb{D}(0, \epsilon)$ for which $\lambda I - A$ is semi-regular for all $\lambda \in \mathbb{D}(0, \epsilon) \setminus \{0\}$.*

Since $\sigma_k(A) \subseteq \sigma_{es}(A) \subseteq \sigma_{se}(A)$, as a straightforward consequence of Theorem 2.1, we easily obtain that these spectra differ from each other on at most countably many isolated points.

**Proposition 2.2.** [1, Corollary 1.45.] *The sets $\sigma_{se}(A) \setminus \sigma_k(A)$ and $\sigma_{es}(A) \setminus \sigma_k(A)$ are at most countable.*

In the present paper we concentrate on classes of quasi-Fredholm operators.

**Definition 2.3.** Let $A \in \mathcal{L}(X)$, $A$ is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

(1) $R(A^n) \cap N(A) = R(A^d) \cap N(A)$ for all $n \geq d$.
(2) $R(A^d) \cap N(A)$ and $R(A^d) + N(A)$ are closed in $X$.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree $d$.

We denote by $q\Phi(X)$ the set of all quasi-Fredholm operators. Examples of quasi-Fredholm operators are semi-regular operators (quasi-Fredholm of degree 0), surjective operators as well as injective operators with closed range, Fredholm operators, semi-Fredholm operators and B-Fredholm operators , Some other examples of quasi-Fredholm operators operators may be found in Mbekhta [19], Labrousse [12] and Berkani [5, 6, 7].

**Definition 2.4.** Let $A \in \mathcal{L}(X)$, the essential quasi-Fredholm spectrum is defined by

$$\sigma_{eq}(A) := \{\lambda \in \mathbb{C} : \lambda - A \notin q\Phi(X)\}$$

Note that the set $q\Phi(X)$ is open (see [12, 7, 11]), consequently the essential quasi-Fredholm spectrum $\sigma_{eq}(A)$ is a compact set of the spectrum $\sigma(A)$ of $A$. $\sigma_{eq}(A)$ may be empty, this is the case where the spectrum $\sigma(A)$ is a finite set of poles of the resolvent.

**Theorem 2.3.** [1, Theorem 1.42.] *Let $A \in \mathcal{L}(X)$, and assume that $A$ is of Kato type of order $d$ with a decomposition $(M, N)$. Then:*

(1) $M \cap N(A) = R(A^n) \cap N(A) = R(A^d) \cap N(A)$ *for every* $n \in \mathbb{N}$, $n \geq d$.
(2) $R(A) + N(A^n) = A(M) \oplus N$ *for every* $n \in \mathbb{N}$, $n \geq d$.
*Moreover* $R(A) + N(A^n)$ *is closed in* $X$.

Note that by results of J.P Labrousse [12], in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators. In the case of Banach spaces the Kato type operator is also quasi-Fredholm, according to the remark following Theorem 3.2.2 in [12] the converse is true when $R(A^d) \cap N(A)$ and $R(A) + N(A^d)$ are complemented in the Banach space $X$. Note that all these sets of spectra defined above in general satisfy the following inclusions

$$\sigma_{eq}(A) \subseteq \sigma_k(A) \subseteq \sigma_{es}(A) \subseteq \sigma_{se}(A), \quad \text{and} \quad \sigma_{ec}(A) \subseteq \sigma_{es}(A) \subseteq \sigma_{se}(A).$$

The reduced minimum modulus of a non-zero operator $A$ is defined by

$$\gamma(A) = \inf_{x \notin N(A)} \frac{\|Ax\|}{\operatorname{dist}(x, N(A))},$$

where $\operatorname{dist}(x, N(A)) = \inf_{y \in N(A)} \|x - y\|$. If $A = 0$, then we take $\gamma(A) = \infty$.
Note that (see [11]): $\gamma(A) > 0 \Leftrightarrow R(A)$ is closed.

Let $M, N$ be two closed linear subspaces of the Banach space $X$ and set

$$\delta(M, N) = \sup\{\operatorname{dist}(x, N) : x \in M, \|x\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, N) = 0$ for any subspace N.

The gap between $M$ and $N$ is defined by $\widehat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$.

$\widehat{\delta}$ is a metric on the set $\mathcal{F}(X)$ of all linear closed subspaces of $X$, and the convergence $M_n \longrightarrow M$ in $\mathcal{F}(X)$ is obviously defined by $\widehat{\delta}(M_n, M) \longrightarrow 0$ as $n \longrightarrow \infty$ in $\mathbb{R}$. Moreover, $(\mathcal{F}(X), \widehat{\delta})$ is complete metric space (see [11]).

Most of the classes of operators, for example in Fredholm theory, require that the operators have closed ranges. Thus it is natural to consider the closed-range spectrum or Goldberg spectrum of an operator $A \in \mathcal{L}(X)$,

$$\sigma_{ec}(A) = \{\lambda \in \mathbb{C} \ ; \ R(\lambda I - A) \text{ is not closed}\}.$$

However, the closed-range spectrum has not good properties:

(1) $\sigma_{ec}(A)$ is not necessarily non-empty. For example, $A = 0$.

(2) $\sigma_{ec}(A)$ may be not closed. There exists an operator $A$ such that $R(A)$ is closed but $R(\lambda I - A)$ is not closed for all $\lambda \in \mathbb{D}(0,1)\setminus\{0\}$. For example, the right shift operator $A$ defined on $\ell^2$ by

$$A(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

(3) It is possible that $R(A^2)$ is closed but $R(A)$ is not. Let $A$ be defined on $\ell^2$ by

$$A(x_1, x_2, x_3, \dots) = (0, x_1, 0, \frac{1}{3}x_2, 0, \frac{1}{5}x_3, 0, \dots)$$

The operator $A$ is compact and $R(A)$ is not closed, $A^2 = 0$ and $R(A^2)$ is closed.

(4) Conversely, it is also possible that $R(A)$ is closed but $R(A^2)$ is not. let $A = \begin{pmatrix} V & I \\ 0 & 0 \end{pmatrix}$ be an operator defined on $\ell^2 \oplus \ell^2$, where $V$ has the following proprieties that $V^2 = 0$ and $R(V)$ is not closed. Then $R(A)$ is closed, $R(A^2)$ is not closed, $A^3 = 0$.

(5) $\sigma_{ec}(A)$ is unstable under nilpotent perturbations. For example, $A = 0$ and $N$ the nilpotent operator defined in (3.). Then $0 \in \sigma_{ec}(A + N)$ but $0 \notin \sigma_{ec}(A)$.

Note that the essentially semi-regular spectrum, which has very nice spectral properties, is not too far from the closed-range spectrum. Clearly, $\sigma_{ec}(A) \subset \sigma_{es}(A)$ and $\sigma_{es}(A)\setminus\sigma_{ec}(A)$ is at most countable. Thus, the essentially semi-regular spectrum can be considered as a nice completion of the closed-range spectrum. However, the spectrum $\sigma_{ec}(A)$ can be used to obtain informations on the location in the complex plane of the various types of essential spectra, Fredholm, Weyl and Browder spectra etc..., for large classes of linear operators arising in applications. For example, integral, difference, and pseudo-differential operators (see [2, 9, 14, 15, 16]).

## 3. Main Results

The essential spectra were studied by many authors (see [9, 14, 15, 16, 17, 20]). Now, the main question is the relationship between them. Motivated by a problem concerning the essential quasi-Fredholm spectrum posed in [13], J. P. Labrousse characterized in the case of Hilbert spaces, a relation between the essential quasi-Fredholm spectrum and the closed-range spectrum.

Now, we study this relation in the case of Banach spaces. We begin with the following preparatory result proved in [4, Theorem 3], which is crucial for our purposes.

**Theorem 3.1.** *Let $A \in \mathcal{L}(X)$. For $\alpha$ a nonzero positive real number, let*

$$\mathcal{R}(\alpha) = \{\lambda \in \mathbb{C} \ ; \ \gamma(\lambda I - A) \geq \alpha\}$$

*If $(\lambda_n)_n \subset \mathcal{R}(\alpha)$ is a non stationary sequence and $\lambda_n \longrightarrow \lambda_0$ in $\mathbb{C}$, then*

(1) $\widehat{\delta}(N(\lambda_n I - A), N(\lambda_0 I - A)) \leq \frac{1}{\alpha}|\lambda_n - \lambda_0|$.
(2) $\lambda_0 \in \mathcal{R}(\alpha)$.
(3) $\lambda_0 I - A$ *is semi-regular.*

Note that this theorem generalizes the result proved by J.P Labrousse [13] to Banach spaces.

**Proposition 3.2.** *If $\lambda \in \sigma_{ec}(A)$ is non-isolated point then $\lambda \in \sigma_{eq}(A)$.*

*Proof.* Let $\lambda \in \sigma_{ec}(A)$ be a non-isolated point. Assume that $\lambda I - A$ is quasi-Fredholm operator. Then there exists an open disc $\mathbb{D}(\lambda, \epsilon)$ such that $\mu I - A$ is semi-regular and thus $R(\mu I - A)$ is closed if $\mu \in \mathbb{D}(\lambda, \epsilon)\setminus\{\lambda\}$. This contradicts our assumption that $\lambda$ non-isolated point.

□

**Theorem 3.3.** *The symmetric difference $\sigma_{eq}(A)\Delta\sigma_{ec}(A)$ is at most countable.*

*Proof.* We have,

$$\sigma_{eq}(A)\Delta\sigma_{ec}(A) = (\sigma_{eq}(A) \cap (\mathbb{C} \setminus \sigma_{ec}(A))) \cup (\sigma_{ec}(A) \cap (\mathbb{C} \setminus \sigma_{eq}(A))).$$

From Proposition 3.2 the set $\sigma_{ec}(A) \setminus \sigma_{eq}(A)$ is at most countable, and $\mathbb{C} \setminus \sigma_{ec}(A) = \bigcup_{m=1}^{\infty} \mathcal{R}(\frac{1}{m})$. Thus

$$\sigma_{eq}(A) \cap (\mathbb{C} \setminus \sigma_{ec}(A)) = \bigcup_{m=1}^{\infty} (\sigma_{eq}(A) \cap \mathcal{R}(\frac{1}{m})).$$

Finally, we prove that the set $\sigma_{eq}(A) \cap \mathcal{R}(\frac{1}{m})$ is at most countable. Let $\lambda_0$ be a non-isolated point of $\sigma_{eq}(A) \cap \mathcal{R}(\frac{1}{m})$. Then there exists $(\lambda_n)_n \subset \sigma_{eq}(A) \cap \mathcal{R}(\frac{1}{m})$ such that $\lambda_n \to \lambda_0$, by Theorem 3.1 $\lambda_0 I - A$ is semi-regular operator, hence $\lambda_0 \notin \sigma_{eq}(A)$. This contradicts the closedness of $\sigma_{eq}(A)$. □

**Proposition 3.4.** *$\sigma_{se}(A) \setminus (\sigma_{eq}(A) \cap \sigma_{ec}(A))$ is at most countable.*

*Proof.* We have

$$\sigma_{se}(A) \setminus (\sigma_{eq}(A) \cap \sigma_{ec}(A)) = (\sigma_{eq}(A)\Delta\sigma_{ec}(A)) \cup \sigma_{se}(A) \setminus (\sigma_{eq}(A) \cup \sigma_{ec}(A))$$

Since the sets $\sigma_{se}(A) \setminus \sigma_{eq}(A)$, $\sigma_{se}(A) \setminus \sigma_{ec}(A)$ are at most countable , Theorem 3.3 implies that $\sigma_{eq}(A)\Delta\sigma_{ec}(A)$ is at most countable, so then we have the result. □

**Proposition 3.5.** *If $\lambda \in \partial\sigma(A)$ is non-isolated point, then $\lambda \in \sigma_{eq}(A)$.*

*Proof.* Let $\lambda \in \partial\sigma(A)$ is non-isolated point, since $\partial\sigma(A) \subseteq \sigma_{se}(A)$, then $\lambda \in \sigma_{se}(A)$ is non-isolated point, hence $\lambda \in \sigma_{eq}(A)$. □

Given $n \in \mathbb{N}$, we denote by $A_n$ the restriction of $A \in \mathcal{L}(X)$ on the subspace $R(A^n)$. According M. Berkani [5], an operator $A$ is said to be semi B-Fredholm (resp. B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm), if for some integer $n \geq 0$ the range $R(A^n)$ is closed and $A_n$, viewed as an operator from the space $R(A^n)$ in to itself, is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $A \in \mathcal{L}(X)$ is said to be B-Browder (resp. upper semi B-Browder, lower semi B-Browder, B-Weyl, upper semi B-Weyl, lower semi B-Weyl ), if for some integer $n \geq 0$ the range $R(A^n)$ is closed and $A_n$ is a Browder operator (resp. upper semi-Browder, lower semi-Browder, Weyl, upper semi-weyl, lower semi-Weyl).

These classes of operators motive the definition of several spectra. The B-Ferdholm spectrum is defined by

$$\sigma_{bf}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not B-Ferdholm}\}$$

the semi B-Fredholm spectrum is defined by

$$\sigma_{sbf}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not semi B-Fredholm}\}$$

the upper semi B-Fredholm spectrum is defined by

$$\sigma_{ubf}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not upper semi B-Fredholm}\}$$

the lower semi B-Fredholm spectrum is defined by

$$\sigma_{lbf}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not lower semi B-Fredholm}\}$$

the B-Browder spectrum is defined by

$$\sigma_{bb}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not B-Browder}\}$$

the upper semi B-Browder spectrum is defined by

$$\sigma_{ubb}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not upper semi B-Browder}\}$$

the lower semi B-Browder spectrum is defined by

$$\sigma_{lbb}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not lower semi B-Browder}\}$$

the B-Weyl spectrum is defined by

$$\sigma_{bw}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not B-Weyl}\}$$

the upper semi B-Weyl spectrum is defined by

$$\sigma_{ubw}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not upper semi B-Weyl}\}$$

the lower semi B-Weyl spectrum is defined by

$$\sigma_{lbw}(A) := \{\lambda \in \mathbb{C} \ : \ \lambda I - A \text{ is not lower semi B-Weyl}\}$$

We have

$$\sigma_{bf}(A) = \sigma_{ubf}(A) \cup \sigma_{lbf}(A), \tag{3.1}$$

$$\sigma_{bw}(A)) = \sigma_{ubw}(A) \cup \sigma_{lbw}(A), \tag{3.2}$$

and

$$\sigma_{eq}(A) \subseteq \sigma_{bf}(A) \subseteq \sigma_{bw}(A) \subseteq \sigma_{bb}(A) = \sigma_{ubb}(A) \cup \sigma_{lbb}(A). \tag{3.3}$$

Note that all the B-spectra are closed subsets of $\mathbb{C}$ (see [7], [12]), and may be empty. This is the case where the spectrum $\sigma(A)$ of $A$ is a finite set of poles of the resolvent.
By (3.1), (3.2) and (3.3), similar statements of Theorem 3.3 hold if, instead of $\sigma_{eq}(A)$, we consider the B-Fredholm spectra.

**Theorem 3.6.** *If $\lambda \in \sigma_{ec}(A)$ is non-isolated point then $\lambda \in \sigma_i(A)$. Moreover, $\sigma_i(A)\Delta\sigma_{ec}(A)$ is at most countable, $i = bf,\ bw,\ bb,\ ubf, lbf, ubw, lbw, ubb, lbb$.*

In the next two theorems we consider a situation which occurs in some concrete cases.

**Theorem 3.7.** *Let $A \in \mathcal{L}(X)$ an operator for which $\sigma_{ec}(A) = \sigma(A)$ and every $\lambda$ is non-isolated in $\sigma(A)$. Then,*

$$\sigma(A) = \sigma_{ec}(A) = \sigma_{eq}(A) = \sigma_{bf}(A) = \sigma_{bw}(A) = \sigma_{bb}(A).$$

*Proof.* Since $\lambda \in \sigma_{ec}(A)$ is non-isolated, according to Theorem 3.3 and Theorem 3.6,

$$\sigma_{ec}(A) = \sigma(A) \subseteq \sigma_{eq}(A) \subseteq \sigma_{bf}(A) \subseteq \sigma_{bw}(A) \subseteq \sigma_{bb}(A) \subseteq \sigma(A).$$

that is, the statement of theorem. □

The approximate point spectrum is defined by

$$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not one to one or } R(\lambda I - A) \text{ is not closed } \}$$

**Theorem 3.8.** *Let $A \in \mathcal{L}(X)$ an operator for which $\sigma_{ap}(A) = \partial\sigma(A)$ and every $\lambda \in \partial\sigma(A)$ is non-isolated in $\sigma(A)$. Then, $\sigma_{ec}(A) \subseteq \sigma_{eq}(A) = \sigma_k(A) = \sigma_{es}(A) = \sigma_{se}(A)$.*

*Proof.* Since $\lambda \in \partial\sigma(A)$ is non-isolated, according to Proposition 3.5 we obtain,

$$\sigma_{ap}(A) = \partial\sigma(A) \subseteq \sigma_{eq}(A) \subseteq \sigma_k(A) \subseteq \sigma_{es}(A) \subseteq \sigma_{se}(A) \subseteq \sigma_{ap}(A).$$

that is,

$$\sigma_{ec}(A) \subseteq \sigma_{ap}(A) = \sigma_{eq}(A) = \sigma_k(A) = \sigma_{es}(A) = \sigma_{se}(A).$$

□

**Example 3.9.** Cesaro operator $C_p$ on the classical Hardy space $H_p(\mathbb{D})$, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$ and $1 < p < \infty$, is defined by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^{\lambda} \frac{f(\lambda)}{1-\lambda}\mathrm{d}\mu, \text{ for all } f \in H_p(\mathbb{D}) \text{ and } \lambda \in \mathbb{D}.$$

The spectrum of the operator $C_p$ is the closed disc $\Gamma_p$ centered at $\frac{p}{2}$ with radius $\frac{p}{2}$ and $\sigma_{ap}(C_p) = \partial\Gamma_p$, see [1]. From Theorem 3.8 we also have

$$\sigma_{ec}(C_p) \subseteq \sigma_{ap}(C_p) = \sigma_{eq}(C_p) = \sigma_k(C_p) = \sigma_{es}(C_p) = \sigma_{se}(C_p) = \partial\Gamma_p.$$

## 4. Application

Let the class $\mathcal{P}_g(X)$ of linear bounded operators on a Banach space $X$ which satisfy a polynomial growth condition. An operator $A$ satisfies this condition if there exists $K > 0$, and $\delta > 0$ for which

$$\|exp(i\lambda A)\| \leq K(1+|\lambda|^{\delta}) \text{ for all } \lambda \in \mathbb{R},$$

Examples of operators which satisfy a polynomial growth condition are hermitian operators on Hilbert spaces, nilpotent and projection operators, algebraic operators with real spectra. It is shown that $\mathcal{P}_g(X)$ coincides with the class of all generalized scalar operators having real spectra. We first note that the polynomial growth condition may be reformulated as follows (see [1]) : $A \in \mathcal{P}_g(X)$ if and only if $\sigma(A) \subseteq \mathbb{R}$ and there is a constant $K > 0$, and $\delta > 0$ such that

$$\left\|(\lambda I - A)^{-1}\right\| \leq K(1+|\mathrm{Im}\lambda|^{-\delta}) \text{ for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Im}\lambda \neq 0, \tag{4.1}$$

We recall (see, e.g. [10]) that for $A \in \mathcal{L}(X)$, the ascent of $A$, is the smallest positive integer $p = p(A)$ such that

$$N(A^p) = N(A^{p+1}).$$

If there is no such integer we set $p(A) = \infty$.
The descent of $A$ is the smallest positive integer $q = q(A)$ such that

$$R(A^q) = R(A^{q+1}).$$

If such an integer does not exist, we put $q(A) = \infty$.

The following proposition establish the finiteness of the ascent of a linear operator $A \in \mathcal{P}_g(X)$.

**Proposition 4.1.** *([1]) Assume that $A \in \mathcal{P}_g(X)$. Then, for every $\lambda \in \sigma(A)$ we have:*

(1) $p(\lambda I - A) < \infty$ .
(2) $\overline{R((\lambda I - A)^p)} = \overline{R((\lambda I - A)^{p+k})}$; $k \in \mathbb{N}$. *and* $p = p(\lambda I - A)$.

**Proposition 4.2.** [4] *Let $A \in \mathcal{P}_g(X)$, we have:*

(1) *If $\lambda \in \sigma(A) \setminus \sigma_{ec}(A)$, then $\lambda$ is a pole of the resolvent $(\lambda I - A)^{-1}$.*
(2) *If $\lambda \in \sigma_{ec}(A)$ and $R((\lambda I - A)^p)$ is closed for some $p \in \mathbb{N}$, then $\lambda$ is a pole of the resolvent $(\lambda I - A)^{-1}$.*

The following theorem illustrate Theorem 3.3 by means of the class of operators which satisfy a polynomial growth condition.

**Theorem 4.3.** *Let $A \in \mathcal{P}_g(X)$, then $\sigma_{eq}(A)\Delta\sigma_{ec}(A)$ is at most countable.*

*Proof.* From the Proposition 4.2, if $\lambda \notin \sigma_{ec}(A)$, then $\lambda$ is a pole of the resolvent $(\lambda I - A)^{-1}$. This implies the $\lambda \notin \sigma_{eq}(A)$ and the set $\sigma_{eq}(A) \setminus \sigma_{ec}(A))$ is empty. Now if $\lambda \in \sigma_{ec}(A)$, we have two cases. First if there exists $p \in \mathbb{N}$ such that $R((\lambda I - A)^p)$ is closed, by the Proposition 4.2 part 2, $\lambda$ is a pole of the resolvent and $\lambda \notin \sigma_{eq}(A)$, thus $\sigma_{ec}(A)) \setminus \sigma_{eq}(A)$ is at most countable . Now if $R((\lambda I - A)^p)$ is not closed for every $p \in \mathbb{N}$, then by [7, Corollary 3.3] $\lambda I - A$ is not quasi-Fredholm operator and $\lambda \in \sigma_{eq}(A)$, the set $\sigma_{ec}(A) \setminus \sigma_{eq}(A)$ is then empty. □

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