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L^2- BOUNDEDNESS AND L^2- COMPACTNESS OF A CLASS OF FOURIER INTEGRAL OPERATORS

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Abstract . In this paper , we study the L^2- boundedness and L^2- compactness of a class of Fourier integral operators . These operators are bounded (respec - t ively compact) if the weight of the amplitude is bounded (respectively tends to 0) .

1 . Introduction

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$(the Schwartz space) , the integral operators

$$F\varphi(x) = \int e^{iS(x,\theta)} a(x,\theta)\mathcal{F}\varphi(\theta)d\theta \tag{1.1}$$

appear naturally in the expression of the solutions of the hyperbolic partial differ - ential equations and in the expression of the $C^\infty-$ solution of the associate Cauchy ' s problem (see [5 , 1 0]) .

If we write formally the Fourier transformation $\mathcal{F}\varphi(\theta)$ in (1 . 1) , we obtain the following Fourier integral operators

$$F\varphi(x) = \int\int e^{i(S(x,\theta)-y\theta)} a(x,\theta)\varphi(y)dyd\theta \tag{1.2}$$

in which appear two $C^\infty-$ functions , the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ and the amplitude a.

Since 1 970 , many efforts have been made by several authors in order to study these type of operators (see , e . g . , [1 , 4 , 6 , 7 , 8 , 1 5]) . The first works on Fourier inte - gral operators deal with lo cal properties . On the other hand , Asada and Fuj iwara have studied for the first time a class of Fourier integral operators defined on R^n.

For the Fourier integral operators , an interesting question is under which condi - tions on a and S these operators are bounded on L^2 or are compact on L^2.

It has been proved in [1] by a very elaborated proof and with some hypothesis on the phase function ϕ and the amplitude a that all operators of the form (2 . 1) (see below) are bounded on L^2. The technique used there is based on the fact that the operators $I(a,\phi)I^*(a,\phi), I^*(a,\phi)I(a,\phi)$ are pseudodifferential and it uses Cald é ron - Vaillancourt ' s theorem (here $I(a,\phi)^*$ is the adjoint of $I(a,\phi)$).

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In this work , we apply the same technique of [1] to est ablish the boundedness and the compactness of the operators (1 . 2) . To this end we give a brief and simple proof for a result of [1] in our framework .

We mainly prove the continuity of the operator F on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a is bounded . Moreover , F is compact on $L^2(\mathbb{R}^n)$ if this weight tends to zero . Using the estimate given in [1 2] for $h-$ pseudodifferential ($h-$ admissible)
operators , we also establish an L^2- estimate of $\| F \|$.

We note that if the amplitude a is j uste bounded , the Fourier integral operator F is not necessarily bounded on $L^2(\mathbb{R}^n)$. Recently , Hasanov [6] and Messirdi - Senoussaoui [1 1] constructed a class of unbounded Fourier integral operators with an amplitude in the H ö rmander ' s class $S^0_{1,1}$ and in $\bigcap_{0<\rho<1} S^0_{\rho,1}$.

To our knowledge , this work constitutes a first attempt to diagonalize the Fourier integral operators on $L^2(\mathbb{R}^n)$(relying on the compactness of these operators) .

Let us now describe the plan of this article . In the second section we recall the continuity of some general class of Fourier integral operators on $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$. The assumptions and preliminaries results are given in the third section . The last section is devoted to prove the main result .

2 . A general class of Fourier integral operators

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we consider the following integral transformations

$$(I(a,\phi)\varphi)(x) = \iint_{\mathbb{R}^n_y \times \mathbb{R}^N_\theta} e^{i\phi(x,\theta,y)} a(x,\theta,y)\varphi(y)dyd\theta \tag{2.1}$$

where , $x \in \mathbb{R}^n, n \in \mathbb{N}^*$ and $N \in \mathbb{N}$(if $N = 0, \theta$ doesn ' t appear in (2 . 1)) .

In general the integral (2 . 1) is not absolutely convergent , so we use the technique of the oscillatory integral developed by H ö rmander in [8] . The phase function ϕ and the amplitude a are assumed to satisfy the following hypothesis :

(H 1) $\phi \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{R})$($\phi$ is a real function)

(H 2) For all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$| \partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha \phi(x,\theta,y) | \le C_{\alpha,\beta,\gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)} + (x,\theta,y)$$

where $\lambda(x,\theta,y) = (1+ | x |^2 + | \theta |^2 + | y |^2)^{1/2}$ is called the weight and

$$(2- | \alpha | - | \beta | - | \gamma |)_+ = \max(2- | \alpha | - | \beta | - | \gamma |, 0)$$

(H 3) There exist $K_1, K_2 > 0$ such that

$$K_1\lambda(x,\theta,y) \le \lambda(\partial_y\phi, \partial_\theta\phi, y) \le K_2\lambda(x,\theta,y), \quad \forall (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$$

(H 3*) There exist $K_1^*, K_2^* > 0$ such that

$$K_1^*\lambda(x,\theta,y) \le \lambda(x, \partial_\theta\phi, \partial_x\phi) \le K_2^*\lambda(x,\theta,y), \quad \forall (x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y.$$

For any open subset Ω of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mu \in \mathbb{R}$ and $\rho \in [0,1]$, we set

$$\Gamma^\mu_\rho(\Omega) = \{a \in C^\infty(\Omega) : \forall (\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha,\beta,\gamma} > 0 : \\ | \partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha a(x,\theta,y) | \le C_{\alpha,\beta,\gamma} \lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\theta,y)\}$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma^\mu_\rho(\Omega) = \Gamma^\mu_\rho$.

 To give a meaning to the right hand side of (2 . 1) , we consider $g \in \mathcal{S}(\mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n), g(0)=1$. If $a \in \Gamma_0^\mu$, we define

$$a_\sigma(x,\theta,y) = g(x/\sigma,\theta/\sigma,y/\sigma)a(x,\theta,y), \quad \sigma>0.$$

Now we are able to st ate the following result .

Theorem 2 . 1 . *If ϕ satisfies (H 1) , (H 2) , (H 3) and (H 3∗), and if $a \in \Gamma_0^\mu$, th en*

1 . For al l $\varphi \in \mathcal{S}(\mathbb{R}^n), \lim_{\sigma\to+\infty}[I(a_\sigma,\phi)\varphi](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We define

$$(I(a,\phi)\varphi)(x) := \lim_{\to_\sigma+\infty} {}^{(I(a} \sigma,\phi)\varphi)(x)$$

2. *$I(a,\phi) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I(a,\phi) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ (here $\mathcal{L}(E)$ is the space of bounded lin ear mapping from E to E and $\mathcal{S}'(\mathbb{R}^n)$ the space of al l distrib utions with t emperate growth on $\mathbb{R}^n$).*

The proof of the above theorem can be found in [7] or in [1 2 , propostion II . 2] .

Example 2 . 2 . Let us give two examples of operators of the form (2 . 1) which satisfy

$$\text{(H1)} - \text{(H3}*) :$$

(1) The Fourier transform $\mathcal{F}\psi(x) = \int_{\mathbb{R}^n} e^{-ixy}\psi(y)dy, \psi \in \mathcal{S}(\mathbb{R}^n)$, (2) Pseudodifferential operators

$$A\psi(x) = (2\pi)^{-n}\int_{\mathbb{R}^2n} e^{i(x-y)\theta}a(x,y,\theta)\psi(y)dyd\theta,$$
$$\text{with} \psi \in \mathcal{S}(\mathbb{R}^n), a \in \Gamma_0^\mu(\mathbb{R}^{3n}).$$

3 . Assumptions and Preliminaries

In this paper we consider the special form of the phase function

$$\phi(x,y,\theta) = S(x,\theta) - y\theta \tag{3.1}$$

where S satisfies

$$\text{(G1)} \quad S \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\theta^n, \mathbb{R}),$$

(G 2) For each $(\alpha,\beta) \in \mathbb{N}^{2n}$, there exist $C_{\alpha,\beta} > 0$, such that

$$| \partial_x^\alpha \partial_\theta^\beta S(x,\theta) | \le C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)_+},$$

(G 3) There exists $C_1 > 0$ such that $| x | \le C_1\lambda(\theta, \partial_\theta S)$, for all $(x,\theta) \in \mathbb{R}^{2n}$,

(G 3∗) There exists $C_2 > 0$, such that $| \theta | \le C_2\lambda(x,\partial_x S)$, for all $(x,\theta) \in \mathbb{R}^{2n}$.

Proposition 3 . 1 . *Let ' s assume that S satisfies (G 1) , (G 2) , (G 3) and (G 3∗). Then the function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ satisfies (H 1) , (H 2) , (H 3) and (H 3∗).*

Proof . (H 1) and (H 2) are trivially satisfied . The condition (G 3) implies

$$\lambda(x,\theta,y) \le \lambda(x,\theta) + \lambda(y) \le C_3(\lambda(\theta,\partial_\theta S) + \lambda(y)), \quad C_3 > 0.$$

Also , we have $\partial_{y_j}\phi = -\theta_j$ and $\partial_{\theta_j}\phi = \partial_{\theta_j}S - y_j$ and so

$$\lambda(\theta, \partial_\theta S) = \lambda(\partial_y \phi, \partial_\theta \phi + y) \le 2\lambda(\partial_y \phi, \partial_\theta \phi, y),$$

which finally gives for some $C_4 > 0$,

$$\lambda(x, \theta, y) \le C_3(2\lambda(\partial_y \phi, \partial_\theta \phi, y) + \lambda(y)) \le \frac{1}{C_4}\lambda(\partial_y \phi, \partial_\theta \phi, y)$$

The second inequality in (H 3) is a consequence of the assumption (G 2) . By a similar argument we can show (H 3$*$). $\square$

 We now introduce the assumption

(G 4) There exists $\delta_0 > 0$ such that

$$\inf_{\theta,x\in\mathbb{R}n} \mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \mid\geq \delta_0.$$

We note that if $\phi(x,y,\theta) = S(x,\theta) - y\theta$, then

$$D(\phi)(x,\theta,y) = (\frac{\partial^2\phi}{\partial x\partial yline - phi_{\partial\theta\partial y}\partial^2}{}^{(x,}_{(x,}\theta^{\theta},y^{y)}_{)} \quad \frac{\partial^2\phi}{phi - line\partial^2\frac{\partial\theta\partial\theta}{\partial x\partial\theta}}{}^{(x,}_{(x,}\theta^{\theta},y^{y)}_{)}) = (^{0}_{-I_n} \quad \frac{\partial 2_S}{\frac{\partial_{\partial x_2^{\partial\theta}}S}{\partial\theta\partial\theta}}{}^{(x,}_{(x,}\theta^{\theta)}_{)})$$

and

$$|\det D(\phi)(x,\theta,y)| = |\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)| \geq \delta_0.$$

Remark 3 . 2 . By the global implicit function theorem (cf . [1 4] , [3 , theorem 4 . 1 . 7])
and using (G 1) , (G 2) and (G 4) , we can easily see that the mappings h_1 and h_2 defined by

$$h_1 : (x,\theta) \to (x, \partial_x S(x,\theta)), \quad h_2 : (x,\theta) \to (\theta, \partial_\theta S(x,\theta))$$

are global diffeomorphism of $\mathbb{R}^{2n}$. Indeed ,

$$h_1'(x,\theta) = (^{In}_{0} \quad \frac{\frac{\partial^2 S}{\partial^{\partial_{\text{two}-x}}{}^2_S}}{\partial x\partial\theta}{}^{(x,}_{(x,}{}^{\theta)}_{\theta)}), \quad h_2'(x,\theta) = (^{0}_{In} \quad \frac{\partial^2 S}{\partial\frac{\partial^{\text{two}-x}\partial S}{\partial\theta^2}\theta}{}^{(x,}_{(x,}{}^{\theta)}_{\theta\,)}).$$

and $\mid \det h_1'(x,\theta) \mid=\mid \det h_2'(x,\theta) \mid=\mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \mid\geq \delta_0 > 0$, for all $(x,\theta) \in \mathbb{R}^{2n}$. Then

$$\begin{aligned} \| (h_1'(x,\theta))^{-1} \| &= \frac{1}{\mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \mid} \| t_{A(x,}\theta) \| \\ \| (h_2'(x,\theta))^{-1} \| &= \frac{1}{\mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta) \mid} \| t_{B(x,}\theta) \|, \end{aligned}$$

where $A(x,\theta), B(x,\theta)$ are respectively the cofactor matrix of $h_1'(x,\theta), h_2'(x,\theta)$. By (G 2) , we know that $\| t_{A(x,}\theta) \|$ and $\| t_{B(x,}\theta) \|$ are uniformly bounded .

Let ' s now assume that S satisfies the following condition which is stronger than (G 2) .

(G 5) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha,\beta} > 0$, such that

$$\mid \partial_x^\alpha\partial_\theta^\beta S(x,\theta) \mid\leq C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)}.$$

Lemma 3 . 3 . *If S satisfies (G 1) , (G 4) and (G 5) , then S satisfies (G 3) and (G 3∗). Also there exists $C_5 > 0$ such that for al l $(x,\theta),(x',\theta') \in \mathbb{R}^{2n}$,*

$$\mid x - x' \mid + \mid \theta - \theta' \mid\leq C_5[\mid (\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta') \mid + \mid \theta - \theta' \mid] \qquad (3.2)$$

Proof . The mappings

$$\mathbb{R}^n \ni \theta \to f_x(\theta) = \partial_x S(x,\theta), \quad \mathbb{R}^n \ni x \to g\theta(x) = \partial_\theta S(x,\theta)$$

are global diffeomorphisms of $R^n_.$ *From* (G 4) and (G 5) , it follows that $\| (f_x^{-1})' \|$,

$\| (g_\theta^{-1})' \|$ and $\| (h_2^{-1})' \|$ are uniformly bounded on $\mathbb{R}^{2n}$. Thus (G 5) and the Taylor ' s theorem lead to the following estimates : There exist $M, N > 0$, such that for all

$$(x, \theta), (x', \theta') \in \mathbb{R}^{2n},$$

$$| \theta | = | f_x^{-1}(f_x(\theta)) - f_x^{-1}(f_x(0)) | \leq M \mid \partial_x S(x, \theta) - \partial_x S(x, 0) | \leq C_6 \lambda(x, \partial_x S),$$

$$\text{with} C_6 > 0;$$
$$| x |=| g_\theta^{-1}(g\theta(\theta)) - g_\theta^{-1}(g\theta(0)) |\leq N \mid \partial_\theta S(x,\theta) - \partial_\theta S(0,\theta) |\leq C_7\lambda(\partial_\theta S,\theta),$$
$$\text{with} C_7 > 0;$$
$$| (x,\theta) - (x',\theta') |=| h_2^{-1}(h_2(x,\theta)) - h_2^{-1}(h_2(x',\theta')) |$$
$$\leq C_5 \mid (\theta, \partial_\theta S(x,\theta)) - (\theta', \partial_\theta S(x',\theta')) |$$

□

When $\theta = \theta'$ in (3 . 2) , there exists $C_5 > 0$, such that for all $(x, x', \theta) \in \mathbb{R}^{3n}$,

$$| x - x' |\leq C_5 \mid (\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta) \mid . \tag{3.3}$$

Proposition 3 . 4 . *If S satisfies (G 1) and (G 5) , then there exists a constant $\epsilon_0 > 0$ such that the phase function ϕ given in (3 . 1) belongs to $\Gamma_1^2(\Omega_{\phi,\epsilon_0})$ where*

$$\Omega_{\phi,\epsilon_0} = \{(x,\theta,y) \in \mathbb{R}^{3n}; \quad | \partial_\theta S(x,\theta) - y |^2 < \epsilon_0(| x |^2 + | y |^2 + | \theta |^2)\}.$$

Proo f − period We have to show that : There exists $\epsilon_0 > 0$, such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$,

$$\text{thereexist} C_{\alpha,\beta,\gamma} > 0 :$$
$$| \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y) |\leq C_{\alpha,\beta,\gamma}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|-|\gamma|)}, \quad \forall (x,\theta,y) \in \Omega_{\phi,\epsilon_0}. \tag{3.4}$$
$$\text{If } | \gamma |= 1, \text{then}$$
$$| \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y) |=| \partial_x^\alpha \partial_\theta^\beta(-\theta) |= \begin{cases} 0 & \text{if } | \alpha |= 0 \\ | \partial_\theta^\beta(-\theta) | & \text{if} \alpha = 0; \end{cases}$$
$$\text{If } | \gamma |> 1, \text{then } | \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y) |= 0.$$

Hence the estimate (3 . 4) is satisfied .

If $| \gamma |= 0$, then for all $\alpha, \beta \in \mathbb{N}^n; | \alpha | + | \beta |\leq 2$, there exists $C_{\alpha,\beta} > 0$ such that

$$| \partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y) |=| \partial_x^\alpha \partial_\theta^\beta S(x,\theta) - \partial_x^\alpha \partial_\theta^\beta (y\theta) |\leq C_{\alpha,\beta}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|)}.$$

If $| \alpha | + | \beta |> 2$, one has $\partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y) = \partial_x^\alpha \partial_\theta^\beta S(x,\theta)$. In Ω_{ϕ,ϵ_0} we have

$$| y |=| \partial_\theta S(x,\theta) - y - \partial_\theta S(x,\theta) |\leq \sqrt{\epsilon_0}(| x |^2 + | y |^2 + | \theta |^2)^{1/2} + C_8\lambda(x,\theta),$$

with $C_8 > 0$. For ϵ_0 sufficiently small , we obtain a constant $C_9 > 0$ such that

$$| y |\leq C_9\lambda(x,\theta), \quad \forall (x,\theta,y) \in \Omega_{\phi,\epsilon_0}. \tag{3.5}$$

This inequality leads to the equivalence

$$\lambda(x,\theta,y) \simeq \lambda(x,\theta) \quad \text{in}\Omega_{\phi,\epsilon_0} \tag{3.6}$$

thus the assumption $(G5)$ and (3 . 6) give the estimate (3.4). □

Using (3 . 6) , we have the following result . **Proposition 3 . 5 .** *If $(x,\theta) \to a(x,\theta)$ belongs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then $(x,\theta,y) \to a(x,\theta)$ belo ngs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n) \cap \Gamma_k^m(\Omega_{\phi,\epsilon_0}), k \in \{0,1\}$.*

4. L^2- BOUNDEDNESS AND L^2- COMPACTNESS OF F

The main result is as follows .

Theorem 4 . 1 . *Let F be the integral operator of distribution kernel*

$$K(x,y)=\int_{\mathbb{R}n} e^{i(S(x,\theta)-y\theta)}a(x,\theta)\widehat{^{d}}\theta \tag{4.1}$$

where $\widehat{d}\theta=(2\pi)^{-n}d\theta, a\in\Gamma_k^m(\mathbb{R}^{2n}_{x},_\theta), k=0,1$ and S satisfies $(G1)$, (G 4) and (G 5) .
Then FF^* and F^*F are pseudodifferential operators with symbol in $\Gamma_k^{2m}(\mathbb{R}^{2n})$, $k=0,1$, given by

$$\sigma(FF^*)(x,\partial_x S(x,\theta))\equiv\mid a(x,\theta)\mid^2\mid(\det\frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)\mid$$
$$\sigma(F^*F)(\partial_\theta S(x,\theta),\theta)\equiv\mid a(x,\theta)\mid^2\mid(\det\frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)\mid$$

we denote here $a\equiv b$ for $a,b\in\Gamma_k^{2p}(\mathbb{R}^{2n})$ if $(a-b)\in\Gamma_k^{2p-2}(\mathbb{R}^{2n})$ and σ s tands for the symbol . Proof . If $u\in\mathcal{S}(\mathbb{R}^n)$, then $Fu(x)$ is given by

$$\begin{aligned} Fu(x) &= \int_{\mathbb{R}n} K(x,y)u(y)dy \\ =&= \int_{\mathbb{R}n\int_{\mathbb{R}n}} \mathbb{R}n^{\theta)_{a(x,\theta)(\int_{\mathbb{R}n}e^{-iy\theta}u(y)dy)}}_{\int^{e^{i(S(x,\theta)-y\theta)}a(x,}_{e^{iS(x,}}\theta)u(y)dy^{\widehat{d}}\theta}\widehat{d}\theta \\ &= \int_{\mathbb{R}n} e^{iS(x,\theta)}a(x,\theta)\mathcal{F}u(\theta)\widehat{^{d}}\theta. \end{aligned} \tag{4.2}$$

Here F is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$(by Theorem 2 . 1) . Let

$$v\in\mathcal{S}(\mathbb{R}^n),\text{then}$$
$$\begin{aligned}\langle Fu,v\rangle L^2(\mathbb{R}n) &= \int_{\mathbb{R}n}(\int_{\mathbb{R}n} e^{iS(x,\theta)}a(x,\theta)\mathcal{F}u(\theta)\widehat{^{d}}\theta)\overline{\quad\quad} v(x)dx \\ &= \int_{\mathbb{R}n}\mathcal{F}u(\theta)(\int_{\mathbb{R}n}\overline{\quad\quad}_{e^{iS_-(x,\theta)}}\overline{\quad\quad}_{a(x,\theta)v(x)dx})\widehat{d}\theta\end{aligned}$$

thus

$$\langle Fu(x),v(x)\rangle L^2(\mathbb{R}n)=(2\pi)^{-n}\langle\mathcal{F}u(\theta),\mathcal{F}((F^*v))(\theta)\rangle L^2(\mathbb{R}n)$$

where

$$\mathcal{F}((F^*v))(\theta)=\int_{\mathbb{R}n} e^{-iS(x-e,\theta)}\overline{\quad\quad} a(\widetilde{x},\theta)v(\widetilde{x})d\widetilde{x}. \tag{4.3}$$

Hence , for all $v\in\mathcal{S}(\mathbb{R}^n)$,

$$(FF^*v)(x)=\int_{\mathbb{R}n}\int_{\mathbb{R}n} e^{i(S(x,\theta)-S(x-e,\theta))}a(x,\theta)\overline{\quad\quad} a(\widetilde{x},\theta)d\widetilde{x}\widehat{d}\theta. \tag{4.4}$$

The main idea to show that FF^* is a pseudodifferential operator , is to use the fact that $(S(x,\theta)-S(\widetilde{x},\theta))$ can be expressed by the scalar product $\langle x-\widetilde{x},\xi(x,\widetilde{x},\theta)\rangle$

after considering the change of variables $(x,\widetilde{x},\theta) \to (x,\widetilde{x},\xi = \xi(x,\widetilde{x},\theta))$.

The distribution kernel of FF^* is

$$K(x,\tilde{x}) = \int_{\mathbb{R}n} e^{i(S(x,\theta)-S(\tilde{x},\theta))} a(x,\theta)\overline{\qquad\quad} a(\tilde{x},\theta)\hat{^{d}}\theta.$$

 We obtain from (3 . 3) that if $| x - \widetilde{x} | \geq \frac{\epsilon}{2}\lambda(x, \widetilde{x}, \theta)$(where $\epsilon > 0$ is sufficiently small) then

$$| (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\widetilde{x}, \theta) | \geq \frac{\epsilon}{2C_5}\lambda(x, \widetilde{x}, \theta). \quad (4.5)$$

Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\omega(x) \geq 0, \quad \forall x \in \mathbb{R}$$
$$\omega(x) = 1 \quad \text{if} x \in [-\frac{1}{2}, \frac{1}{2}]$$
$$\text{supp}\omega \subset] - 1, 1[$$

and setting

$$b(x, \tilde{x}, \theta) := a(x, \theta) \text{———} a(\tilde{x}, \theta) = b_{1,\epsilon}(x, \tilde{x}, \theta) + b_{2,\epsilon}(x, \tilde{x}, \theta)$$
$$b_{1,\epsilon}(x, \tilde{x}, \theta) = \omega(\frac{| x - \tilde{x} |}{\epsilon\lambda(x, \tilde{x}, \theta)})b(x, \tilde{x}, \theta)$$
$$b_{2,\epsilon}(x, \tilde{x}, \theta) = [1 - \omega(\frac{| x - \tilde{x} |}{\epsilon\lambda(x, \tilde{x}, \theta)})]b(x, \tilde{x}, \theta).$$

We have $K(x, \widetilde{x}) = K_{1,\epsilon}(x, \widetilde{x}) + K_{2,\epsilon}(x, \widetilde{x})$, where

$$K_{j,\epsilon}(x, \tilde{x}) = \int_{\mathbb{R}n} e^{i(S(x,\theta)-S(\tilde{x},\theta))} b_{j,\epsilon}(x, \tilde{x}, \theta) \widehat{^d}\theta, \quad j = 1, 2.$$

We will study separately the kernels $K_{1,\epsilon}$ and $K_{2,\epsilon}$.

On the support of $b_{2,\epsilon}$, inequality (4 . 5) is satisfied and we have

$$K_{2,\epsilon}(x, \widetilde{x}) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed , using the oscillatory integral method , there is a linear partial differential operator L of order 1 such that

$$L(e^{i(S(x,\theta)-S(\tilde{x},\theta))}) = e^{i(S(x,\theta)-S(\tilde{x},\theta))}$$

where

$$L = -i \mid (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\widetilde{x}, \theta) \mid^{-2} \sum_{l=1}^{n} [(\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\widetilde{x}, \theta)]\partial_{\theta_l}.$$

The transpose operator of L is

$$t_L = \sum_{l=1}^{n} F_l(x, \widetilde{x}, \theta)\partial_{\theta_l} + G(x, \widetilde{x}, \theta)$$

$$\text{where} F_l(x, \widetilde{x}, \theta) \in \Gamma_0^{-1}(\Omega_\epsilon), G(x, \widetilde{x}, \theta) \in \Gamma_0^{-2}(\Omega_\epsilon),$$

$$F_l(x, \widetilde{x}, \theta) = i \mid (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\widetilde{x}, \theta) \mid^{-2} ((\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\widetilde{x}, \theta)),$$

$$G(x, \widetilde{x}, \theta) = i \sum_{l=1}^{n} \partial_{\theta_l}[\mid (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\widetilde{x}, \theta) \mid^{-2} ((\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\widetilde{x}, \theta))],$$

$$\Omega_\epsilon = \{(x, \tilde{x}, \theta) \in \mathbb{R}^{3n} : \mid \partial_\theta S(x, \theta) - \partial_\theta S(\tilde{x}, \theta) \mid > \frac{\epsilon}{2C_5} \lambda(x, \tilde{x}, \theta)\}.$$

On the other hand we prove by induction on q that

$$({}^tL)^q b_{2,\epsilon}(x, \tilde{x}, \theta) = \sum_{\mid \gamma \mid \leq q, \gamma \in \mathbb{N}n} g_{\gamma,q}(x, \tilde{x}, \theta)\partial_\theta^\gamma b_{2,\epsilon}(x, \tilde{x}, \theta), \quad \gamma_g^{(q)} \in \Gamma_0^{-q}(\Omega_\epsilon),$$

 and so ,

$$K_{2,\epsilon}(x,\tilde{x}) = \int_{\mathbb{R}n} e^{i(S(x,\theta)-S(\tilde{x},\theta))}({}^tL)^q b_{2,\epsilon}(x,\tilde{x},\theta)\hat{d}\theta.$$

Using Leibnitz ' s formula , (G 5) and the form $({}^tL)^q$, we can choose q large enough such that for all $\alpha,\alpha',\beta,\beta' \in \mathbb{N}^n$, $\exists C_{\alpha,\alpha'},\beta,\beta' > 0$,

$$\sup_{,\overset{x-e}{x} \in \mathbb{R}n} | x^\alpha \widetilde{x}^{\alpha'}_{\partial_x}{}^{\beta}\partial^{\beta'}_{x-e} K_{2,\epsilon}(x,\widetilde{x}) | \leq C_{\alpha,\alpha'},\beta,\beta'.$$

Next , we study K_1^ϵ : this is more difficult and depends on the choice of the parameter ϵ. It follows from Taylor ' s formula that

$$S(x,\theta) - S(\widetilde{x},\theta) = \langle x-\widetilde{x}, \xi(x,\widetilde{x},\theta)\rangle \mathbb{R}n,$$
$$\xi(x,\widetilde{x},\theta) = \int_0^1 (\partial_x S)(\widetilde{x} + t(x-\widetilde{x}),\theta)dt.$$

We define the vectorial function

$$\widetilde{\xi_\epsilon}(x,\widetilde{x},\theta) = \omega(\frac{| x-\tilde{x} |}{2\epsilon\lambda(x,\tilde{x},\theta)}\Big)\xi(x,\widetilde{x},\theta) + (1-\omega(\frac{| x-\tilde{x} |}{2\epsilon\lambda(x,\tilde{x},\theta)}))(\partial_x S)(\widetilde{x},\theta).$$

We have

$$\widetilde{\xi_\epsilon}(x,\widetilde{x},\theta) = \xi(x,\widetilde{x},\theta) \mathrm{onsupp} b_{1,\epsilon}.$$

Moreover , for ϵ sufficiently small ,

$$\lambda(x,\theta) \simeq \lambda(\widetilde{x},\theta) \simeq \lambda(x,\widetilde{x},\theta)\mathrm{onsupp} b_{1,\epsilon}. \tag{4.6}$$

Let us consider the mapping

$$\mathbb{R}^{3n} \ni (x,\widetilde{x},\theta) \to (x,\widetilde{x},\widetilde{\xi_\epsilon}(x,\widetilde{x},\theta)) \tag{4.7}$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \partial_x\widetilde{\xi_\epsilon} & \partial_{x-e}\widetilde{\xi_\epsilon} & \partial_\theta\widetilde{\xi_\epsilon} \end{pmatrix}.$$

We have

$$\begin{aligned} &\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta) \\ &= \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta) + \omega(\frac{| x-\tilde{x} |}{2\epsilon\lambda(x,\tilde{x},\theta)}\Big)(\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)) \\ &- \frac{| x-\tilde{x} |}{2\epsilon\lambda(x,\tilde{x},\theta)}\frac{\partial\lambda}{\partial\theta_i}(x,\tilde{x},\theta)\lambda^{-1}(x,\tilde{x},\theta)\omega'(\frac{| x-\tilde{x} |}{2\epsilon\lambda(x,\tilde{x},\theta)}\Big)(\xi_j(x,\widetilde{x},\theta) - \frac{\partial S}{\partial x_j}(\widetilde{x},\theta)). \end{aligned}$$

Thus , we obtain

$$\begin{aligned}
&|\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)| \\
&\leq |\omega(\frac{\mid x-\tilde{x}\mid}{2\epsilon\lambda(x,\tilde{x},\theta)})||\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)| \\
&+\lambda^{-1}(x,\tilde{x},\theta)|\omega'(\frac{\mid x-\tilde{x}\mid}{2\epsilon\lambda(x,\tilde{x},\theta)})||\xi_j(x,\widetilde{x},\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)|.
\end{aligned}$$

 Now it follows from (G 5) , (4 . 6) and Taylor ' s formula that

$$|\frac{\partial \xi_j}{\partial \theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)|\le\int_0^1|\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)|dt$$
$$\le C_{10}\mid x-\widetilde{x}\mid\lambda^{-1}(x,\tilde{x},\theta),\quad C_{10}>0 \qquad (4.8)$$

$$|\xi_j(x,\widetilde{x},\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)|\le\int_0^1|\frac{\partial S}{\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)|dt \qquad (4.9)$$
$$\le C_{11}\mid x-\widetilde{x}\mid,\quad C_{11}>0.$$

From (4 . 8) and (4 . 9) , there exists a positive constant $C_{12}>0$ such that

$$\mid\frac{\partial\widetilde{\xi}_{\epsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\mid\le C_{12}\epsilon,\quad\forall i,j\in\{1,...,n\}. \qquad (4.10)$$

If $\epsilon<\frac{\delta_0}{2C-e},$ then (4 . 1 0) and (G 4) yields the estimate

$$\delta_0/2\le-\widetilde{C}_\epsilon+\delta_0\le-\widetilde{C}_\epsilon+\det\frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\le\det\partial_\theta\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta), \qquad (4.11)$$

with $\widetilde{C}>0$ If ϵ is such that (4 . 6) and (4 . 1 1) hold , then the mapping given in (4 . 7) is a global diffeomorphism of $\mathbb{R}^{3n}$. Hence there exists a mapping

$$\theta:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\ni(x,\widetilde{x},\xi)\to\theta(x,\widetilde{x},\xi)\in\mathbb{R}^n$$

such that

$$\begin{aligned}\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta(x,\widetilde{x},\xi))&=\xi\\ \theta(x,\widetilde{x},\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta))&=x\\ \partial^\alpha\theta(x,\widetilde{x},\xi)=\mathcal{O}(1),\quad\forall\alpha&\in\mathbb{N}^{3n}\setminus\{0\}\end{aligned} \qquad (4.12)$$

If we change the variable ξ by $\theta(x,\widetilde{x},\xi)$ in $K_{1,\epsilon}(x,\widetilde{x})$, we obtain

$$K_{1,\epsilon}(x,\widetilde{x})=\int_{\mathbb{R}n}e^{i\langle x-\tilde{x},\xi\rangle}b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)|\hat{^d}\xi. \qquad (4.13)$$

From (4 . 1 2) we have , for $k=0,1,$ that $b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\mid\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\mid$ belongs

$$\text{to}\Gamma_k^{2m}(\mathbb{R}^{3n})\text{if}a\in\Gamma_k^m(\mathbb{R}^{2n}).$$

Applying the stationary phase theorem (c . f . [1 2]) to 4 . 1 3 , we obtain the expres - sion of the symbol of the pseudodifferential operator FF^*,

$$\sigma(FF^*)=b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)|\mid e-x=x+R(x,\xi)$$

where $R(x,\xi)$ belongs to $\Gamma_k^{2m-2}(\mathbb{R}^{2n})$ if $a\in\Gamma_k^m(\mathbb{R}^{2n}),k=0,1.$

For $\tilde{x}=x$, we have $b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))=\mid a(x,\theta(x,x,\xi))\mid^2$ where $\theta(x,x,\xi)$ is the inverse of the mapping $\theta\to\partial_x S(x,\theta)=\xi$. Thus

$$\sigma(FF^*)(x,\partial_x S(x,\theta))\equiv\mid a(x,\theta)\mid^2\mid\det\frac{\partial^2 S}{\partial\theta\partial x}(x,\theta)|^{-1}_{.}$$

 From (4 . 2) and (4 . 3) , we obtain the expression of $F^*F : \forall v \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}(F^*F)\mathcal{F}^{-1})v(\theta) = \int_{\mathbb{R}n} e^{-iS(x,\theta)}\overline{\quad\quad} a(x,\theta)(F(\mathcal{F}^{-1}v))(x)dx$$

$$= \int_{\mathbb{R}n} e^{-iS(x,\theta)}\overline{\quad\quad} a(x,\theta)(\int_{\mathbb{R}n} e^{iS(x,^{theta-e})} a(x,\widetilde{\theta})(\mathcal{F}(\mathcal{F}^{-1}v))(^{\widetilde{\theta}})^{\widetilde{d\theta}})dx$$

$$= \int_{\mathbb{R}n}\int_{\mathbb{R}n} e^{-i(S(x,\theta)-S(x,^{\tilde{\theta}}))}\overline{\quad\quad}_a(x,\theta)a(x,\widetilde{\theta})v(^{\tilde{\theta}})^{\widetilde{d\theta}}dx.$$

Hence the distribution kernel of the integral operator $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is

$$\widetilde{K}(\theta,\widetilde{\theta}) = \int_{\mathbb{R}n} e^{-i(S(x,\theta)-S(x,^{\tilde{\theta}}))}\overline{\quad\quad} a(x,\theta)a(x,\tilde{\theta})^{\widehat{d}}x.$$

We remark that we can deduce $\widetilde{K}(\theta,\widetilde{\theta})$ from $K(x,\widetilde{x})$ by replacing x by θ. On the other hand , all assumptions used here are symmetrical on x and θ; therefore , $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is a nice pseudodifferential operator with symbol

$$\sigma(\mathcal{F}(F^*F)\mathcal{F}^{-1})(\theta,-\partial_\theta S(x,\theta)) \equiv\mid a(x,\theta)\mid^2 \mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)|^{-1}_.$$

Thus the symbol of F^*F is given by (c . f . [9])

$$\sigma(F^*F)(\partial_\theta S(x,\theta),\theta) \equiv\mid a(x,\theta)\mid^2 \mid \det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)|^{-1}_.$$

□

Corollary 4 . 2 . *Let F be the integral operator with the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}n} e^{i(S(x,\theta)-y\theta)} a(x,\theta)^{\widehat{d}}\theta$$

where $a \in \Gamma_0^m(\mathbb{R}^{2n}_{x,\theta})$ and S satisfies (G 1) , (G 4) and (G 5) . Then , we have :

(1) For any m such that $m \leq 0, F$ can be extended as a bounded lin ear mapping

$$onL^2(\mathbb{R}^n)$$

(2) For any m such that $m < 0, F$ can be extended as a compact operator on

$$L^2(\mathbb{R}^n).$$

$Proo\ f-period$ It follows from Theorem 4 . 1 that F^*F is a pseudodifferential operator with

$$\text{symbolin}\Gamma_0^{2m}(\mathbb{R}^{2n}).$$

(1) If $m \leq 0,$ the weight $\lambda^{2m}(x,\theta)$ is bounded , so we can apply the Cald é ron - Vaillancourt theorem (see [2 , 12 , 13]) for F^*F and obtain the existence of a positive constant $\gamma(n)$ and a integer $k(n)$ such that

$$\parallel (F^*F)u \parallel L^2(\mathbb{R}n) \leq \gamma(n)Qk(n)^{(\sigma(FF^*))\|}u \parallel L^2(\mathbb{R}n), \quad \forall u \in \mathcal{S}(\mathbb{R}^n)$$

where

$$Qk(n)^{(\sigma(FF^*))} =\mid \alpha \mid + \sum_{|\beta|\leq} k(n)(\theta^{\sup}_{,x}) \in \mathbb{R}^2 n|\partial_x^\alpha \partial_\theta^\beta \sigma(FF^*)(\partial_\theta S(x,\theta),\theta)|$$

Hence , for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\| Fu \| \, L^2(\mathbb{R}n) \leq \| F^*F \|^{1/2}_{\mathcal{L}(L^2(\mathbb{R}^n))} \| u \| \, L^2(\mathbb{R}n) \leq (\gamma(n) Q_{k(n)}(\sigma(FF^*)))^{1/2} \| u \| \, L^2(\mathbb{R}n)\cdot$$

Thus F is also a bounded linear operator on $L^2(\mathbb{R}^n)$. (2) If $m < 0, \lim_{|x|+|\theta|\to+\infty} \lambda^m(x,\theta) = 0$, and the compactness theorem (see [1 2 ,
1 3]) show that the operator F^*F can be extended as a compact operator on $L^2(\mathbb{R}^n)$.

 Thus , the Fourier integral operator F is compact on $L^2(\mathbb{R}^n)$. Indeed , let $(\varphi_j)_j \in \mathbb{N}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\| F^*F - \sum_{j=1}^{n} \langle\varphi_j, .\rangle F^*F\varphi_j \| \to 0 \quad \text{as} n \to +\infty.$$

Since F is bounded , for all $\psi \in L^2(\mathbb{R}^n)$,

$$\|F\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle F\varphi_j\|2 \le \|F^*F\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle F^*F\varphi_j\|\|\psi - \sum_{j=1}^{n} \langle\varphi_j, \psi\rangle\varphi_j\|,$$

it follows that

$$\| F - \sum_{j=1}^{n} \langle\varphi_j, .\rangle F\varphi_j \| \to 0 \quad \text{as} n \to +\infty$$

□

Example 4 . 3 . We consider the function given by

$$S(x, \theta) = \sum_{| \alpha | + | \beta |= 2, \alpha, \beta \in \mathbb{N}n} C_{\alpha,\beta} x^\alpha \theta^\beta, \quad \text{for}(x, \theta) \in \mathbb{R}^{2n}$$

where $C_{\alpha,\beta}$ are real constants . This function satisfies (G 1) , (G 4) and (G 5) .

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