

On the Lack of Ideal Observability in Hilbert Space

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Abstract

Some remarks are given on the lack of ideal observability for linear systems in Hilbert space with bounded operators, under the action of a disturbance generating a measured output on some finite interval of time. A criterion is established in this context by means of a range inclusion of operators according to a well known Douglas theorem. The ideal observability subspace is described as the intersection of a special collection of Kalman observable subspaces.

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1 Introduction

Let us consider the observed system

$$x'(t) = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Hx(t), t \in [0, T]; \quad (2)$$

where $x(t)$ is the state vector, and $y(t)$ the output observed on a finite interval of time $[0, T]$ under the action of a disturbance $u(\cdot)$. Unless otherwise stated, the following assumptions (A1)-(A3) will be made throughout this paper:

- (A1) The (unknown) function $u(\cdot)$ is assumed to be (locally) integrable in the sense of Bochner, i.e. $u(\cdot) \in L^1([0, T] ; U)$,
- (A2) The spaces of phase, output and disturbance, noted X , Y and U are real separable Hilbert spaces,
- (A3) The operators A , B and H are linear and continuous, i.e. $A \in \mathcal{L}(X, X) \equiv \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$ and $H \in \mathcal{L}(X, Y)$, with H non invertible and B with finite range.

The notations A^* , $\text{Im } A$, $\ker A$ will concern respectively the adjoint, the range and the null subspace of the operator A . We shall use sometimes $y(\cdot)$ instead of $\{y(t), t \in [0, T]\}$, "a.e." instead of "almost every where", and "iff" for "if and only if". For any subspace E , the scalar product " $\langle z, E \rangle = 0$ " will mean " $\langle z, e \rangle = 0$, for all $e \in E$ ".

Using the known equation of motion (1) with its given coefficients A , B and H , and assuming that (2) has been measured for (1) under the action of the perturbation $u(\cdot)$, the problem is to rebuild the trajectory $x(\cdot)$ on some time interval $[0, T]$ solely from the available information $y(\cdot)$, for every possible $y(\cdot)$.

Somewhat different (but finely equivalent) definitions have been used here and there in $X = \mathbb{R}^n$ for their connection with the approach followed to solve the problem, as in [4, 10, 11]. For it's convenience, the definition in the next section will be expressed in terms of ideally observable directions.

Some fundamental results have been established in [1] through the extension to Hilbert spaces of the finite dimensional approach developed in [4] and based on "geometric" arguments. Some other remarks have also been given in [2] concerning the ideal-observability problem in Banach space.

This work was suggested to us by a significant theorem obtained for $X = \mathbb{R}^n$ and exposed in [11]. One purpose of this paper is to generalize such a result to Hilbert space and bounded operators, with however a (rather realistic) restriction on the transmission mechanism B . Meanwhile, such a generalization will permit to characterize the lack of ideal observability under the assumptions (A1)-(A3), and to describe the ideal observability subspace as the intersection of a "family" of Kalman observability subspaces. Also, and contrary to what occurs in finite dimension, such a description will confirm (by other arguments than those utilized in [1]) that the notion of ideal observability does not have significance in the Hilbert X when H is reduced in assumption (A3) to a simple functional on X .

Let us now recall some preliminaries needed in the following sections.

2 Preliminary Notes

We first notice that the observability of the couple(1)–(2) includes the systems in the form:

$$x'(t) = Ax(t) + B_1u_1(t) + B_2u_2(t)$$

$$y(t) = Hx(t), t \in [0, T];$$

where $u_1(\cdot)$ is a control known to the observer, while $u_2(\cdot)$ is a perturbation on the system. The known input part $u_1(\cdot)$ and the unknown input part $u_2(\cdot)$ can be reduced to the case of completely unknown $u(\cdot)$.

Definition 1 *The vector $\xi \in X$ will be called an ideally observable direction for the system (1) – (2) iff the following implication is true on $[0, T]$ for any perturbation $u(\cdot)$:*

$$y(t, x_0, u(\cdot)) \equiv 0 \implies \langle x(t, x_0, u(\cdot)), \xi \rangle \equiv 0. \quad (3)$$

In this definition, $x(t, x_0, u(\cdot))$ stands for the solution of equation (1) generated by the perturbation $u(\cdot)$ with the initial condition $x_0 = x(0)$, whereas $y(t, x_0, u(\cdot))$ is the corresponding output. The formulation (3) means that the trajectory $x(\cdot)$ can be recognized along the direction ξ from the knowledge of $y(\cdot)$ on $[0, T]$.

Definition 2 *The ideal observability subspace $\mathfrak{S} \subseteq X$ for the system (1) – (2) is the set of all the directions $\xi \in X$ verifying (3).*

Such a subset \mathfrak{S} is in fact a closed subspace[2], and this is a natural reason to give the following definition.

Definition 3 *The system (1) – (2) will be said ideally observable iff $\mathfrak{S} = X$.*

The subspace \mathfrak{S} has been described in [1] as the greatest (within the meaning of inclusion) subspace of X , on which the orthogonal projection of the state function $x(\cdot)$ can be recognized on $[0, T]$ uniquely from the output $y(\cdot)$. Its expression is given there as the following closed linear span of an increasing sequence of subspaces $\{X_i\}_{i=0}^{\infty}$:

$$\mathfrak{S} = \overline{\text{Sp}}\{X_i, i \in \mathbb{N} \mid X_0 = \text{Im } H^*, X_{i+1} = X_i + A^*(X_i \cap \ker B^*)\}. \quad (4)$$

When $\mathfrak{S} = X$, which means $\mathfrak{S}^\perp = \{0\}$, where \mathfrak{S}^\perp represents the (closed) orthogonal complement of \mathfrak{S} in the Hilbert X , we get exactly the situation

where the determination of the whole state $x(t)$ can be done on the interval $[0, T]$. Such a determination may be understood as a representation as a Fourier series (see [1]) in some orthonormal basis of the separable Hilbert space X .

Moreover, as an applied extension of some conditioned and controlled invariance concepts introduced in [3], we mention the two following "dual" invariance properties for \mathfrak{S} (which contains $X_0 = \text{Im } H^*$), and \mathfrak{S}^\perp (which is contained in $\ker H$):

$$A^*(\mathfrak{S} \cap \ker B^*) \subseteq \mathfrak{S}; \quad (5)$$

$$A\mathfrak{S}^\perp \subseteq \overline{\mathfrak{S}^\perp + \text{Im } B}. \quad (6)$$

Under assumption (A3), the inclusion (6) becomes:

$$A\mathfrak{S}^\perp \subseteq \mathfrak{S}^\perp + \text{Im } B. \quad (7)$$

Besides, it's worth of note that when $u(\cdot)$ is not a disturbance, but a control at the observer's disposal, then the concept of ideal observability is reduced to the well known Kalman observability. The system (1) – (2) is then Kalman observable iff one of the two equivalent statements is true:

$$\overline{\text{Sp}}(\text{Im } H^*, \text{Im } A^*H^*, \text{Im } A^{*2}H^*, \dots) = X, \quad (8)$$

$$\bigcap_{i \in \mathbb{N}} \ker HA^i = \{0\}. \quad (9)$$

These last two relations show that Kalman observability does not depend on B ; which allows to consider $B = 0$ without loss of generality as far as $u(\cdot)$ remain a known function. Therefore, the ideal observability of the system (1) – (2) implies obviously its Kalman observability. Of course, the converse does not hold in general.

Remark 1 *The inclusion (6) derives from (5) by taking the orthogonal complement in both sides. The sequence of subspaces $\{X_i\}_{i \in \mathbb{N}}$ in expression (4) can also be derived independently from (3) by successive differentiations and by continuity of the scalar product. Likewise, the inclusion (5) may derive directly from (3) as shown in the following proposition.*

Proposition 1 *The subspace of ideal observability \mathfrak{S} contains the subspace $\text{Im } H^*$ and verifies the invariance property:*

$$A^*(\mathfrak{S} \cap \ker B^*) \subseteq \mathfrak{S}.$$

Proof. It's trivial that $y(t) = Hx(t) = \langle x(t), \text{Im } H^* \rangle \equiv 0$ implies $\langle x(t), \xi \rangle \equiv 0$ for any $\xi \in \text{Im } H^*$; and thus $\text{Im } H^* \subseteq \mathfrak{I}$.

For the inclusion $A^*(\mathfrak{I} \cap \ker B^*) \subseteq \mathfrak{I}$, it's sufficient to establish the implication:

$$y(t) \equiv 0 \implies \langle x(t), A^*(\mathfrak{I} \cap \ker B^*) \rangle \equiv 0.$$

By definition of \mathfrak{I} , and by continuity of the scalar product, we get by successive differentiations the following implications:

$$\{y(t) \equiv 0\} \implies \{\langle x(t), \mathfrak{I} \rangle \equiv 0\} \implies \{\langle x'(t), \mathfrak{I} \rangle = 0; \text{ a.e. in } [0, T]\};$$

whence:

$$\langle Ax(t) + Bu(t), \mathfrak{I} \rangle = 0; \text{ a.e. in } [0, T].$$

Therefore:

$$\langle x(t), A^*\mathfrak{I} \rangle + \langle u(t), B^*\mathfrak{I} \rangle = 0; \text{ a.e. in } [0, T];$$

and hence $\langle x(t), A^*(\mathfrak{I} \cap \ker B^*) \rangle \equiv 0$. ■

3 Main Results

3.1 A necessary and sufficient condition for non-ideal observability

The aim of this section concerns the connection between the ideal observability of the perturbed system (1) – (2) and its Kalman observability when the function $u(\cdot)$ takes the form $u(t) = Mx(t)$. It will be shown in the main proposition(2) that the lack of ideal observability does happen if and only if there exists an operator $M \in \mathcal{L}(X, U)$ for which the system

$$x'(t) = (A + BM)x(t) \quad (10)$$

$$y(t) = Hx(t) \quad (11)$$

can not be Kalman observable. In other words, the ideal observability of (1) – (2) is fulfilled if and only if $K_M(H) = X$ for every $M \in \mathcal{L}(X, U)$, where the subspace $K_M(H)$ is defined in accordance with (8) by:

$$K_M(H) = \overline{\text{Sp}}(\text{Im } H^*, \text{Im}(A + BM)^*H^*, \text{Im}((A + BM)^*)^2H^*, \dots). \quad (12)$$

The proof will be based on Douglas theorem [8] on the range inclusion of operators on Hilbert space. For our purpose, we give just a short version of this theorem.

Theorem 1 (Douglas theorem) *If S and T are (linear and bounded) operators on a Hilbert space Z , then the following are equivalent:*

- (i) $\text{Im } S \subseteq \text{Im } T$,
- (ii) $S = TC$ for some (linear and bounded) operator C on Z .

Proposition 2 *The two following statements are equivalent:*

- (i) *The system (1) – (2) is not ideally observable;*
- (ii) *The system*

$$\begin{cases} x'(t) = (A + BM_0)x(t) \\ y(t) = Hx(t) \end{cases}$$

is not Kalman-observable for some bounded linear operator $M_0 \in \mathcal{L}(X, U)$.

Proof. The implication (ii) \implies (i) is immediate. In fact, by contradiction, if the system (1) – (2) were ideally observable, and thus with an unknown function $u(t)$, then it would be a fortiori Kalman observable with any feedback $u(t) = Mx(t)$ for which $M \in \mathcal{L}(X, U)$.

The implication (i) \implies (ii) will be proved by invoking Douglas theorem. Let us suppose that (1) – (2) is not ideally observable, meaning $\mathfrak{S}^\perp \neq \{0\}$, and let us then prove the existence of an operator M_0 for which the system $x'(t) = (A + BM_0)x(t)$ can not be Kalman observable on $[0, T]$ from $y(t) = Hx(t)$. Precisely, in accordance with the criterion (9), we will show that:

$$\bigcap_{i \in \mathbb{N}} \ker H(A + BM_0)^i \neq \{0\}. \quad (13)$$

If P denotes the orthogonal projector on the subspace \mathfrak{S}^\perp , then the inclusion (7) becomes $AP(\mathfrak{S}^\perp) \subseteq P(\mathfrak{S}^\perp) + \text{Im } B$, or $\text{Im } AP \subseteq \text{Im } P + \text{Im } B$, and thus:

$$\text{Im } AP \subseteq \text{Im } [P, B], \quad (14)$$

with $AP \in \mathcal{L}(X, X)$, and where $[P, B] \in \mathcal{L}(X \times U, X)$ is defined by $[P, B] \begin{bmatrix} x \\ u \end{bmatrix} = Px + Bu$ for all $x \in X$ and $u \in U$. Now, according to Douglas theorem, the inclusion (14) ensures the existence of an operator $C \in \mathcal{L}(X, X \times U)$ such that $Cx = (C_1x, C_2x)$ for $x \in X$, and for which we get:

$$AP = [P, B]C = PC_1 + BC_2. \quad (15)$$

The relation (15) gives $APx = PC_1x + BC_2x$ for every $x \in X$, and particularly:

$$PC_1x = (A - BC_2)x \in \mathfrak{I}^\perp; \text{ for all } x \in \mathfrak{I}^\perp. \quad (16)$$

So, with $\mathfrak{I}^\perp \subseteq \ker H$, we get from (16):

$$H(A - BC_2)x = 0; \text{ for all } x \in \mathfrak{I}^\perp. \quad (17)$$

Now, since $\mathfrak{I}^\perp \neq \{0\}$, we can take an element $z_0 \neq 0$ in \mathfrak{I}^\perp for which:

$$Hz_0 = 0 \quad (\text{since } \mathfrak{I}^\perp \subseteq \ker H),$$

$$H(A - BC_2)z_0 = 0 \quad (\text{from (17)}).$$

Similarly, since $z_1 = PC_1z_0 = (A - BC_2)z_0 \in \mathfrak{I}^\perp$ in accordance with (16), then $H(A - BC_2)z_1 = 0$ once more from (17); which gives in return:

$$H(A - BC_2)z_1 = H(A - BC_2)^2z_0 = 0.$$

By induction, the resulting equalities $\{H(A - BC_2)^iz_0 = 0, i \in \mathbb{N}\}$ lead to $z_0 \in \bigcap_{i \in \mathbb{N}} \ker H(A - BC_2)^i$; which is the desired result (13) with $M_0 = -C_2$. ■

The following corollary(1) states a connection between the subspace of ideal observability \mathfrak{I} and the Kalman observability subspaces $K_M(H)$ for all $M \in \mathcal{L}(X, U)$. It may be considered as an add-on to the various properties of the ideal observability subspace \mathfrak{I} . The equality $\bigcap_{M \in \mathcal{L}(X, U)} K_M(H) = X$ in corollary(2) is another criterion for the ideal observability of the perturbed system (1)–(2).

Corollary 1 *The ideal observability subspace for the perturbed system (1)–(2) verifies the equality*

$$\mathfrak{I} = \bigcap_{M \in \mathcal{L}(X, U)} K_M(H)$$

where $K_M(H)$ denotes the Kalman-observability subspace defined by (12).

Proof. The proof of $\mathfrak{I} \subseteq \left(\bigcap_{M \in \mathcal{L}(X, U)} K_M(H) \right)$ derives simply from proposition(2). In fact, let $\xi \in \mathfrak{I}$, so that (3) holds true for every possible $u(\cdot) \in L^1([0, T]; U)$, and in particular for every $u(t) = Mx(t)$. Thus, ξ is a Kalman observable direction for the system (10) – (11), meaning that $\xi \in K_M(H)$ for all $M \in \mathcal{L}(X, U)$. Conversely, if $\xi \in \bigcap_{M \in \mathcal{L}(X, U)} K_M(H)$, then it is Kalman observable with $u(t) = Mx(t)$ for all $M \in \mathcal{L}(X, U)$, and thus $\xi \in \mathfrak{I}$ from proposition(2); whence $\left(\bigcap_{M \in \mathcal{L}(X, U)} K_M(H) \right) \subseteq \mathfrak{I}$. ■

Corollary 2 *The perturbed system (1) – (2) is ideally observable if and only if*

$$\bigcap_{M \in \mathcal{L}(X,U)} K_M(H) = X.$$

Proof. The system (1) – (2) is ideally observable iff $\mathfrak{I} = X$, and, from corollary(1), it is so iff $\mathfrak{I} = \bigcap_{M \in \mathcal{L}(X,U)} K_M(H) = X$. ■

3.2 On the lack of ideal observability with a scalar output

Under assumptions (A1)-(A3), we consider in this section the particular situation where the operator H is a simple functional on X . In this case, $Y = \mathbb{R}$ and the system (1) – (2) is observed by means of a scalar output $y(t) = Hx(t) \equiv \langle x(t), h \rangle$ on $[0, T]$, where $h \in X$ is the Riesz representation of H .

Then, a following explicit description of the ideal observability subspace \mathfrak{I} will show that \mathfrak{I} is finite dimensional, and the state $x(\cdot)$ can thus only be recognized along a finite number of directions. So, \mathfrak{I} can't cover X , and the ideal observability of (1)–(2) can never arise (unless $B = 0$). When $B \neq 0$, the proof given below will show explicitly and in accordance with proposition(2) the existence of a control in the form $u(t) = M_0x(t)$ for which there is no Kalman observability.

Proposition 3 *Under assumptions (A1)-(A3), with no restriction on the range of B , the system (1) – (2) with a scalar output is ideally observable if and only if the two following conditions are simultaneously checked:*

$$\overline{\mathbf{Sp}}(h, A^*h, A^{*2}h, \dots) = X, \quad (\text{C1})$$

$$B = 0. \quad (\text{C2})$$

If $B \neq 0$, the ideal observability subspace \mathfrak{I} is reduced to the finite dimensional subspace given by the expression:

$$\mathfrak{I} = \mathbf{Sp}(h, A^*h, A^{*2}h, \dots, A^{*k_0}h), \quad (18)$$

where k_0 is the smallest integer for which $HA^{k_0}B \neq 0$, with $HA^iB = 0$ for $i = 0, 1, \dots, k_0 - 1$.

Proof. Notice that condition (C1) means $\bigcap_{i=0}^{\infty} \text{Ker } HA^i = \{0\}$.

Now, if the two conditions are checked, (C1) means that (1)-(2) is Kalman observable; and (C2) means the absence of any disturbance. Thus, the system is ideally observable.

Reciprocally, ideal observability implies condition (C1).

For (C2), we shall prove by contradiction that, if $B \neq 0$, then condition (C1) can't be achieved, and \mathfrak{I} is reduced to the finite dimensional subspace in (18). So, let us suppose $B \neq 0$, which implies the existence of $k \in \mathbb{N}$ such that $HA^k B \neq 0$ (if not there would be $\text{Im } B \subseteq \bigcap_{i=0}^{\infty} \text{Ker } HA^i = \{0\}$, whence $B = 0$). Now, let us take the smallest integer k_0 for which $HA^{k_0} B \neq 0$ with $HA^i B = 0, i = 0, 1, \dots, k_0 - 1$, and choose $u_0 \in U$ for which $HA^{k_0} Bu_0 \neq 0_{\mathbb{R}}$. Let us consider the control $u(\cdot)$ expressed by:

$$u(t) = M_0 x(t) = -\frac{HA^{k_0+1}x(t)}{HA^{k_0}Bu_0} u_0 \left(= -u_0 \frac{HA^{k_0+1}x(t)}{HA^{k_0}Bu_0} \right).$$

For such a control, where $x \in X \longmapsto M_0 x = -u_0 \frac{HA^{k_0+1}x}{HA^{k_0}Bu_0}$ means clearly that $M_0 \in \mathcal{L}(X, U)$, the system (1) – (2) with a scalar output becomes:

$$x'(t) = (A + BM_0)x(t) \quad (19)$$

$$y(t) = Hx(t) = \langle x(t), h \rangle, \quad t \in [0, T]. \quad (20)$$

Taking into account the condition $HA^i B = 0, 0 \leq i < k_0$, we get successively:

$$H(A + BM_0) = HA + HBM_0 = HA,$$

$$H(A + BM_0)^2 = HA(A + BM_0) = HA^2 + HABM_0 = HA^2, \dots,$$

$$H(A + BM_0)^{k_0} = HA^{k_0-1}(A + BM_0) = HA^{k_0}.$$

Besides, by definition of M_0 , the relation $HA^{k_0+1}x + HA^{k_0}BM_0x = 0$ is true for all $x \in X$; which gives $H(A + BM_0)^{k_0+1} = HA^{k_0+1} + HA^{k_0}BM_0 = 0$, and leads to $H(A + BM_0)^{k_0+p} = 0$ for all $p \geq 1$. Consequently, according to (12), the Kalman observability subspace for (19) – (20) takes the form:

$$K_{M_0}(h) = \mathbf{Sp}(h, A^*h, A^{*2}h, \dots, A^{*k_0}h); \quad (21)$$

and thus, condition (C1) fails. Finally, for the equality $\mathfrak{I} = K_{M_0}(h)$, the inclusion $\mathfrak{I} \subseteq K_{M_0}(h)$ comes from corollary(1); and $K_{M_0}(h) \subseteq \mathfrak{I}$ derives from (3), with $HA^i B = 0, 0 \leq i < k_0$ and by successive differentiations (as done in the proof of proposition(1)). ■

Remark 2 Proposition(3) fails for $Y = \mathbb{R}^m$ with $m > 1$. An example given in [1] proves that two outputs (say $Y = \mathbb{R}^2$) may well be sufficient for the ideal observability of the system (1)-(2). Proposition(3) could be seen as an extension of a finite dimensional approach proposed in [10].

4 Conclusion

Taking into account the fact that much has been written on similar subjects in finite and infinite dimensional spaces, the reader is only referred to some selected works (regardless of their priority among too many others in literature) for their direct or indirect relationship with this paper. For instance, [6, 9, 12] deal with different observability matters, and for excellent expositions of different subjects related to this paper, we can't forget neither [7] for the infinite-dimensional theory nor [5, 13] for the finite dimensional aspects of control.

Concerning the concept of ideal observability for the system (1) – (2) in finite dimensional spaces (say $X = \mathbb{R}^n$) as in [4, 10, 11], different (but finely equivalent) definitions are used here and there according to the approach followed to solve the question. In this work, the statement of the problem, based on definition 1, leads to the obtained results practically without recourse to any explicit expression of the subspace \mathfrak{S} . Under assumptions (A1)-(A3), the lack of ideal observability is stated by means of an operational approach, with a "range inclusion" of operators instead of a geometric "subspace inclusion". Another criterion of ideal observability is also established through the expression of the ideal observability subspace \mathfrak{S} as the intersection of a special collection of Kalman observability subspaces. The given results show also that the size (positive) of the time interval does not play any role. Furthermore, a part of these results may be renewed almost just as they are in Banach space. Nevertheless, further extensions to unbounded operators would be of interest for systems generated by partial differential equations, and others with delay.

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