

RELATIONSHIP BETWEEN THE GOLDBERG SPECTRUM AND THE B-FREDHOLM SPECTRA

M. BENCHARRAT AND B. MESSIRDI

ABSTRACT. A classical result of J. Ph. Labrousse (Rev. Roumaine Math. Pures Appl. 25, 1391-1394, 1980) concerning the symmetric difference between the essential quasi-Fredholm spectrum and the Goldberg spectrum of closed operators in Hilbert spaces is extended to the case of B-Fredholm spectra. The obtained results are used to describe the essential spectrum and some B-Fredholm spectra of some transport operators.

1. TERMINOLOGY AND INTRODUCTION

Let \mathcal{H} be complex Hilbert space and let T be closed, densely defined linear operator on \mathcal{H} . We denote by $\mathcal{D}(T) \subset \mathcal{H}$ its domain, $R(T)$ its range, and $N(T)$ its null space. We denote by $\mathcal{C}(\mathcal{H})$ the set of all closed, densely defined linear operators. Let I denote the identity operator in \mathcal{H} . T is called a Kato type operator if we can write $T = T_1 \oplus T_0$ where T_0 is a nilpotent operator and T_1 is a semi-regular one. In 1958 Kato proved that a closed semi-Fredholm operator is of Kato type. In 1987 J.P Labrousse [15] studied and characterized the so-called quasi-Fredholm operators, in the case of Hilbert spaces, and he proved that this class coincides with the set of Kato type operators. In 1999 M. Berkani [5] studied a class of bounded linear quasi-Fredholm operators acting on a Banach space called B-Fredholm operators and characterized a B-Fredholm operator as the direct sum of a nilpotent operator and a Fredholm operator. Recently in [8] Berkani extended this characterization of B-Fredholm bounded operators to the class of B-Fredholm closed linear operators acting on a Hilbert space \mathcal{H} and study its properties.

2010 *Mathematics Subject Classification.* 47A10, 47A55, 47G20.

Key words and phrases. B-Fredholm operators; Quasi-Fredholm operators; Essential gausi-Fredholm spectrum, B-Fredholm spectrum, Transport operator.

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Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in \mathcal{C}(\mathcal{H})$ on the subspace $R(T^n)$. According to Berkani [8], T is said to be semi B-Fredholm (resp. B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n , viewed as a operator from the space $R(T^n)$ into itself, is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $T \in \mathcal{C}(\mathcal{H})$ is said to be B-Browder (resp. upper semi B-Browder, lower semi B-Browder, B-Weyl, upper semi B-Weyl, lower semi B-Weyl), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder operator (resp. upper semi-Browder, lower semi-Browder, Weyl, upper semi-weyl, lower semi-Weyl). T is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

- (1) $R(T^n) \cap N(T) = R(T^d) \cap N(T)$ for all $n \geq d$.
- (2) $R(T^d) \cap N(T)$ and $R(T^d) + N(T)$ are closed in \mathcal{H} .

These classes of operators motive the definition of several spectra. The B-Ferdholm spectrum is defined by

$$\sigma_{bf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Ferdholm}\}$$

the semi B-Fredholm spectrum is defined by

$$\sigma_{sbf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi B-Fredholm}\}$$

the upper semi B-Fredholm spectrum is defined by

$$\sigma_{ubf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Fredholm}\}$$

the lower semi B-Fredholm spectrum is defined by

$$\sigma_{lbf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Fredholm}\}$$

the B-Browder spectrum is defined by

$$\sigma_{bb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}$$

the upper semi B-Browder spectrum is defined by

$$\sigma_{ubb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\}$$

the lower semi B-Browder spectrum is defined by

$$\sigma_{lbb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\}$$

the B-Weyl spectrum is defined by

$$\sigma_{bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}$$

the upper semi B-Weyl spectrum is defined by

$$\sigma_{ubw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\}$$

the lower semi B-Weyl spectrum is defined by

$$\sigma_{lbw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}$$

while the quasi-Fredholm spectrum is defined by

$$\sigma_{qf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm}\}$$

We have

$$\sigma_{bf}(T) = \sigma_{ubf}(T) \cup \sigma_{lbf}(T) \quad (1.1)$$

$$\sigma_{bw}(T) = \sigma_{ubw}(T) \cup \sigma_{lbw}(T) \quad (1.2)$$

and

$$\sigma_{qf}(T) \subseteq \sigma_{bf}(T) \subseteq \sigma_{bw}(T) \subseteq \sigma_{bb}(T) = \sigma_{ubb}(T) \cup \sigma_{lbb}(T). \quad (1.3)$$

Note that all the B-spectra are closed subsets of \mathbb{C} (see [6], [15]), and may be empty. This is the case where the spectrum $\sigma(T)$ of T is a finite set of poles of the resolvent.

The first main question motivated by J.P Labrousse [16], in the Hilbert spaces, is the relationship between the essential quasi-Fredholm spectrum and the Goldberg spectrum, a subset of the spectrum containing all the complex numbers λ such that $R(\lambda - T)$ is not closed, noted $\sigma_{ec}(T)$ (see [10]), he proved that the symmetric difference of $\sigma_{ec}(T)$ and $\sigma_{qf}(T)$ is at most countable. The main purpose of this paper, is to extend the above result to the case of B-Fredholm spectra of closed linear operators.

The second main question is to describe, by the use the stability under finite-rank perturbations, the essential quasi-Fredholm and some B-Fredholm spectra of the one-dimensional transport operator with abstract boundary conditions on the Hilbert space. More precisely, we consider the unbounded operator

$$\begin{aligned} A_H\psi(x, \xi) &= -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \alpha(\xi)\psi(x, \xi) + \int_{-1}^1 \kappa(x, \xi, \xi')\psi(x, \xi') d\xi' \\ &= T_H\psi(x, \xi) + K\psi(x, \xi) \end{aligned}$$

with the boundary conditions

$$\psi^i = H(\psi^o),$$

where H is bounded linear operator defined on suitable boundary spaces and $\alpha(\cdot) \in L^\infty(-1, 1)$. Here $x \in (-a, a)$ and $\xi \in (-1, 1)$ and $\psi(x, \xi)$ represents the angular density of particles (for instance gas molecules, photons, or neutrons) in a homogeneous slab of thickness $2a$. The functions $\alpha(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are called, respectively, the collision frequency and the scattering kernel. Our analysis is based essentially on Theorem 2.5, Theorem 2.8 and the knowledge of the essential spectra of T_0 where T_0 (*i.e.*, $H = 0$) denotes the streaming operator with vacuum boundary conditions. We prove that, in the Hilbert space setting, if the classes of boundary and collision operators

are finite Rank operators then essential quasi-Fredholm and the B-Fredholm spectra of the operators T_0 and A_H coincide.

The outline of this work is as follows. In Section 2 we extend to the B-Fredholm spectra of closed linear operators in Hilbert spaces, the results concerning the symmetric difference between the essential quasi-Fredholm spectrum and the Goldberg spectrum of a closed operator proved by J.P Labrousse [16]. Section 3 is devoted to apply the obtained results to investigate the essential quasi-Fredholm spectrum and some B-Fredholm spectra of one-dimensional transport equation with general boundary conditions.

2. MAIN RESULTS

In the following, we consider the finite rank perturbations of a B-Fredholm (resp. quasi-Fredholm) operator T and their effect on the B-Fredholm (resp. quasi-Fredholm) spectrum. Recall that a linear bounded operator F is called a finite-rank operator if $\dim R(F) < \infty$.

Theorem 2.1. [12] *Let $T \in \mathcal{C}(\mathcal{H})$ and F a finite-rank operator on \mathcal{H} . Then $\sigma_{qf}(T + F) = \sigma_{qf}(T)$.*

This relation examined in [8] for the B-Fredholm spectrum.

Theorem 2.2. [8] *Let $T \in \mathcal{C}(\mathcal{H})$ be a B-Fredholm operator with $D(T)$ dense in \mathcal{H} and let F be a finite-rank operator. Then $T + F$ is a B-Fredholm operator and $ind(T + F) = ind(T)$, where $ind(T)$ means the index of T .*

In the sequel we need the following product theorem of two B-Fredholm operators.

Theorem 2.3. [9, Theorem 1.1] *If $S, U, V, T \in \mathcal{C}(\mathcal{H})$ are commuting operators such that $US + VT = I$ and if S, T are B-Fredholm operators. Then the product ST is a B-Fredholm operator and $ind(ST) = ind(T) + ind(S)$, where $ind(T)$ means the index of T .*

Proposition 2.4. *Let $T \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \rho(T)$. Then*

$$\mu \in \sigma_{bf}(T) \text{ if and only if } \mu \neq \lambda \text{ and } (\mu - \lambda)^{-1} \in \sigma_{bf}((\lambda I - T)^{-1}).$$

Proof. We start from the identity

$$(\lambda I - T)^{-1} - (\mu - \lambda)^{-1}I = -(\mu - \lambda)^{-1}(\mu I - T)(\lambda I - T)^{-1}$$

Since $(\lambda I - T)^{-1}$ is a bounded invertible operator commute with T , it follows from the Theorem 2.3 that $(\lambda I - T)^{-1} - (\mu - \lambda)^{-1}I$ is B-Fredholm if and only if $(\mu I - T)$ is B-Fredholm. This is equivalent to the statement of the theorem. \square

Note that in applications (transport operators, operators arising in dynamic populations, etc.) we deal with operators A and B such that $B = A + K$ where $A \in \mathcal{C}(\mathcal{H})$ and K is, in general, a closed (or closable) A -defined linear operator (i.e. $D(A) \subset D(B)$). For some physical conditions, we have information about the operator $(\lambda I - A)^{-1} - (\lambda I - B)^{-1}$ ($\lambda \in \rho(A) \cap \rho(B)$). So the following useful stability results.

Theorem 2.5. *Let $T, S \in \mathcal{C}(\mathcal{H})$. If $\lambda \in \rho(T) \cap \rho(S)$, such that $(\lambda I - T)^{-1} - (\lambda I - S)^{-1}$ is a finite rank operator, then*

$$\sigma_{bf}(T) = \sigma_{bf}(S).$$

Proof. The assumptions of the Theorem and the Theorem 2.2 implies that $\sigma_{bf}((\lambda I - T)^{-1}) = \sigma_{bf}((\lambda I - S)^{-1})$, and by Proposition 2.4 we have $\sigma_{bf}(T) = \sigma_{bf}(S)$. \square

Remark 2.6. Since a similar of Theorem 2.2 and Theorem 2.3 hold for the B-Weyl and quasi-Fredholm operators, then a similar statements of Proposition 2.4 hold if, instead of $\sigma_{bf}(T)$, we consider $\sigma_i(T)$, for $i = bw, qf$, and we have the following results.

Proposition 2.7. *Let $T \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \rho(T)$. Then*

$\mu \in \sigma_i(T)$ if and only if $\mu \neq \lambda$ and $(\mu - \lambda)^{-1} \in \sigma_i((\lambda I - T)^{-1})$, $i = bw, qf$.

Theorem 2.8. *Let $T, S \in \mathcal{C}(H)$. If $\lambda \in \rho(T) \cap \rho(S)$, such that $(\lambda I - T)^{-1} - (\lambda I - S)^{-1}$ is a finite rank operator, then*

$$\sigma_i(T) = \sigma_i(S), \quad i = bw, qf.$$

Note that most of the classes of linear operators, for example in Fredholm theory, require that the linear operators have closed range. Thus it is natural to consider the Goldberg spectrum or closed-range spectrum of the linear operator T , as follows

$$\sigma_{ec}(T) = \{\lambda \in \mathbb{C} ; R(\lambda - T) \text{ is not closed}\}.$$

Note that the Goldberg spectrum spectrum is a part of the spectrum has not good properties even for bounded operators T (See [4, p. 7-8]). However, the spectrum $\sigma_{ec}(T)$ can be used to obtain informations on the location in the complex plane of the various types of essential spectra, Fredholm, Weyl and Browder spectra etc..., for large classes of linear operators arising in applications. For example, integral, difference, and pseudo-differential operators (see [10] and the references therein).

In the following theorem, J. P. Labrousse in [16] characterized in the case of Hilbert spaces, a relation of the essential quasi-Fredholm spectrum and the Goldberg spectrum.

Theorem 2.9. *If $\lambda \in \sigma_{ec}(T)$ is non-isolated point then $\lambda \in \sigma_{qf}(T)$. Moreover, $\sigma_{qf}(T) \Delta \sigma_{ec}(T)$ is at most countable, where Δ is the symmetric difference of the sets $\sigma_{qf}(T)$ and $\sigma_{ec}(T)$.*

By the inclusions (1.1), (1.2) and (1.3), similar statements of Theorem 2.9 hold if, instead of $\sigma_{qf}(T)$, we consider the B-Fredholm spectra

Theorem 2.10. *If $\lambda \in \sigma_{ec}(T)$ is non-isolated point then $\lambda \in \sigma_i(T)$. Moreover, $\sigma_i(T) \Delta \sigma_{ec}(T)$ is at most countable, $i = bf, bw, bb, ubf, lbf, ubw, lbw, ubb, lbb$.*

In the next theorem we consider a situation which occurs in some concrete cases.

Theorem 2.11. *Let $T \in \mathcal{C}(\mathcal{H})$ an operator for which $\sigma_{ec}(T) = \sigma(T)$ and every λ is non-isolated in $\sigma(T)$. Then*

$$\sigma(T) = \sigma_{ec}(T) = \sigma_{qf}(T) = \sigma_{bf}(T) = \sigma_{bw}(T) = \sigma_{bb}(T).$$

Proof. Since $\lambda \in \sigma_{ec}(T)$ is non-isolated, according to Theorem 2.9 and Theorem 2.10,

$$\sigma_{ec}(T) = \sigma(T) \subseteq \sigma_{qf}(T) \subseteq \sigma_{bf}(T) \subseteq \sigma_{bw}(T) \subseteq \sigma_{bb}(T) \subseteq \sigma(T).$$

that is, the statement of theorem. \square

3. APPLICATION TO TRANSPORT EQUATIONS

In this section, we shall apply the results of last section to describe the quasi-Fredholm spectrum and some B-Fredholm spectra of the following one-dimensional transport operator acting on $X_2 = L^2([-a, a] \times [-1, 1], dx d\xi)$, $a > 0$:

$$A_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \alpha(\xi) \psi(x, \xi) + \int_{-1}^1 \kappa(x, \xi, \xi') \psi(x, \xi') d\xi' \quad (3.1)$$

with the boundary conditions

$$\psi^i = H(\psi^o). \quad (3.2)$$

Where $\alpha(\cdot) \in L^\infty(-1, 1)$ and H is bounded linear operator from X_2^0 to X_2^i , with

$$X_2^o := L^2[\{-a\} \times [-1, 0], |\xi| d\xi] \times L^2[\{a\} \times [0, 1], |\xi| d\xi] := X_{1,2}^o \times X_{2,2}^o$$

and

$$X_2^i := L^2[\{-a\} \times [0, 1], |\xi| d\xi] \times L^2[\{a\} \times [-1, 0], |\xi| d\xi] := X_{1,2}^i \times X_{2,2}^i$$

respectively equipped with the norms

$$\begin{aligned} \|\psi^o\|_{X_2^o} &= \left(\|\psi_1^o\|_{X_{1,2}^o}^2 + \|\psi_2^o\|_{X_{2,2}^o}^2 \right)^{\frac{1}{2}} \\ &= \left[\int_{-1}^0 |\psi(-a, \xi)|^2 |\xi| d\xi + \int_0^1 |\psi(a, \xi)|^2 |\xi| d\xi \right]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \|\psi^i\|_{X_2^i} &= \left(\|\psi_1^i\|_{X_{1,2}^i}^2 + \|\psi_2^i\|_{X_{2,2}^i}^2 \right)^{\frac{1}{2}} \\ &= \left[\int_0^1 |\psi(-a, \xi)|^2 |\xi| d\xi + \int_{-1}^0 |\psi(a, \xi)|^2 |\xi| d\xi \right]^{\frac{1}{2}}. \end{aligned}$$

The operator A_H describes the transport of particle (neutrons, photons, molecules of gas, etc.) in the domain Ω . The function ψ represents the number (or probability) density of particles having the position x and the velocity ξ . The functions $\alpha(\cdot)$ and $\kappa(\cdot, \cdot)$ are called, respectively, the collision frequency and the scattering kernel. Let us first introduce the functional space \mathcal{W}_2 defined by:

$$\mathcal{W}_2 = \left\{ \psi \in X_2 : \xi \frac{\partial \psi}{\partial x} \in X_2 \right\}.$$

It is well-known that any function ψ in \mathcal{W}_2 has traces on $-a$ and a in X_2^o and X_2^i . They are denoted, respectively by ψ^o and ψ^i , and represent the outgoing and the incoming fluxes.

The streaming operator T_H associated with the boundary condition (3.2) is:

$$\left\{ \begin{array}{l} T_H : \mathcal{D}(T_H) \subseteq X_2 \longrightarrow X_2 \\ \psi \longrightarrow T_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \alpha(\xi) \psi(x, \xi) \\ \mathcal{D}(T_H) = \{ \psi \in \mathcal{W}_2 \text{ such that } \psi^i = H(\psi^o) \} \end{array} \right.$$

The spectrum of the operator T_0 (i.e., $H = 0$) was analyzed in [18]. In particular we have

$$\sigma(T_0) = \sigma_{ec}(T_0) = \sigma_c(T_0) = \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\lambda^* \}, \quad (3.3)$$

where $\sigma_c(T_0)$ is the continuous spectrum of T_0 and $\lambda^* = \liminf_{|\xi| \rightarrow 0} \alpha(\xi)$, for more detail see [17] and [18].

Combining the equality in Theorem 2.11 with Eq. (3.3) we obtain

$$\sigma_{ei}(T_0) = \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\lambda^* \}, \quad i \in \{qf, bf, bw, bb\} \quad (3.4)$$

Let us now consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi \quad (3.5)$$

where φ is a given element of X_2 and the unknown ψ must be found in $\mathcal{D}(T_H)$. For $\operatorname{Re}\lambda + \lambda^* > 0$, where $\lambda^* = -\liminf_{|\xi| \rightarrow 0} \alpha(\xi)$, the solution of (3.5) is formally given by

$$\psi(x, \xi) = \begin{cases} \psi(-a, \xi) e^{-\frac{(\lambda+\alpha(\xi))|a+x|}{|\xi|}} \\ \quad + \frac{1}{|\xi|} \int_{-a}^x e^{-\frac{(\lambda+\alpha(\xi))|x-x'|}{|\xi|}} \varphi(x', \xi) dx' & \text{if } 0 < \xi < 1, \\ \psi(a, \xi) e^{-\frac{(\lambda+\alpha(\xi))|a-x|}{|\xi|}} \\ \quad + \frac{1}{|\xi|} \int_x^a e^{-\frac{(\lambda+\alpha(\xi))|x-x'|}{|\xi|}} \varphi(x', \xi) dx' & \text{if } -1 < \xi < 0. \end{cases}$$

where

$$\begin{cases} \psi(a, \xi) = \psi(-a, \xi) e^{-\frac{2a(\lambda+\alpha(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda+\alpha(\xi))|a-x|}{|\xi|}} \varphi(x, \xi) dx \\ \psi(-a, \xi) = \psi(a, \xi) e^{-\frac{2a(\lambda+\alpha(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda+\alpha(\xi))|a+x|}{|\xi|}} \varphi(x, \xi) dx \end{cases}$$

In the sequel we shall consider the following operators:

$$\begin{cases} M_\lambda : X_p^i \longrightarrow X_p^o, M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \text{ where} \\ M_\lambda^+ u(-a, \xi) = u(-a, \xi) e^{-\frac{2a}{|\xi|}(\lambda+\alpha(\xi))} & \text{if } 0 < \xi < 1 \\ M_\lambda^- u(a, \xi) = u(a, \xi) e^{-\frac{2a}{|\xi|}(\lambda+\alpha(\xi))} & \text{if } 0 < \xi < 1 \\ B_\lambda : X_p^i \longrightarrow X_p; B_\lambda u = \chi_{(-1,0)}(\xi) B_\lambda^- u + \chi_{(0,1)}(\xi) B_\lambda^+ u \text{ where} \\ (B_\lambda^+ u)(-a, \xi) = u(-a, \xi) e^{-\frac{(\lambda+\alpha(\xi))|a+x|}{|\xi|}}, & \text{if } 0 < \xi < 1 \\ (B_\lambda^- u)(-a, \xi) = u(-a, \xi) e^{-\frac{(\lambda+\alpha(\xi))|a-x|}{|\xi|}}, & \text{if } -1 < \xi < 0 \end{cases}$$

$$\left\{ \begin{array}{l} G_\lambda : X_p \longrightarrow X_p^o, G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \text{ where} \\ G_\lambda^+ \varphi = \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda+\alpha(\xi))}{|\xi|}|a-x|} \varphi(x, \xi) dx, \text{ if } 0 < \xi < 1 \\ G_\lambda^- \varphi = \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda+\alpha(\xi))}{|\xi|}|a+x|} \varphi(x, \xi) dx, \text{ if } -1 < \xi < 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} C_\lambda : X_p \longrightarrow X_p; C_\lambda \varphi = \chi_{(-1,0)}(\xi) C_\lambda^+ \varphi + \chi_{(0,1)}(\xi) C_\lambda^- \varphi \text{ where} \\ C_\lambda^- \varphi = \frac{1}{|\xi|} \int_{-a}^x e^{-\frac{(\lambda+\alpha(\xi))}{|\xi|}|x'-x|} \varphi(x', \xi) dx' \text{ if } 0 < \xi < 1 \\ C_\lambda^+ \varphi = \frac{1}{|\xi|} \int_x^a e^{-\frac{(\lambda+\alpha(\xi))}{|\xi|}|x'-x|} \varphi(x', \xi) dx' \text{ if } -1 < \xi < 0 \end{array} \right.$$

where $\chi_{(-1,0)}$ and $\chi_{(0,1)}$ denote, respectively the characteristic functions of the intervals $(-1, 0)$ and $(0, 1)$. The operators M_λ , B_λ , G_λ and C_λ are bounded on their respective domains respectively, by $e^{-2a(\operatorname{Re}\lambda + \lambda^*)}$, $[2(\operatorname{Re}\lambda + \lambda^*)]^{-\frac{1}{2}}$, $[(\operatorname{Re}\lambda + \lambda^*)]^{-\frac{1}{2}}$ and $[(\operatorname{Re}\lambda + \lambda^*)]^{-1}$. We define the real λ_0 by

$$\lambda_0 = \begin{cases} -\lambda^*, & \text{if } \|H\| \leq 1 \\ \frac{1}{2a} \log \|H\| - \lambda^* & \text{if } \|H\| > 1 \end{cases}$$

It follows from the norm estimate of M_λ that, for $\operatorname{Re}\lambda > \lambda_0$, $\|M_\lambda H\| < 1$ and consequently

$$\psi_0 = \sum_{n=0}^{+\infty} (M_\lambda H)^n G_\lambda \varphi \quad (3.6)$$

On the other hand, we have

$$\begin{aligned} \psi &= B_\lambda H \psi_0 + C_\lambda \varphi \\ &= (B_\lambda H \sum_{n=0}^{+\infty} (M_\lambda H)^n G_\lambda + C_\lambda) \varphi \end{aligned}$$

Hence, $\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > \lambda_0\} \subset \rho(T_H)$ and for $\operatorname{Re}\lambda > \lambda_0$

$$(\lambda - T_H)^{-1} = B_\lambda H (I - M_\lambda H)^{-1} G_\lambda + C_\lambda \quad (3.7)$$

Theorem 3.1. *Suppose that the boundary operator H is a finite rank operator, then*

$$\sigma_i(T_H) = \sigma_i(T_0), \quad i = bf, bw, bb, qf.$$

Proof. If $\operatorname{Re}\lambda > \lambda_0$, then $\lambda \in \rho(T_H) \cap \rho(T_0)$ and

$$(\lambda - T_H)^{-1} - (\lambda - T_0)^{-1} = D_\lambda,$$

where

$$D_\lambda = B_\lambda H \sum_{n=0}^{+\infty} (M_\lambda H)^n G_\lambda.$$

Since H is a finite rank operator, then D_λ is finite rank operator. This implies, by Theorem 2.5 and Theorem 2.8, the statement of theorem. \square

The transport operator (3.1) can be formulated as follows $A_H = T_H + K$, where K denotes the following collision operator

$$\begin{cases} K : X_2 \longrightarrow X_2 \\ \psi \longrightarrow \int_{-1}^1 \kappa(x, \xi, \xi') \psi(x, \xi') d\xi' \end{cases}$$

and κ is of the form:

$$\kappa(x, \xi, \xi') = \sum_{i=1}^n \alpha_i(x) f_i(\xi) g_i(\xi') \quad (3.8)$$

where $\alpha_i(\cdot) \in L^2([-a, a], dx)$, $f_i(\cdot) \in L^2([-1, 1], d\xi)$ and $g_i(\cdot) \in L^2([-1, 1], d\xi')$.

Theorem 3.2. *Suppose that the collision operator in the form (3.8) on X_2 , then*

$$\sigma_i(A_H) = \sigma_i(T_H), \quad i = bf, bw, bb, qf.$$

Furthermore, if the boundary operator H is a finite rank operator then

$$\sigma_i(A_H) = \sigma_i(T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq -\lambda^*\}, \quad i = bf, bw, bb, qf.$$

Proof. By virtue of the relation (3.8) the rang of operator K is contained in the subspace of X_2 spanned by the families $\{\alpha_1(\cdot)f_1(\cdot), \dots, \alpha_n(\cdot)f_n(\cdot)\}$. Then K is of finite rank operator. Now by Theorem 2.5 and Theorem 2.8 we have

$$\sigma_i(A_H) = \sigma_i(T_H + K) = \sigma_i(T_H), \quad i = bf, bw, bb, qf.$$

Furthermore, if the boundary operator H is a finite rank operator, the desired result follows from the relation (3.4) and Theorem 3.1. \square

Remark 3.3. As an example of the boundary operator H to be a finite rank operator, we can take H as a projection operator on the subspace spanned by a finite orthonormal basis of $X_{1,2}^i \times X_{2,2}^i$ and equal the null operator otherwise.

Acknowledgement. The authors are grateful to anonymous referee for valuable remarks and comments, which significantly contributed to the quality of the paper.

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(Received: December 3, 2015)

(Revised: April 6, 2016)

Mohammed Benharrat
 Département de Mathématiques
 et Informatique
 Ecole Nationale Polytechnique d'Oran
 B.P. 1523 Oran-El M'Naouar
 31000 Oran
 Algérie
 mohammed.benharrat@enp-oran.dz

Bekkai Messirdi
Département de Mathématiques
Université d'Oran 1
31000 Oran
Algérie
messirdi.bekkai@univ-oran.dz