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On Closed Range Operators in Hilbert Space

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Abstract

Let $C(H)$ be the set of all linear closed operators densely defined on a Hilbert space H. This paper is devoted to the study of the equivalent statements for which an operator A belongs to $C(H)$ has a closed range.

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1 Introduction

Let $A \in C(H)$. $D(A)$, $N(A)$ and $R(A)$ denotes respectively the domain, null space and the range of A. $A^*$is the adjoint operator of A. $B(H)$ is the space of all linear bounded operators on H. $C(H)$ is the set of all all linear closed operators densely defined On H

We recall that every operator $A \in C(H)$ can be regarded as a bounded operator A_0 from the Hilbert space $D(A)$ into H when $D(A)$ is equipped with the graph inner product $\langle x, y\rangle_A = \langle x, y\rangle + \langle Ax, Ay\rangle$, $x, y \in D(A)$. It is, in particular, known that there is a connexion between the closedness of $R(A)$ and some topological properties of A_0.

It is also well known that the knowledge whether the range of A is closed or not is particularly important when one want to solve the operator equation $Ax = y$ in H. In fact, by vertue of the identity $\overline{R(A)} = N(A^*)^{\perp}$, between the closure $\overline{R(A)}$ of $R(A)$ and the orthogonal subspace $N(A^*)^{\perp}$ of $N(A^*)$, we see that the closedness of $R(A)$ implies that an equation $Ax = y$ is solvable if the space of such y is the orthogonal complement of the solutions to the homogeneous equation $A^*z = 0$. This last information is very useful since oftentimes $N(A^*)$ is finite dimensional. In the infinite dimensional case the equation can be solved by reduceding the problem to a sequence of finite matrix equations (see e.g. Groetsch [3]).

The closedness of $R(A)$ is equivalent to A regular in the sense that A admits a bounded generalized inverse $A^+ \in C(H)$ for which $A = AA^+A$, so that if $y = Ax$ can be solved then $x = A^+y$ is a solution.

There are many important applications of the closedness of the range in the spectral study of differential operators and also in the context of perturbation theory (see e.g. [2]).

In this paper we establish a certain number of results concerning the closedness of $R(A)$ if $A \in C(H)$ by using different concepts in basic operator theory : Adjoint operators, closedness and lower semiboundedness of operators, reduced minimum modulus, generalized inverse, Hyers-Ulam stability, bounded transform R_A and S_A, spectrum, and regular operators. The methods discussed here give a rather complete list concerning the closedness of the operator ranges, they complete certain results of [5]. These characterizations are afterward applied to multiplication operators.

In another paper in preparation we give sufficient conditions for which the closedness of the range be invariant under addition, composition and limits.

2 Preliminaries

If $A \in C(H)$, the reduced minimum modulus of A is defined by

$$\gamma(A) = \inf \{\|Ax\| \ ; \ x \in \mathcal{C}(A), \|x\| = 1\}$$

where the subspace $\mathcal{C}(A) = D(A) \cap N(A)^{\perp}$ is the carrier of A (note that $D(A)$ is the orthogonal direct sum of $N(A)$ and $\mathcal{C}(A)$).

We say A has the Hyers-Ulam stability if there exists a constant $K > 0$ satisfying the following property :

For any $\varepsilon > 0$ and $x \in H$ with $\|Ax\| \leq \varepsilon$, there exists $x_0 \in H$ such that $Ax_0 = 0$ and $\|x - x_0\| \leq K\varepsilon$.

The positive number $K > 0$ is called the Hyers-Ulam stability constant for A and denote K_A the infimum of all Hyers-Ulam stability constants for A.

If $A \in C(H)$, then $(I + A^*A)$ is bijective densely defined operator on H, let now R_A denote the operator $(I + A^*A)^{-1}$. The selfadjoint positive definite operator R_A has a unique positive definite selfadjoint square root, which we denote by $S_A = \sqrt{R_A}$.

In the begining, we have to remind different properties of the operators R_A and S_A.

Proposition 1 *(see [1]) Let $A \in C(H)$, we have the following results:*

1) *If $u \in D(A)$, $R_{A^*}Au = AR_A$ and $S_{A^*}Au = AS_Au$*

2) $(AR_A)^* = A^*R_{A^*}$ *and* $(AS_A)^* = A^*S_{A^*}$

3) $\|R_Au\|^2 + \|AR_Au\|^2 = \langle u, R_Au\rangle$ *and* $\|S_Au\|^2 + \|AS_Au\|^2 = \|u\|^2$, *for all* $u \in H$

Then, R_A, $AR_A, S_A, AS_A \in B(H)$; $\|R_A\| \leq 1$, $\|AR_A\| \leq \frac{1}{2}$, $\|S_A\| \leq 1$, $\|AS_A\| \leq 1$.

4) $N(AR_A) = N(A), N(AS_A) = N(A),\ R(S_A) = D(A)$.

3 Closedeness of the range

In this section we collect several known results founded in the mathematical litterature about the closedness of the range of an unbounded operator on a Hilbert space and we include other equivalent new results related to R_A and S_A.

Theorem 2 *For $A \in C(H)$, the following statements are equivalent:*

1) $R(A)$ *is closed*

2) $R(A^*)$ *is closed*

3) $A_0 = A_{|C(A)}$ *has a bounded inverse*

4) $\|Ax\| \geq m\|x\|$, *for all $x \in C(A)$ with $m > 0$*

5) $\gamma(A) > 0$

6) A^+*is bounded*

7) $\gamma(A) = \gamma(A^*)$

8) $R(A^*A)$ *is closed*

9) $R(AA^*)$ *is closed*

10) 0 *is not an accumulation point of the spectrum $\sigma(A^*A)$ of A^*A or equivalently, there exists $\delta > 0$ such that* $\sigma(A^*A) \subset \{0\} \cup [\delta, +\infty[$

11) $R(AR_A)$ *is closed*

12) $R(A^*R_{A^*})$ *is closed*

13) *A has the Hyers-Ulam stability*

14) *A_0 has the Hyers-Ulam stability*

15) $H = N(A^*) \oplus R(A)$

16) $R(AS_A)$ *is closed*

17) $R(A^*S_{A^*})$ *is closed*

18) $0 \in reg\ (A)$

where $reg(A)$ is the set of $\lambda \in \mathbb{C}$ such that $(A - \lambda I)$ admits a generalized analytic resolvent operator in a neighborhood of 0

19) *A is regular (in the sense that $N(A^n) \subseteq R(A)$; for all $n \in \mathbb{N}$)*

20) $0 \in reg\ (A^*)$

21) A^* *is regular*

Proof. For the equivalences (1) to (12) we can consult the works of Kulkarni, Nair and Ramesh [5] and those of Hirasawa and Miura [4] for (13), (14). The equivalence between (1) and (15) is directly proved in [2] .

Now, we show the equivalence between 16) and 17) and the precedent properties. Clearly from 2) in proposition 1, 16) is equivalent to 17) if we use the equivalence between 1) and 2).

We have:

$$\|AS_Av\|^2 \geq \gamma^2(A) \ \ \|S_Av\|^2 = \gamma^2(A)\left[\|v\|^2 - \|AS_Av\|^2\right] \text{ , for all } v \in H$$

Then

$$\gamma(AS_A) \geq \frac{\gamma(A)}{\sqrt{1+\gamma^2(A)}}$$

This inequality shows that $\gamma(AS_A) > 0$ if $\gamma(A) > 0$.

Now from the properties 3) and 4) in proposition 1, we obtain that $\gamma(AS_A) \leq \gamma(A)$. Therefore the closedness of $R(AS_A)$ implies that $R(A)$ is closed too.

For the equivalence between the closedness of $R(A)$ and (18) to (21) we can see the works of Labrousse and Mbekhta [6]. ■

4 Application to Multiplication Operators

Let (X,μ) be a measurable space with μ being a positive σ-finite measure on X, ϕ a measurable complex function on X and the multiplication operator $T_\phi\varphi = \phi\varphi$ by ϕ. T_ϕ is an unbounded densely defined operator on $L^2(X,\mu)$ with domain $D(T_\phi) = \{\varphi \in L^2(X,\mu) \ ; \ \phi\varphi \in L^2(X,\mu)\}$. Recall that T_ϕ is bounded if and only if ϕ is a.e. bounded on X, in this case $\|T_\phi\| = \|\phi\|_{L^\infty(X)} = ess \sup\limits_{x\in X} |\phi(x)|$.

Let $E = \{x \in X \ ; \ \phi(x) = 0\}$ and $L = \{\varphi \in L^2(X,\mu) \ ; \ \varphi(x) = 0$ for every $x \in E\}$.

Then, $\overline{R(T_\phi)} = L$.

Indeed, obviously $R(T_\phi) \subset L$ and L is closed in $L^2(X,\mu)$. Thus, $\overline{R(T_\phi)} \subset L$.

Conversely, let us pose:

$$\begin{cases} F = \{x \in X \ ; \ \phi(x) \neq 0\} \\ F_n = \{x \in X \ ; \ |\phi(x)| \geq \frac{1}{n}\} \text{ for every } n \in \mathbb{N}^* \end{cases}$$

and define $f = \frac{1}{\phi}\chi$ and $f_n = \frac{1}{\phi}\chi_n$ where χ and χ_n are the characteristic functions of F and F_n respectively.

Then f and f_n are measurable functions on X.

Let $\varphi \in L$. Since $|f_n| \leq n$, we have $f_n\varphi \in L^2(X,\mu)$ and $\phi f_n\varphi \in L^2(X,\mu)$, thus $f_n\varphi \in D(T_\phi)$ for every $n \in \mathbb{N}^*$.

Also $T_\phi(f_n\varphi) = \phi f_n \varphi = \varphi\chi_n \in R(T_\phi)$ for every $n \in \mathbb{N}^*$.

Now, since $F_n \subset F_{n+1}$ for every $n \in \mathbb{N}^*$, then $\bigcup_{n=1}^{\infty} F_n = F$ and by using the dominated convergence theorem we obtain that $\lim_{n \longrightarrow +\infty} \varphi\chi_n = \varphi$ in $L^2(X,\mu)$, so that $\varphi \in \overline{R(T_\phi)}$.

We have directly:

1) $R(T_\phi) = R(T_\phi^*) = R(T_\phi^* T_\phi) = \overline{R(T_\phi)} = L$ where T_ϕ^* and $T_\phi^* T_\phi = T_\phi T_\phi^*$ are the multiplication operators on $L^2(X,\mu)$ by the conjugate function $\overline{\phi}$ of ϕ and $|\phi|^2 = \phi\overline{\phi}$ respectively.

2) $\gamma(T_\phi) = \gamma(T_\phi^*) \geq \inf_{x \in X \setminus E} |\phi(x)| = \delta \geq 0$.

3) $T_\phi^+ \varphi = f\varphi$ defined on $L^2(X,\mu)$.

4) $\sigma(T_\phi^* T_\phi) \subseteq \{0\} \bigcup [\delta^2, +\infty[$ (since $T_\phi^* T_\phi$ is semi-bounded).

5) $R(T_\phi R_{T_\phi}) = R(T_\phi^* R_{T_\phi^*}) = R(T_\phi S_{T_\phi}) = R(T_\phi^* S_{T_\phi^*}) = \overline{R(T_\phi)} = L$, since $T_\phi R_{T_\phi}, T_\phi^* R_{T_\phi^*}, T_\phi S_{T_\phi}$ and $T_\phi^* S_{T_\phi^*}$ are the multiplication operators on $L^2(X,\mu)$ by $\frac{\phi}{1+|\phi|^2}, \frac{\overline{\phi}}{1+|\phi|^2}, \frac{\phi}{\sqrt{1+|\phi|^2}}$ and $\frac{\overline{\phi}}{\sqrt{1+|\phi|^2}}$, respectively.

6) If δ is strictly positive then T_ϕ has a Hyers-Ulam stability with a Hyers-Ulam stability constant $K_{T_\phi} = \frac{1}{\delta}$. The corresponding operator $(T_\phi)_0 = T_\phi$ defined on $D(T_\phi)$ equipped with the graph inner product has also a Hyers-Ulam stability with a Hyers-Ulam stability constant $K_{(T_\phi)_0} = \sqrt{1+\delta^{-2}}$.

7) $0 \in reg(T_\phi)$ and T_ϕ^* is also regular as soon as δ is strictly positive.

To get back all the properties of the theorem 1 on T_ϕ, it is then sufficient to verify that the following statements are equivalent:

a) $R(T_\phi)$ is closed

b) $R(T_\phi) = L$

c) $\exists \delta > 0$ such that $|\phi| \geq \delta$ a.e. on $X \setminus E$

d) $\exists \delta > 0$ such that $\sigma(T_\phi) \bigcap \{z \in \mathbb{C} \; ; \; |z| < \delta\} \subseteq \{0\}$

Indeed, the equality $\overline{R(T_\phi)} = L$ proves the equivalence of *a*) and *b*).

We obtain the equivalence of *c*) and *d*) as a consequence of the fact that the spectrum $\sigma(T_\phi)$ of T_ϕ is the essential range of the function ϕ.

So it remains to show the equivalence of *b*) and *c*).

Suppose *c*) holds, then $|f| \leq \frac{1}{\delta}$ a.e. on X. Hence for $\varphi \in L$, we have $f\varphi \in L^2(X,\mu)$ and $T_\phi(f\varphi) = \varphi \in R(T_\phi)$. Hence $R(T_\phi) = L$.

Now assume that *b*) holds and define the operator Λ on $L^2(X,\mu)$ by $\Lambda\varphi = f\varphi$. Observe that $\Lambda = T_\phi^+$ and thus Λ is bounded since $R(T_\phi) = L$ is closed. Then, f is essentially bounded on X and $\|\Lambda\| = \|f\|_{L^\infty(X)}$. Consequently, *c*) follows by taking $\delta = \frac{1}{\|f\|_{L^\infty(X)}}$.

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