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On a General Feshbach Method[1]

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Abstract

We present a general reduction Feshbach method that can be applied in particular to the spectral study of operators including the hamiltonian of the Born-Oppenheimer approximation and also to Fredholm operators.

This method reduces the study on the first case to the one of a pseudodifferential matrix, and in the second case to a matrix acting between two finite dimensional euclidian spaces. The stability theorem of the Fredholm character and the index is originally obtained as a consequence of this reduction.

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1. Introduction

The M. Born and R. Oppenheimer approximation's technique [3] has been of interest to many authors (see [8], [9], [13], [14]), see also [1] and [5] for physics motivations.

Broadly speaking, one studies the behaviour of a N-body system when we send the mass M of some particles (the nuclear mass for the molecule case), to $+\infty$. Thus we are lead to the study of a cut out Hamiltonian of the type:

$$P(h) = -h^2\Delta_x + h^2 p(\partial_y) + Q(x)\ ,\ Q(x) = -\Delta_y + V(x,y)$$

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on $L^2(\mathbb{R}^n_x \times \mathbb{R}^p_y)$, when h tends to 0. Here $p(\partial_y)$ which stands for the isotropic term, is a second order operator that will not be of importance for us in the sequel and then it will be ignored. h is proportionnal to $\frac{1}{\sqrt{M}}$, V is the sum of all the interactions between the particles, and $Q(x)$ is the electronic Hamiltonian defined on $L^2(\mathbb{R}^p_y)$.

M. Born and R. Oppenheimer realized that the study of $P(h)$ can be approximately reduced, modulo $\mathcal{O}(h^3)$ when h is small enough, to the one of the family of operators $-h^2\Delta_x + \lambda_j(x)$, $j = 1, ..., N$, on $\bigoplus_1^N L^2(\mathbb{R}^n_x)$ where $\lambda_1(x) < \lambda_2(x) \leq ... \leq \lambda_N(x)$ are the (so called electronic levels) first N discrete eigenvalues of $Q(x)$.

In fact, we are interested in the spectral properties of $P(h)$ near a fixed energy level E_0. The method allow us to deal only with $\lambda_j(x)$ verifying $\inf_{x\in\mathbb{R}^n} \lambda_j(x) \leq E_0$. The minimums of $\lambda_j(x)$ can be physically interpreted as the equilibrum positions of the nuclei around which those will gravitate in a molecular system.

The spectral study of $P(h)$ can be reduced to that of semiclassical analytic pseudodifferential matrix operator $F(z)$ on $\bigoplus_1^N L^2(\mathbb{R}^n_x)$. The principal part of $F(z)$ is equal modulo $\mathcal{O}(h^4)$ to the diagonal matrix $diag(-h^2\Delta_x + \lambda_j(x))_{1\leq j\leq N}$ corresponding to the original intuition of M. Born and R. Oppenheimer. If σ stands for the spectrum, we have the following equivalence:

$$z \in \sigma(P(h)) \Longleftrightarrow z \in \sigma(F(z))$$

The way in which the spectral reduction of $P(h)$ and this equivalence are obtained relies on the construction of a matrix operator, the so-called Grushin operator, acting on a greater space by means of the eigenfunctions of $Q(x)$ associated with the eigenvalues $\lambda_1(x), \lambda_2(x), ..., \lambda_N(x)$.

This is the Feshbach method introduced by Feshbach in [6] which is also used in several situations to study the eigenvalues and resonances of $P(h)$ in the semiclassical limit where the potential function $V(x, y)$ is smooth (see. [4], [7], [9], [11]) and recently in [13] and [14] where $V(x, y)$ has singular Coulombic interactions. WKB type expansions for the eigenvalues and resonances of $P(h)$ are obtained by virtue of the Feshbach method and the pseudodifferential operators calculus (see. [2], [12]).

In the same spirit, another result of reduction has been established by B. Messirdi and G. Djellouli in [10] for Fredholm type operators. They give the Feshbach method in a slightly different way that can be applied to the Fredholm class operators. The idea was to construct a Grushin operator by means only of the nullity and the deficiency (index). The eigenvalues or the eigenfunctions are not used in this case. They show then in a simple and original manner the stability theorem for Fredholm operators also a result of spectral reduction of these operators.

Here we plan to present this reduction scheme that can be applied in the first case to a general class of operators of the type $P(x, y, hD_x, D_y)$ which in particular contain the Hamiltonians of the Born-Oppenheimer approximation and in the second case to Fredholm operators.

We describe in this framework some results of Messirdi [9], Messirdi-Senoussaoui [13] and Messirdi-Senoussaoui-Djellouli [14] concerning the study of the discrete spectrum and the resonances of molecular systems and the results of Messirdi-Djellouli [10] about the stability of the Fredholm character and the index under a small perturbation by virtue of the Feshbach reduction method.

2. Feshbach method for a general Hamiltonians

We study the discrete spectrum of a general class of Born-Oppenheimer hamiltonians of the type;

$$P(h) = -h^2\Delta_x + P(x, y, D_y) \text{ on } L^2(\mathbb{R}^n_x \times \mathbb{R}^p_y), \quad n; p \in \mathbb{N}^*$$

when h tends to 0^+, and $P(x, y, D_y) = Q(x)$ is a pseudodifferential operator on $L^2(\mathbb{R}^p_y)$ satisfying the following assumptions:

(H1) For all $x \in \mathbb{R}^n$, $Q(x)$ is selfadjoint on $L^2(\mathbb{R}^p_y)$ with $x-$independant domain D and $Q(x) \in C_b^\infty(\mathbb{R}^n, \mathcal{L}(D, L^2(\mathbb{R}^p_y)))$. ($C_b^\infty$ stands for the space of functions that are uniformly bounded together with all their derivatives and $\mathcal{L}(H_1, H_2)$ denotes the space of bounded linear operators from a Hilbert space H_1 to another Hilbert space H_2).

(H2) For any $x \in \mathbb{R}^n$, the spectrum $\sigma(Q(x))$ of $Q(x)$ has two disjoint components :

$$\sigma(Q(x)) = \{\lambda_1(x), \lambda_2(x), ..., \lambda_N(x)\} \cup \sigma_0(x)$$

where $\lambda_1(x) < \lambda_2(x) \leq ... \leq \lambda_N(x)$ depends continuously on x and are separated by a gap from $\sigma_0(x)$:

$$\inf_{x\in\mathbb{R}^n} \sigma_0(x) > \sup_{x\in\mathbb{R}^n} \{\lambda_1(x), \lambda_2(x), ..., \lambda_N(x)\} + \delta \quad , \quad \delta > 0$$

In order to avoid regularity problems at infinity of the values $(\lambda_j(x))_{1\leq j\leq N}$, we shall in addition assume that

$$\inf_{j\neq k,\ |x|\geq C} |\lambda_j(x) - \lambda_k(x)| \geq \frac{1}{C} \quad , \quad C > 0$$

Examples:

1) $Q(x) = -\Delta_y + (1+x^2)^{2l}\sum_{j=1}^{p}\mu_j y_j^2$, $y = (y_1,...,y_p)$, $x^2 = \sum_{i=1}^{n} x_i^2$ if $x = (x_1,...,x_n)$ and $\mu_j > 0$ for all $j \in \{1,...p\}$. $Q(x)$ satisfies the assumptions (H1) and (H2) with $D = H^2(\mathbb{R}_y^p) \cap \{\varphi \in L^2(\mathbb{R}_y^p)\ ;\ y_j^2\varphi \in L^2(\mathbb{R}_y^p),\ j = 1,...,p\}$, $\lambda_\alpha(x) = \sum_{j=1}^{p}\sqrt{\mu_j}(2\alpha_j+1)(1+x^2)^l$, $\alpha = (\alpha_1,...,\alpha_p) \in \mathbb{N}^p$ and $\sigma_0(x) = \{\lambda_\alpha(x)\ ;\ \alpha \in \mathbb{N}^p \setminus I_N\}$ where $I_N \subset \mathbb{N}^p$, $cardinal(I_N) = N$.

2) $Q(x) = -\Delta_y + V(x,y)$ when the interaction $V(x,y)$ is real and regular in the sense that $V \in C_b^\infty(\mathbb{R}^n, \mathcal{L}(H^2(\mathbb{R}_y^p), L^2(\mathbb{R}_y^p)))$ and satisfying the conditions (H2). ◄

Furthermore, under the assumptions (H1) and (H2), one can use the constructions made in [Me] and obtain an orthonormalized family $\{v_1(x),...,v_N(x)\}$ in $L^2(\mathbb{R}_y^p)$ such that:

- $\forall j \in \{1,...,N\}$, $v_j(x) \in C_b^\infty(\mathbb{R}_x^n, D)$
- $\{v_1(x),...,v_N(x)\}$ is a basis of $\bigoplus_1^N \ker(Q(x) - \lambda_j(x))$, $\forall x \in \mathbb{R}^n$

If $\pi(x) = \sum_{j=1}^{N}\langle ., v_j(x)\rangle_{L^2(\mathbb{R}_y^p)} v_j(x)$ and $\widehat{\pi}(x) = 1 - \pi(x)$ then (H2) ensures in particular that

$$\widehat{\pi}(x)Q(x)\widehat{\pi}(x) - z > 0 \quad , \quad \forall z \in \mathbb{C} \text{ such that } \operatorname{Re} z < \inf_{x\in\mathbb{R}^n}\sigma_0(x) \tag{2.1}$$

Define the following operators:

$$R^- : \bigoplus_1^N L^2(\mathbb{R}_x^n) \longrightarrow L^2(\mathbb{R}_x^n \times \mathbb{R}_y^p) \tag{2.2}$$

$$u^- = (u_1^-,...,u_N^-) \longmapsto R^-u^- = \sum_{j=1}^{N} u_j^- v_j(x)$$

and

$$R^+ = (R^-)^* : L^2(\mathbb{R}_x^n \times \mathbb{R}_y^p) \longrightarrow \bigoplus_1^N L^2(\mathbb{R}_x^n)$$

$$u \longmapsto R^+u = \bigoplus_{j=1}^{N}\langle u, v_j(x)\rangle_{L^2(\mathbb{R}_y^p)} \tag{2.3}$$

We then consider a Grushin problem that will lead to the Feshbach reduction. For $z \in \mathbb{C}$ define

$$\mathcal{P}(z) = \begin{pmatrix} (P(h) - z) & R^- \\ R^+ & 0 \end{pmatrix} \tag{2.4}$$

from $H^2(\mathbb{R}^n_x, D)\bigoplus(\bigoplus_1^N L^2(\mathbb{R}^n_x))$ into $L^2(\mathbb{R}^n_x \times \mathbb{R}^p_y)\bigoplus(\bigoplus_1^N H^2(\mathbb{R}^n_x))$.

The matrix $\mathcal{P}(z)$ can be considered as a $h-$pseudodifferential operator in x associated with the operator-valued symbol

$$p(x\ \xi; z) = \begin{pmatrix} (\xi^2 + Q(x) - z) & R^- \\ R^+ & 0 \end{pmatrix} \tag{2.5}$$

with $p(x\ \xi; z) \in C^\infty(T^*\mathbb{R}^n_x, \mathcal{L}(D\bigoplus(\bigoplus_1^N L^2(\mathbb{R}^n_x), L^2(\mathbb{R}^n_x\times\mathbb{R}^p_y)\bigoplus(\bigoplus_1^N L^2(\mathbb{R}^n_x))$.

Thanks to (2.1) the symbol $p(x\ \xi; z)$ is invertible for any $(x, \xi) \in T^*\mathbb{R}^n_x$ and its inverse is given by

$$q(x\ \xi; z) = \begin{pmatrix} \widehat{\pi}(x)(\xi^2 + \widehat{\pi}(x)Q(x)\widehat{\pi}(x) - z)^{-1}\widehat{\pi}(x) & R^- \\ R^+ & (z - \xi^2 - \lambda_j(x))_{1\leq j\leq N} \end{pmatrix} \tag{2.6}$$

By virtue of (H1) one can consider the weyl quantification $\mathcal{Q}(z)$ of $q(x, \xi; z)$ defined on $L^2(\mathbb{R}^n_x, D)$ by the oscillatory integral

$$\mathcal{Q}(z) = Op_h^w(q(x\ \xi; z))u(x) = (2\pi h)^{-n/2}\int e^{i(x-y)\xi/h}q(\frac{x+y}{2}\ \xi; z)u(y)dyd\xi$$

and use the composition formula of $h-$pseudodifferential operators.

It then follows that

$$\begin{cases} \mathcal{P}(z)\mathcal{Q}(z) = 1 + \mathcal{O}(h) \\ \mathcal{Q}(z)\mathcal{P}(z) = 1 + \mathcal{O}(h) \end{cases}$$

uniformly with respect to h.

Consequently, if h is small enough $\mathcal{P}(z)$ is invertible and its inverse is given by the Neumann series.

Writing

$$\mathcal{P}^{-1}(z) = \begin{pmatrix} a(z) & a^+(z) \\ a^-(z) & a^{-+}(z) \end{pmatrix}$$

We see that $a(z)$, $a^\pm(z)$ and $a^{-+}(z)$ are $h-$pseudodifferential operators analytic on z. The principal symbol of $a^{-+}(z)$ is $(z - \xi^2 - \lambda_j(x))_{1\leq j\leq N}$.

The principal interest in having considered the operator $\mathcal{P}(z)$ is that it reduces the spectral study of $P(h)$ of that of a matrix operator much simpler and only acting on the variable x as it is shown in the following proposition.

Prposition 2.1: We have the following equivalence:

$$z \in \sigma(P(h)) \Longleftrightarrow z \in \sigma(F(z))$$

where $F(z) = z - a^{-+}(z)$, *called the Feshbach operator, is a* $N \times N$ *matrix of bounded* $h-$*pseudodifferential operators on* $\bigoplus_1^N L^2(R_x^n)$ *with principal symbol* $diag(\xi^2 + \lambda_j(x))_{1 \le j \le N}$.

Proof: We have

$$(P(h) - z)u = v \Longleftrightarrow \mathcal{P}(z)(u \bigoplus 0) = (v \bigoplus R^+ u) \tag{2.7}$$

$$\Longleftrightarrow (u \bigoplus 0) = \mathcal{P}^{-1}(z)(v \bigoplus R^+ u)$$

$$\Longleftrightarrow \left\{ \begin{array}{c} a(z)v + a^+(z)R^+u = u \\ a^-(z)v + a^{-+}(z)R^+u = 0 \end{array} \right.$$

and

$$a^{-+}(z)\alpha = \beta \Longleftrightarrow \mathcal{P}^{-1}(z)(0 \bigoplus \alpha) = (a^+(z)\alpha \bigoplus \beta) \tag{2.8}$$

$$\Longleftrightarrow (0 \bigoplus \alpha) = \mathcal{P}(z)(a^+(z)\alpha \bigoplus \beta)$$

$$\Longleftrightarrow \left\{ \begin{array}{l} (P(h) - z)a^+(z)\alpha + R^-\beta = 0 \\ R^+a^+(z)\alpha = \alpha \end{array} \right.$$

If $z \notin \sigma(P)$, we obtain from (2.8)

$$a^+(z)\alpha = -(P(h) - z)^{-1}R^-\beta \quad \text{and} \quad \alpha = -R^+(P(h) - z)^{-1}R^-\beta$$

thus $z \notin \sigma(F(z))$ and

$$(z - F(z))^{-1} = R^+(z - P(h))^{-1}R^- \tag{2.9}$$

Conversely, if $z \notin \sigma(F(z))$ then (2.7) gives

$$R^+u = -(z - F(z))^{-1}a^-(z)v \quad \text{and} \quad u = [a(z) - a^+(z)(z - F(z))^{-1}a^-(z)]v$$

Therefore, $z \notin \sigma(P(h))$ and

$$(z - P(h))^{-1} = a^+(z)(z - F(z))^{-1}a^-(z) - a(z) \tag{2.10}$$

◀

these formulae show that since $a(z)$, $a^{\pm}(z)$ and $a^{-+}(z)$ depend analytically on z , then a singularity of $(z - P(h))^{-1}$ is necessarily a singularity of $(z - F(z))^{-1}$ and conversely.

3. Eigenvalues of polyatomic molecules

A potential $\frac{1}{|x\pm y|}$ is not within the framework in a natural way in the pseudodifferential calculus since the derivatives of order k of such potential will have a singularity stronger than k (of order $\frac{1}{|x\pm y|^{k+1}}$). We will show that, modulo a change of variables, the use of such a computation and the Feshbach reduction in the Coulombian case are still possible.

Now if $P(h) = -h^2\Delta_x + Q(x)$ on $L^2(\mathbb{R}^{3n}_x \times \mathbb{R}^{3p}_y)$ where the real potential function$V(x,y)$ may have Coulomb-type singularities $\pm\frac{1}{|y_l - x_j|}$, $+\frac{1}{|x_j - x_k|}$, $x_j, y_l \in \mathbb{R}^3$, $x = (x_1, ..., x_n) \in \mathbb{R}^{3n}$, $y = (y_1, ..., y_p) \in \mathbb{R}^{3p}$, the eigenvalues and eigenfunctions of $Q(x) = -\Delta_y + V(x,y)$ are only C^2 with respect to the x $-$variables.

Neverthless, by introducing some $x-$dependent changes in the $y-$variables that will regularize $Q(x)$ and permit an adaptable semiclassical pseudodifferential calculus (see [13], [14]).

Indeed, if $x_0, x \in \mathbb{R}^{3n} \setminus \mathcal{C}$ where $\mathcal{C}$ denotes the collision set :

$$\mathcal{C} = \{x = (x_1, ..., x_n) \in \mathbb{R}^{3n} \;\; ; \exists j \neq k \;, \; x_j = x_k\}$$

and x is close enough to x_0, we consider the change $y \longrightarrow y'$ defined by:

$$y = F_0(x, y') = (F_{x_0}(x, y'_1), ..., F_{x_0}(x, y'_p))$$

where

$$y_l = F_{x_0}(x, y'_l) = y'_l + \sum_{j=1}^{n} (x_j - x_j^0) f_j(y'_l) \quad , \quad l \in \{1, ..., p\} \tag{3.1}$$

$x_0 = (x_1^0, ..., x_n^0)$ and $f_j \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $f_j(x_0^k) = \delta_{jk}$ (kroenecker symbol). In particular, we have $F_{x_0}(x, x_j^0) = x_j$, $j \in \{1, ..., n\}$.

We associate with (3.1) the unitary operator on $L^2(\mathbb{R}^3)$ defined by

$$\mathcal{U}_{x_0}\varphi(x, y'_l) = \varphi(x, F_{x_0}(x, y'_l)) \left|\det(\partial_{y'_l} F_{x_0}(x, y'_l))\right|^{1/2} \;\; , \;\; l \in \{1, ..., p\} \tag{3.2}$$

It is then easy to observe that for any $j \in \{1, ..., n\}$ and $l \in \{1, ..., p\}$

$$\mathcal{U}_{x_0} \frac{1}{|y_l \pm x_j|} \in C^\infty(\Omega_{x_0}, \mathcal{L}(H^2(\mathbb{R}^3), L^2(\mathbb{R}^3)) \tag{3.3}$$

where Ω_{x_0} is a small neighborhood of x_0. It also follows that

$$V(x, F_0(x, y'))(-\Delta_y + 1)^{-1} \in C^\infty(\Omega_{x_0}, \mathcal{L}(L^2(\mathbb{R}^{3p}_y)) \tag{3.4}$$

We only consider the first electronic level $\lambda_1(x)$ $(N = 1)$ which is assumed to be separated by a gap from the rest of the spectrum of $Q(x)$ and that $\lambda_1^{-1}(]-\infty, E_0])$ is compact where E_0 is a fixed energy such that $E_0 < \inf\{\sigma(Q(x))\backslash\{\lambda_1(x)\}\}$.

By compacity, we cover $\lambda_1^{-1}(]-\infty, E_0])$ by a finite number of neighborhoods $(\Omega_l)_{1\leq l\leq M}$ associated with unitary transformations $\mathcal{U}_l$ of the type (3.2) satisfying (3.3) and (3.4).

Setting $W = \mathbb{R}_x^{3n} \smallsetminus \bigcup_{l=1}^{M}\Omega_l$, one can modify $P(h)$ near W such that the modified operator $\widehat{P}(h)$ becomes smooth with respect to x in W and $\sigma(\widehat{P}(h)) = \sigma(P(h)) + \mathcal{O}(e^{-\varepsilon/h})$, $\varepsilon > 0$ (see [13], [14]).

Setting also $\Omega_0 = \mathbb{R}_x^{3n}\smallsetminus\lambda_1^{-1}(]-\infty, E_0])$ and considering the localized matrix operator $\widehat{\mathcal{P}}(z)$ associated with $\widehat{P}(h)$ as in (2.4)

$$\widehat{\mathcal{P}}(z) = \sum_{l=0}^{M} \chi_l(x)\mathcal{U}_l\chi_l(x)\widetilde{P}_l(z)\chi_l(x)\mathcal{U}_l^{-1}\chi_l(x) \tag{3.5}$$

where $(\chi_l^4)_{0\leq l\leq M}$ is an adapted partition of unity of $(\Omega_l)_{0\leq l\leq M}$ and $\widetilde{P}_l(z)$'s are smooth pseudodifferential operators with operator-valued symbol while $\mathcal{U}_0$ is the identity.

By the techniques developped in the second section we get that $\widehat{\mathcal{P}}(z)$ is a smooth invertible $h-$pseudodifferential operator. Writing

$$\widehat{\mathcal{P}}^{-1}(z) = \begin{pmatrix} \widehat{a}(z) & \widehat{a}^{+}(z) \\ \widehat{a}^{-}(z) & \widehat{a}^{-+}(z) \end{pmatrix}$$

and using the fact that the transformations $\mathcal{U}_l$ only act on the $y-$variables, we obtain

Theorem 3.1: *$\widehat{F}(z) = z - \widehat{a}^{-+}(z)$ is a smooth $h-$pseudodifferential operator, with principal symbol $\xi^2 + \lambda_1(x)$. Moreover,*

$$z \in \sigma(\widehat{P}(h)) \Longleftrightarrow z \in \sigma(\widehat{F}(z)$$

For more detail about these results, the interested reader may consult [8], [13] and [14].

4. Resonances of polyatomic molecules

In this Section we describe the results of [14] concerning the resonances of the hamiltonian $P(h) = -h^2\Delta_x - \Delta_y + V(x,y)$ on $L^2(\mathbb{R}_x^{3n} \times \mathbb{R}_y^{3p})$ where

$$V(x,y) = \sum_{j,k=1}^{n} \frac{\alpha_{jk}}{|x_j - x_k|} + \sum_{\substack{1\leq l\leq p \\ 1\leq k\leq n}} \frac{\alpha_{lk}^{\pm}}{|y_l \pm x_k|} + \sum_{\substack{l,q=1 \\ l\neq q}}^{p} \frac{\beta_{lq}}{|y_l - y_q|} \tag{4.1}$$

$x = (x_1, ..., x_n) \in \mathbb{R}_x^{3n}$, $y = (y_1, ..., y_p) \in \mathbb{R}_y^{3p}$, $\alpha_{jk}, \alpha_{lk}^{\pm}$ and β_{lq} are real constants and $\alpha_{jk} > 0$, $\forall j, k \in \{1, ..., n\}$. $W(x) = \sum_{j,k=1}^{n} \frac{\alpha_{jk}}{|x_j - x_k|}$ denotes the interactions between the nuclei of the molecule and $(V(x,y) - W(x))$ the nuclei-electrons and electrons-electrons interactions, $\alpha_{lk}^{\pm}$ indicates the charges of the molecule in particular if $\alpha_{lk}^{+} = \alpha_{lk}^{-}$ the molecule is symmetric.

$P(h)$ with domain $H^2(\mathbb{R}_x^{3n} \times \mathbb{R}_y^{3p})$ is self adjoint in $L^2(\mathbb{R}_x^{3n} \times \mathbb{R}_y^{3p})$ and one can define the resonances of $P(h)$ as the discrete eigenvalues of the distorded Hamiltonian $P_\mu(h)$ ($\mu \in \mathbb{C}$ small enough, $\operatorname{Im} \mu > 0$) obtained by the analytic distorsion (see [14]):

$$x_j \longrightarrow x_j + \mu\omega(x_j) \quad , \quad 1 \leq j \leq n \tag{4.2}$$

$$y_l \longrightarrow y_l + \mu\omega(y_l) \quad , \quad 1 \leq l \leq p$$

where ω is a smooth vector field of $\mathbb{R}^3$, $\omega = 0$ near the collision set $\mathcal{C}$ of all nuclei of the molecule and ω is the identity far from $\mathcal{C}$. $P_\mu(h)$ is defined by $U_\mu P(h) U_\mu^{-1}$ such that

$$U_\mu \varphi(x,y) = \varphi((x_1 + \mu\omega(x_1), ..., x_n + \mu\omega(x_n)\ , y_1 + \mu\omega(y_1), ..., y_p + \mu\omega(y_p))\, |J|^{1/2} \tag{4.3}$$

where $J = J(x,y,\mu) = \prod_{j=1}^{n} \det(1 + \mu D\omega(x_j))\ \prod_{l=1}^{p} \det(1 + \mu D\omega(y_l))$ is the Jacobian of the transformation (4.2).

Under this distorsion the domains and the singularities of $V(x,y)$ are not changed and $P(h)$ can be analytically extended to small complex values of μ.

However, the technique of the thirt Section is not sufficient in this case since the classically allowed region with respect to x, $\lambda_1^{-1}(]-\infty, E_0])$ is now unbounded if the scattering energy level $E_0 \in \sigma_{ess}(P(h)) \backslash \{\text{thresholds of } P(h)\}$.

We assume (H2) to $\sigma(Q(x)) \cap\ ]-\infty, E_0]$ for all x outside $\mathcal{C}$ and also the existence of a constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}_x^{3n} \backslash \mathcal{C}} \widetilde{\lambda}_N(x) \leq C$$

where $\widetilde{\lambda}_N(x) = \lambda_N(x) - W(x)$ (this last assumption is automatically satisfied if e.g. $\alpha_{lk}^{\pm} \leq 0$).

Thus, one must first make a change of variables whose purpose is to localize in a compact region the $x-$dependent singularities with respect to y. After that the previous idea can be adapted to $P_\mu(h)$ and the study of $P_\mu(h)$ can be reduced to the one of a matrix of $h-$pseudodifferential operators on $L^2(\mathbb{R}_x^{3n})$.

More precisely, we costruct a change of type

$$y = (y_1, ..., y_p) \longrightarrow y' = (\tau_x(y_1), ..., \tau_x(y_p))$$

for $x \in \mathbb{R}_x^{3n} \setminus \mathcal{C}$ such that

$$\tau_x(y_l) = \frac{y_l}{|x|} \quad \text{if} \quad |y_l| \leq |x|$$

$$\tau_x(y_l) = Ay_l \quad \text{if} \quad |y_l| \geq 2A\,|x| \tag{4.4}$$

$l \in \{1, ..., p\}$, and $A > 0$ is fixed large enough.

The singularities become localized in the ball $|y'| \leq 1$ and $\widehat{P}_\mu(h)$ the regularization of $P_\mu(h)$ in the elliptic region is now smooth. One can then use the techniques of Section 3 and by the Feshbach method we get:

Theorem 4.1: *For any complex number z close enough to E_0 there exists a family of $N \times N$ matrices $\widehat{F}_\mu(z)$ of $h-$pseudodifferential operators on R_x^{3n} depending analitically on μ (complex small enough),* $\operatorname{Im}\mu > 0$, *such that*

$$z \in \sigma(\widehat{P}_\mu(h)) \Longleftrightarrow z \in \sigma(\widehat{F}_\mu(z))$$

5. Feshbach reduction of Fredholm operators

We present here the Feshbach method that can be applied to the class of Fredholm operators (see [10]). Our trick is to reduce the problem to the inversion of a Grushin problem using no spectral tools.

Let $(P_z)_{z\in\omega}$ be a continuous family on an open complex set ω of linear operators between two complex Hilbert spaces H_1 and H_2.

We assume that for a certain point z_0 of ω, P_{z_0} is a Fredholm operator with index $(n_0 - d_0)$, n_0 is the nullity and d_0 the deficiency of P_{z_0}.

By analogy with the Grushin operator introduced precedently we set

$$\mathcal{P}(z) = \begin{pmatrix} P_z & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

which is acting on $H_1 \bigoplus \mathbb{C}^{d_0}$ into $H_2 \bigoplus \mathbb{C}^{n_0}$ where R_0^+ and R_0^- are bounded linear operators respectively from H_1 to $\mathbb{C}^{n_0}$ and from $\mathbb{C}^{d_0}$ to H_2, with maximum ranks such that

$$\begin{cases} \operatorname{Ran}P_{z_0} \bigoplus \operatorname{Ran}R_0^- = H_2 \\ R_0^+ \,|_{\ker P} \;\; \text{is invertible} \end{cases}$$

Accordingly, $\mathcal{P}(z)$ becomes invertible for z in an enough small complex neighborhood Ω of z_0 contained in ω. Its inverse is given by

$$\mathcal{P}^{-1}(z) = \begin{pmatrix} a(z) & a^+(z) \\ a^-(z) & a^{-+}(z) \end{pmatrix}$$

where $a(z), a^{\pm}(z)$ and $a^{-+}(z)$ are bounded and depends continuously on z in Ω, $a(z) \in \mathcal{L}(H_1, H_2)$, $a^+(z) \in \mathcal{L}(\mathbb{C}^{n_0}, H_1)$, $a^-(z) \in \mathcal{L}(H_2, \mathbb{C}^{d_0})$ and $a^{-+}(z) \in \mathcal{L}(\mathbb{C}^{n_0}, \mathbb{C}^{d_0})$.

Furthermore, $a^{-+}(z_0) = 0$ and $a^{\pm}(z)$ remains with maximal ranks in Ω.

Now, for a given g in H_2, we have in view of (2.7)

$$P_z f = g \Longleftrightarrow \begin{cases} a(z)g + a^+(z)R_0^+ f = f \\ a^-(z)g + a^{-+}(z)R_0^+ f = 0 \end{cases} \tag{5.1}$$

and thus the equation $P_z f = g$ admits a solution f in H_1 if and only if $a^-(z)g \in \mathrm{Ran}(a^{-+}(z))$. This ensures that the map $\rho(\overset{\bullet}{g}) = \widehat{a^-(z)g}$ is an isomorphism from the quotient space $H_2/\mathrm{Ran}P_z$ into $\mathbb{C}^{d_0}/\mathrm{Ran}(a^{-+}(z))$ where $\bullet$ and $\wedge$ are respectively the equivalence classes in $H_2/\mathrm{Ran}P_z$ and $\mathbb{C}^{d_0}/\mathrm{Ran}(a^{-+}(z))$.

Consequently,

$$\dim(H_2/\mathrm{Ran}P_z) = \dim(\mathbb{C}^{d_0}/\mathrm{Ran}(a^{-+}(z))) < +\infty \quad , \quad \forall z \in \Omega \tag{5.2}$$

Conversly, if $\alpha \in \mathbb{C}^{n_0}$ and $\beta \in \mathbb{C}^{d_0}$, then using (2.8) we have

$$a^{-+}(z)\alpha = \beta \Longleftrightarrow \begin{cases} P_z a^+(z)\alpha + R_0^- \beta = 0 \\ R_0^+ a^+(z)\alpha = \alpha \end{cases} \tag{5.3}$$

We deduce that $\dim(\mathrm{Ran}a^+(z)) = n_0$ and $a^+(z)$ is an isomorphism from $\ker a^{-+}(z)$ into $\ker P_z$.

Finally,

$$\dim(\ker P_z) = \dim(\ker a^{-+}(z)) < +\infty \quad , \quad \forall z \in \Omega \tag{5.4}$$

We then have established the following stability result:

Theorem 5.1: *For any z in Ω, P_z is a Fredholm operator with index independent of z.*

$$\mathrm{index}(P_z) = \mathrm{index}(a^{-+}(z)) = n_0 - d_0 \quad , \forall z \in \Omega \tag{5.5}$$

Remark 5.2: *Now If Ω is an open complex connected neighborhood of 0 verifying the foregoing conditions, $H_1 \subset H_2$ with a dense inclusion and $P_z = P - z$ where P is a Fredholm operator such that $n_0 = \dim(\ker P) < +\infty$ and $d_0 = \dim(H_2/RanP) < +\infty$. We also suppose that P_z is invertible for at least a point of Ω different from 0. Then $a^{-+}(z)$ is also invertible at the same point by virtue of (5.2) and (5.5) and $n_0 = d_0$. We get a reduction result as follows*

$$z \in \sigma(P) \Longleftrightarrow \det(a^{-+}(z)) = 0$$

The identities (2.9) and (2.10) are also obtained here.

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