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Existence and Nonexistence Results of a Problem Involving the Pseudobilaplacian Operator

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Abstract. In this paper we obtain some existence and nonexistence results of a problem involving the pseudobilaplacian operator. By the sub and supersolutions techniques we prove that the problem admits a positive solution when it is subhomogeneous. In the case where it is superhomogeneous, we prove the existence of nontrivial solutions. The nonexistence result follows from a Pohozaev-type identity.

Key Words and Phrases. Pseudobilaplacian operator, Sub-Supersolution method, Mountain-Pass lemma, Palais-Smale condition, Pohozaev-type Identity.

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1. Introduction

This paper deals with the existence and nonexistence of nontrivial solutions of the problem,

$$\text{(P)} \qquad \begin{cases} \Delta_p^2 u - g(x,u) = 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\boldsymbol{R}^N$, $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-bilaplacian operator, $N > 2p > 4$ and $g(x,u)$ is a given function that we will specify later.

Existence and nonexistence of solutions of problems involving the bilaplacian operator have been a subject of active research during the last decade. As example, we quote the works of Bernis et al. [1], Dràbek and Ôtani [3], Edmunds et al. [5], Van der Vorst [10] ...

Borrowing ideas from de Thelin [9] who consider the existence and nonexistence of solutions of a problem involving the p-laplacian operator, we extend the study to our problem (P). By the sub and supersolutions techniques we prove that (P) admits a positive solution when it is subhomogeneous. In the case where it is superhomogeneous, we prove the existence of nontrivial solutions. Finally we give a nonexistence result of nontrivial solutions.

Notation 1. $L^p(\Omega)$, $1 \le p \le \infty$, denote Lebesgue spaces, the norm L^p is denoted by $\|.\|_p$ for $1 \le p \le \infty$.

Set $E = W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$ the Banach space, with the norm $\|u\| = \|\Delta u\|_p$ for $1 \le p < \infty$.

λ_1 is the first eigenvalue of the problem

$$(\mathrm{VP}(\Omega)) \qquad \begin{cases} \Delta_p^2 u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and ϕ_1 is the eigenfunction associated to λ_1 with $\phi_1 > 0$ and $\|\phi_1\| = 1$.

Let $R > 0$ such that $\bar{\Omega} \subsetneq B(O,R)$. Set $\alpha = \min_{\bar{\Omega}} \phi_{1,R} > 0$ where $\phi_{1,R}$ is the first eigenfunction of $\mathrm{VP}(B(O,R))$.

Our main results are given as follows:

Theorem 1. *Suppose that there exist γ_1 and $\gamma_2 : 1 < \gamma_1 < \gamma_2 < p$ and $\mu_1 > 0$, $\mu_2 > 0$, $\mu \ge 0$ such that*

i) *For all $x \in \Omega$, $u \ge 0 : \mu_1 u^{\gamma_1 - 1} \le g(x,u)$,*

ii) *For all $x \in \Omega$, $u \ge 0 : g(x,u) \le \mu + \mu_2 u^{\gamma_2 - 1}$.*

Then (P) *has at least a positive solution in E.*

Theorem 2. *Suppose*

*g*1) *$g(x,u) > 0$ for $u > 0$, $x \in \Omega$,*

*g*2) *$g(x,u) = o(u^{p-1})$ if $u \to 0$, uniformly in x,*

*g*3) *there exist $a \in L^\infty(\Omega)$, $a(x) > 0$ in Ω and $\gamma \in]p, p^*[$ with $p^* = Np/(N-2p)$ such that $g(x,u) = a(x)u^{\gamma-1}$, for u large enough.*

Then (P) *has at least a nontrivial solution in E.*

Theorem 3. *Let $\Omega \subset \mathbf{R}^N$ be a bounded, smooth and strictly starshaped domain, and $g(x,u) = |u|^{\gamma-2}u$, with $\gamma \ge p^* = Np/(N-2p)$. Then, problem* (P) *has no positive solution in E.*

The organization of the paper is as follows: In section 2, we recall some preliminaries. In section 3, we give an existence result of positive solutions by sub-super solutions method. In section 4, by variational methods we obtain an existence result of nontrivial solutions when (P) is superhomogeneous and subcritical. Finally in the last section, we give a nonexistence result of positive solutions when (P) is supercritical.

2. Preliminaries

We start by giving some definitions (see for example J. L. Lions [7])

Definition 1 ([2]). $(u_0, u^0) \in (W^{2,p}(\Omega))^2$ is called a sub-supersolutions of (P) if

$$\begin{cases} \Delta_p^2 u^0 - g(x,u^0) \ge 0 \ge \Delta_p^2 u_0 - g(x,u_0) & \text{in } \Omega \\ u^0 \ge u_0 & \text{in } \Omega \\ u^0 \ge 0 \ge u_0 & \text{on } \partial\Omega \\ \dfrac{\partial u^0}{\partial n} \le 0 \le \dfrac{\partial u_0}{\partial n} & \text{on } \partial\Omega. \end{cases}$$

Definition 2 ([7]). An operator $A : E \to E'$ (dual of E) is hemicontinuous if

$$\lambda \mapsto \langle A(u_1 + \lambda u_2), u_3 \rangle_{E',E}$$

is continuous from $\boldsymbol{R}$ to $\boldsymbol{R}$, for all $u_1, u_2, u_3 \in E$.

Definition 3 ([7]). An operator $A : E \to E'$ is said of calculus of variations if it is bounded and if it can be represented by $A(u) = \tilde{A}(u,u)$ where $\tilde{A} : (u,\hat{u}) \mapsto \tilde{A}(u,\hat{u})$ is an operator from $E \times E$ to E' satisfying the following properties:

d_1) For all $u \in E$, $\hat{u} \mapsto \tilde{A}(u,\hat{u})$ is hemicontinuous, bounded and satisfying $\langle \tilde{A}(u,u) - \tilde{A}(u,\hat{u}), u - \hat{u} \rangle \ge 0$.

d_2) For all $\hat{u} \in E$, $u \mapsto \tilde{A}(u,\hat{u})$ is hemicontinuous and bounded.

d_3) If $(u_n)_{n\in N}$ converges weakly to u in E and if $\langle \tilde{A}(u_n,u_n) - \tilde{A}(u_n,u), u_n - u \rangle \to 0$ as $n \to \infty$, then, for all $\hat{u}$ in E, the sequence $\tilde{A}(u_n,\hat{u})$ converges weakly to $\tilde{A}(u,\hat{u})$ in E'.

d_4) If $(u_n)_{n\in N}$ converges weakly to u in E and if $\tilde{A}(u_n,\hat{u})$ converges weakly to φ in E' then $\langle \tilde{A}(u_n,\hat{u}), u_n \rangle_{E',E} \to \langle \varphi, u \rangle_{E',E}$ as $n \to \infty$.

Proposition 1 ([7]). *Let E be a Banach space, and let A be a coercive operator of calculus of variations. Then, for all $f \in E'$, the equation $A(u) = f$, admits at least a solution $u \in E$.*

3. Existence results by sub-supersolutions method

In this section we construct explicitly a sub and supersolutions of (P). Next we show that the truncated problem ($\tilde{\text{P}}$) associated to (P) admits at least a positive solution. Finally, we prove that any solution of ($\tilde{\text{P}}$) is also solution of (P).

3.1. Existence of sub and supersolutions

For the proof of theorem 1, we require the following lemmas.

Lemma 1. *Under the hypothesis* ii) *of theorem* 1 *and for all* $M > 0$, (P) *has a supersolution* u^0 *such that* $u^0(x) \geq M$, *for all* $x \in \Omega$.

Proof. Let $R > 0$ be a real number such that $\bar{\Omega} \subsetneq B(O, R)$ and set $u^0 = A\phi_{1,R}$, where A is a positive constant, then $\Delta_p^2 u^0 = A^{p-1}\lambda_{1,R}\phi_{1,R}$. From ii), we obtain

$$\Delta_p^2 u^0 - g(x, u^0) \geq A^{p-1}\lambda_{1,R}\phi_{1,R}^{p-1} - \mu_2 A^{\gamma_2 - 1}\phi_{1,R}^{\gamma_2 - 1} - \mu := K(A).$$

Since $\gamma_2 < p$, $\lim_{A \to +\infty} K(A) = +\infty$. Thus there exists an $A_0 > 0$ such that $K(A) \geq 0$, for all $A \geq A_0$.

We have also $\partial u^0/\partial n = A(\partial\phi_{1,R}/\partial n) < 0$ on $\partial\Omega$ since $\partial\phi_{1,R}/\partial n < 0$ on $\partial\Omega$ and $u^0 > 0$ on $\partial\Omega$ and if we take again $A\alpha \geq M$ then $u^0(x) \geq M$.

Thus, for $A \geq \max(M/\alpha, A_0)$, $u^0 = A\phi_{1,R}$ is supersolution of (P) satisfying $u^0(x) \geq M$, $\forall x \in \Omega$. □

Lemma 2. *Under the hypothesis* i) *of theorem* 1 *and for all* $M > 0$, (P) *has a subsolution* u_0 *such that* $0 \leq u_0(x) \leq M$, *for all* $x \in \Omega$.

Proof. We purpose to construct a subsolution under the form $u_0 = \varepsilon\phi_1$, with $\varepsilon > 0$. We know that

$$\Delta_p^2 u_0 - g(x, u_0) \leq \varepsilon^{p-1}\lambda_1\phi_1^{p-1} - \mu_1\varepsilon^{\gamma_1 - 1}\phi_1^{\gamma_1 - 1} := G(\varepsilon)$$

As $\gamma_1 < p$, then there exists a positive constant ε_0 such that $G(\varepsilon) \leq 0$ for $\varepsilon < \varepsilon_0$, and if

$$\varepsilon\|\phi_1\|_\infty \leq M, \qquad \text{i.e.} : \varepsilon \leq \frac{M}{\|\phi_1\|_\infty} = \varepsilon_1,$$

and as $\phi_1 = 0$ on $\partial\Omega$ and $\partial\phi_1/\partial n = 0$ on $\partial\Omega$, then, for $\varepsilon \leq \varepsilon_2 = \min(\varepsilon_0, \varepsilon_1)$, $u_0 = \varepsilon\phi_1$ is a subsolution of (P) verifying $u_0(x) \leq M$, $\forall x \in \Omega$. □

3.2. The truncated problem

Taking as a starting point the work of Deuel-Hess [4], we define the truncated problem $(\tilde{\mathrm{P}})$ associated to (P) by:

Find $u \in E$ such that

$$(\tilde{\mathrm{P}}) \qquad \Delta_p^2 u - \tilde{g}(x, u) = -\gamma(x, u) \qquad \text{in } \Omega,$$

where

$$\tilde{g}(x, u) = \begin{cases} g(x, u^0) & \text{if } u \geq u^0 \\ g(x, u) & \text{if } u_0 \leq u \leq u^0 \\ g(x, u_0) & \text{if } u \leq u_0 \end{cases} \tag{3.1}$$

and

(3.2) $$\gamma(x,u) = -(u_0 - u)_+^{p-1} + (u - u^0)_+^{p-1} \qquad \text{with } w_+ = \max(w,0).$$

We define an operator A on E and a semilinear application $a(.,.)$ on $E \times E$, associated to $(\tilde{\mathrm{P}})$, by

(3.3) $$u \mapsto A(u) = \Delta_p^2 u - \tilde{g}(x,u) + \gamma(x,u) \qquad \text{from } E \text{ into } E'$$

and

$$a(.,.) : (u,\hat{u}) \mapsto a(u,\hat{u}) = \int_\Omega A(u)\hat{u}\,dx \qquad \text{from } E \times E \text{ into } \boldsymbol{R}$$

i.e. $a(u,\hat{u}) = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta \hat{u}\,dx - \int_\Omega \tilde{g}(x,u)\hat{u}\,dx + \int_\Omega \gamma(x,u)\hat{u}\,dx.$

Let us give some auxiliary results.

Lemma 3. *The operator A is bounded.*

Proof. Using the Hölder inequality, we obtain

(3.4) $$\left|\int_\Omega |\Delta u|^{p-2}\Delta u \Delta \hat{u}\,dx\right| \le \left(\int_\Omega |\Delta u|^p dx\right)^{(p-1)/p} \left(\int_\Omega |\Delta \hat{u}|^p dx\right)^{1/p}.$$

Thus, for all $\hat{u} \in E \backslash \{0\}$

(3.5) $$\|\hat{u}\|^{-1}\left|\int_\Omega |\Delta u|^{p-2}\Delta u \Delta \hat{u}\,dx\right| \le \|u\|^{p-1}.$$

The applications A_1 and A_2 defined on E by

$$\langle A_1(u);\hat{u}\rangle_{E',E} = \int_\Omega \tilde{g}(x,u)\hat{u}\,dx \qquad \text{and} \qquad \langle A_2(u);\hat{u}\rangle_{E',E} = \int_\Omega \gamma(x,u)\hat{u}\,dx$$

are bounded in E'. Indeed, for all $\hat{u} \in E$

$$\left|\int_\Omega \tilde{g}(x,u)\hat{u}\,dx\right| \le \int_\Omega |\tilde{g}(x,u)|\,|\hat{u}|dx,$$

the application $x \mapsto \tilde{g}(x,u(x))$ is bounded in $\boldsymbol{R}$ (by construction).

Therefore there exists $C > 0$ such that: $|\langle A_1(u);\hat{u}\rangle_{E',E}| \le C\|\hat{u}\|$.

Thus for $\hat{u} \in E\backslash\{0\}$, we have

(3.6) $$\|\hat{u}\|^{-1}|\langle A_1(u);\hat{u}\rangle_{E',E}| \le C.$$

Similarly,

$$|\langle A_2(u);\hat{u}\rangle_{E',E}| \le \int_\Omega |\gamma(x,u)|\,|\hat{u}|dx.$$

The relation (3.2) implies that for any x in Ω,

$$|\gamma(x,u)| \le 2^{p-2}\max(|u^0|,|u|)^{p-1} + 2^{p-2}\max(|u_0|,|u|)^{p-1}.$$

Thus

$$\int_\Omega |\gamma(x,u)|\,|\hat{u}|dx \le C\|u\|^{p-1}\|\hat{u}\|,$$

where C is a positive constant.

Consequently, for all $\hat{u}$ in $E\backslash\{0\}$,

$$\|\hat{u}\|^{-1}|\langle A_2(u);\hat{u}\rangle_{E',E}| \le C\|u\|^{p-1}. \tag{3.7}$$

It result from (3.5), (3.6) and (3.7) that the operator A is bounded. □

The operator A is of calculus of variations and coercive.

We define

$$\langle\tilde{A}(u_1,u_2),\hat{u}\rangle_{E',E} := \int_\Omega |\Delta u_2|^{p-2}\Delta u_2 \Delta\hat{u}\,dx - \int_\Omega \tilde{g}(x,u_1)\hat{u}\,dx + \int_\Omega \gamma(x,u_1)\hat{u}\,dx,$$

for all $u_1, u_2, \hat{u} \in E$.

Lemma 4. *The operator $\tilde{A}$ satisfies the properties $d_1), \dots, d_4)$ of definition* 3.

Proof. Consider a sequence of real numbers $(\lambda_n)_{n\in\boldsymbol{N}}$ such that $\lim_{n\to+\infty} \lambda_n = \lambda \in \boldsymbol{R}$, we claim that:

$$\langle\tilde{A}(u_1,u_2+\lambda_n\tilde{u}_2),\hat{u}\rangle_{E',E} \to \langle\tilde{A}(u_1,u_2+\lambda\tilde{u}_2),\hat{u}\rangle_{E',E} \qquad \text{when } n\to\infty.$$

In fact, if we put $F_p(s) = |s|^{p-2}s$ with $p>2$ one has

$$\lim_{n\to+\infty} F_p[\Delta(u_2(x)+\lambda_n\tilde{u}_2(x))]\Delta\hat{u}(x) = F_p[\Delta(u_2(x)+\lambda\tilde{u}_2(x))]\Delta\hat{u}(x) \qquad \text{p.p } x\in\Omega,$$

and

$$|F_p[\Delta(u_2(x)+\lambda_n\tilde{u}_2(x))]\Delta\hat{u}(x)| \le (|\Delta u_2(x)| + M_\lambda|\Delta\tilde{u}_2(x)|)^{p-1}|\Delta\hat{u}(x)|. \tag{3.8}$$

where M_λ is a positive constant resulting from the convergence of $(\lambda_n)_{n\in\boldsymbol{N}}$.

Thus from the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_\Omega F_p[\Delta(u_2+\lambda_n\tilde{u}_2)]\Delta\hat{u}\,dx = \int_\Omega F_p[\Delta(u_2+\lambda\tilde{u}_2)]\Delta\hat{u}\,dx.$$

Using again the dominated convergence theorem, we get

$$\lim_{n\to+\infty}\int_\Omega \tilde{g}(x,u_1+\lambda_n\tilde{u}_1)\hat{u}(x)dx = \int_\Omega \tilde{g}(x,u_1+\lambda\tilde{u}_1)\hat{u}(x)dx$$

and

$$\lim_{n\to+\infty}\int_\Omega \gamma(x,u_1+\lambda_n\tilde{u}_1)\hat{u}(x)dx = \int_\Omega \gamma(x,u_1+\lambda\tilde{u}_1)\hat{u}(x)dx.$$

We know that, for all $u, \hat{u} \in E$

$$\langle \Delta_p^2 u - \Delta_p^2 \hat{u}; u - \hat{u}\rangle_{E',E} \geq 0.$$

Thus

$$\langle \tilde{A}(u,u) - \tilde{A}(u,\hat{u}), u - \hat{u}\rangle_{E',E} \geq 0.$$

By the same argument we have d_2).

For the condition d_3), let $(u_n)_{n\in \boldsymbol{N}}$ be a weakly converging sequence to $u \in E$ and suppose that

$$\langle \tilde{A}(u_n,u_n) - \tilde{A}(u_n,u), u_n - u\rangle_{E',E} \to 0.$$

Then for all $\hat{u} \in E$, $(\tilde{A}(u_n,\hat{u}))_{n\in \boldsymbol{N}}$ converges weakly to $\tilde{A}(u,\hat{u})$ in E'.

According to the definition of $\tilde{A}$ it suffices to prove

(3.9) $\quad \tilde{g}(x,u_n) \to \tilde{g}(x,u)$ in $L^{p'}(\Omega)$ and $\gamma(x,u_n) \to \gamma(x,u)$ in $L^{p'}(\Omega)$,

where p' is the conjugate of p.

We know that the embedding of E in $L^p(\Omega)$ is compact, then we can extract a subsequence still denoted $(u_n)_{n\in \boldsymbol{N}}$ such that

$$u_n(x) \to u(x) \qquad \text{p.p in } \Omega \text{ when } n \to +\infty.$$

From the construction of $\tilde{g}$, we deduce

(3.10) $$\tilde{g}(x,u_n(x)) \to \tilde{g}(x,u(x)) \qquad \text{p.p in } \Omega, \text{ as } n \to \infty.$$

and

(3.11) $$|\tilde{g}(x,u_n(x))| \leq h(x) \qquad \text{p.p in } \Omega,\ \forall n \in \boldsymbol{N},$$

where h is an application from Ω to $\boldsymbol{R}$ depending on u^0, μ, μ_2 and γ_2.

Thus

$$\lim_{n\to\infty}\int_\Omega \tilde{g}(x,u_n)\hat{u}\,dx = \int_\Omega \tilde{g}(x,u)\hat{u}\,dx.$$

Similarly for γ.

To show d_4).

Let $(u_n)_{n\in \boldsymbol{N}}$ be a weakly converging sequence to $u \in E$ and suppose that

(3.12) $\quad (\tilde{A}(u_n,\tilde{u}))_{n\in \boldsymbol{N}}$ converges weakly to φ in E', where $\tilde{u} \in E$.

It suffices to prove

$$a) \quad \lim_{n\to+\infty}\int_\Omega F_p(\Delta\tilde{u})\Delta u_n\,dx = \int_\Omega F_p(\Delta\tilde{u})\Delta u\,dx,$$

$$b) \quad \lim_{n\to+\infty}\int_\Omega \tilde{g}(x,u_n)u_n\,dx = \int_\Omega \tilde{g}(x,u)u\,dx,$$

and

$$c) \quad \lim_{n\to+\infty}\int_\Omega \gamma(x,u_n)u_n\,dx = \int_\Omega \gamma(x,u)u\,dx.$$

In fact, we can write

$$\int_\Omega F_p(\Delta\tilde{u})\Delta u_n\,dx = \langle \Delta_p^2\tilde{u}, u_n\rangle_{E',E}.$$

Thus

$$\lim_{n\to+\infty}\int_\Omega F_p(\Delta\tilde{u})\Delta u_n\,dx = \lim_{n\to+\infty}\langle \Delta_p^2\tilde{u}, u_n\rangle_{E',E} = \langle \Delta_p^2\tilde{u}, u\rangle.$$

Using the Sobolev embedding and the Hölder inequality, we obtain

$$\left|\int_\Omega \tilde{g}(x,u_n)(u_n-u)dx\right| \le \|\tilde{g}(x,u_n)\|_{p'}\|u_n-u\|_p.$$

Similarly for $\gamma(.,u)$, we have

$$\begin{aligned}\left|\int_\Omega \gamma(x,u_n)(u_n-u)dx\right| &\le 2^{p-1}\int_\Omega (|u^0(x)|^{p-1}+|u_n|^{p-1})|u_n-u|dx\\ &\le 2^{p-1}(\|u^0\|_p^{p-1}+C)\|u_n-u\|_p,\end{aligned}$$

where C is a positive constant. From the compactness of the embedding $E \hookrightarrow L^p(\Omega)$, we get $b)$ and $c)$.

According to $a)$, $b)$ and $c)$ we can write

$$\lim_{n\to+\infty}\langle \tilde{A}(u_n,\tilde{u}); u_n\rangle_{E',E} = \lim_{n\to+\infty}\langle \tilde{A}(u_n,\tilde{u}); u\rangle_{E',E}, \qquad \forall \tilde{u}\in E. \tag{3.13}$$

(3.12) implies that

$$\lim_{n\to+\infty}\langle \tilde{A}(u_n,\tilde{u}); u\rangle_{E',E} = \langle \varphi, u\rangle_{E',E}.$$

Thus

$$\lim_{n\to+\infty}\langle \tilde{A}(u_n,\tilde{u}); u_n\rangle_{E',E} = \langle \varphi, u\rangle_{E',E}.$$

□

Lemma 5. *The application A is coercive.*

Proof. Show that

$$\frac{a(u,u)}{\|u\|} \to +\infty \qquad \text{when } \|u\| \to +\infty.$$

From the construction of the functions $\tilde{g}$ and γ, we can write

$$-\int_\Omega \tilde{g}(x,u)u\,dx \geq -M_1\|u\|,$$

where M_1 is a positive constant depending on $\|u_0\|$, μ and μ_2, and

$$\int_\Omega \gamma(x,u)u\,dx \geq D_1\|u\|_p^p - D_2\|u\|_p^{p-1},$$

where D_1 and D_2 are positive constants. Putting $\hat{u} = u$ in the definition of the form $a(.,.)$, we obtain

$$a(u,u) \geq Q_1\|u\|^p + D_1\|u\|_p^p - D_2\|u\|_p^{p-1} - M_1\|u\|.$$

Using the Sobolev embedding we have

$$a(u,u) \geq Q_1\|u\|^p - \tilde{D}\|u\|^{p-1} - M_1\|u\|,$$

where Q_1 and $\tilde{D}$ are positive constants. Thus

$$\frac{a(u,u)}{\|u\|} \to +\infty \qquad \text{when } \|u\| \to +\infty \ (\text{since } p > 2). \qquad \square$$

Lemma 6. (P) *admits at least one solution in E.*

Proof. From lemmas 3, 4 and 5 we deduce that

(3.14) For all $g_1 \in E'$, $\exists u^* \in E$ such that $A(u^*) = g_1$.

In particular for $g_1 = 0$, there exists u^* such that

$$\Delta_p^2 u^* - \tilde{g}(x,u^*) = -\gamma(x,u^*). \qquad \square$$

3.3. Existence results for positive solutions of (P)

Let us show that:

$$u_0 \leq u^* \leq u^0 \qquad \text{in } \Omega.$$

First we prove that

$$u^* \leq u^0.$$

Put $\hat{u} := (u^* - u^0)_+$.

Multiplying the equation of $(\tilde{\mathrm{P}})$ by $\hat{u}$ and integrating on Ω, we have

$$\int_\Omega F_p(\Delta u^*)\Delta\hat{u}\,dx - \int_\Omega \tilde{g}(x,u^*)\hat{u}\,dx + \int_\Omega \gamma(x,u^*)\hat{u}\,dx = 0. \tag{3.15}$$

According to the definition of u^0, we get

$$\int_\Omega (F_p(\Delta u^0)\Delta\hat{u} - g(x,u^0)\hat{u})dx \geq 0. \tag{3.16}$$

Substracting (3.16) from (3.15), we obtain

$$\begin{aligned}\int_\Omega &(F_p(\Delta u^*) - F_p(\Delta u^0))\Delta(u^* - u^0)_+ dx \\ &- \int_\Omega (\tilde{g}(x,u^*) - g(x,u^0))(u^* - u^0)_+ dx \\ &+ \int_\Omega [-(u_0 - u^*)_+^{p-1} + (u^* - u^0)_+^{p-1}](u^* - u^0)_+ dx \leq 0.\end{aligned} \tag{3.17}$$

Now from the monotonicity of the functional $u \mapsto \int_\Omega |\Delta u|^p dx/p$, we have

$$\int_\Omega (F_p(\Delta u^*) - F_p(\Delta u^0))\Delta(u^* - u^0)_+ dx \geq 0,$$

and by construction of $\tilde{g}$, we get

$$\int_\Omega (\tilde{g}(x,u^*) - \tilde{g}(x,u^0))(u^* - u^0)_+ dx = 0,$$

then (3.17) becomes

$$\int_\Omega (u^* - u^0)_+^p dx \leq 0.$$

Thus $u^* \leq u^0$ in Ω.

Similarly, we prove $u_0 \leq u^*$ in Ω.

4. Existence results by the variational method

The energy functional J associated to (P) is defined on E by

$$J(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx - \int_\Omega G(x,u)dx, \qquad \text{where } G(x,u) = \int_0^u g(x,s)ds.$$

The nontrivial critical points of J are the solutions of (P).

To prove the theorem 2, we use the three following lemmas.

Set $J(u) := A(u) - B(u)$ with $A(u) := \int_\Omega |\Delta u|^p dx/p$ and $B(u) := \int_\Omega G(x,u)dx$, where $G(x,u) = \int_0^u g(x,s)ds$.

Lemma 7. *J is $C^1(E, \boldsymbol{R})$.*

Proof. It is enough to show that A and B are of class C^1 on E.

According to an algebraic relation of Simon [8] and the Hölder inequality one has

$$\begin{aligned}\|A'(u_1) - A'(u_2)\|_{E'} &\le \|F_p(\Delta u_1) - F_p(\Delta u_2)\|_{p'} \\ &\le C\|\Delta u_1 - \Delta u_2\|_p(\|\Delta u_1\|_p + \|\Delta u_2\|_p)^{p-2}.\end{aligned}$$

Therefore A is of class C^1 on E. Since g is subcritical, then B is of class C^1. □

Lemma 8. *J satisfies the Palais-Smale conditions.*

Proof. Let (u_n) be a Palais-Smale sequence in E (i.e. $J(u_n)$ is bounded and $J'(u_n) \to 0$), then (u_n) is bounded in E. In fact,

$$\int_\Omega (|\Delta u_n|^p - |u_n|^\gamma)dx \to 0, \qquad \text{as } n \to \infty \tag{4.1}$$

and

$$\int_\Omega \left(\frac{1}{p}|\Delta u_n|^p - \frac{1}{\gamma}|u_n|^\gamma\right)dx \text{ is bounded,} \tag{4.2}$$

combining (4.1) and (4.2) we obtain that

$$\left(\frac{1}{p} - \frac{1}{\gamma}\right)\int_\Omega |\Delta u_n|^p dx \text{ is bounded,}$$

then (u_n) is bounded when $\gamma \neq p$.

The embedding $E \hookrightarrow L^\gamma(\Omega)$ is compact for $\gamma \in]p, p^*[$, then there exists a subsequence, still denoted by (u_n), which converges strongly in $L^\gamma(\Omega)$.

Set

$$I_{n,m} = \int_\Omega [F_p(\Delta u_n) - F_p(\Delta u_m)](\Delta u_n - \Delta u_m)dx, \qquad \text{with } m, n \in \boldsymbol{N}.$$

Then we can write $I_{n,m}$ in the form

$$I_{n,m} = \langle J'(u_n) - J'(u_m), u_n - u_m\rangle + \int_\Omega [g(x,u_n) - g(x,u_m)](u_n - u_m)dx.$$

From the algebraic relation [8], we get

$$|\Delta u_n - \Delta u_m|^p \leq C[F_p(\Delta u_n) - F_p(\Delta u_m)](\Delta u_n - \Delta u_m),$$

thus

$$\|u_n - u_m\|^p \leq CI_{n,m},$$

and therefore

$$\lim_{n,m\to+\infty} \|u_n - u_m\| = 0.$$

Then, (u_n) converges strongly in E. □

Lemma 9. *J satisfies the geometric conditions.*

Proof. Since $J(0) = 0$ and $J(\underline{u}) > 0$ for an element $\underline{u} \geq 0$; $\underline{u} \not\equiv 0$, then

$$\lim_{t\to+\infty} J(t\underline{u}) = -\infty.$$

Indeed, let t be positive sufficiently large number

$$J(t\underline{u}) = At^p - Bt^\gamma;$$

where

$$A := \frac{1}{p}\int_\Omega |\Delta \underline{u}|^p dx, \qquad \text{and} \qquad B := \frac{1}{\gamma}\int_\Omega a(x)|\underline{u}|^\gamma dx$$

and since $\gamma > p$, we get

$$\lim_{t\to+\infty} J(t\underline{u}) = -\infty.$$

Thus there exists $v_0 = t_0\underline{u}$ such that $J(v_0) = 0$. From hypothese g_2) and g_3) we have

$$\int_\Omega G(u)dx \leq \varepsilon\|u\|^p + C_\varepsilon\|u\|^\gamma, \qquad \text{with } C_\varepsilon \text{ is a positive constant where } \varepsilon > 0,$$

then

$$J(u) \geq \frac{1}{p}\|u\|^p - (\varepsilon\|u\|^p + C_\varepsilon\|u\|^\gamma).$$

For $\|u\|$ enough small, one obtains $J(u) \geq \beta > 0$. □

Proof of Theorem 2. From the lemmas 7, 8 and 9, we deduce that there exists $u \in E\backslash\{0\}$ such that

$$\int_\Omega F_p(\Delta u).\Delta v\, dx - \int_\Omega g(x,u)v\, dx = 0 \qquad \forall v \in E.$$

5. Nonexistence results of positive solutions of (P)

By a Pohozaev type identity we show the nonexistence of positive solution of (P) in the case $g(u) = u^{\gamma-1}$, $\gamma \geq p^* = Np/(N-2p)$ and Ω is a strictly starshaped domain.

First we need the following Lemma.

Lemma 10. *Let $u \in E$ be a positive solution of* (P).
Then the following identity holds

$$\left(\frac{N-2p}{p}\right)\int_\Omega |\Delta u|^p dx - N\int_\Omega G(u)dx = -\left(\frac{p-1}{p}\right)\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma,$$

where $\vec{n}$ is the exterior normal to $\partial\Omega$ and $G(u) = \int_0^u g(s)ds$.

Proof. Multiplying the equation of (P) by the inner product $x.\nabla u$ and integrating on Ω, we obtain

$$\int_\Omega g(u)(x.\nabla u)dx = \int_\Omega \Delta w(x.\nabla u)dx \qquad \text{where } w = F_p(u).$$

Set

$$A_1(u) := \int_\Omega g(u)(x.\nabla u)dx \qquad \text{and} \qquad A_2(u) := \int_\Omega \Delta w(x.\nabla u)dx.$$

Calculation of $A_1(u)$

$$\int_\Omega g(u)(x.\nabla u)dx = \sum_{j=1}^N \int_\Omega g(u)\left(x_j \frac{\partial u}{\partial x_j}\right)dx = \sum_{j=1}^N \int_\Omega \frac{\partial}{\partial x_j}(G(u))x_j\,dx. \tag{5.1}$$

According to the divergence theorem, we write

$$\sum_{j=1}^N \int_\Omega \frac{\partial}{\partial x_j}(G(u))x_j\,dx = -\sum_{j=1}^N \int_\Omega G(u)dx + \sum_{j=1}^N \int_{\partial\Omega} G(u)x_j\vec{n}_j\,d\sigma = -N\int_\Omega G(u)dx + \int_{\partial\Omega} G(u)(x.\vec{n})d\sigma. \tag{5.2}$$

Since

$$u = 0 \quad \text{on } \partial\Omega \qquad \text{implies} \qquad G(u) = 0 \quad \text{on } \partial\Omega,$$

and taking account of (5.1) and (5.2), we deduce that

$$A_1(u) = -N\int_\Omega G(u)dx = \frac{-N}{\gamma}\int_\Omega |u|^\gamma dx. \tag{5.3}$$

Calculation of $A_2(u)$

In [6] Mitidieri established the following relation:

$$\int_\Omega (\Delta v(x.\nabla u) + \Delta u(x.\nabla v))dx = \int_{\partial\Omega}\left(\frac{\partial v}{\partial \vec{n}}(x.\nabla u) + \frac{\partial u}{\partial \vec{n}}(x.\nabla v) - (\nabla u.\nabla v)(x.\vec{n})\right)d\sigma$$
$$+ (N-2)\int_\Omega \nabla u.\nabla v\, dx.$$

Let us apply to our case with $v = w = F_p(u)$, we get

$$A_2(u) = \int_\Omega\left(\frac{\partial w}{\partial \vec{n}}(x.\nabla u) + \frac{\partial u}{\partial \vec{n}}(x.\nabla w)\right)d\sigma - \int_{\partial\Omega}(\nabla u.\nabla w)(x.\vec{n})d\sigma - \int_\Omega \Delta u(x.\nabla w)$$
$$+ (N-2)\int_\Omega \nabla u.\nabla w\, dx.$$

Put

$$H_1 = \int_{\partial\Omega}\left(\frac{\partial w}{\partial \vec{n}}(x.\nabla u) + \frac{\partial u}{\partial \vec{n}}(x.\nabla w)\right)d\sigma, \qquad H_2 = \int_{\partial\Omega}(\nabla u.\nabla w)(x.\vec{n})d\sigma,$$

$$H_3 = \int_\Omega \Delta u(x.\nabla w)dx \qquad \text{and} \qquad H_4 = \int_\Omega \nabla u.\nabla w\, dx.$$

The fact that $u = \nabla u = 0$ on $\partial\Omega$,
we get

$$H_1 = \int_{\partial\Omega}\left(\frac{\partial w}{\partial \vec{n}}(x.\nabla u) + \frac{\partial u}{\partial \vec{n}}(x.\nabla w)\right)d\sigma = 0 \tag{5.4}$$

and

$$H_2 = \int_{\partial\Omega}(\nabla u.\nabla w)(x.\vec{n})d\sigma = 0. \tag{5.5}$$

Calculation of H_3

We have

$$H_3 = \int_\Omega \Delta u(x.\nabla w)dx = \left(\frac{p-1}{p}\right)\int_\Omega x.[\nabla|\Delta u|^p]dx,$$

and applying the divergence theorem, we obtain

$$H_3 = \left(\frac{p-1}{p}\right)\int_{\partial\Omega}|\Delta u|^p(x.\vec{n})d\sigma - \frac{N(p-1)}{p}\int_\Omega |\Delta u|^p dx. \tag{5.6}$$

Calculation of H_4

Applying the generalized Green's formula, we obtain

$$H_4 = \int_\Omega (\nabla u.\nabla w)dx = -\int_\Omega w\Delta u\, dx,$$

therefore

$$H_4 = -\int_\Omega |\Delta u|^p dx. \tag{5.7}$$

Now from the relations (5.4), (5.5), (5.6) and (5.7) we obtain

$$A_2(u) = -\frac{(p-1)}{p}\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma + \frac{N(p-1)}{p}\int_\Omega |\Delta u|^p dx - (N-2)\int_\Omega |\Delta u|^p dx.$$

Thus

$$A_2(u) = -\left(\frac{N-2p}{p}\right)\int_\Omega |\Delta u|^p dx - \frac{(p-1)}{p}\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma, \tag{5.8}$$

and from the relations (5.3) and (5.8) we deduce that

$$\left(\frac{N-2p}{p}\right)\int_\Omega |\Delta u|^p dx - N\int_\Omega G(u)dx = -\frac{(p-1)}{p}\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma. \tag{5.9}$$

Proof of Theorem 3. We proceed by contradiction.

The lemma 10 gives

$$\left(\frac{N-2p}{p}\right)\int_\Omega |\Delta u|^p dx - N\int_\Omega G(u)dx = -\frac{(p-1)}{p}\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma.$$

Owing to the fact that

$$G(u) = \frac{|u|^\gamma}{\gamma}$$

we obtain

$$\left(\frac{N-2p}{p} - \frac{N}{\gamma}\right)\int_\Omega |u|^\gamma dx = -\frac{(p-1)}{p}\int_{\partial\Omega} |\Delta u|^p (x.\vec{n})d\sigma,$$

and since the domain Ω is strictly starshaped then,

$$\frac{N-2p}{p} - \frac{N}{\gamma} < 0,$$

what contradicts the fact that $\gamma \geq p^* = Np/(N-2p)$. □

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