RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ ABDELHAMID IBN BADIS DE MOSTAGANEM
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UNIVERSITE
Abdelhamid Ibn Badis
MOSTAGANEM

THÈSE PRÉSENTÉE POUR L'OBTENTION DU DIPLÔME DOCTORAT LMD EN MATHÉMATIQUES

Option:
Analyse Fonctionnelle

Intitulée:

LES INCLUSIONS DIFFÉRENTIELLES FRACTIONNAIRES

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Acknowlegement

First of all, i would like to thank my thesis director **Dr. Belarbi Hamani Samira***, Professor at the University of Mostaganem, for her choise of the subject of my thesis and also for her advices, her encouragements and her disposition throughout the preparation of my thesis.*

I also want to thank **Dr. Saada Hamouda***, Professor at the University of Mostaganem, who gave us his time and agree to chair the jury. My thanks also go to* **Dr. Wafaa BATAT***, Professor at the National Polytechnic School of Oran, and to* **Dr. Mouffak BENCHOHRA***, Professor at the University of Sidi Bel-Abbès, who have honored us in accepting to participate in the jury.*

I want to thank all the members of the laboratory of pure and applied mathematics of Mostaganem University. I do not forget to thank Pr. Johnny Henderson and Pr. John R. Graef for their collaborations.

Finally, i thank all those who contributed to the realization of this work, especially, my parents, my brothers Ahmed and Taha Yacine, my sister Nor el houda, my uncle Mahmoud and my friend Mounir.

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2019

Abstract

Fractional differential inclusions

This thesis deals with the existence of solutions for several classes of initial and boundary value problems to differential inclusions of fractional order. For this, we shall use three fixed points theorems. The first fixed point theorem is the nonlinear alternative of Leray-schauder, this theorem is used when the set-valued map is convex. The second fixed point theorem is the theorem of Banach for contraction multivalued maps due to Covitz and Nadler, we use this theorem when the set-valued map is noncovex. While the third theorem is the Mönch fixed point theorem combined with the technique of measure of noncompactness, this theorem was generalized in the set-valued version by D. O'Regan and R. Precup in 2000.

Key words and phrases: Existence of solutions, initial and boundary value problems, differential inclusions of fractional order, the nonlinear alternative of Leray-schauder, convex, Banach, noncovex, Mönch, measure of noncompactness.

Résumé

Les inclusions différentielles fractionnaires

Dans cette thèse, on s'intéresse à résoudre des différentes classes de problèmes initiales et aussi aux limites pour les inclusions différentielles d'ordre fractionnaire. Pour cela on va utiliser trois théorèmes de points fixes dans leurs versions multivoques. Le premier théorème de point fixe utilisé est l'alternative nonlinéaire de Leray-Schauder, ce théorème est appliqué quand l'application multivoque est convexe. le deuxième théorème de point fixe est le théorème de Banach pour les applications multivoques contractantes et on l'applique quand l'application multivoque est non convexe. Le trosième théorème de point fixe est le théorème de Mönch combiné avec la mesure de non compacité de Kuratowski, ce théorème a été généralisé dans sa version multivoque par D. O'Regan et R. Precup en 2000.

Phrases et mots clés: Résoudre, problèmes initiales et aussi aux limites, les inclusions différentielles d'ordre fractionnaire, l'alternative nonlinéaire de Leray-Schauder, convexe, Banach, non convexe, Mönch, mesure de non compacité.

الملخص

الإحتواءات التفاضلية الكسرية

في هذه الأطروحة، قُمنا بعرض مجموعة من المشكلات ذات الشروط الإبتدائية و كذلك النهائية للإحتواءات التفاضلية ذات الرتب الكسرية و برهنا على أنها تقبل حلولا، وهذا باستعمال ثلاث نظريات مختلفة للنقطة الثابتة. النظرية الأولى المُستعملة هي المتناوبة الغير خطية لليراي و شاودر، و تُستعمل عادة لما تكون الدالة متعددة الأشكال محدبة. بينما النظرية الثانية هي نظرية بانج و تُستعمل عندما تكون الدالة متعددة الأشكال غير محدبة. بينما النظرية الأخيرة هي نظرية مونك ممزوجة مع نظرية القياس لكوراتسكي، حيث أن هذه النظرية تم تعميمها من طرف د. أوريغان و ر. بريكوب في سنة 2000.

جُمل و كلمات مفتاحية: المشكلات ذات الشروط الإبتدائية و النهائية، الإحتواءات التفاضلية ذات الرتب الكسرية، تقبل حلولا، المتناوبة الغير خطية لليراي و شاودر، محدبة، بانج، غير محدبة، مونك، نظرية القياس.

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Contents

Introduction

In recent years, the domaine of fractional calculus has grown considerably and differential equations of fractional order became a valuable tools in the mathematical modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. In the monographs of Hilfer [55], Kilbas *et al.* [61], Podlubny [74], Momani *et al.* [68], Samko *et al* [76], Delbosco and Rodino [34], Diethelm *et al* [35, 36, 37], we can find the background mathematics and various applications of fractional calculus.

Recently, many researchers paid attention to existence and uniqueness of solutions of initial and boundary value problems for fractional differential equations (see [1], [6], [7], [15], [16], [17] , [41]-[47], [57], [61]). In particular, anti-periodic, integral and nonlocal boundary value problems constitute an important class of boundary value problems and receiving considerable recent attention.

Anti-periodic boundary conditions occur in mathematical modelling of many physical processes and many researchers investigate the existence of solutions of this class of boundary value problem; see the articles of Ahmad *et al.* [8] and Chen *et al.* [31]. As examples of this research, the authors in [8] used the Banach fixed point theorem to investigate existence and uniqueness of solutions for integro-differential equations of fractional order $\alpha \in (1, 2]$ with anti-periodic boundary conditions. In [4], the authors investigated existence of solutions for an anti-periodic boundary value problem for a fractional differential equation of order $\alpha \in (2, 3]$ by using the Banach fixed point theorem. Many authors used the Schauder fixed point theorem to investigate the existence of solutions for anti-periodic fractional differential equations (see[7], [31]).

The Caputo fractional derivative is very useful in many applied problems, because it saisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as the same data for boundary conditions.

The Hadamard fractional derivative was introduced by Hadamard in 1892 [50], this derivative differs from the Caputo derivative in two ways; the first way is that its kernal contains a logarithmic function of arbitrary exponent, and the second way is that the Hadamard dervative of a constant does not equal to 0.

In this thesis, we investigate the existence of solutions to many problems of fractional differential inclusions, for this, we use several fixed point theorems. This thesis is arranged as follows:

In the first chapter, we give some notations and definitions concerned the fractional calculus and the set-valued maps also we recall some fixed point theorems. In the first section of this chapter we give some notations. The second section is devoted to the fractional calculus. In the third section we shall be concerned by the set-valued maps theory. In the last section we recall some fixed point theorems on the set valued version.

In the second chapter, we present our first main result. In the section 2.1 we study the convexe case, our approach here is based upon the nonlinear alternative of Leray-Schauder. In the section 2.2 we study the nonconvex case, here our result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. In the section 2.3 we give an example to illustrate our main results.

In the third chapter we shall be concerned by several problems for fractional differential inlusions, for each problem, we present two existence results, one relies on the nonlinear alternative of Leray-Schauder type, while the other is based upon the Banach fixed point theorem for contraction multivalued. In the section 3.1 we study an initial value problem of order $\alpha \in (0, 1]$. In the section 3.2 we present our main results for a nonlinear boundary problem for fractional differential inclusions. In the section 3.3 we study a boundary value problem of fractional differential inclusions with nonlocal multi-point boundary conditions of order $\alpha \in (1, 2]$. In the section 3.4 we are interesting by a neutral functional differential inclusions with Hadamard type derivative, in the last, we give an example to illustrate the abstract theory.

In the fourth chapter we use the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness to investigate the existence of solutions for different fractional inclusions. Recently, this has proved to be a valued tool in solving fractional differential equation and inclusions in Banach spaces; for details, see the papers of Laosta *et al* [63], Agarwal *et al.* [2] and Benchohra *et al.* [18], [19], [20]. In the section 4.1 we discuss the existence of solutions for a fractional inclusion of order $\alpha \in (1, 2]$. In the last we give an example to illustrate our main results. In the section 4.2 we announce the theorem of existence of solutions for a nonlinear boundary value problem for fractional differential inclusions with integral boundary conditions. In the section 4.3 we present another existence result for the fractional inclusion given in the section 3.1. In the section 4.4 we study a boundary value problem of fractional differential inclusions with Hadamard type derivative in Banach spaces with integral boundary conditions.

Key words and phrases: Fractional calculus, anti-periodic boundary conditions, Banach fixed point theorem, existence, uniqueness, Caputo fractional derivative, Hadamard fractional derivative, convex, nonconvex, initial value problem, set-valued maps, fixed points.

AMS (MOS) Subject Classifications: 26A33, 34A60, 43A08, 34A60, 34B15.

Chapter 1

Preliminaries

In this chapter, we give some notations and definitions and we recall some properties of the fractional calculus and the theory of set-valued map. Also we give some fixed point theorems on the multivalued version.

1.1 Notations and definitions

Let $(E, |\cdot|)$ be a Banach space, we set
$C(J, E)$ the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $L^1(J, E)$ the Banach Bochner integrable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

The space $AC(J, E)$ is the space of functions $y : J \to E$ that are absolutely continuous.
And $AC^1(J, E)$ is the space of functions $y : J \to E$ which are derivables and have a continuous first derivative.
For any Banach space $(X, \|\cdot\|)$, we set

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},$$

$$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},$$

$$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$$

$$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}.$$

For a given set V of functions $u : J \mapsto E$, we set

$$V(t) = \{u(t) : u \in V\}, t \in J$$

and

$$V(J) = \{u(t) : u \in V(t), t \in J\}$$

1.2 Some properties of fractional calculus

1.2.1 Introduction

Gottfried Wilhelm Leibniz was the first mathematician who gave a sense to the expression $\dfrac{d^n f}{dx^n}$ which means the derivative of order n of the real function f, such that n is integer. In 1695, G.W.Leibniz and the frensh mathematician Marquis De L'Hospital discussed the posibility to generalize the concept of the derivative of order n, given by leibniz, to the case when n is non integer. This great idea dont appears easy to realize! because like its known the following expression

$$\frac{d^\alpha}{dx^\alpha} e^{\lambda x} = \lambda^\alpha e^{\lambda x} \tag{1.1}$$

is valid when α is integer but it is not the case when α is not integer, this is called the Leibniz' paradox.
By defining and using his Gamma function Γ, which is a generalization of the product $1.2...n = n!$, Euler was able to introduce the following fractional derivative

$$\frac{d^\alpha}{dx^\alpha} x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha, \beta \in \mathbb{Q} \tag{1.2}$$

unfortunately, this don't resolve completely the Leibniz' paradox. Indeed, by using this fractional derivative, we have

$$\frac{d^\alpha}{dx^\alpha} e^x = \frac{d^\alpha}{dx^\alpha} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{k!\Gamma(1-\alpha+\beta)} x^{k-\alpha} \neq e^x \tag{1.3}$$

Motivating by Leibniz' paradox many researchers (Fourier, Liouville, Rieman, Grunwald, Letnikov,...) have contributed with different ideas, which developed the sciense of Fractional Calculus. Nowadays, we distinguish different fractional integrals and derivatives.

1.2.2 Fractional integral and derivative of Riemann-Liouville

As we know, the integration of order n (n is integer) of the function f is given by

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n) dt_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt \tag{1.4}$$

by using the function Γ, we can give to this formula a sens when n is non integer.

Definition 1.1 *([61], [76]) Let $h \in L^1([a,b], \mathbb{R})$. The left sided fractional integral of Riemann-Liouville of order α is defined by*

$$(I_a^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $\alpha > 0$*. When* $a = 0$*, we write*

$$I^{\alpha}h(t) = h(t) * \varphi_{\alpha}(t)$$

where

$$\varphi_{\alpha}(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0, \end{cases}$$

and

$$\varphi_{\alpha} \to \delta(t) \text{ as } \alpha \to 0$$

where δ *is the delta function.*

Example 1.2 *([76], section 2.5) Let* $h(t) = (t-a)^{\beta}$ *where* $t > a$ *and* $\beta > -1$*, then we have*

$$(I_a^{\alpha}h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^{\beta} ds,$$

by setting $y = \dfrac{s-a}{t-a}$*, we find*

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^{\beta} ds &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t(1-y) - a(1-y))^{\alpha-1} y^{\beta} (t-a)^{\beta} (t-a) dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 ((1-y)(t-a))^{\alpha-1} y^{\beta} (t-a)^{\beta} (t-a) dy \\ &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-y)^{\alpha-1} y^{\beta} dy \end{aligned}$$

Since

$$B(\alpha, \beta+1) = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}, \text{ (B is the Beta function, see the Annex)}$$

we find

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^{\beta} ds = \frac{\Gamma(\beta+1)(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

Example 1.3 *Let* $h(t) = \exp(\lambda t)$ *where* $\lambda > 0$*, then we have*

$$\begin{aligned} I_a^{\alpha} \exp(\lambda t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \exp(\lambda s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sum_{k \geq 0} \frac{(\lambda s)^k}{k!} ds \\ &= \sum_{k \geq 0} \frac{\lambda^k}{k!\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} s^k ds \end{aligned}$$

Now, by using the Example 1.2*, we find*

$$I_a^\alpha \exp(\lambda t) = \lambda^{-\alpha} \sum_{k\geq 0} \frac{(\lambda t)^{\alpha+k}}{\Gamma(\alpha+k+1)}$$

Definition 1.4 *([61], [76]) Let h a function given on the interval $[a,b]$. The left-handed fractional derivative of Riemann-Liouville of order α, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Theorem 1.5 *([76], section* 2.3*) Let $h \in C([a,b], \mathbb{R})$. For all $\alpha > 0$ and $\beta > 0$, we have*

$$I^\alpha I^\beta h = I^{\alpha+\beta} h$$

Proof:
First of all, we have

$$I^\alpha I^\beta h = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha-1} ds \int_a^s (s-x)^{\beta-1} h(x)dx$$

By Fubini's theorem, we find

$$\begin{aligned} I^\alpha I^\beta h &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^s (t-s)^{\alpha-1} h(x)(s-x)^{\beta-1} dsdx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(h(x) \int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1} ds \right) dx \end{aligned}$$

by setting $s = x + z(t-x)$, we find

$$\begin{aligned} \int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1} ds &= \int_0^1 (t-x-z(t-x))^{\alpha-1}(z(t-x))^{\beta-1}(t-x)dz \\ &= (t-x)^{\alpha+\beta-1} \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz \\ &= (t-x)^{\alpha+\beta-1} \mathrm{B}(\alpha,\beta). \end{aligned}$$

Hence

$$\begin{aligned} I^\alpha I^\beta h &= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t h(x)(t-x)^{\alpha+\beta-1} dx \\ &= I^{\alpha+\beta} h \end{aligned}$$

□

Theorem 1.6 *([76], Theorem* 2.4*) Let h be a summable function. then we have*

$$D_{a+}^\alpha I_a^\alpha h = h$$

Proof:
Proceding with the same manier like in Theorem (1.5), we find

$$\begin{aligned} D^{\alpha}_{a+}I^{\alpha}_{a}h &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-\alpha-1}ds\int_{a}^{s}(s-x)^{\alpha-1}h(x)dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}\int_{a}^{s}(t-s)^{n-\alpha-1}h(x)(s-x)^{\alpha-1}dsdx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}\left(h(x)\int_{x}^{t}(t-s)^{n-\alpha-1}(s-x)^{\alpha-1}ds\right)dx \end{aligned}$$

by setting $s=x+z(t-x)$, we find

$$\begin{aligned} \int_{x}^{t}(t-s)^{n-\alpha-1}(s-x)^{\alpha-1}ds &= \int_{0}^{1}(t-x-z(t-x))^{n-\alpha-1}(z(t-x))^{\alpha-1}(t-x)dz \\ &= (t-x)^{n-1}\int_{0}^{1}(1-z)^{n-\alpha-1}z^{\alpha-1}dz \\ &= (t-x)^{n-1}\mathrm{B}(n-\alpha,\alpha). \end{aligned}$$

Hence

$$\begin{aligned} D^{\alpha}_{a+}I^{\alpha}_{a}h &= \frac{B(n-\alpha,\alpha)}{\Gamma(\alpha)\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}h(x)(t-x)^{n-1}dx \\ &= \frac{1}{\Gamma(n)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}h(x)(t-x)^{n-1}dx. \end{aligned}$$

Now, by using the equation (1.4) we find that

$$D^{\alpha}_{a+}I^{\alpha}_{a}h = h$$

□

1.2.3 Fractional derivative of Caputo

Definition 1.7 *[61] Let h be a function given on the interval $[a,b]$. The left-sided fractional derivative of Caputo of order α, is defined by*

$$({}^{c}D^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

Here $n=[\alpha]+1$ and $\alpha>0$.

Remark 1.8 *([61], Lemma 2.2) The fractional derivative of Riemann-Liouville and the fractional derivative of Caputo are connected with each other by the following relation:*

$$({}^{c}D^{\alpha}_{a+}h)(t) = D^{\alpha}_{a+}\left[h(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k}\right]$$

To prove this we use the following representation of the fractional derivative of Riemann-Liouville ([76], Theorem 2.2):

$$\begin{aligned}(D^{\alpha}_{a+}h)(t) &= \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}+\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}h^{(n)}(s)ds\\ &= \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}+({}^cD^{\alpha}_{a+}h)(t)\end{aligned}$$

or

$$({}^cD^{\alpha}_{a+}h)(t)=D^{\alpha}_{a+}\left[h(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right]$$

Example 1.9 *([61], Property 2.16) Let $h(t)=(t-a)^{\beta}$ where $t>a$ and $\beta>-1$, then we have*

$$({}^cD^{\alpha}_{a+}h)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}\frac{d^n}{ds^n}(s-a)^{\beta}ds.$$

Since

$$({}^cD^{\alpha}_{a+}h)(t)=I_a^{n-\alpha}\left(\frac{d^n}{dt^n}h(t)\right)$$

then

$$\begin{aligned}({}^cD^{\alpha}_{a+}h)(t) &= I_a^{n-\alpha}\left(\frac{d^n}{dt^n}(t-a)^{\beta}\right)\\ &= I_a^{n-\alpha}\left(\frac{\Gamma(\beta+1)(t-a)^{\beta-n}}{\Gamma(\beta-n+1)}\right)\\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}I_a^{n-\alpha}\left((t-a)^{\beta-n}\right)\end{aligned}$$

Now by using the result in Example (1.2) we find

$${}^cD^{\alpha}_{a+}(t-a)^{\beta}=\frac{\Gamma(\beta+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$$

1.2.4 The Hadamard Fractional integral and derivative

Definition 1.10 *([61],[76]) Let h be a real function defined on $[a,+\infty)$, such that $a\geq 0$. The Hadamard fractional integral of order α of h is defined by*

$$({}^HI^{\alpha}h)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s)}{s}ds,\ \alpha>0,$$

provided the integral exists.

Definition 1.11 *([61], [76]) Let h be a real function defined on $[a,+\infty)$, such that $a \geq 0$. The α Hadamard fractional-order derivative of h is defined by*

$$({}^H D^\alpha h)(t) \;=\; \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s}ds.$$

Here $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α and $\log(.)=\log_e(.)$.

Example 1.12 *([61], Property 2.24) Let $h(t)=\left(\log\frac{t}{a}\right)^\beta$ where $t>a>0$ and $\beta>-1$, then we have*

$$({}^H I^\alpha h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^\beta \frac{s}{ds},$$

by setting, $\left(\log\frac{s}{a}\right)=y\left(\log\frac{t}{a}\right)$, *we find*

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^\beta \frac{s}{ds} &= \frac{1}{\Gamma(\alpha)}\int_0^1 (\log t(1-y)-\log a(1-y))^{\alpha-1} y^\beta (\log t-\log a)^\beta \times \\
&\times\ (\log t-\log a)dy \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 ((1-y)(\log t-\log a))^{\alpha-1} y^\beta (\log t-\log a)^\beta \times \\
&\times\ (\log t-\log a)dy \\
&= \frac{(\log t-\log a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-y)^{\alpha-1}y^\beta dy \\
&= \frac{\Gamma(\beta+1)\left(\log\frac{t}{a}\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}
\end{aligned}$$

Remark 1.13 *[76] From the fractional integral of Hadamard, we can see the following relations:*

- $$({}^H I^\alpha h)(e^t) = \frac{1}{\Gamma(\alpha)}\int_{e^a}^{e^t} (t-s)^{\alpha-1}h(e^s)ds = (I^\alpha_{e^a} h)(e^t)$$
- $$x\frac{d}{dx}({}^H I^{\alpha+1}h)(t) = ({}^H I^\alpha h)(t)$$

1.3 Set-valued maps

Definition 1.14 *A set valued map (also called multivalued map) $F: X \mapsto Y$ is an application which associate with any $x\in X$ a subset $F(x)$ which belongs to $\mathcal{P}(Y)$, where X and Y are two sets.*

The set valued map has a great important in many fields of mathematics, as exemples

- **Quasi variational problems** (as example, see [70]): in this kind of problems the set valued map is called the variational selection S and it is depends with the own solution of the problem, as example, we give the following quasi variational problem: find u,

$$u \in S(u) \quad \text{and } f(u,w) \leq 0, \quad \forall w \in S(u)$$

- **Differential inclusions:** in this type of problems we make different conditions on the set valued map in order to find the solution. As exemple we give the following differential inclusions:

$$y'(t) \in F(t, y(t)), \text{ for almost all } t \in J \tag{1.5}$$

  Where F it's a set valued map.

- **Ill posed problems:** In this well known problems, the set valued map allows us to get the unique solution of the problem without restriction of the map.

For more details on multivalued maps see for example the book of Aubin and Frankowska [11].

1.3.1 Definitions

Definition 1.15 *[11] The domain of a set-valued map F is the subset of elements $x \in X$ such that $F(x)$ is not empty:*

$$Dom(F) = \{x \in X | F(x) \neq \emptyset\}$$

Definition 1.16 *[11] Let X and Y be metric spaces. A set valued map F from X to Y is characterized by its graph $Graph(F)$, the subset of the product space $X \times Y$, defined by*

$$Graph(F) = \{(x,y) \in X \times Y | y \in F(x)\}$$

Definition 1.17 *[16] Let X, Y be nonempty sets and $F : X \mapsto P(Y)$, then*

(1) The single-valued operator $f : X \mapsto Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.

(2) The set of selections of F is defined by

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \ a.e. \ t \in J\}.$$

Definition 1.18 *[11] A multivalued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(X)$ is convex (closed) for all $x \in X$.*

Definition 1.19 *[16] A multivalued map $F : J \to P_{cl}(\mathrm{I\!R})$ is said to be measurable if for every $y \in \mathrm{I\!R}$, the function:*

$$t \to d(y, F(t)) = inf\{|y - z| : z \in F(t)\}$$

is measurable.

1.3.2 Continuity of set-valued maps

Let X be a topological space and let $F : X \mapsto \mathcal{P}(X)$ be a multivalued map.

Definition 1.20 *[46]*

(1) F is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty subset of X, and for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subset N$.

(2) F is called lower semi-continuous on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty subset of X, and for each open set N such that $N \cap F(x_0) \neq \emptyset$, there exists an open neighborhood N_0 of x_0 such that

$$x \in N_0 \Rightarrow F(x) \cap N \neq \emptyset$$

Now we give the equivalent defintions of semi continuity in metric spaces X and Y.

Definition 1.21 *[11]*

(1) A set valued map $F : X \mapsto Y$ is called upper semicontinuous at $x \in Dom(F)$ if and only if for any neighborhood U of $F(x)$

$$\exists \mu > 0 \quad \text{such that} \quad \forall x' \in B_X(x, \mu), F(x') \subset U$$

(2) A set-valued map $F : X \mapsto Y$ is called lower semicontinuous at $x \in Dom(F)$ if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in Dom(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.

Definition 1.22 *[11] The set-valued map F is continuous at x if it is both upper semi-continuous and lower semicontinuous at x, and that it is continuous if and only if it is continuous at every point of $Dom(F)$*

Remark 1.23 *([11], P* 39*) There is no relation between the upper semi continuity and the lower semi continuity.*

Proposition 1.24 *([11], Proposition (1.4.8)) The graph of an upper semi-continuous set-valued maps $F : X \to Y$ with closed domain and closed values is closed. The converse is true if we assume that Y is compact.*

Now we give some important definitions that will serve in the remainder of this thesis.

Definition 1.25 *A set valued operator F is called completely continuous if $F(B)$ is relatively compact for every $B \in P_B(X)$.*

Definition 1.26 *[16] A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:*

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$.

(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

Definition 1.27 *([21], P 132) Let (X,d) be a metric space induced from the normed space $(X, |\cdot|)$. The Hausdorff-Pompeiu metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ is given by:*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

Definition 1.28 *[38] A multivalued operator $N : X \to P_{cl}(X)$ is called*

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in X.$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

1.4 Some fixed point theorems

In this section we present the set-valued versions of well known fixed point theorems. Let us start by the Nonlinear alternative of Leray-Schauder type.

Theorem 1.29 *[43] Let X be a Banach space and C a nonempty closed convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to P_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(1) T has fixed points in $\overline{U}$, or

(2) There exist $u \in \partial U$ and $\lambda \in [0,1]$ with $u \in \lambda T(u)$.

Theorem 1.30 *(Schaefer theorem, [53]) Let X be a Banach space and $N : X \mapsto X$ completely continuous operator. If the set*

$$E(N) = \{x \in X : x = \lambda N x \ \ for \ \ \lambda \in [0,1].\}$$

is bounded, then N has fixed points.

Now we give the theorem of Covitz and Nadler concerning the multivalued contraction.

Theorem 1.31 *[32] Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let us now recall Mönch's fixed point theorem.

Theorem 1.32 ([71], Theorem 3.2) *Let K be a closed and convex subset of a Banach space E, U be a relatively open subset of K, and $N : \overline{U} \mapsto \mathcal{P}(K)$. Assume that graph$N$ is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

(1) $M \subset \overline{U}$, $M \subset conv(x_0 \cup N(M))$, $\overline{M} = \overline{C}$, with C a countable subset of M, implies $\overline{M}$ is compact;

(2) $x \notin (1-\lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U}/U, \lambda \in (0, 1)$

Then there exists $x \in \overline{U}$ with $x \in N(x)$

Chapter 2

Boundary value problem for fractional differential inclusions with Caputo type derivative

[1] This chapter is concerned with the existence of solutions the following boundary value problems (BVP for short), for fractional order differential inclusions

$$^cD^\alpha y(t) \in F(t, y), \ \text{ for almost all } t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{2.1}$$

$$y(T) + y(0) = b \int_0^T y(s)ds, \quad bT \neq 2, \tag{2.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $b \in \mathbb{R} - \{\frac{T}{2}\}$.

We shall present two existence results for the problem (2.1)-(2.2), when in one case, the right hand side is convex valued, and in the other case, nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type, while the other is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

2.1 The convex case

In this section, we shall give a theorem of exitence of solutions for the problem (2.1)-(2.2). Let us start by the following lemma:

Lemma 2.1 *([81]) Let $\alpha \geq 0$. Then the differential equation*

$$^cD^\alpha h(t) = 0 \tag{2.3}$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, \ldots, n-1$, $n = [\alpha] + 1$.

[1] N. Guerraiche, S. Hamani and J. Henderson, Boundary value Problems for Differential Inclusions with Integral and Anti-periodic Conditions, *Communications on Applied Nonlinear Analysis.* **23** (2016), No. 3, 33 - 46.

Lemma 2.2 *([61]) Let $\alpha \geq 0$. If $h(t) \in AC^n[a,b]$ or $h(t) \in C^n[a,b]$, then*

$$I^{\alpha}\, {}^cD^{\alpha} h(t) = h(t) - c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \tag{2.4}$$

$c_i \in \mathbb{R}$, $i = 0, \ldots, n-1$, $n = [\alpha] + 1$.

Now, we give the definition of a solution to the problem (2.1)-(2.2).

Definition 2.3 *A function $y \in AC([1,T], E)$ is said to be a solution of (2.1)-(2.2) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that ${}^cD^{\alpha} y(t) = v(t)$ on J, and the condition $y(T) + y(0) = b\int_0^T y(s)ds$, where $bT \neq 2$, is satisfied.*

Lemma 2.4 *Let $0 < \alpha \leq 1$ and $bT \neq 2$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \int_0^T G(t,s) h(s) ds \tag{2.5}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{b(T-s)^{\alpha}}{(2-bT)\Gamma(\alpha+1)} - \dfrac{(T-s)^{\alpha-1}}{(2-bT)\Gamma(\alpha)}, & 0 \leq s < t \leq T, \\ \dfrac{b(T-s)^{\alpha}}{(2-bT)\Gamma(\alpha+1)} - \dfrac{(T-s)^{\alpha-1}}{(2-bT)\Gamma(\alpha)}, & 0 \leq t \leq s < T, \end{cases} \tag{2.6}$$

if and only if y is a solution of the fractional BVP

$${}^cD^{\alpha} y(t) = h(t), \ \text{for almost each } t \in J = [0,T], \quad 0 < \alpha \leq 1, \tag{2.7}$$

$$y(T) + y(0) = b\int_0^T y(s)ds, \quad bT \neq 2. \tag{2.8}$$

Proof: Assume y satisfies (2.7), then Lemma 2.2 implies that

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - c_0. \tag{2.9}$$

By (2.12),

$$c_0 = \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \frac{b}{2}\int_0^T y(s) ds, \tag{2.10}$$

Hence

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \frac{b}{2}\int_0^T y(s) ds, \tag{2.11}$$

Integrating Equation(2.11), we have

$$\int_0^T y(s)ds = \frac{2}{(2-bT)\Gamma(\alpha+1)}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}h(s)ds - \frac{1}{(2-bT)\Gamma(\alpha)}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds$$

the last implies

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)}h(s)ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &+ \frac{b}{(2-bT)\Gamma(\alpha+1)}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}h(s)ds - \frac{b}{(4-2bT)\Gamma(\alpha)}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &= \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha)}h(s)ds + \frac{b}{(2-bT)\Gamma(\alpha+1)}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}h(s)ds \\ &- \frac{1}{(2-bT)\Gamma(\alpha)}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds. \end{aligned}$$

hence we get equation (2.5), where G defined in (2.6).

Conversely, it is clear that if y satisfies (2.5), then equations (2.7) and (2.12) hold. □

Now we introduce the following hypotheses which are assumed hereafter:

(H1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multi-valued map.

(H2) There exist $p \in C(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

(H3) There exists $l \in L^1(J, \mathbb{R}^+)$, with $I^{\alpha}l$ is bounded, such that

$$H_d(F(t,u), F(t,\bar{u})) \le l(t)|u-\bar{u}| \qquad \text{for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \le l(t), \;\; a.et \in J.$$

(H4) There exists a number $M > 0$ such that

$$\frac{M}{\psi(M)\|I^{\alpha}p\|_{\infty} + \frac{|b|}{|2-bT|}\psi(M)(I^{\alpha+1}p)(T) + \frac{\psi(M)}{|2-bT|}(I^{\alpha}p)(T)} > 1. \tag{2.12}$$

Theorem 2.5 *Assume that hypotheses (H1)-(H4) are satisfied. Then the BVP (2.1)-(2.2) has at least one solution.*

Proof: Transform the problem (2.1)-(2.2) into a fixed point problem. Consider the multivalued operator,

$$N(y)=\left\{h\in C(J,\mathrm{I\!R}):\begin{array}{rcl} h(t) & = & \displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+\frac{b}{2-bT}\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}v(s)ds\\ & - & \displaystyle\frac{1}{2-bT}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds,\quad v\in S_{F,y}\end{array}\right\}.$$

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y\in C(J,E)$.*

Indeed, if $h_1,h_2\in N(y)$, then there exist $v_1,n_2\in S_{F,y}$ such that, for each $t\in J$, we have

$$\begin{array}{rcl} h_i(t) & = & \displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_i(s)ds+\frac{b}{2-bT}\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}v_i(s)ds\\ & & \displaystyle-\frac{1}{2-bT}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v_i(s)ds.\end{array}$$

For $i=1,2$, let $0\le d\le 1$. Then, for each $t\in J$, we have

$$\begin{array}{rcl} (dh_1+(1-d)h_2)(t) & = & \displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[dv_1(s)+(1-d)v_2(s)\right]ds\\ & & \displaystyle+\frac{b}{2-bT}\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}\left[dv_1(s)+(1-d)v_2(s)\right]ds\\ & & \displaystyle-\frac{1}{2-bT}\int_0^T\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[dv_1(s)+(1-d)v_2(s)\right]ds.\end{array}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1+(1-d)h_2\in N(y).$$

Step 2: *N maps bounded sets into bounded sets in $C(J,\mathrm{I\!R})$.*

Let $B_{\mu_*}=\{y\in C(J,\mathrm{I\!R}):\|y\|_\infty\le\mu_*\}$ be a bounded set in $C(J,\mathrm{I\!R})$ Then for each $h\in N(y)$, there exists $v\in S_{F,y}$ such that

$$\begin{array}{rcl} h(t) & = & \displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds+\frac{b}{2-bT}\int_0^T\frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}v(s)ds\\ & + & \displaystyle\frac{1}{2-bT}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds.\end{array}$$

By (H2), we have, for each $t\in J$

$$\begin{aligned}
|h(t)| &\leq \int_0^t \frac{(t-s)^\alpha - 1}{\Gamma(\alpha)}|v(s)|ds + \frac{|b|}{|2-bT|}\int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}|v(s)|ds \\
&- \frac{1}{|2-bT|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\psi(|y(s)|)ds + \frac{|b|}{|2-bT|}\int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}p(s)\psi(|y(s)|)ds \\
&+ \frac{1}{|2-bT|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\psi(|y(s)|)ds \\
&\leq \psi(\mu^*)I^\alpha(p)(t)_\infty + \frac{|b|}{|2-bT|}\psi(\mu^*)(I^{\alpha+1}p)(T) + \frac{\psi(\mu^*)}{|2-bT|}(I^\alpha p)(T).
\end{aligned}$$

Thus

$$\|h\|_\infty \leq \psi(\mu^*)\|I^\alpha p\|_\infty + \frac{|b|}{|2-bT|}\psi(\mu^*)(I^{\alpha+1}p)(T) + \frac{\psi(\mu^*)}{|2-bT|}(I^\alpha p)(T) := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{μ^*} be bounded set of $C(J,\mathbb{R})$ as in Step 2. Let $y \in B_{\mu^*}$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &= |\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]v(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}v(s)ds| \\
&\leq \frac{\|p\|_\infty\psi(\mu^*)}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds \\
&\quad + \frac{\|p\|_\infty\psi(\mu^*)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\leq \frac{\|p\|_\infty\psi(\mu^*)}{\Gamma(\alpha+1)}\left[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha\right] + \frac{\|p\|_\infty\psi(\mu^*)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in J$

$$\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds + \frac{b}{2-bT}\int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)}v_n(s)ds \\
&\quad - \frac{1}{2-bT}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds.
\end{aligned}$$

We must show that there exists $v_* \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds + \frac{b}{2-bT}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)} v_*(s)ds \\ &\quad -\frac{1}{2-bT}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds. \end{aligned}$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \ \text{ as } \ m \to \infty.$$

and

$$v_*(t) \in F(t, y_*(t)), \ \text{ a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))).$$

We obtain an analogues relation by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_n(s) - v_*(s)|ds \\ &+ \frac{b}{(2-bT)\Gamma(\alpha+1)}\int_0^T (T-s)^{\alpha}|v_n(s) - v_*(s)|ds \\ &+ \frac{1}{(2-bT)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|v_n(s) - v_*(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} l(s)ds\|y_{n_m} - y_*\|_\infty \\ &+ \frac{b}{(2-bT)\Gamma(\alpha+1)}\int_0^T (T-s)^{\alpha} l(s)ds\|y_{n_m} - y_*\|_\infty \\ &+ \frac{1}{(2-bT)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha} l(s)ds\|y_{n_m} - y_*\|_\infty. \end{aligned}$$

Hence

$$
\begin{aligned}
\|h_n(t)-h_*(t)\|_\infty &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)ds\|y_{n_m}-y_*\|_\infty\\
&+ \frac{b}{(2-bT)\Gamma(\alpha+1)}\int_0^T (T-s)^{\alpha}l(s)ds\|y_{n_m}-y_*\|_\infty\\
&+ \frac{1}{(2-bT)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}l(s)ds\|y_{n_m}-y_*\|_\infty \to 0,
\end{aligned}
$$

as $m\to\infty$

Step 5: *A priori bounds on solutions.*

Let y be such that $y\in\lambda N(y)$ with $\lambda\in(0,1]$. Then there exists $v\in S_{F,y}$ such that, for each $t\in J$

$$
\begin{aligned}
h(t) &= \int_0^t \frac{\lambda(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds + \frac{\lambda b}{2-bT}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}v(s)ds\\
&- \frac{\lambda}{2-bT}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds.
\end{aligned}
$$

This implies by $(H2)$ that, for each $t\in J$, we have

$$
\begin{aligned}
|y(t)| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds + \frac{|b|}{|2-bT|}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}|v(s)|ds\\
&+ \frac{1}{|2-bT|}\int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds\\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\psi(|y(s)|)ds + \frac{|b|}{|2-bT|}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)}p(s)\psi(|y(s)|)ds\\
&+ \frac{1}{|2-bT|}\int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\psi(|y(s)|)ds\\
&\le \psi(\|y\|_\infty)I^\alpha(p)(t) + \frac{|b|}{|2-bT|}\psi(\|y\|_\infty)(I^{\alpha+1}p)(T) + \frac{\psi(\|y\|_\infty)}{|2-bT|}(I^\alpha p)(T).
\end{aligned}
$$

Thus

$$
\frac{\|y\|_\infty}{\psi(\|y\|_\infty)\|I^\alpha(p)\|_\infty + \frac{|b|}{|2-bT|}\psi(\|y\|_\infty)(I^{\alpha+1}p)(T) + \frac{\psi(\|y\|_\infty)}{|2-bT|}(I^\alpha p)(T)} < 1.
$$

Then by (H4) and condition (2.12), there exists $M>0$ such that $\|y\|_\infty\neq M$. Let $U=\{y\in C(J,\mathbb{R}) : \|y\|_\infty<M\}$. The operator $N:\overline{U}\to\mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N(y)$ for some $\lambda\in(0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that N has a fixed point $y\in\overline{U}$ which is a solution of the problem (2.1)-(2.2). This completes the proof. □

2.2 The nonconvex case

We present now the second existece result for the problem (2.1)-(2.2) with a nonconvex valued right hand side. Our considerations are based on Theorem 1.31 (that is, the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [32]). We need the following hypothese:

(H5) $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ has the property that $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$.

Theorem 2.6 *Assume that (H3) and (H5) are satisfied. If*

$$\|I^{\alpha}l\|_{\infty} + \frac{|b|}{|2-bT|}(I^{\alpha+1}l)(T) + \frac{1}{|2-bT|}(I^{\alpha}l)(T) < 1 \tag{2.13}$$

then the BVP (2.1)-(2.2) has at least one solution on J.

Remark 2.7 *For each $y \in C(J, \mathbb{R})$, the set $S_{F,y}$ is nonempty by assumption $(H5)$. Thus F has a measurable selection ([30], Proposition III.6).*

Proof: We shall show that N satisfies the assumptions of Theorem 1.31. The proof will be given in two steps.

Step 1: *$N(y) \in \mathcal{P}_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$.*

Indeed, let $(y_n)_{n\geq 0} \subset N(y)$ be such that $y_n \to \bar{y}$ in $C(J, \mathbb{R})$. Then, $\bar{y} \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds + \frac{b}{2-bT}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)} v_n(s)ds \\ &\quad - \frac{1}{2-bT}\int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds. \end{aligned}$$

Using the fact that F has compact values and (H3), we may pass to a subsequence if necessary to get that (v_n) converges weakly to v in $L^1_w(J, \mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem implies that (v_n) converges strongly to v and hence $v \in S_{F,y}$. Then for each $t \in J$,

$$\begin{aligned} y_n(t) \;\to\; \bar{y}(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \frac{b}{2-bT}\int_0^T \frac{(T-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)ds \\ &\quad - \frac{1}{2-bT}\int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds. \end{aligned}$$

So, $\bar{y} \in N(y)$.

Step 2: *There exists $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty$ for each $y, \bar{y} \in C(J, \mathbb{R})$.*

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds + \frac{b}{2-bT} \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} v_1(s)ds \\ &\quad - \frac{1}{2-bT} \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds. \end{aligned}$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y})(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable ([30], proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J$

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

For $v_2 \in J$, we define

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds + \frac{b}{2-bT} \int_0^T \frac{(T-s)^\alpha}{\Gamma(\alpha+1)} v_2(s)ds \\ &\quad - \frac{1}{2-bT} \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds. \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)|ds \\ &\quad + \frac{|b|}{|2-bT|\Gamma(\alpha+1)} \int_0^T (T-s)^\alpha |v_1(s) - v_2(s)|ds \\ &\quad + \frac{1}{|2-bT|\Gamma(\alpha)} \int_0^T (T-s)^\alpha |v_1(s) - v_2(s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s)|y(s) - \bar{y}(s)|ds \\ &\quad + \frac{|b|}{|2-bT|\Gamma(\alpha+1)} \int_0^T (T-s)^\alpha |y(s) - \bar{y}(s)| l(s)ds \\ &\quad + \frac{1}{|2-bT|\Gamma(\alpha)} \int_0^T (T-s)^\alpha |y(s) - \bar{y}(s)| l(s)ds. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \le \left[\|I^\alpha l\|_\infty + \tfrac{|b|}{|2-bT|}(I^{\alpha+1}l)(T) + \frac{1}{|2-bT|}(I^\alpha l)(T)\right]\|y-\bar{y}\|_\infty.$$

We obtain an analogous relation by interchanging the roles of y and $\bar{y}$, and it follows that

$$H_d(N(y), N(\bar{y})) \le \left[\|I^\alpha l\|_\infty + \frac{|b|}{|2-bT|}(I^{\alpha+1}l)(T) + \frac{1}{|2-bT|}(I^\alpha l)(T)\right]\|y-\bar{y}\|_\infty.$$

So by (2.13), N is a contraction and thus, by Theorem 1.31, N has a fixed point y which is a solution to (2.1)-(2.2). The proof is complete. □

2.3 An example

We end this chapter by giving an example. We apply Theorem 3.11, to the the following fractional differential inclusion,

$$^cD^\alpha y(t) \in F(t, y(t)), \text{ for almost all } t \in J = [0,1], 0 < \alpha \le 1, \tag{2.14}$$

$$y(1) + y(0) = \int_0^1 y(s)ds, \tag{2.15}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. Set

$$F(t,y) = \{v \in \mathbb{R} : f_1(t,y) \le v \le f_2(t,y)\}$$

where $f_1, f_2 : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [0,1]$, $f_1(t,\cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t,y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [0,1]$, $f_2(t,\cdot)$ is upper semi-continuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t,y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C([0,1], \mathbb{R}^+)$ and $\psi : [0,\infty) \mapsto (0,\infty)$ continuous and nondecreasing such that

$$\max(|f_1(t,y)|, |f_2(t,y)| \le p(t)\psi(|y|), \quad t \in [0,1], \text{ and all } y \in \mathbb{R}.$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous. Assume there exists a number $M > 0$ such that

$$\frac{M}{\psi(M)[\|I^\alpha p\|_\infty + (I^{\alpha+1}p)(T) + (I^\alpha p)(T)]} > 1. \tag{2.16}$$

Since all the conditions of Theorem 3.11 are satisfied, problem (2.14)-(2.15) has at least one solution y on $[0,1]$.

Chapter 3

Problems for fractional differential inclusions with Hadamard and Caputo type derivatives

In this chapter, we investigate the existence of solutions for fractional differential inclusions. We will give two results of existence of solutions for each problem.

3.1 Initial value problem of fractional functional differential inclusions

[1] This section deals with the existence of solution for the following initial value problems (IVP for short), for fractional order differential functional inclusions:

$$^{H}D^{\alpha}y(t) \in F(t, y_t), \text{ for almost each } t \in J = [1, T], 0 < \alpha \leq 1, \tag{3.1}$$

$$y(t) = \varphi(t) \quad t \in [1 - r, 1], \tag{3.2}$$

Where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\varphi \in C([1 - r, 1], \mathbb{R})$ with $\varphi(1) = 0$. For any function y defined on $[1 - r, T]$ and any $t \in J$, we denote by y_t the element of $C([1 - r, 1], \mathbb{R})$ and is defined by

$$y_t = y(t + \theta), \ \theta \in [-r, 0]$$

Hence $y_t(.)$ represents the history of the state from times $t - r$ up to the present time t.

In this section, we shall discuss the existence result when the right hand side is convex as well as nonconvex valued.

[1]N. Guerraiche, S. Hamani and J. Henderson, Initial Value Problems for Fractional Functional Differential Inclusions with Hadamard type derivative, *Archivum Mathematicum.* **52** (2016), 263 - 273.

3.1.1 The convex case

Let us start by the following definition.

Definition 3.1 *A function $y \in C([1-r,T],\mathbb{R})$ which is absolutly continuous in the interval $[1,T]$, is said to be a solution of (3.1)-(3.2), if there exists a function $v \in L^1([1,T],\mathbb{R})$ with $v(t) \in F(t,y_t)$, for a.e $t \in [1,T]$, such that*

$$^{H}D^{\alpha}y(t) = v(t), \quad a.e \;\; t \in [1,T], 0 < \alpha < 1,$$

and the function y satisfies condition (3.2).

Now we give the following auxiliary lemma.

Lemma 3.2 *Let $h : [1,+\infty) \to \mathbb{R}$ be continuous functions. A function y is a solution of the fractional equation*

$$y(t) = \begin{cases} \varphi(t) & if \;\; t \in [1-r,1] \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds & if \;\; t \in [1,T] \end{cases} \tag{3.3}$$

if and only if y is a solution of the nonlinear fractional problem

$$^{H}D^{\alpha}y(t) = h(t) \;\; for \; a.e. \;\; t \in J = [1,T], \quad 0 < \alpha \leq 1, \tag{3.4}$$

$$y(t) = \varphi(t) \quad t \in [1-r,1], \tag{3.5}$$

Proof: Applying the Hadamard fractional integral of order α to both sides of (3.4), we have

$$y(t) = {}^{H}I^{\alpha}h(t). \tag{3.6}$$

and by (3.5), we get (3.3).
Conversely, it is clear that if y satisfies equation (3.3), then equations (3.4) and (3.5) hold. □

Let us introduce the following hypotheses:

(H6) There exist $p \in C([1,T],\mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|_C) \text{ for } t \in [1,T] \text{ and each } u \in C([1-r,1],\mathbb{R})$$

(H7) There exists an number $M > 0$ such that

$$\frac{M}{\dfrac{\psi(M)\|p\|_{\infty}}{\Gamma(\alpha+1)}(\log T)^{\alpha}} > 1 \tag{3.7}$$

Theorem 3.3 *Assume that the hypotheses (H1), (H3), (H6) and (H7) hold, then the IVP (3.1)-(3.2) has at least one solution on* $[1-r,T]$.

proof : Transform the problem (3.1)-(3.2) into a fixed point problem. consider the multivalued operator .

$$N_1(y)(t)=\left\{h\in C([1-r,T],\mathbb{R})\ h(t)=\begin{cases}\varphi(t) & \text{if } t\in[1-r,1]\\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds & \text{if } t\in[1,T]\end{cases};v\in F(t,y_t)\right\}.$$

We shall prove that N_1 has at least a fixed point.

The operator $N_1:C([1-r,T],\mathbb{R})\mapsto\mathcal{P}(C([1-r,T],\mathbb{R}))$ is completely continuous and upper semicontinuous, the proof of this is similar to that of Theorem 3.11.

Now, let y be such that $y\in\lambda N_1(y)$ with $\lambda\in(0,1]$. Then there exists $v\in S_{F,y}$ such that, for each $t\in[1,T]$

$$h(t)=\frac{\lambda}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds.$$

This implies by $(H6)$ that, for each $t\in[1,T]$, we have

$$\begin{aligned}|y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{p(s)\psi(\|y_s\|_C)}{s}ds\\ &\leq \frac{\psi(\|y\|_{[1-r,T]})\|p\|_\infty}{\Gamma(\alpha+1)}(\log T)^\alpha.\end{aligned}$$

Thus

$$\frac{\|y\|_{[1-r,T]}}{\psi(\|p\|_\infty\|y\|_{1-r,T})\dfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}}<1.$$

Then by condition (3.7), there exist $M>0$ such that $\|y\|_\infty\neq M$.
Let $U=\{y\in C(J,\mathbb{R}):\|y\|_\infty<M\}$. The operator $N_1:\overline{U}\to\mathcal{P}(C([1-r,T],\mathbb{R}))$ is upper semi continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N_1(y)$ for some $\lambda\in(0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder , we deduce that N_1 has a fixed point $y\in\overline{U}$ which is a solution of the problem (2.1)-(2.2). □

3.1.2 The nonconvex case

We present now the second existence result for the problem (3.1)-(3.2). Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler. We anounce the following theorem of existence of solutions.

Theorem 3.4 *Assume (H3) and (H5) hold. If*

$$\frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} < 1 \tag{3.8}$$

then the IVP (3.1)-(3.2) has at least one solution on $[1-r, T]$.

Proof : We shall show that N_1 defined above is a contraction.

We can show that $N_1(y) \in \mathcal{P}_{cl}(C([1-r,T],\mathrm{I\!R}))$ for each $y \in C([1-r,T],\mathrm{I\!R})$. We must show that there exist $\gamma < 1$ such that

$$H_d(N_1(y), N_1(\bar{y})) \leq \gamma \|y - \bar{y}\|_{[1-r,T]} \text{ for each } y, \bar{y} \in C([1-r,T],\mathrm{I\!R}).$$

Let $y, \bar{y} \in C([1-r,T],\mathrm{I\!R})$ and $h_1 \in N_1(y)$. Then, there exist $v_1 \in F(t, y_t)$ such that for each $t \in [1,T]$

$$y_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_1(s)}{s} ds$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y})(t)) \leq l(t)|y(t) - \bar{y}(t)|$$

Hence, there exist $w \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|, t \in [1,T]$$

Consider $U : [1,T] \to \mathcal{P}(\mathrm{I\!R})$ given by

$$U(t) = \{w \in \mathrm{I\!R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable ([30], proposition III.4), there exists a function $v_2(t)$ which is measurable selection for V. So, $v_2 \in F(t, \bar{y}_t)$, and for each $t \in [1,T]$

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, t \in [1,T]$$

Let us define for each $t \in [1,T]$

$$y_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_2(s)}{s} ds$$

Then for each $t \in [1,T]$

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |l(s)||y_s - \bar{y}_s| ds \\
&\leq \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty
\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$ it follows that

$$H_d(N_1(y), N_1(\bar{y})) \leq \frac{\|l\|_\infty (\log T)^\alpha}{\Gamma(\alpha+1)} \|y - \bar{y}\|_{[1-r,T]}$$

So by (3.8), N_1 is a contaction and thus, by Lemma (1.31), N_1 has a fixed point y which is solution to (3.1)-(3.2). The proof is complete. □

3.2 Nonlinear boundary problem for fractional differential inclusions

[2] In this section we are concerned with the existence of solutions for the following nonlinear fractional differential inclusion with integral boundary value conditions

$$^H D^r y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \quad 2 < r \leq 3, \tag{3.9}$$

$$y(1) = y''(1) = 0, \tag{3.10}$$

$$y(T) = \int_1^T g(s, y(s)) ds, \tag{3.11}$$

where $^H D^r$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given function.

In this section, we shall present two existence results for the problem (2.1)-(2.5), when in one case, the right hand side is convex valued, and in the other case, nonconvex valued.

3.2.1 The convex case

Let us start by defining what we mean by a solution of the problem (3.9)-(3.11)

Definition 3.5 *A function $y \in AC^1([1, T], \mathbb{R})$ is said to be a solution of (3.9)-(3.11) if there exist a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that $^H D^r y(t) = v(t)$ on J, and the conditions (3.10) and (3.11) are satisfied.*

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[2] N. Guerraiche, S. Hamani and J. Henderson, Nonlinear boundary value Problems for Hadamard fractional differential inclusions with integral boundary conditions, *Advances in Dynamical Systems and Applications,* **12** (2017), No. 2, 107-121.

Lemma 3.6 *Let h, $\rho : [1, +\infty) \to \mathbb{R}$ be continuous functions . A function y is a solution of the fractional equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T \rho(s) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s) \frac{ds}{s} \right] \end{aligned} \tag{3.12}$$

if and only if y is a solution of the nonlinear fractional problem

$$^H D^r y(t) = h(t), \text{ for a.e. } t \in J = [1, T], \quad 2 < r \le 3, \tag{3.13}$$

$$y(1) = y''(1) = 0, \tag{3.14}$$

$$y(T) = \int_1^T \rho(s) ds. \tag{3.15}$$

Proof: Applying the Hadamard fractional integral of order r to both sides of (3.13), we have

$$y(t) = c_1 (\log t)^{r-1} + c_2 (\log t)^{r-2} + c_3 + {}^H I^r h(t). \tag{3.16}$$

First of all, from $y(1) = 0$ we have $c_3 = 0$.

Now by differentiating y, we have

$$\begin{aligned} y'(t) &= c_1 (r-1) \frac{(\log t)^{r-2}}{t} \\ &+ c_2 (r-2) \frac{(\log t)^{r-3}}{t} + \frac{(r-1)}{t\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-2} h(s) \frac{ds}{s}. \end{aligned} \tag{3.17}$$

Differentiating y for the second time, we find

$$\begin{aligned} y''(t) &= c_1 (r-1)(r-2) \frac{(\log t)^{r-3}}{t^2} - c_1 (r-1) \frac{(\log t)^{r-2}}{t^2} \\ &+ c_2 (r-2)(r-3) \frac{(\log t)^{r-4}}{t^2} - c_2 (r-2) \frac{(\log t)^{r-3}}{t^2} \\ &+ \frac{(r-1)(r-2)}{t^2 \Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-3} h(s) \frac{ds}{s} \\ &- \frac{(r-1)}{t^2 \Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-2} h(s) \frac{ds}{s}. \end{aligned} \tag{3.18}$$

Using the conditions (3.14) and (3.15), we find

$$c_2 = 0$$

and

$$c_1 = \frac{\displaystyle\int_1^T \rho(s) ds - {}^H I^r h(T)}{(\log T)^{r-1}}.$$

Hence we get equation (3.12).

Conversely, it is clear that if y satisfies equation (3.12), then equations (3.13)-(3.15) hold. □

Now we give the following hypotheses:

(H8) There exists $k > 0$ such that

$$\|g(t,y)\| \leq k, \text{ for each, } (t,y) \in J \times \mathbb{R}$$

(H9) There exists a number $M > 0$ such that

$$\frac{M}{2\frac{(\log T)^r \psi(M)}{\Gamma(r+1)}\|p\|_{L^1} + (T-1)k} > 1. \tag{3.19}$$

Theorem 3.7 *Assume that (H1)-(H3) and (H8)-(H9) hold. Then the problem (3.9)-(3.11) has at least one solution on J.*

Proof. Transform the problem (3.9)-(3.11) into a fixed point problem. Consider the multivalued operator,

$$N_2(y) = \left\{ h \in C(J,\mathbb{R}) : \begin{array}{rcl} h(t) & = & \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ & + & \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds \right. \\ & - & \left. \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right], v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 4.11, the fixed points of N are solutions to (3.9)-(3.11).

It is clear that $N_2 : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous and upper semi continuous, the proof of this is similar to that given above, so we omit the details.

Now we shall find an open set U such that there is no $y \in \partial U$ with $y \in \lambda N_4(y)$ for some $\lambda \in (0,1]$.

Let y be such that $y \in \lambda N_2(y)$ with $\lambda \in (0,1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{array}{rcl} h(t) & = & \frac{\lambda}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ & + & \lambda\frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right]. \end{array}$$

This implies by $(H2)$ that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T \|g(s, y(s))\| ds + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \right] \\
&\leq \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s)\psi(|y(s)|) ds + \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s)\psi(|y(s)|) ds \\
&\leq 2\frac{(\log T)^r \psi(\mu^*)}{\Gamma(r+1)} \int_1^T p(s) ds + (T-1)k.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{2\frac{(\log T)^r \psi(\|y\|_\infty)}{\Gamma(r+1)} \|p\|_{L^1} + (T-1)k} < 1.$$

Then by condition (3.19), there exists $M > 0$ such that $\|y\|_\infty \neq M$. Let $U = \{y \in C(J, \mathbb{R}) : \|y\|_\infty < M\}$. The operator $N_2 : \overline{U} \to \mathcal{P}(C(J, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N_2(y)$ for some $\lambda \in (0, 1]$. From the above, we deduce that N_2 has a fixed point $y \in \overline{U}$ which is a solution of the problem (3.9)-(3.11). This completes the proof. □

3.2.2 The nonconvex case

We present now a result for the problem (3.9)-(3.11) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem given in Theorem 1.31.

Theorem 3.8 *Assume (H3) and (H5) hold:*

If

$$2\frac{(\log T)^r}{\Gamma(r+1)} \|l\|_{L^1} < 1 \tag{3.20}$$

then the problem (3.9)-(3.11) has at least one solution on J.

Proof : We shall show that N_2 satisfies the assumptions of Theorem 1.31.

We can show that $N_2(y) \in \mathcal{P}_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$.

Now let show that there exists $\gamma < 1$ such that

$$H_d(N_2(y), N_2(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty, \text{ for each } y, \bar{y} \in C(J, \mathbb{R}).$$

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N_2(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_1(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T g(s, y(s)) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_1(s) \frac{ds}{s} \right]. \end{aligned}$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y})(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable ([30], proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J$

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Let us define for each $v_2 \in J$,

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_2(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T g(s, y(s)) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_2(s) \frac{ds}{s} \right]. \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \right] \\ &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \right] \\ &\leq \left[2 \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T l(s) ds \right] \|y - \bar{y}\|_\infty. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \le \left[2\frac{(\log T)^r}{\Gamma(r+1)}\|l\|_{L^1}\right]\|y-\bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N_2(y), N_2(\bar{y})) \le \left[2\frac{(\log T)^r}{\Gamma(r+1)}\|l\|_{L^1}\right]\|y-\bar{y}\|_\infty.$$

So by (3.20), N_2 is a contraction and thus, by Theorem 1.31, N_2 has a fixed point y which is solution to (3.9)-(3.11). The proof is complete. □

3.3 Boundary value problem of fractional differential inclusions with nonlocal multi-point boundary conditions

[3] This section deals with the existence of solutions to boundary value problem for fractional order differential inclusions. We consider the boundary value problem

$$^cD^\alpha y(t) \in F(t, y(t)), \text{ for almost each } t \in J = [0,T], 1 < \alpha \le 2, \tag{3.21}$$

$$y(0) = y^* + g(y), \quad ^cD^p y(T) = \sum_{i=1}^{m} \lambda_i^c D^p y(\mu_i) \;\; 0 < p < 1, \tag{3.22}$$

Where $^cD^\alpha$ and $^cD^p$ is the Caputo fractional derivatives, $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $y^* \in \mathbb{R}$ $\lambda_i \in \mathbb{R}$, $0 < \mu_i < T$ $i = 1, ..., m$, $m \ge 2$ and $g : C(J, \mathbb{R}) \mapsto \mathbb{R}$ a continuous function.

In this section, and by using the same fixed points theorems like in the sections above, we shall present two existence results for the problem (3.21)-(3.22), when in one case, the right hand side is convex valued, and in the other case, nonconvex valued.

3.3.1 The convex case

Let us start by defining what we mean by a solution of the problem (3.21)-(3.22)

Definition 3.9 *Let* $\sum_{i=1}^{m} \lambda_i {\mu_i}^{1-p} \neq T^{1-p}$. *A function* $y \in AC^1([0,T], \mathbb{R})$ *is said to be a solution of (3.21)-(3.22) if there exist a function* $v \in L^1(J, \mathbb{R})$ *with* $v(t) \in F(t, y(t))$, *for a.e.* $t \in J$, *such that* $^cD^\alpha y(t) = v(t)$ *on* J, *and the conditions* $y(0) = y^* + g(y)$ *and* $^cD^p y(T) = \sum_{i=1}^{m} \lambda_i^c D^p y(\mu_i)$, *are satisfied.*

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Lemma 3.10 *Let $h : [0, +\infty) \to \mathbb{R}$ be a continuous function. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= y^* + g(y) + t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}h(s)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}h(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p})} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \end{aligned} \tag{3.23}$$

if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = h(t), \ \textit{for a.e. } t \in J = [0, T], \quad 1 < \alpha \leq 2, \tag{3.24}$$

$$y(0) = y^* + g(y) \quad ^cD^p y(T) = \sum_{i=1}^m \lambda_i^c D^p y(\mu_i) \quad 0 < p < 1, \tag{3.25}$$

Proof: Assume y satisfies (3.24), then lemma (3.10) implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \tag{3.26}$$

It is clear that for $t = 0$ we find $c_0 = y^* + g(y)$. Now let us find c_1
Derivating the equation (3.26), we find

$$^cD^p y(t) = \frac{c_1 t^{1-p}}{\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha-p)}\int_0^t (t-s)^{\alpha-p-1}h(s)ds$$

for $t = T$, we have

$$^cD^p y(T) = \frac{c_1 T^{1-p}}{\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha-p)}\int_0^T (T-s)^{\alpha-p-1}h(s)ds$$

using the condition $^cD^p y(T) = \sum_{i=1}^m \lambda_i^c D^p y(\mu_i)$, we have

$$\begin{gathered} \frac{c_1 T^{1-p}}{\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha-p)}\int_0^T (T-s)^{\alpha-p-1}h(s)ds = \\ \sum_{i=1}^m \left[\frac{\lambda_i c_1 \mu_i^{1-p}}{\Gamma(\alpha-p)} + \frac{\lambda_i}{\Gamma(\alpha-p)}\int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}h(s)ds\right] \end{gathered}$$

finally a simple calculus gives

$$c_1 = \frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}h(s)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}h(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p})}$$

Hence we get equation (3.23). Conversely, it is clear that if y satisfies equation (3.23), then equations (3.24)-(3.25) hold. □

Theorem 3.11 *Assume(H1), (H3) and the following hypotheses hold:*
(H10) There exist $z \in C(J, \mathbb{R}^+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\|F(t,u)\|_{\mathcal{P}} \leq z(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

(H11) There exists a constant $M_1 > 0$ with

$$|g(y)| \leq M_1, \quad \text{for all } y \in C(J, \mathbb{R})$$

(H12) There exists a number $M_2 > 0$ such that

$$\frac{M_2}{|y^*| + M_1 + T\frac{\Gamma(2-p)\psi(M_2)\left[(I^{\alpha-p}z)(T) - \sum_{i=1}^m \lambda_i (I^{\alpha-p}z)(\mu_i)\right]}{|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} + \psi(M_2)\|I^\alpha z\|} > 1. \tag{3.27}$$

Then the BVP (3.21)-(3.22) has at least one solution on J.

Proof : Transform the problem (3.21)-(3.22) into a fixed point problem. We consider the multivalued operator,

$$N_3(y) = \left\{ h \in C(J, \mathbb{R}) : \begin{array}{rcl} h(t) &=& y^* + g(y) + t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}h(s)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i-s)^{\alpha-p-1}h(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p})} \\ &+& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds, \quad v \in S_{F,y} \end{array} \right\}.$$

We shall show that N_3 satisfies the assumptions of the nonlinear alternative of Leray-Schauder.

We can show that N_3 is completely continuous and upper semicontinuous. It remains to find an open set U such that there is no $y \in \partial U$ with $y \in \lambda N_4(y)$ for some $\lambda \in (0, 1]$.

Let y be such that $y \in \lambda N_3(y)$ with $\lambda \in (0, 1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$

$$h(t) = \lambda y^* + \lambda g(y) + t\frac{\lambda\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}h(s)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}h(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p})}$$
$$+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

This implies by $(H10)$ that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\leq \lambda|y^* + g(y)| + \lambda t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}|v(s)|ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}|v(s)|ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\
&+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\
&\leq |y^* + g(y)| + t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}|v(s)|ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}|v(s)|ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\
&\leq |y^*| + M_1 + T\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}z(s)\psi(|y(s)|)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}z(s)\psi(|y(s)|)ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)\psi(|y(s)|)ds \\
&\leq |y^*| + M_1 + T\frac{\Gamma(2-p)\psi(\|y\|_\infty)\left[(I^{\alpha-p}z)(T) - \sum_{i=1}^m \lambda_i (I^{\alpha-p}z)(\mu_i)\right]}{|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\
&+ \psi(\|y\|_\infty)(I^\alpha z)(t)
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{|y^*| + M_1 + T\frac{\Gamma(2-p)\psi(\|y\|_\infty)\left[(I^{\alpha-p}z)(T) - \sum_{i=1}^m \lambda_i (I^{\alpha-p}z)(\mu_i)\right]}{|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} + \psi(\|y\|_\infty)\|I^\alpha z\|} < 1$$

Then by condition (3.27), there exist $M > 0$ such that $\|y\|_\infty \neq M$. Let $U = \{y \in C(J, \mathbb{R}) : \|y\|_\infty < M_2\}$. The operator $N_3 : \overline{U} \to \mathcal{P}(C(J, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N_3(y)$ for some $\lambda \in (0, 1]$. As a consequence, we deduce that N_3 has a fixed point $y \in \overline{U}$ which is a solution of the problem (3.21)-(3.22). This completes the proof.

□

3.3.2 The nonconvex case

We present now a result for the problem (3.21)-(3.22) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler.

Theorem 3.12 *Assume (H3) and (H5) hold*

If

$$T\frac{\Gamma(2-p)\left[I^{\alpha-p}l)(T)ds + \sum_{i=1}^{m}\lambda_i(I^{\alpha-p}(l)(\mu_i)\right]}{|\sum_{i=1}^{m}\lambda_i{\mu_i}^{1-p} - T^{1-p}|} + \|I^\alpha l\|_\infty < 1 \tag{3.28}$$

then the BVP (3.21)-(3.22) has at least one solution on J.

Proof : We shall show that N_3 satisfies the assumptions of Theorem (1.31).

It is clear that $N_3(y) \in \mathcal{P}_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$.

Let us now show that There exists $\gamma < 1$ such that

$$H_d(N_3(y), N_3(\bar{y})) \leq \gamma\|y - \bar{y}\|_\infty, \text{ for each } y, \bar{y} \in C(J, \mathbb{R}).$$

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N_3(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= y^* + g(y) + t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}v_1(s)ds - \sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}(\mu_i - s)^{\alpha-p-1}v_1(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^{m}\lambda_i{\mu_i}^{1-p} - T^{1-p})} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_1(s)ds \end{aligned}$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y})(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t))$ such that
$$|v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|,\ t \in J.$$
Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by
$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le l(t)|y(t) - \bar{y}(t)|\}$$
Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable ([30], proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J$
$$|v_1(t) - v_2(t)| \le l(t)|y(t) - \bar{y}(t)|,\ t \in J.$$
Let us define for each $v_2 \in J$,
$$\begin{aligned} h_2(t) &= y^* + g(y) + t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}v_2(s)ds - \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i - s)^{\alpha-p-1}v_2(s)ds\right]}{\Gamma(\alpha-p)(\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p})} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_2(s)ds \end{aligned}$$
Then for each $t \in J$,
$$\begin{aligned} |h_1(t) - h_2(t)| &\le t\frac{\Gamma(2-p)\left[\left|\int_0^T (T-s)^{\alpha-p-1}(v_1(s)-v_2(s))ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i-s)^{\alpha-p-1}(v_2(s)-v_1(s))ds\right|\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_1(s) - v_2(s)|ds \\ &\le t\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}|v_1(s)-v_2(s)|ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i-s)^{\alpha-p-1}|v_2(s)-v_1(s)|ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_1(s) - v_2(s)|ds \\ &\le T\frac{\Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}|y(s)-\bar{y}(s)|l(s)ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i-s)^{\alpha-p-1}|y(s)-\bar{y}(s)|l(s)ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s) - \bar{y}(s)|l(s)ds \\ &\le T\frac{\|y-\bar{y}\|_\infty \Gamma(2-p)\left[\int_0^T (T-s)^{\alpha-p-1}l(s)ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} (\mu_i-s)^{\alpha-p-1}l(s)ds\right]}{\Gamma(\alpha-p)|\sum_{i=1}^m \lambda_i {\mu_i}^{1-p} - T^{1-p}|} \\ &+ \frac{\|y-\bar{y}\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)ds. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \left[T \frac{\Gamma(2-p)\left[I^{\alpha-p}l)(T)ds + \sum_{i=1}^{m} \lambda_i (I^{\alpha-p}(l)(\mu_i) \right]}{|\sum_{i=1}^{m} \lambda_i {\mu_i}^{1-p} - T^{1-p}|} + \|I^\alpha l\|_\infty \right] \|y - \bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N_3(y), N_3(\bar{y})) \leq \left[T \frac{\Gamma(2-p)\left[I^{\alpha-p}l)(T)ds + \sum_{i=1}^{m} \lambda_i (I^{\alpha-p}(l)(\mu_i) \right]}{|\sum_{i=1}^{m} \lambda_i {\mu_i}^{1-p} - T^{1-p}|} + \|I^\alpha l\|_\infty \right] \|y - \bar{y}\|_\infty.$$

So by (3.28), N_3 is a contraction and thus, by Theorem (1.31) of Covitz and Nadler, N_3 has a fixed point y which is solution to (3.21)-(3.22). The proof is complete. □

3.4 Neutral functional differential inclusions with Hadamard type derivative

[4] We end this chapter by invesigate the existence of solutions to Neutral fractional functional differential inclusions given by

$$^H D^\alpha[y(t) - g(t, y(t))] \in F(t, y(t)), \text{ for almost each } t \in J = [1, T], 1 < \alpha \leq 2, \quad (3.29)$$

$$y(1) = g(1, y_1) = 0, \quad y(T) = y_T, \quad (3.30)$$

Where $^H D^\alpha$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $y_T \in \mathbb{R}$, and $g : J \times \mathbb{R} \mapsto \mathbb{R}$ is a given function such that $g(1, y_1) = 0$.

The first result given here, is when the right hand side is convex valued, and it is relies on the nonlinear alternative of Leray-Schauder, while the second result (nonconvex case) is based upon the fixed point theorem due to Covitz and Nadler.

3.4.1 The convex case

Let us start by defining what we mean by a solution of the problem (3.29)-(3.30)

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Definition 3.13 *A function $y \in AC^1([1,T],\mathbb{R})$ is said to be a solution of (3.29)-(3.30) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that ${}^H D^\alpha[y(t) - g(t,y(t))] = v(t)$ on J, and the conditions $y(1) = g(1,y_1) = 0$ and $y(T) = y_T$ are satisfied.*

Lemma 3.14 *Let $h : [1,+\infty) \to E$ be a continuous function. A function y is a solution of the fractional equation*

$$\begin{aligned} y(t) &= g(t,y(t)) + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}\right] \end{aligned} \tag{3.31}$$

if and only if y is a solution of the fractional IVP

$${}^H D^r[y(t) - g(t,y(t))] = h(t), \text{ for a.e. } t \in J = [1,T], \quad 1 < r \le 2, \tag{3.32}$$

$$y(1) = g(1,y_1) = 0, \;\; y(T) = y_T. \tag{3.33}$$

Proof: Applying the Hadamard fractional integral of order r to both sides of (3.32), we have

$$y(t) - g(t,y(t)) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} + {}^H I^r h(t). \tag{3.34}$$

From $y(1) = g(1,y_1) = 0$, we have $c_2 = 0$ and

$$c_1 = \frac{1}{(\log T)^{r-1}}[y_T - g(T,y(T)) - {}^H I^r h(T)]$$

Hence we get equation (3.31). Conversely, it is clear that if y satisfies equation (3.31), then equations (3.32)-(3.33) hold. □

Theorem 3.15 *Assume (H1)-(H3) and the following hypotheses hold:*
(H13) There exists a nonnegative constant C such that:

$$|g(t,y) - g(t,y')| \le C\|y - y'\|_\infty \quad \forall y, y' \in C(J,\mathbb{R})$$

(H14) the function g is continuous, and for any bounded set B in $C(J,\mathbb{R})$, the set $\{t \mapsto g(t,y(t)) : y \in B\}$ is equicontinuous in $C(J,\mathbb{R})$, and there exist constants $0 \le d_1 < 1, d_2 \ge 0$ such that

$$|g(t,u)| \le d_1\|u\|_\infty + d_2, \quad t \in J, u \in C(J,\mathbb{R}).$$

(H15) There exists a number $M > 0$ such that

$$\frac{(1-2d_1)M}{2\dfrac{(\log T)^r\|p\|_\infty\psi(M)}{\Gamma(r+1)} + |y_T| + 2d_2} > 1. \tag{3.35}$$

Then the BVP (3.29)-(3.30) has at least one solution on J.

Proof : We consider the operator $N_3 : C(J,\mathrm{I\!R}) \mapsto \mathcal{P}(C(J,\mathrm{I\!R}))$ defined by

$$N_4(y) = \left\{ h \in C(J,\mathrm{I\!R}) : \begin{array}{lll} h(t) & = & g(t,y(t)) + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} \\ & + & \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s) \frac{ds}{s} \right], \\ & & v \in S_{F,y} \end{array} \right\}.$$

and we shall show that the operator N_3 has at least a fixed point. We should prove the following steps.

Step 1: *$N_4(y)$ is convex for each $y \in C(J,\mathrm{I\!R})$.*

Indeed, if h_1, h_2 belong to $N_4(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) &= g(t,y(t)) + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \right]. \end{aligned}$$

For $i = 1, 2$, let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2] \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} [dv_1 + (1-d)v_2] \frac{ds}{s} \right] \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N_4(y).$$

Step 2: *N_4 maps bounded sets into bounded sets in $C(J,\mathrm{I\!R})$.*

Let $B_{\mu_*} = \{y \in C(J,\mathrm{I\!R}) : \|y\|_\infty \leq \mu_*\}$ be a bounded set in $C(J,\mathrm{I\!R})$ and $y \in B_{\mu_*}$. Then for each $h \in N_3(y)$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= g(t,y(t)) + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s) \frac{ds}{s} \right]. \end{aligned}$$

By (H2) and (H14), we have, for each $t \in J$

$$\begin{aligned} |h(t)| &\leq |g(t, y(t))| + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[|y_T| + |g(T, y(T))| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \right] \\ &\leq d_1 \|y\|_\infty + d_2 + \frac{(\log T)^r}{\Gamma(r+1)} \int_1^t p(s)\psi(|y(s)|) ds \\ &+ \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s)\psi(|y(s)|) ds + |y_T| + d_1 \|y\|_\infty + d_2 \\ &\leq 2 \frac{(\log T)^r \psi(\mu^*)}{\Gamma(r+1)} \int_1^T p(s) ds + |y_T| + 2d_1 \|y\|_\infty + 2d_2 \end{aligned}$$

Thus

$$\|h\|_\infty \leq 2 \frac{(\log T)^r \psi(\mu^*)}{\Gamma(r+1)} \|p\|_\infty + |y_T| + 2d_1 \|y\|_\infty + 2d_2 := \ell$$

Step 3: *N_4 maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$*

Let $t_1, t_2 \in J$, $t_1 < t_2$, and let B_{μ^*} be bounded set of $C(J, \mathbb{R})$ as in Step 2. Let $y \in B_{\mu^*}$ and $h \in N_4(y)$. Then

$$\begin{aligned} |h(t_2) - h(t_1)| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{v(s)}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \Bigg| \\ &\leq \frac{p(s)\psi(|y(s)|)}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{ds}{s} \\ &+ \frac{p(s)\psi(|y(s)|)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{\|p\|_\infty \psi(\mu^*)}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{ds}{s} \\ &+ \frac{\|p\|_\infty \psi(\mu^*)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $N_4 : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Step 4: *N_4 has a closed graph.*

Let $y_n \to y_*$ $h_n \in N_4(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N_4(y_*)$. $h_n \in N_4(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in [1,T]$

$$\begin{aligned} h_n(t) = g(t, y_n(t)) &+ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T, y_n(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right]. \end{aligned}$$

We must show that there exists $v_* \in S_{F,y}$ such that, for each $t \in [1,T]$,

$$\begin{aligned} h(t) = g(t, y_*(t)) &+ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_*(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - g(T, y_*(T)) - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_*(s) \frac{ds}{s} \right]. \end{aligned}$$

Since $F(t, \dot{)}$ is upper semi continuous then for every $\epsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1), \quad \text{a.e } t \in [1,T]$$

Since $F(.,.)$ has compact values, then there exists a subsequence $v_{n_m}(.)$ such that

$$v_{n_m}(.) \to v_*(.) \text{ as } m \to \infty$$

and

$$v_*(t) \in F(t, y_*(t)), \text{ a.e} t \in [1,T]$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq |g(t, y_n)(t) - g(t, y_*(t))| + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[|g(T, y_n)(T) - g(T, y_*(T))| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \right] \\ &\leq |g(t, y_n) - g(t, y_*)| + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[|g(T, y_n)(T) - g(T, y_*(T))| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \right] \end{aligned}$$

Hence

$$\begin{aligned}\|h_n(t)-h_*(t)\|_\infty &\leq C\|y_{n_m}-y_*\|_\infty+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}l(s)\frac{ds}{s}\|y_{n_m}-y_*\|_\infty\\ &+\frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[C\|y_{n_m}-y_*\|_\infty+\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}l(s)\frac{ds}{s}\|y_{n_m}-y_*\|_\infty\right]\to 0\end{aligned}$$

as $m\to\infty$

Step 5: *A priori bounds on solutions.*

Let y be such that $y\in\lambda N_4(y)$ with $\lambda\in(0,1]$. Then there exists $v\in S_{F,y}$ such that, for each $t\in J$

$$\begin{aligned}h(t) &= \lambda g(t,y_t)+\frac{\lambda}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}\\ &+\frac{\lambda(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T-g(T,y(T))\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v(s)\frac{ds}{s}\right]\end{aligned}$$

This implies by (H2) and (H14) that, for each $t\in J$, we have

$$\begin{aligned}|y(t)| &\leq |g(t,y(t))|+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\\ &+\frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[|y_T|+|g(T,y(T))+\frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v(s)\frac{ds}{s}|\right]\\ &\leq d_1\|y\|_\infty+d_2+2\frac{(\log T)^r\|p\|_\infty\psi(\|y(s)\|_\infty)}{\Gamma(r+1)}+|y_T|+d_1\|y\|_\infty+d_2\end{aligned}$$

i.e

$$(1-2d_1)\|y\|_\infty\leq 2d_2+2\frac{(\log T)^r\|p\|_\infty\psi(\|y(s)\|_\infty)}{\Gamma(r+1)}+|y_T|$$

Thus

$$\frac{(1-2d_1)\|y\|_\infty}{2\dfrac{(\log T)^r\|p\|_\infty\psi(\|y\|_\infty)}{\Gamma(r+1)}+|y_T|}<1.$$

Then by condition (3.35), there exist $M>0$ such that $\|y\|_\infty\neq M$. Let $U=\{y\in C(J,\mathbb{R}):\|y\|_\infty<M\}$. The operator $N_4:\overline{U}\to\mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N_4(y)$ for some $\lambda\in(0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that N_4 has a fixed point $y\in\overline{U}$ which is a solution of the problem (3.29)-(3.30). This completes the proof. □

3.4.2 The nonconvex case

We present now a result for the problem (3.29)-(3.30) with a nonconvex valued right hand side.

Theorem 3.16 *Assume (H3), (H5), (H10) and (H14) hold*

If

$$2\frac{(\log T)^r}{\Gamma(r+1)}\|l\|_{L^1} + C < 1 \tag{3.36}$$

then the BVP (3.29)-(3.30) has at least one solution on J.

Proof : We shall show that N_4 satisfies the assumptions of Theorem 1.31.

It is clear that $N_4(y) \in \mathcal{P}_{cl}(C(J,\mathbb{R}))$ for each $y \in C(J,\mathbb{R})$.

Now we shall prove that there exists $\gamma < 1$ such that

$$H_d(N_4(y), N_4(\bar{y})) \leq \gamma \|y_t - \bar{y}_t\|_\infty$$

for each $y_t, \bar{y}_t \in C(J,\mathbb{R})$.

Let $y, \bar{y} \in C(J,\mathbb{R})$ and $h_1 \in N_4(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= g(t,y(t)) + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_1(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - g(T,y(T)) - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_1(s)\frac{ds}{s}\right] \end{aligned}$$

From (H3) it follows that

$$H_d(F(t,y(t)), F(t,\bar{y}(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t,\bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|,\ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\bar{y}(t))$ is measurable ([30], proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t,\bar{y}(t))$, and for each $t \in J$

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|,\ t \in J.$$

Let us define for each $v_2 \in J$,

$$\begin{aligned} h_2(t) &= g(t,\bar{y}(t)) + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_2(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - g(T,\bar{y}(T) - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_2(s)\frac{ds}{s}\right] \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\le |g(t,y(t)) - g(t,\bar{y}(t)| + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v_1(s) - v_2(s)|\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[|g(T,y(T)) - g(T,\bar{y}(T)| + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |v_1(s) - v_2(s)|\frac{ds}{s}\right] \\ &\le C\|y-\bar{y}\|_\infty + \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)|l(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[C\|y-\bar{y}\|_\infty + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |y(s) - \bar{y}(s)|l(s)\frac{ds}{s}\right] \\ &\le \left[2\frac{(\log T)^r}{\Gamma(r+1)}\int_1^T l(s)ds + C\right]\|y-\bar{y}\|_\infty. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \le \left[2\frac{(\log T)^r}{\Gamma(r+1)}\|l\|_{L^1} + C\right]\|y-\bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N_4(y), N_4(\bar{y})) \le \left[2\frac{(\log T)^r}{\Gamma(r+1)}\|l\|_{L^1} + C\right]\|y-\bar{y}\|_\infty.$$

So by (3.36), N_4 is a contraction and has a fixed point y which is solution to (3.29)-(3.30). The proof is complete. □

3.4.3 An example

In this section we give an example to our result introduced above in theorem (3.16). Let us consider the following neutral fractional problem:

$$^{H}D^{\frac{3}{2}}[y(t) - g(t,y(t))] \in F(t,y(t)), \text{ for almost each } t \in J = [1,e],, \tag{3.37}$$

$$y(1) = g(1,y_1) = 0 \quad y(e) = y_T, \tag{3.38}$$

Set

$$F(t,y) = \{v \in \mathbb{R} : f_1(t,y) \le v \le f_2(t,y)\}$$

Where $f_1, f_2 : [1,T] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1,T]$, $f_1(t,.)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t,y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1,T]$, $f_2(t,.)$ is upper semi-continuous (i.e, the set the set $\{y \in \mathbb{R} : f_2(t,y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). It is clear that F is compact and convex valued, and it is upper semi-continuous.
There exists $l \in L^1(J, \mathbb{R}^+)$, with $I^{\frac{3}{2}} l < \infty$, such that

$$H_d(F(t,u), F(t,\bar{u})) \leq l(t)|u - \bar{u}| \qquad \text{for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq l(t), \quad \text{a.e } t \in J.$$

$F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$ and let

$$g(t, y(t)) = \frac{\|y\|_\infty}{2(1 + \|y\|_\infty)} = \frac{y}{2(1+y)}, \quad (t,y) \in [1,e] \times (e, \infty).$$

So we have For $y, \bar{y} \in (e, \infty)$ and $t \in J$

$$|g(t,y) - g(t,\bar{y})| = \frac{1}{2}\left|\frac{y}{1+y} - \frac{\bar{y}}{1+\bar{y}}\right| = \frac{|y-\bar{y}|}{2(1+y)(1+\bar{y})} \leq \frac{1}{2}|y - \bar{y}|$$

then g is a contraction. All the hypothesis of theorem (3.16) are verified, so when the condition

$$\frac{(\log e)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\|l\|_{L^1} + C = \frac{4}{3\sqrt{\pi}}\|l\|_{L^1} + \frac{1}{2} \leq 1.$$

is verified, by theorem (3.16), the problem (3.37)-(3.38) has at least one solution on J.
□

Chapter 4

Problems for fractional differential inclusions in Banach spaces

In this chapter, by using the Mönch's fixed point theorem, we investigate the existence of solutions to fractional differential inclusions.

4.1 Boundary value problem for fractional differential inclusions

[1] In this section, we are concerned with the existence of solutions to Boundary value problems (BVP for short) for fractional order differential inclusions. In particular, we consider the Boundary value problem

$$^{H}D^{\alpha}y(t) \in F(t, y(t)), \quad \text{for a.e } \ t \in J = [1, T], 1 < \alpha \leq 2, \tag{4.1}$$

$$y(1) = 0, \quad y(T) = y_T, \tag{4.2}$$

Where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $(E, |\cdot|)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T] \times E \to \mathcal{P}(E)$ is a multivalued map and $y_T \in \mathbb{R}$.
In what follows, we present an existence result for the problem (4.1)-(4.2), when the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. We include an example to illustrate our main results.

4.1.1 Main results

First of all, we recall the definitions of the Kuratowski measure of noncompacteness an summarize the main properties of this measure.

[1]J R. Graef, N. Guerraiche and S. Hamani, Boundary value problems for fractional differential inclusions with Hadamrd type derivatives in Banach spaces, *Stud. Univ. Babes-Bolyai. Math,* **62** (2017), No. 4, 427-438.

Definition 4.1 *([9, 12]) Let X be a Banach space and let Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\beta : \Omega_X \to [0, \infty)$ defined by*

$$\beta(B) = \inf\{\epsilon > 0 \, : B \subset \bigcup_{j=1}^{m} B_j \text{ and } diam(B_j) \leq \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [9, 12]:

(1) $\beta(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).

(2) $\beta(B) = \beta(\overline{B})$.

(3) $A \subset B$ implies $\beta(A) \leq \beta(B)$.

(4) $\beta(A + B) \leq \beta(A) + \beta(B)$.

(5) $\beta(cB) = |c|\beta(B), c \in \mathbb{R}$.

(6) $\beta(conB) = \beta(B)$.

Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively.

Now we give an important theorem and lemma.

Theorem 4.2 *([52], [[71], Theorem 1.3]) Let E be a Banach space and C be a countable subset of $L^1(J, E)$ such that there exists $h \in L^1(J, \mathbb{R}_+)$ with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$, where. Then the function $\varphi(t) = \beta(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\beta\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \beta(C(s))ds.$$

Lemma 4.3 *([[63], Lemma 2.6]) Let J be a compact real interval. Let F be a Caratheodory multivalued map, and let θ be a linear continuous map from $L^1(J, E) \mapsto C(J, E)$. Then the operator*

$$\theta \circ S_{F,y} : L^1(J, E) \mapsto P_{cp,c}(C(J, E)), \quad y \mapsto (\theta \circ S_{F,y})(y) = \theta(S_{F,y})$$

is a closed graph operator in $L^1(J, E) \times C(J, E)$

Now, we define what we mean by a solution of the problem (4.1)-(4.2)

Definition 4.4 *A function $y \in AC^1([1, T], E)$ is said to be a solution of (4.1)-(4.2) if there exist a function $v \in L^1(J, E)$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that ${}^H D^\alpha y(t) = v(t)$ on J, and the conditions $y(1) = 0$ and $y(T) = y_T$ are satisfied.*

Lemma 4.5 *Let $h : J \to E$ be an integrable function. A function y is a solution of the fractional equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}\right] \end{aligned} \tag{4.3}$$

if and only if y is a solution of the fractional BVP

$$^{H}D^{r}y(t) = h(t), \text{ for a.e. } t \in J = [1,T], \quad 1 < r \leq 2, \tag{4.4}$$

$$y(1) = 0, \;\; y(T) = y_T. \tag{4.5}$$

Proof: Applying the Hadamard fractional integral of order r to both sides of (4.4), we have

$$y(t) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} +^{H} I^{r}h(t). \tag{4.6}$$

From (4.5), we have $c_2 = 0$ and

$$c_1 = \frac{1}{(\log T)^{r-1}}[y_T -^{H} I^{r}h(T)]$$

Hence, we obtain (4.3). Conversely, it is clear that if y satisfies equation (4.3), then equations (4.4)-(4.5) hold. □

Theorem 4.6 *Let $R > 0$, $B = \{x \in E : \|x\| \leq R\}$, $U = \{x \in C(J,E) : \|x\|_\infty < R\}$, and assume that:*

(H4.1) $F : J \times E \to \mathcal{P}_{cp,c}(E)$ is a Carathéodory multi-valued map;

(H4.2) For each $R > 0$, there exists a function $p \in L^1(J,E)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v|, v(t) \in F(t,y)\} \leq p(t)$$

for each $(t,y) \in J \times E$ with $|y| \leq R$, and

$$\lim_{\mathbb{R}\mapsto\infty} \inf \frac{\int_J p(t)dt}{R} = \delta < \infty$$

(H4.3) There exists a Carathéodory function $\psi : J \times [0,2R] \mapsto \mathbb{R}_+$ such that

$$\beta(F(t,M)) \leq \psi(t,\beta(M)), \text{ a.e. } t \in J \text{ and each } M \subset B,$$

(H4.4) The function $\varphi = 0$ is the unique solution in $C(J, [0, 2R])$ of the inequality

$$\varphi(t) \le 2\left\{\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s} + \left[\frac{(\log t)^{r-1}}{(\log T)^{r-1}\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s}\right]\right\}.$$

for $t \in J$.

Then the BVP (4.1)-(4.2) has at least one solution in $C(J, B)$, provided that

$$\delta < \frac{\Gamma(r+1)}{(\log T)^r}$$

Proof. We wish to transform the problem (4.1)-(4.2) into a fixed point problem, so consider the multivalued operator

$$S_1(y) = \left\{ h \in C(J, \mathbb{R}) : \begin{array}{rcl} y(t) & = & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ & + & \dfrac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \dfrac{1}{\Gamma(r)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right], \quad v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 4.5, the fixed points of S_1 are solutions to (4.1)–(4.2). We shall show that S_1 satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps. First note that $\overline{U} = C(J, B)$.

Step 1: *$S_1(y)$ is convex for each $y \in C(J, B)$.*

Take $h_1, h_2 \in S_1(y)$; then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{array}{rcl} h_i(t) & = & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_i(s)\frac{ds}{s} \\ & + & \dfrac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \dfrac{1}{\Gamma(r)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_i(s)\frac{ds}{s}\right]. \end{array}$$

For $i = 1, 2$ Let $0 \le d \le 1$; then for each $t \in J$,

$$\begin{array}{rcl} (dh_1 + (1-d)h_2)(t) & = & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s} \\ & + & \dfrac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \dfrac{1}{\Gamma(r)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s}\right] \end{array}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$S_1(M)$ is relatively compact for each compact $M \subset \overline{U}$.*

Let $M \subset \overline{U}$ be a compact set and let (h_n) by any sequence of elements of $S_1(M)$. We will show that (h_n) has a convergent subsequence by using the Arzela-Ascoli criterion of compactness in $C(J, B)$. Since $h_n \in S_1(M)$ there exist $y_n \in M$ and $v_n \in S_{F,y}$ such that

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right]. \end{aligned}$$

For $n \geq 1$. Using Theorem (4.2) and the properties of the measure of noncompactness of Kuratowski, we have

$$\begin{aligned} \beta(\{h_n(t)\}) &\leq 2 \left\{ \frac{1}{\Gamma(r)} \int_1^t \beta\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} : n \geq 1\right\}\right) \right. \\ &+ \left. \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[-\frac{1}{\Gamma(r)} \int_1^T \beta\left(\left\{\left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} : n \geq 1\right\}\right) \right] \right\}. \end{aligned} \tag{4.7}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\beta(v_n(s) : n \geq 1) = 0$ for a.e. $s \in J$. Furthermore

$$\beta\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s}\right\}\right) = \left(\log \frac{t}{s}\right)^{r-1} \frac{1}{s} \beta(\{v_n(s) : n \geq 1\}) = 0,$$

and

$$\beta\left(\left\{\left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s}\right\}\right) = \left(\log \frac{T}{s}\right)^{r-1} \frac{1}{s} \beta(\{v_n(s) : n \geq 1\}) = 0$$

for a.e. $t, s \in J$. Hence, from this and (4.7), $\{h_n(t) : n \geq 1\}$ is relatively compact in B, for each $t \in J$. In addition, for each t_1 and t_2 from J, $t_1 < t_2$, we have

$$\begin{aligned}
|h_n(t_2) - h_n(t_1)| &= \left| \frac{1}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{v_n(s)}{s} ds \right. \\
&+ \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{v_n(s)}{s} ds \\
&+ \left. \frac{(\log t_2 - \log t_1)}{\log T} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right] \right| \\
&\leq \frac{p(t)}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1} \right] \frac{ds}{s} \\
&+ \frac{p(t)}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\
&+ \frac{(\log t_2 - \log t_1)}{\log T} \left| y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \frac{ds}{s} \right|.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that$\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J, B)$.

Step 3: *S_1 has a closed graph*

Let $y_n \to y_*, h_n \in S_1(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in S_1(y_*)$. We can verify this step by adding the Lipschitz hypothes in the Theorem of existence of solutions.

Step 4: *M is relatively compact in $C(J, B)$*

Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an argument similar to the one used in Step 2 shows that $S_1(M)$ is equicontinuous. Then, since $M \subset conv(\{0\} \cup S_1(M))$, we see that M is equicontinuous as well. To apply the Arzela-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup S_1(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset S_1(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned}
h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right].
\end{aligned}$$

From $\overline{M} \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, from the properties of the Kuratowski measure of noncompactness, we have

$$\beta(M(t)) \leq \beta(\overline{C(t)} \leq \beta(H(t)) = \beta(\{h_n((t) : n \geq 1\}).$$

Using (4.7) and the fact that $v_n(s) \in M(s)$, we obtain

$$\begin{aligned}
\beta(M(t)) &\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n\geq 1\right\}\right)\\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[-\frac{1}{\Gamma(r)}\int_1^T \beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n\geq 1\right\}\right)\right]\Bigg\}\\
&\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} \beta(M(s))\frac{ds}{s}\\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[-\frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \beta(M(s))\frac{ds}{s}\right]\Bigg\}\\
&\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} \psi(s,\beta(M(s)))\frac{ds}{s}\\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[-\frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \psi(s,\beta(M(s)))\frac{ds}{s}\right]\Bigg\}.
\end{aligned}$$

We also have that the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J,[0,2R])$. Consequently, by $(H4.4)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M is relatively compact in $C(J,B)$.

Step 5:

Let $h \in S_1(y)$ with $y \in U$. we calim that $S_1(\overline{U}) \subset \overline{U}$, if this were not the case, then in view of (H4.2), there exists a function $v \in S_{F,y}$ and $p \in L^1(J,E)$ such that

$$\begin{aligned}
h(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s}\\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right],
\end{aligned}$$

and

$$\begin{aligned}
R &\leq \|S_1(y)\|_{\mathcal{P}}\\
&\leq \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[|y_T| + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\right]\\
&\leq \frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds + \frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)ds|\\
&\leq 2\frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds
\end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \mapsto \infty$, we conclude that

$$2\left[\frac{(\log T)^r}{\Gamma(r+1)}\right]\delta \geq 1$$

which contradicts (3.5). Hence $S_1(\overline{U}) \subset \overline{U}$.

As a consequence of steps $1-5$ and Theorem (4.6), we conclude that S_1 has a fixed point $y \in C(J, B)$ which is a solution of problem (4.1)-(4.2).

4.1.2 An example

We conclude this paper whith an example to illustratee our main result, namely, Theorem 4.6 above.

Consider the fractional differential inclusions

$$^{H}D^r y(t) \in F(t, y(t)), \quad \text{for almost each } \ t \in J = [1, e], 1 < r \leq 2, \tag{4.8}$$

$$y(1) = 0 \quad y(e) = y_T, \tag{4.9}$$

here $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map satisfying

$$F(t, y) = \{v \in E : f_1(t, y) \leq v \leq f_2(t, y)\}$$

Where $f_1, f_2 : [1, e] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1, e]$, $f_1(t, .)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1, T]$, $f_2(t, .)$ is upper semi-continuous (i.e, the set the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there is a function $p \in L^1(J, \mathbb{R}))$ such that

$$\begin{aligned} \|F(t, u)\|_{\mathcal{P}} &= \sup\{|v|, v(t) \in F(t, y)\} \\ &= \max(|f_1(t, y)|, |f_2(t, y)|) \leq p(t), \quad \text{for each } t \in [1, e], y \in \mathbb{R}. \end{aligned}$$

It is clear that F is compact and convex valued, and it is upper semi-continuous. And for $(t, y) \in J \times \mathbb{R}$ with $|y| \leq R$, we have

$$\lim_{R \mapsto \infty} \inf \frac{\int_0^e p(t)dt}{R} = \delta < \Gamma(r+1);$$

Finally, we assume that there exists a Carathéodory function $\psi : J[0, 2R] \mapsto \mathbb{R}_+$ such that

$$\beta(F(t, M)) \leq \psi(t, \beta(M)), \text{ a.e. } t \in J \text{ and each } M \subset B = \{x \in \mathbb{R} : |x| \leq R\},$$

and $\varphi = 0$ is the unique solution in $C(J, [0, 2R])$ of the inequality

$$\begin{aligned} \varphi(t) \ &\leq \ 2\left[\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} \psi(s,\varphi(s))\frac{ds}{s}\right. \\ &+ \left.\frac{(\log t)^{r-1}}{(\log T)^{r-1}\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} \psi(s,\varphi(s))\frac{ds}{s}\right]. \end{aligned}$$

for $t \in J$.

Since all the conditions of Theorem 4.6 are satisfied, problem (4.8)-(4.9) has at least one solution y on $[1, e]$.

4.2 Nonlinear boundary value problem for fractional differential inclusions with integral boundary conditions

[2] In this section, we retake the problem given in section (3.2) and we investigate the existence of solution to this problem by using the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. We consider the Boundary Value Problem

$$^{H}D^{r}y(t) = h(t), \text{ for a.e. } t \in J = [1, T], \quad 2 < r \leq 3, \tag{4.10}$$

$$y(1) = y''(1) = 0 \tag{4.11}$$

$$y(T) = \int_1^T g(s, y(s)ds \tag{4.12}$$

Where $^{H}D^{r}$ is the Hadamard fractional derivative, $F : [1, T] \times E \to \mathcal{P}(E)$ is a multivalued map, $\mathcal{P}(E)$ is the family of all nonempty subsets of E and and $g : [1, T] \times E \to E$ is a function given.

4.2.1 Main results

Let us start by defining what we mean by a solution of the problem (4.10)-(4.12)

Definition 4.7 *A function $y \in AC^1([1, T], E)$ is said to be a solution of (4.10)-(4.12) if there exist a function $v \in L^1(J, E)$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that $^{H}D^{\alpha}y(t) = v(t)$ on J, and the conditions (4.11) and (4.12) are satisfied.*

[2] N. Guerraiche and S. Hamani, Nonlinear value problem of fractional differential inclusions with Hadamard type derivative and integral boundary value conditions in Banach spaces, submitted

Lemma 4.8 *Let $h : J \to E$ be an integrable function. A function y is a solution of the fractional equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s}\right] \end{aligned} \tag{4.13}$$

if and only if y is a solution of the nonlinear fractional problem

$$^{H}D^{r}y(t) = h(t), \text{ for a.e. } t \in J = [1,T], \quad 2 < r \leq 3, \tag{4.14}$$

$$y(1) = y''(1) = 0 \tag{4.15}$$

$$y(T) = \int_1^T g(s, y(s)ds \tag{4.16}$$

Proof: Already proved. □

Theorem 4.9 *Let $B = \{x \in E : \|x\| \leq R\}$, $U = \{x \in C(J,E) : \|x\|_\infty < R\}$, Assume (H4.1)-(H4.3) and the following hypotheses hold:*

(H4.5) There exists $k > 0$ such that

$$\|g(t,y)\|_\infty \leq K, \text{ for each, } (t,y) \in J \times E$$

(H4.6) The function $\varphi = 0$ is the unique solution in $C(J,[0,2R])$ of the inequality

$$\begin{aligned} \varphi(t) &\leq 2\left[\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s}\right. \\ &+ \left.\frac{(\log t)^{r-1}}{(\log T)^{r-1}\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s}\right]. \end{aligned}$$

for $t \in J$.

Then the BVP (4.10)-(4.12) has at least one solution on $C(J,B)$, provided that

$$\delta < \frac{\Gamma(r+1)}{(\log T)^{r}} \tag{4.17}$$

Proof. Transform the problem (4.10)-(4.12) into a fixed point problem. Consider the multivalued operator,

$$S_2(y) = \left\{ h \in C(J,E) : \begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right], v \in S_F \end{aligned}\right.$$

We shall show that S_2 satisfies the assumptions of Mönch's fixed point theorem. We note that $\overline{U} = C(J, B)$.

It can be shown, like in the proof of Theorem (4.6), that S_2 has a closed graph and maps compact sets into relatively compact sets.

It remains to show that M is relatively compact in $C(J, B)$ and $S_2(\overline{U}) \subset \overline{U}$.

Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. By using similar arguments like in the proof of Theorem (4.6), we can show that $S_2(M)$ and M are equicontinuous. Now, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup S_2(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset S_2(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T g(s, y(s)) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \right]. \end{aligned}$$

From $\overline{M} \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, and according to Theorem (4.2), we have

$$\beta(M(t)) \leq \beta(\overline{C(t)} \leq \beta(H(t)) = \beta(\{h_n((t) : n \geq 1\}).$$

Using the properties of the measure of noncompactness of Kuratowski, we obtain

$$\begin{aligned} \beta(M(t)) &\leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \beta\left(\left\{ \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} : n \geq 1 \right\} \right) \right. \\ &+ \left. \frac{1}{\Gamma(r)} \int_1^T \beta\left(\left\{ \left(\log \frac{T}{s}\right)^{r-1} v_n(s) \frac{ds}{s} : n \geq 1 \right\} \right) \right\} \\ &\leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \beta(M(s)) \frac{ds}{s} \right. \\ &+ \left. \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \beta(M(s)) \frac{ds}{s} \right\} \\ &\leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \beta(M(s))) \frac{ds}{s} \right. \\ &+ \left. \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s, \beta(M(s))) \frac{ds}{s} \right\}. \end{aligned}$$

Also, the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J, [0, 2R])$. Consequently by $(H4.6)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M

is relatively compact in $C(J,E)$.

Now let $h \in S_2(y)$ with $y \in U$. We claim that $S_2(\overline{U}) \subset \overline{U}$, because if it were not true, then there exists a function $v \in S_{F,y}$ and $p \in L^1(J,E)$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right], \end{aligned}$$

and

$$\begin{aligned} R &\leq \|N(y)\|_{\mathcal{P}} \\ &\leq \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T |g(s,y(s))|ds + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\right] \\ &\leq k(T-1) + \frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds + \frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)ds| \\ &\leq k(T-1) + 2\frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)ds \end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \mapsto \infty$, we conclude that

$$\delta\left[\frac{(\log T)^r}{\Gamma(r+1)}\right] \geq 1$$

which contradicts (4.17). Hence $S_2(\overline{U}) \subset \overline{U}$. From the above, we conclude that the problem (4.10)-(4.12) has at least one solution $y \in C(J,B)$ which is a fixed point of S_2.

4.3 Initial value problem of fractional functional differential inclusions with Hadamard type derivative in Banach spaces

[3] In this section, we retake the problem given in section (3.1) and we present existence results to this problem when the right hand side is convex valued. Our approach is based upon the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. We consider the Initial value problem

$$^{H}D^{\alpha}y(t) \in F(t,y_t), \text{ for almost each } t \in J=[1,T], 0<\alpha\leq 1, \tag{4.18}$$

[3] J R. Graef, N. Guerraiche and S. Hamani, Initial value problem of fractional functional differential inclusions with Hadamard type derivative, *Surveys in mathematics and its applications*, Vol **13**, p27-40, 2018

$$y(t) = \varphi^*(t) \quad t \in [1-r, 1], \tag{4.19}$$

Where ${}^H D^\alpha$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\varphi^* \in C([1-r, 1], \mathbb{R})$ with $\varphi^*(1) = 0$.

4.3.1 Main results

Definition 4.10 *A function $y \in AC([1-r, T], \mathbb{R})$ is said to be a solution of (4.18)-(4.19), if there exists a function $v \in L^1([1, T], \mathbb{R})$ with $v(t) \in F(t, y_t)$, for a.e $t \in [1, T]$, such that*

$$ {}^H D^\alpha y(t) = v(t), \quad a.e\ t \in [1, T], 0 < \alpha < 1,$$

and the function y satisfies the condition (4.19).

Lemma 4.11 *Let $h : [1, +\infty) \to \mathbb{R}$ be continuous functions. A function y is a solution of the fractional equation*

$$y(t) = \begin{cases} \varphi^*(t) & \text{if } t \in [1-r, 1] \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds & \text{if } t \in [1, T] \end{cases} \tag{4.20}$$

if and only if y is a solution of the nonlinear fractional problem

$$ {}^H D^\alpha y(t) = v(t) \ \text{ for a.e. } \ t \in J = [1, T], \quad 0 < \alpha \le 1, \tag{4.21}$$

$$y(t) = \varphi^*(t) \quad t \in [1-r, 1], \tag{4.22}$$

Proof: Already proved. □

Theorem 4.12 *Let $B = \{x \in E : \|x\| \le R\}$, $U = \{x \in C(J, E) : \|x\|_\infty < R\}$, Assume (H4.1)-(H4.3) and the following hypothes hold:*

(H4.7) The function $\varphi = 0$ is the unique solution in $C(J, [0, 2R])$ of the inequality

$$\varphi(t) \ \le \ 2\left[\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \varphi(s, \varphi(s)) \frac{ds}{s}\right].$$

for $t \in J$.

Then the BVP (4.18)-(4.19) has at least one solution on $C(J, B)$, provided that

$$\delta < \frac{\Gamma(r+1)}{(\log T)^r} \tag{4.23}$$

Proof. We wish to transform the problem (4.18)-(4.19) into a fixed point problem. Consider the multivalued operator,

$$S_3(y)(t) = \left\{ h \in C([1-r,T],\mathbb{R})\ h(t) = \begin{cases} \varphi^*(t) & \text{if } t \in [1-r,1] \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds & \text{if } t \in [1,T] \end{cases} ; v \in F(t,y_t) \right\}.$$

It is clear that the fixed points of S_3 are solutions to (4.18)–(4.19). In the following, we shall show that S_3 satisfies the assumptions of Mönch's fixed point theorem. We note that $\overline{U} = C(J,B)$.

First we can show that S_3 has a closed graph and maps compact sets into relatively compact sets.

Let us now prove that M is relatively compact in $C(J,B)$.

Suppose $M \subset U$, $M \subset conv(\{0\} \cup S_3(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. As in the above (Theorem (4.6)), we can see that $S_3(M)$ is equicontinuous and also we can deduce that M is equicontinuous too. It remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup S_3(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset S_3(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) \;=\; \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s}.$$

From $\overline{M} \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, from the properties of the Kuratowski measure of noncompactness, we have

$$\beta(M(t)) \leq \beta(\overline{C(t)} \leq \beta(H(t)) = \beta(\{h_n((t) : n \geq 1\}).$$

From the fact that $v_n(s) \in M(s)$, we obtain

$$\begin{aligned} \beta(M(t)) &\leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \beta\left(\left\{ \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} : n \geq 1 \right\} \right) \right\} \\ &\leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \beta(M(s)) \frac{ds}{s} \right\} \\ &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \beta(M(s))) \frac{ds}{s}. \end{aligned}$$

Also, the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J,[0,2R])$. Consequently by $(H4.7)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M is relatively compact in $C(J,E)$.

Now we shall prove that $S_3(\overline{U}) \subset \overline{U}$.

Let $h \in S_3(y)$ with $y \in U$. We claim that $S_3(\overline{U}) \subset \overline{U}$, because by (H4.2), if it were not true then there exists a function $v \in S_{F,y}$ and $p \in L^1(J,E)$ such that

$$h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s}.$$

and

$$\begin{aligned} R &\leq \|S_3(y)\|_{\mathcal{P}} \\ &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \\ &\leq \frac{(\log T)^r}{\Gamma(r+1)} \int_1^t p(s) ds \end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \mapsto \infty$, we conclude that

$$\left[\frac{(\log T)^r}{\Gamma(r+1)}\right] \delta \geq 1$$

which contradicts (4.23). Hence $S_3(\overline{U}) \subset \overline{U}$. From the above, we conclude that S_3 has a fixed point $y \in C(J,B)$ which is a solution of the problem (4.18)-(4.19).

4.4 Boundary value problem of fractional differential inclusions with Hadamard type derivative in Banach spaces with integral boundary conditions

[4] In this section, we are concerned to investigate the existence of solutions for another Boundary value problems (BVP for short), for fractional order differential inclusions. We consider the boundary value problem

$${}^H D^\alpha y(t) \in F(t, y(t)), \text{ for almost each } t \in J = [1, e], 1 < \alpha \leq 2, \tag{4.24}$$

$$y(1) = 0 \quad y(e) = I^\mu y(\epsilon) \quad 1 < \epsilon < e, \tag{4.25}$$

Where ${}^H D^\alpha$ is the Hadamard fractional derivative, $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and I^μ is the Hadamard fractional integral of order μ.
In this section, we shall present existence results for the problem (4.24)-(4.25), when the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness.

[4] N. Guerraiche and S. Hamani, Boundary value problem of fractional differential inclusions with Hadamart type derivative in Banach spaces with integral boundary conditions, ROMAI J., v. 13, no. **2** (2017), 69-84.

4.4.1 Main results

Let us start by defining what we mean by a solution of the problem (4.24)-(4.25)

Definition 4.13 *A function $y \in AC^1([1,T],E)$ is said to be a solution of (4.24)-(4.25) if there exist a function $v \in L^1(J,E)$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that ${}^H D^{\alpha} y(t) = v(t)$ on J, and the conditions $y(1) = 0$ and $y(e) = I^{\mu} y(\epsilon)$ are satisfied.*

Lemma 4.14 *Let $h : [1,+\infty) \to E$ be a continuous function. A function y is a solution of the fractional equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{\Omega} \left[\frac{1}{\Gamma(r+\mu)} \int_1^{\epsilon} \left(\log \frac{\epsilon}{s}\right)^{\mu+r-1} h(s) \frac{ds}{s} - \frac{1}{\Gamma(r)} \int_1^{e} \left(\log \frac{e}{s}\right)^{r-1} h(s) \frac{ds}{s} \right] \end{aligned} \tag{4.26}$$

where

$$\Omega = \frac{1}{\dfrac{1}{\Gamma(\mu)} \displaystyle\int_1^{\mu} \left(\log \frac{\epsilon}{s}\right)^{\beta-1} (\log s)^{r-1} \frac{ds}{s}}$$

if and only if y is a solution of the fractional BVP

$${}^H D^r y(t) = h(t), \text{ for a.e. } t \in J = [1,T], \quad 1 < r \leq 2, \tag{4.27}$$

$$y(1) = 0, \;\; y(e) = I^{\mu} y(\epsilon). \tag{4.28}$$

Proof: Applying the Hadamard fractional integral of order r to both sides of (4.27), we have

$$y(t) = c_1 (\log t)^{r-1} + c_2 (\log t)^{r-2} +^H I^r h(t). \tag{4.29}$$

From $y(1) = 0$, we have $c_2 = 0$ and

$${}^H I^r h(e) + c_1 = I^{\mu} \left({}^H I^r h(t) + c_1 (\log t)^{r-1}\right)(\mu) = I^{r+\mu} h(t) + \frac{c_1}{\Gamma(\mu)} \int_1^{\epsilon} \left(\log \frac{\epsilon}{s}\right)^{\mu-1} (\log s)^{r-1} \frac{ds}{s}$$

wich gives

$$c_1 = \frac{1}{\dfrac{1}{\Gamma(\mu)} \displaystyle\int_1^{\epsilon} \left(\log \frac{\epsilon}{s}\right)^{\mu-1} (\log s)^{r-1} \frac{ds}{s}} \left[I^{r+\mu} h(\epsilon) - I^r h(e) \right]$$

Hence we get equation (4.26). Conversely, it is clear that if y satisfies equation (4.26), then equations (4.27)-(4.28) hold. □

Theorem 4.15 *Let $B = \{x \in E : \|x\| \leq R\}$, $U = \{x \in C(J,E) : \|x\|_{\infty} < R\}$, Assume (H4.1)-(H4.3) and the following hypothes hold:*

(H4.8) The function $\varphi = 0$ is the unique solution in $C(J,[0,2R])$ of the inequality

$$\begin{aligned}\varphi(t) &\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s}\\ &+ \frac{(\log t)^{r-1}}{\Omega}\Bigg[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon}\left(\log\frac{\epsilon}{s}\right)^{\mu+r-1}\psi(s,\varphi(s))\frac{ds}{s}\\ &- \frac{1}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}\psi(s,\varphi(s))\frac{ds}{s}\Bigg\}\Bigg].\end{aligned}$$

for $t \in J$.

Then the BVP (4.24)-(4.25) has at least one solution on $C(J,B)$, provided that

$$\delta < \left[\frac{(\log T)^r\left((\log\mu)^{\mu+r}-1\right)}{|\Omega|\Gamma(r+1)}\right] \tag{4.30}$$

Proof. Transform the problem (4.24)-(4.25) into a fixed point problem. Consider the multivalued operator,

$$S_4(y) = \left\{ h \in C(J,\mathrm{I\!R}) : \begin{array}{rcl} y(t) &=& \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s}\\ &+& \frac{(\log t)^{r-1}}{\Omega}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon}\left(\log\frac{\epsilon}{s}\right)^{\mu+r-1} v(s)\frac{ds}{s}\right.\\ &-& \left.\frac{1}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1} v(s)\frac{ds}{s}\right], \quad v \in S_{F,y}\end{array}\right\}.$$

From Lemma (4.14), the fixed points of S_4 are solutions to (4.24)–(4.25). We shall show that S_4 satisfies the assumptions of Mönch's fixed point theorem.

We can show that S_3 has a closed graph and maps compact sets into relatively compact sets.

Now we shall prove that M is relatively compact in $C(J,B)$.

Suppose $M \subset U$, $M \subset conv(\{0\} \cup S_4(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. By using similar arguments, we can show that $S_4(M)$ and M are equicontinuous. To apply the Arzela-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup S_4(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset S_4(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned}h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\\ &+ \frac{(\log t)^{r-1}}{\Omega}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon}\left(\log\frac{\mu}{s}\right)^{\mu+r-1} v_n(s)\frac{ds}{s} - \frac{1}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right].\end{aligned}$$

From $\overline{M} \subset \overline{C} \subset \overline{conv}(\{0\} \cup H))$, and according to the properties of the Kuratowski measure of noncompactness, we have

$$\beta(M(t)) \leq \beta(\overline{C(t)} \leq \beta(H(t)) = \beta(\{h_n((t) : n \geq 1\}).$$

Now, since $v_n(s) \in M(s)$, we have

$$\begin{aligned}
\beta(M(t)) &\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n \geq 1\right\}\right) \\
&+ \frac{(\log t)^{r-1}}{\Omega}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon} \beta\left(\left\{\left(\log\frac{\epsilon}{s}\right)^{\mu+r-1} v_n(s)\frac{ds}{s} : n \geq 1\right\}\right)\right. \\
&- \left.\frac{1}{\Gamma(r)}\int_1^{e} \beta\left(\left\{\left(\log\frac{e}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n \geq 1\right\}\right)\right]\Bigg\} \\
&\leq 2\Bigg\{\frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} \psi(s, \beta(M(s)))\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{\Omega}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon} \left(\log\frac{\mu}{s}\right)^{\mu+r-1} \psi(s, \beta(M(s)))\frac{ds}{s}\right. \\
&- \left.\frac{1}{\Gamma(r)}\int_1^{e} \left(\log\frac{e}{s}\right)^{r-1} \psi(s, \beta(M(s)))\frac{ds}{s}\right]\Bigg\}.
\end{aligned}$$

Also, the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J, [0, 2R])$. Consequently by $(H4.8)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M is relatively compact in $C(J, E)$.

Let us now prove that $S_4(\overline{U}) \subset \overline{U}$.

Let $h \in S_4(y)$ with $y \in U$. Since $|y(s)| \leq R$ and by $(H4.15)$, we have $S_4(\overline{U}) \subset \overline{U}$, because if it were not true, then there exists a function $v \in S_{F,y}$ and $p \in L^1(J, E)$ such that

$$\begin{aligned}
h(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{\Omega}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon} \left(\log\frac{\epsilon}{s}\right)^{r+\mu-1} v(s)\frac{ds}{s} - \frac{1}{\Gamma(r)}\int_1^{e} \left(\log\frac{e}{s}\right)^{r-1} v(s)\frac{ds}{s}\right],
\end{aligned}$$

and

$$
\begin{aligned}
R &\leq \|S_4(y)\|_{\mathcal{P}} \\
&\leq \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}|v(s)|\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{|\Omega|}\left[\frac{1}{\Gamma(r+\mu)}\int_1^{\epsilon}\left(\log\frac{\epsilon}{s}\right)^{\mu+r-1}|v(s)|\frac{ds}{s}-\frac{1}{\Gamma(r)}\int_1^{e}\left(\log\frac{e}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\right] \\
&\leq \frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)ds \\
&+ \frac{(\log t)^{r-1}}{|\Omega|}\left[\frac{(\log\epsilon)^{\mu+r}}{\Gamma(r+\mu+1)}\int_1^{\epsilon}p(s)ds-\frac{1}{\Gamma(r+1)}\int_1^{e}p(s)ds\right] \\
&\leq \frac{(\log T)^r((\log\epsilon)^{\mu+r}-1)}{|\Omega|\Gamma(r+1)}\int_1^{e}p(s)ds
\end{aligned}
$$

Dividing both sides by R and taking the lower limits as $R\mapsto\infty$, we conclude that

$$\left[\frac{(\log T)^r\left((\log\mu)^{\mu+r}-1\right)}{|\Omega|\Gamma(r+1)}\right]\delta\geq 1$$

which contradicts (4.30). Hence $S_4(\overline{U})\subset\overline{U}$. As a consequence of the above, we conclude that the problem (4.24)-(4.25) has at least one solution $y\in C(J,B)$ which is a fixed point of S_4.

Annex

Definition 4.16 *The Mellin transform of a function* $\varphi(t)$ *of a real variable* $t \in \mathbb{R}^+ = (0, \infty)$ *is defined by*

$$(M\varphi)(p) = M[\varphi(t)](p) = \varphi^*(s) = \int_0^\infty t^{s-1}\varphi(t)dt \quad s \in \mathbb{C}$$

Definition 4.17 *[61] The Euler Gamma function* $\Gamma(z)$ *is defined by the so-called Euler integral of the second kind*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad re(z) > 0$$

This integral is convegent for all complex number $z \in \mathbb{C}$ $re(z) > 0$. *It follows that the Gamma function is the Mellin transform of the exponential function.*

Property 1 *From the definition of the Gamma function, we can find*

- $$\Gamma(n) = (n-1)!$$
- $$\Gamma(z+1) = z\Gamma(z) \quad rez > 0$$

Definition 4.18 *[76] The Beta function* $B(z, w)$ *is defined by the so-called Euler integral of the first order*

$$B(z, w) = \int_0^1 x^{z-1}(1-x)^{w-1}dx, \quad re(w) > 0$$

It is connected with the Gamma function by the relation

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

Definition 4.19 *The Mittag-Leffler function* $E_\alpha(z)$ *is defined by*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)} \quad (re(z) > 0)$$

From the definition of Mittag-Leffler function E_α, we have

- $$E_1(z) = e^z \text{ and } E_2(z) = cosh(\sqrt{z}).$$

- $$\left(\frac{d}{dz}\right)^n E_n(\lambda z^n) = \lambda E_n(\lambda z^n) \quad (n \in \mathbb{N};\ \lambda \in \mathbb{R})$$

Theorem 4.20 (Mazur)*[80] Let $\{x_n\}$ be a weakly convergent sequence to x in a Banach space E. then, there is a sequence of convex combination of elements of $\{x_n\}$ which converges strongly to x.*

Conclusion:

In this thesis, we studied the existence of solutions for certain fractional differential inclusions. For this, we are based on three well known fixed point theorems, and we got some original results which are given in the chapters 2-4.

Indeed, in the second and the third chapter, we gave sufficient conditions for the existence of solutions to fractional differential inclusions by involving the fractional derivatives of Caputo and Hadamard, and we gave rigorous demonstrations of the theorems of existence of solutions based on the fixed point theorems of Leray-Schauder and Covitz and Nadler and taking into account the convexity and the nonconvexity of the set valued map. And to show the effectiveness of our results we were able to give some examples.

Then, in the fourth chapter, by using the Mönch's fixed point theorem combined with the technique of mesure of noncompactness of Kuratowski, we gave several results of existence of solutions for different differential fractional inclusions. We mention that the difference between these three techniques lies in the sufficient conditions we put in the theorems of existence of solutions. Finally, in the future we will study other differential inclusions by involving other fractional derivatives like the fractional derivative of Caputo-Hadamard and by using some others techniques and theorems like the Bressan Colombo selection theorem.

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