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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A NEUMANN BOUNDARY-VALUE PROBLEM

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Abstract. In this article, we show the existence and uniqueness of positive solutions for perturbed Neumann boundary-value problems of second-order differential equations. We use a fixed point theorem for general α-concave operators.

1. Introduction

This article is devoted to the existence and uniqueness of positive solutions for the perturbed Neumann boundary-value problem

$$\begin{gathered} u''(t) + m^2 u(t) = f\big(t, u(t)\big) + g(t), \quad 0 < t < 1 \\ u'(0) = u'(1) = 0, \end{gathered} \tag{1.1}$$

where m is a positive constant, $f : [0,1] \times [0, s_0) \to [0, +\infty)$ and $g : [0,1] \to [0, +\infty)$ are given continuous functions with g not identically equal to 0 and s_0 is a given positive constant.

The study of nonlinear differential equations is a question of great importance and still relevant. These equations arise not only in mathematics fields but also in other branches of science. Many works have been devoted to this subject and significant results have been obtained via fixed point theory in Banach spaces, see [1, 5, 7].

The Neumann boundary value problems become one of the main concerns for this kind of differential equations, we cite for example [2, 3, 6, 8, 9, 10, 11, 12, 13]. Many attempts have been made to develop criteria which guarantee the existence and uniqueness of positive solutions to these problems see [3, 7]. Krasnoselskii [7] studied the u_0-concave operator with $u_0 > \theta$. Chen [3] established fixed point theorems for α-sublinear mapping where $\alpha \in (0, 1)$.

The problem

$$\begin{gathered} -u''(t) + m^2 u(t) = f_1(t, u), \quad 0 < t < 1 \\ u'(0) = u'(1) = 0. \end{gathered} \tag{1.2}$$

has been studied, for m a positive constant.

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Using the fixed point theorem for increasing α-concave operators, Zhang and Zhai in [13], obtained the existence and uniqueness of positive solutions for problem (1.2) with $f_1(t,u(t)) = f(t,u(t)) + g(t)$, under certain conditions on f and g.

The same problem with $f_1(t,u) = |u|^p f(t)$, $p > 1$, $m > 0$ and f a positive continuous and symmetric function, has been considered by Bensedik and Bouchekif in [2]. They established the existence, uniqueness and symmetry of positive solutions by using a fixed point theorem of Krasnoselskii in a cone (see [5, 7]). Mays and Norbury [8] studied problem (1.2) with $f_1(t,u(t)) = u^2(1+\sin t)$ by using analytical and numerical methods.

To our knowledge, only a few results are known about problem (1.1). Recently, Zhai and Cao [11] presented the concept of α-u_0-concave operator which generalizes the previous concepts. More explicitly they gave some new existence and uniqueness theorems of fixed points for α-u_0-concave increasing operators in ordered Banach spaces. Zhang and Zhai [13] proved the existence of a unique positive solution in a certain cone under sufficient conditions on f and g, for $m \in (0, \pi/2)$.

A natural and interesting question is whether results concerning the positive solutions of (1.1) with $m \in (0, \pi/2)$ remain valid for an arbitrary positive constant m. The response is affirmative.

Before giving our main result, we state here some definitions, notation and known results. For more details, the reader can consult the books [5, 7].

Let $(E, \|\cdot\|)$ be a real Banach space and K be a cone in E. The cone K defines a partial ordering in E through $x \le y \Leftrightarrow y - x \in K$, $\forall x, y \in E$.

K is said to be normal if there exists a positive constant N such that for any $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where θ denotes the zero element in E. Given $h > \theta$ (i.e. $h - \theta \in K$ and $h \ne \theta$), we denote by K_h the set

$$\{u \in K : \exists \lambda(u); \mu(u) > 0;\ u - \mu(u)h \in K \text{ and } \lambda(u)h - u \in K\}.$$

We recall the fixed point theorem for general α-concave operators which is the main tool for proving the existence and uniqueness of positive solutions in K_h for the problem $u = Au + u_0$ where u_0 is given. We start by the following definition.

Definition 1.1. The operator $A : K_h \to K_h$ is said to be a general α-concave operator if: For any $u \in K_h$ and $t \in [0,1]$, there exists $\alpha(t) \in (0,1)$ such that $A(tu) \ge t^{\alpha(t)} A(u)$.

Theorem 1.2 ([12]). *Assume that the cone K is normal and the operator A satisfies the following conditions:*

- (A1) *$A : K_h \to K_h$ is increasing*
- (A2) *For any $u \in K_h$ and $t \in [0,1]$, there exists $\alpha(t) \in (0,1)$ such that $A(tu) \ge t^{\alpha(t)} A(u)$*
- (A3) *There exists a constant $l \ge 0$ such that $u_0 \in [\theta, lh]$.*

Then the operator equation $u = Au + u_0$ has a unique solution in K_h.

By a positive solution of (1.1), we understand a function $u(t) \in C^2([0,1])$, which is positive for $0 < t < 1$, and satisfies the differential equation and the boundary conditions in (1.1).

In this paper, the Banach space $E = C([0,1])$ is endowed with the norm $|u|_0 := \max_{t\in[0,1]} |u(t)|$. Let

$$K = \{u \in E : u(t) \ge 0 \text{ for } t \in [0,1]\},$$

the normal cone of normality constant 1, and

$$K_h := \{u \in K : \exists\lambda(u); \mu(u) > 0 \text{ such that } \mu(u)h \le u \le \lambda(u)h\}$$

where $h \in E$ is a given strictly positive function.

Let m be a positive number, and m_1 chosen arbitrarily in $(0, \pi/2)$ such that $m^2 = m_1^2 + m_2^2$. Consider the following assumptions:

(F1) $f(t, s)$ is increasing in $s \in (0, s_0)$ for fixed t in $[0, 1]$ and $f'_s(t, 0) = +\infty$;

(F2) For any $\gamma \in (0, 1)$, $s \in (0, s_0)$ there exists $\varphi(\gamma) \in (\gamma, 1]$ such that

$$f(t, \gamma s) \ge \varphi(\gamma)f(t, s), \quad \text{for } t \in [0, 1].$$

(G1) There exists $s_1 \in (0, s_0)$ such that

$$|g|_0 \le \big(m_1 \sin m_1 + m_2^2\big)s_1 - f(t, s_1) \quad \forall t \in [0, 1].$$

Note that for large s, there is no condition assumed on f. This is in contrast with most of the papers cited above, concerning similar problems. Now, we give our main result.

Theorem 1.3. *Assume that* (F1), (F2), (G1) *hold. Then* (1.1) *with* $m > 0$ *has a unique solution in* K_h*, where*

$$h(t) = \cos m_1 t \cos m_1(1 - t), \quad t \in [0, 1],$$
$$m_1 \in (0, \pi/2) \quad \textit{such that } m^2 = m_1^2 + m_2^2.$$

This work is organized as follows. In Section 2, we introduce the modified problem, Section 3 is concerned with the existence and uniqueness result.

2. Modified problem

Let $G_m(t, s)$ be the Green's function for the boundary-value problem

$$u''(t) + m^2u(t) = 0, \quad 0 < t < 1$$
$$u'(0) = u'(1) = 0.$$

Explicitly, G_m is given as [4]

$$G_m(t, s) = \frac{1}{m \sin m}\begin{cases} \cos ms \cos m(1 - t), & \text{if } 0 \le s \le t \le 1 \\ \cos mt \cos m(1 - s), & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Before formulating the modified problem, we recall the results of Zhang and Zhai [13]. They studied the problem

$$\begin{gathered} u''(t) + m^2u(t) = \widetilde{f}(t, u(t)) + g(t), \quad 0 < t < 1 \\ u'(0) = u'(1) = 0. \end{gathered} \tag{2.1}$$

Under the following hypothesis:

(H1) $\widetilde{f}(t, s)$ is increasing in $s \in \mathbb{R}^+$for fixed t.

(H2) For any $\gamma \in (0, 1)$, there exists $\varphi_1(\gamma) \in (\gamma, 1]$ such that

$$\widetilde{f}(t, \gamma s) \ge \varphi_1(\gamma)\widetilde{f}(t, s), \quad \text{for } t \in [0, 1].$$

(H3) For any $t \in [0, 1]$, $\widetilde{f}(t, \cos^2 m) > 0$, for $m \in (0, \pi/2)$.

The following result is obtained in [13].

Theorem 2.1. *Assume that* (H1), (H2), (H3) *hold. Then,* (2.1) *with* $m \in (0, \pi/2)$ *has a unique solution in* K_h*, where*

$$h(t) = \cos mt \cos m(1-t), \quad t \in [0,1].$$

The solution in the above theorem is represented as

$$u(t) = \int_0^1 G_m(t,s) f(s,u(s)) ds + \int_0^1 G_m(t,s) g(s) ds.$$

Our idea is to use Theorem 2.1 by introducing the modified problem below that reduces problem (1.1) to $m_1 \in (0, \pi/2)$:

$$\begin{gathered} u''(t) + m_1^2 u(t) = \tilde{f}(t, u(t)) + g(t), \quad 0 < t < 1 \\ u'(0) = u'(1) = 0 \end{gathered} \tag{2.2}$$

where

$$\tilde{f}(t,s) = \begin{cases} f(t,s) - m_2^2 s & \text{if } s \le s_2 \\ \mu(t,s_2) s^{\alpha} & \text{if } s \ge s_2, \end{cases}$$

with $\mu(t, s_2) = (f(t, s_2) - m_2^2 s_2) s_2^{-\alpha}$ is a positive continuous function for $t \in [0,1]$, $\alpha \in (0,1)$ fixed and $s_2 = \min(s_1, s_2')$ where s_2' will be defined later.

To prove existence and uniqueness of solutions for the modified problem (2.2) we apply theorem 2.1. First, we show that (H1) remains valid for $\tilde{f}(t,s)$. Indeed by hypothesis (F1) there exists $s_2' > 0$ such that

$$\frac{f(t,r_2) - f(t,r_1)}{r_2 - r_1} \ge m^2, \quad \text{for } 0 \le r_1 < r_2 \le s_2'. \tag{2.3}$$

Thus $\tilde{f}(t,s)$ is increasing in $s \in \mathbb{R}^+$ for $t \in [0,1]$.

Next, we prove that (H2) holds. We know that for any $\gamma \in (0,1)$, there exists $\varphi(\gamma) \in (\gamma, 1]$ such that $f(t, \gamma s) \ge \varphi(\gamma) f(t,s)$, for $t \in [0,1]$. So, for $s \le s_2$, we have

$$\begin{aligned} \tilde{f}(t,\gamma s) &= f(t,\gamma s) - m_2^2 \gamma s \\ &\ge \varphi(\gamma) f(t,s) - m_2^2 \gamma s \\ &\ge \varphi(\gamma) \tilde{f}(t,s). \end{aligned}$$

For $s \ge s_2$, we have

$$\tilde{f}(t,\gamma s) = \mu(t,s_2)(\gamma s)^{\alpha}(t) = \gamma^{\alpha} \tilde{f}(t,s).$$

Choosing $\varphi_1(\gamma) = \min(\varphi(\gamma), \gamma^{\alpha})$ which satisfy $\varphi_1(\gamma) \in (\gamma, 1]$, thus we obtain the desired result.

Finally, it's clear that $\tilde{f}(t, \cos^2 m_1) > 0$. Thus, we conclude that problem (2.2) admits a unique solution $\tilde{u}$ in K_h.

3. Existence and uniqueness results

To conclude that $\tilde{u}$ is also a solution of the problem (1.1), it suffices to prove that $|\widetilde{u}|_0 \le s_2$. The solution $\widetilde{u}$ is given by

$$\widetilde{u}(t) = \int_0^1 G_{m_1}(t,s)[\widetilde{f}(s, \widetilde{u}(s)) + g(s)] ds.$$

Observe that $|G_{m_1}(t,r)| \le (m_1 \sin m_1)^{-1}$ for all $t, r \in [0,1]$. Therefore, we obtain the estimate

$$|\widetilde{u}|_0 \le \overline{\mu}(s_2)(m_1 \sin m_1)^{-1} |\widetilde{u}|_0^{\alpha} + |g|_0 (m_1 \sin m_1)^{-1},$$

where $\overline{\mu}(s_2) := \max_{t\in[0,1]}(f(t,s_2) - m_2^2 s_2)s_2^{-\alpha}$.

Let $\psi(s) := s - \overline{\mu}(s_2)(m_1 \sin m_1)^{-1}s^\alpha - |g|_0(m_1 \sin m_1)^{-1}$. We have $|\widetilde{u}|_0 \le s_2$ if $\psi(s_2) \ge 0$, which follows from conditions (2.3) and (G1). Thus $\tilde{u}$ is also the unique solution of the problem (1.1) in K_h with $h(t) = \cos m_1 t \cos m_1(1-t)$, $t \in [0,1]$.

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