PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

ABDELHAMID IBN BADIS UNIVERSITY OF MOSTAGANEM

THESIS

for the degree of

DOCTOR OF SCIENCES

Speciality: Mathematics

Option: Fractional Calculus

Presented by

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Title

ON CERTAIN CLASSES OF SEQUENTIAL NONLINEAR BOUNDARY PROBLEMS OF ARBITRARY ORDER

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defended publicly in front of the jury members:

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Thanks

I express my deep gratitude to Mr. Zoubir DAHMANI, Professor at the University of Mostaganem, who guided me during my thesis. I thank him for the help he gave me, for his great patience and his encouragement to
finish this work. I would also like to express my thanks to Madam Samira HAMANI, Professor at the University of Mostaganem for having done me the honor of chairing the jury for this thesis. I would also like to thank the Professors Abdelkader SNOUCI and Mohamed HOUAS for agreeing to judge my work and to be members of the jury. This work could not have been completed without the help of many people. May those I forget here forgive me, but I address a special thought to the teachers who contributed to my training in graduation and post-graduation.

I dedicate this work to: My dear parents, my brothers, my sisters, my wife, my children Abderrezak, Haroune, Bouchra and all my family in general as well as to my close entourage and to anyone who supported me, helped or contributed to me or away to this work.

Résumé

Dans ce projet de thèse, on s"intéresse à l"étude de certaines classes d"équations différentielles séquentielles non linéaires d"ordre non entier. On propose d"abord d'explorer le domaine des inégalités intégrales selon les deux approches: Hadamard, Riemann-Liouville. Puis, en utilisant la théorie des opérateurs non linéaires, la théorie des points fixes ainsi que les approches de Caputo, Riemann-liouville et de Hadamard, on établit des nouveaux résultats sur l"existence et l"unicité des solutions pour les classes séquentielles qu'on a considérées. D"autres résultats d"existence seront aussi établies, et des conditions suffisantes sur "ces existences" sont discutées. Des applications sur nos résultats sont aussi discutées dans cette thèse.

Mots clés: Riemann-Liouville, séquentiel, point fixe, Hadamard, existence .

Abstract

In this thesis, we are concerned with the study of certain classes of nonlinear sequential differential equations of arbitrary order. We first propose to explore the domain of integral inequalities according to the two approaches: Hadamard, Riemann-Liouville. Then, using the theory of nonlinear operators, the theory of fixed points as well as the derivative approaches of Caputo, Riemann-liouville and Hadamard, we establish new results on the existence and uniqueness of the solutions for the considered sequential classes. Other existence results are also established. Some other sufficient conditions for the existence of results for our considered classes also imposed. Each chapter is illustrated by some examples that show the applicability of the obtianed results.

Keywords: Riemann-Liouville, sequential, fixed point, Hadamard, existence.

General Introduction

It is know that the notion of sequential differential equations have not a sense in the standard case of derivation. This is because the commutativity and semi-groupe properties are satisfied for these equations. But the above notions can be applied for differential equations of fractional orders under some particular conditions.
In this project we are concerned with sequential fractional differential problems.

This thesis consists of four chapters organised as follows :

In the first chapter we present some concepts, definitions, lemmas and some basic results of fractional calculus used in this work.

In the second chapter, we give two lemmas useful in our work as follows:

Lemma : Let φ a function and let $\Psi : [0;T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bounded function. Then the operator defined on $C\left([0;T], \mathbb{R}\right)$ by

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) = \left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-u)^{\alpha-\beta-1}\right. \\
&\times \int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) du \\
&+\frac{1}{\left|I^\gamma \psi(\rho)\right|} \left\{ \int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} \chi(\xi) \left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^\xi (\xi-u)^{\alpha-\beta-1} \right.\right. \\
&\left.\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) \right) du \right) d\xi \right. \\
&+\frac{|b_2| \, \Gamma(\alpha-2\beta)}{|c_2|} \int_0^T (T-u)^{\alpha-\beta-1} \\
&\times \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) \right) du
\end{aligned}$$

$$+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.$$

$$\times\left(\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)ds+k\varphi(u)\right)du\right)d\xi$$

$$+\frac{|b_1|}{|\Delta|\Gamma(\alpha-\beta)}\int_0^T(T-u)^{\alpha-\beta-1}$$

$$\left.\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)ds+k\varphi(u)\right)du\right]$$

$$+\frac{1}{\Delta I^{\gamma}h(\rho)}\left[|c_2|\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\psi(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right.$$

$$\left.\left(\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)ds+k\varphi(u)\right)du\right)d\xi$$

$$+|b_2|\Gamma(\alpha-2\beta)\int_0^T(T-u)^{\alpha-2\beta-1}$$

$$\left.\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)ds+k\varphi(u)\right)du\right]$$

$$\left.\left.\times\left(|c_1|\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\,d\xi+(a_1+b_1)\right)\right\}\right)$$

maps bounded sets into bounded in $C([0;T],\mathbb{R})$.

The second lemma is about the boundedness of the operator defined by

$$\begin{aligned}
F(u,v)(t) \;:\;&= \frac{1}{\Sigma}\left[-\Lambda_3\theta_2 + \left(\Lambda_3 - \Lambda_4\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right.\\
&\times\left(\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\phi\left(b,u(b),v(b),{}^H D^{\alpha_2}v(b)\right)\right)\right.\\
&\left.\left.+\lambda_2^H I^{\beta_2}\omega\left(b,u(b),v(b),{}^H D^{\beta_1}u(b)\right)\right)+\lambda_2\frac{\theta_1}{\gamma_2}\right]\\
&+\frac{1}{\Sigma}\left[(\Lambda_1+\Lambda_4)\theta_4 - \left(\Lambda_1 + \Lambda_2\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right.\\
&\times\left(\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\omega\left(b,u(b),v(b),{}^H D^{\beta_1}u(b)\right)\right)\right.\\
&\left.\left.+\lambda_4 I^{\alpha_1}\phi\left(b,u(b),v(b),{}^H D^{\alpha_2}v(b)\right)\right)+\lambda_3\frac{\theta_3}{\gamma_3}\right]\\
&+{}^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\phi\left(t,u(t),v(t),{}^H D^{\alpha_2}v(t)\right)\right)
\end{aligned}$$

such that ϕ, $\omega : [a;b]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ are two given functions.
Lemma : If ϕ and ω are two bounded functions, then the above operator maps bounded sets into bounded sets.

In the third chapter, we are concerned with the study of the following problem of sequential type:

$$\begin{cases}
\left({}^cD^{\alpha} + k{}^cD^{\beta}\right)x(t) = f\left(t,x(t),D^{\beta}x(t)\right),\; 1<\alpha\leq 2,\; 0<\beta\leq 1,\; 0\leq t\leq T,\\
a_1x(0)+b_1x(T) = c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)g(\xi)d\xi,\\
a_2D^{\beta}x(0)+b_2D^{\beta}x(T) = c_2\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)h(\xi)d\xi.
\end{cases}$$

Taking into account that ${}^cD^{\alpha},{}^c D^{\beta}$ denote the Caputo derivatives, with, $1<\alpha\leq 2, 0<\beta\leq 1$ $,0<\mu,\rho\leq T$ and $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$, $f:[0;T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is continuous and k,γ are positive real constants.
We prove two existence results for this problem, the first result is based on the Banach

contraction principle, and the second one uses the fixed point theorem of Schaefer to proven the existence and uniqueness of solutions.

It's in the fourth chapter where we study the following problem:

$$\begin{cases} {}^{C}D^{\alpha_1 H}D^{\beta_1}x\left(t\right)=f\left(t,x\left(t\right),y\left(t\right),{}^{H}D^{\alpha_2}y\left(t\right)\right), \ a\leq t\leq b, \\ {}^{H}D^{\beta_2 C}D^{\alpha_2}y\left(t\right)=g\left(t,x\left(t\right),{}^{H}D^{\beta_1}x\left(t\right),y\left(t\right)\right), \ \ a\leq t\leq b, \\ \gamma_1 x\left(a\right)+\gamma_2^C D^{\alpha_2}y\left(a\right)=\theta_1, \ \lambda_1 x\left(b\right)+\lambda_2^C D^{\alpha_2}y\left(b\right)=\theta_2, \\ \gamma_3 y\left(a\right)+\gamma_4^H D^{\beta_1}x\left(a\right)=\theta_3, \ \lambda_3 y\left(b\right)+\lambda_4^H D^{\beta_1}x\left(b\right)=\theta_4, \end{cases}$$

where ${}^{C}D^{\alpha_i},{}^{H}D^{\beta_i}$ denote the Caputo and Hadamard fractional derivatives of orders α_i and β_i, respectively with, $0<\alpha_i,\beta_i\leq 1$, $i=\overline{1,2}$ and γ_i, λ_i,θ_i, $\left(i=\overline{1,4}\right)$ are real numbers such that γ_2,γ_2 are no zero numbers, $a,b\in\mathbb{R}$ with $a>0$, and $f,g:[a;b]\times\mathbb{R}^3\longrightarrow\mathbb{R}$ are two given functions.

After giving the general solution, we use the Banach contraction principle to prove our first main result.
The second main result concerns the existence and uniqueness of solutions for this problem. It will be established via Scaefer theorem.

This thesis is achieved by a conclusion and some perspectives.

Contents

Chapter 1

Basic Concepts on Fractional Calculus

In this chapter we recall some definitions, notions, properties and results on the different approaches of fractional derivation and some results which will be useful in the rest of this thesis. The first section includes a brief reminder on special functions. In the second section we present some approaches of integration of arbitrary order and its properties. We conclude the chapter with a section reserved for the different lemmas and fixed point theorems used in this work. These definitions, notations and properties can be found in references [29], [30], [35], [37], [40].

1.1 Notations and Basic Definitions

We recall the notations used in the following chapters, as well as some definitions and properties of fractional calculus.

The sets of real and complex numbers are denoted respectively $\mathbb{R}$ and $\mathbb{C}$. We note $\mathbb{N}$ the set of integers $\{1, 2, 3, ..., ...\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

1.1.1 Some Functional Spaces

We denote by Ω, the finite interval of the form $[a, b]$ $(0 \leq a < b < \infty)$.

Définition 1.1.1 *The space of continuous functions $f : \Omega \longrightarrow \mathbb{R}$ is noted $C(\Omega)$ and*

$$\|f\|_{C(\Omega)} = \sup_{x\in\Omega} |f(x)|$$

Définition 1.1.2 *Either $k \in \mathbb{Z}_+$, one notes $C^k(\Omega)$ the space of k-times continuously differentiable functions on Ω, and*

$$\|f\|_{C^k(\Omega)} = \sum_{\alpha=0}^{k} \left\| f^{(k)} \right\|_{C(\Omega)}.$$

Définition 1.1.3 *Soit $p \in [1, +\infty]$, we define $L^p(\Omega)$ the set of function classes $f : \Omega \longrightarrow \mathbb{C}$ measurable such as*

$$\begin{aligned} \|f\|_p &= \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}}, p < +\infty \\ \|f\|_\infty &= ess \sup_{x \in \Omega} |f(x)|. \end{aligned}$$

1.1.2 Absolutely Continuous Functions

Définition 1.1.4 *the space of absolutely continuous functions on Ω, denoted by $AC(\Omega)$ is consisting of functions that are primitive Lebesgue-summable i..e*

$$f \in AC(\Omega) \Leftrightarrow \exists \varphi \in L^1(\Omega) \text{ such that } f(x) = c + \int_a^x \varphi(t) dt.$$

Théorème 1.1 *The space $AC(\Omega)$ coincides with the space of the primitives of summable Lebesgue functions, i.e.*

$$f \in AC(\Omega) \Leftrightarrow f(x) = c + \int_a^x \varphi(t) dt, \quad \left(\varphi \in L^1(\Omega) \right)$$

Définition 1.1.5 *Let $n \in \mathbb{N}$. We note $AC^n(\Omega)$, the function space $f : \Omega \longrightarrow \mathbb{C}$ $(n-1)$ $-$times continuously differentiable on Ω such as $f^{(n-1)} \in AC(\Omega)$, i.e. :*

$$AC^n(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{C}, f^{(k)} \in C(\Omega), k \in \{0, 1, .., n-1\} \text{ et } f^{(n-1)} \in AC(\Omega) \right\}.$$

In particular $AC^1(\Omega) = AC(\Omega)$.

A characterization of the functions of this space is given by the following lemma:

Lemma 1.1 *Let $n \in \mathbb{N}$. Then $f \in AC^n(\Omega)$ if the function f is represent in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t) dt + \sum_0^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

1.2 Special Functions

In this section, we present the basics of the special functions used in the other chapters. We give here some properties on the gamma and beta functions, these functions play the most important role in the theory of differentiation and integration of arbitrary order and in the theory of fractional differential equations. For more details see for example the references [24], [39], [40].

Définition 1.2.1 *Let $z \in \mathbb{R}$ such as $z > 0$. The Gamma function, noted Γ is defined by*

$$\Gamma(z) := \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

with $\Gamma(1) = 1$.

Proposition 1.2.1 *The Gamma function is well defined on $\mathbb{R}^+$.*

Proof. We write $\Gamma(z)$ under the form

$$\Gamma(z) := \int_0^{+\infty} e^{-t} t^{z-1} dt = \int_0^1 e^{-t} t^{z-1} dt + \int_1^{+\infty} e^{-t} t^{z-1} dt = I_1 + I_2,$$

such as $I_1 = \int_0^1 e^{-t} t^{z-1} dt$ and $I_2 = \int_1^{+\infty} e^{-t} t^{z-1} dt$.

We have

$$I_1 = \int_0^1 e^{-t} t^{z-1} dt < \int_0^1 t^{z-1} dt = \frac{1}{z},$$

from where I_1 is convergent for $0 < z \leq 1$. Let us study the convergence of I_2. We have

$$\frac{t^{z-1}}{e^{-\frac{t}{2}}} \leq 1 \text{ because } \lim_{t \longrightarrow +\infty} \frac{t^{z-1}}{e^{-\frac{t}{2}}} = 0.$$

Then

$$I_2 = \int_1^{+\infty} e^{-t} t^{z-1} dt < \int_1^{+\infty} e^{-\frac{t}{2}} dt = 2e^{-\frac{1}{2}}.$$

Hence the Gamma function is defined for every $z > 0$.

Proposition 1.2.2 *Let $z \in \mathbb{R}$ such as $z > 0$, then the Gamma function check the following properties*

i) $\Gamma(z+1) = z\Gamma(z)$

ii) $\Gamma(n+1) = n!, \forall n \in \mathbb{Z}_+$.

Proof. i) Using integration by part, we have

$$\begin{aligned}\Gamma(z+1) &= \int_0^{+\infty} e^{-t}t^z dt = -e^{-t}t^z\Big|_0^{\infty} + \int_0^{+\infty} ze^{-t}t^{z-1}dt \\ &= z\int_0^{+\infty} e^{-t}t^{z-1}dt = z\Gamma(z).\end{aligned}$$

ii) Using the property i), we will have

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1!, \\ \Gamma(3) &= 2\Gamma(2) = 2\times 1! = 2!, \\ \Gamma(4) &= 3\Gamma(3) = 3\times 2! = 3!, \\ &\dots \\ &\dots \\ &\dots \\ \Gamma(n+1) &= n\Gamma(n) = n(n-1)! = n!.\end{aligned}$$

Proposition 1.2.3 *We have the following properties*

i) $\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt$

ii) $\dfrac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\dfrac{z}{n}\right)e^{-\frac{z}{n}}$, ($\gamma$ is the Euler-Mascheroni constant)

iii) $\Gamma(z) = \lim_{n\longrightarrow+\infty}\dfrac{n!\ n^z}{z(z+1)\dots(z+n)}$, $z \neq 0, -1, -2, \dots$

Proof. See [40].

Définition 1.2.2 *Let $\alpha, \beta \in \mathbb{R}$ such as $\alpha, \beta > 0$. The Beta function, noted $B(\alpha,\beta)$ is defined by*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Proposition 1.2.4 *Let $\alpha, \beta \in \mathbb{R}$ such as $\alpha, \beta > 0$. Then*

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.1}$$

Proof. we put $t=\dfrac{s}{s+1}$ and so we get $dt=\dfrac{ds}{(s+1)^2}$. Then

$$\begin{aligned}
B(\alpha,\beta) &= = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \\
&= \int_0^{+\infty}\left(\frac{s}{s+1}\right)^{\alpha-1}\left(1-\frac{s}{s+1}\right)^{\beta-1}\frac{ds}{(s+1)^2} \\
&= \int_0^{+\infty}\frac{s^{\alpha-1}}{(s+1)^{\alpha-1}}\left(\frac{1}{(s+1)^{\beta-1}}\right)\left(\frac{1}{(s+1)^2}\right)ds \\
&= \int_0^{+\infty}\frac{s^{\alpha-1}}{(s+1)^{\alpha+\beta}}ds.
\end{aligned}$$

From where,

$$\int_0^{+\infty} u^{\alpha-1}(u+1)^{-\alpha-\beta}\,du = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.2}$$

Remark 1.2.1 *The Beta function verifies the property of symmetry, i.e.*

$$B(\alpha,\beta) = B(\beta,\alpha).$$

Indeed, using the relation (1.1) *and change the order of* α *and* β.

1.3 Some Approaches of Integration of Arbitrary Order

There are several mathematical definitions of fractional order integration, in this section we will introduce the fractional integration operators that are more used in our work and their definitions and characteristics, for more details about these integrals please browse the references [19, 29, 31, 37]

1.3.1 Approache of Riemann-Liouville

Définition 1.3.1 *Let* $f\in L^1(\Omega)$. *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of* f *is given by*

$$\left({}^{RL}I_a^{\alpha}f\right)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\,ds,\ t>a. \tag{1.3}$$

Remark 1.3.1 *i) In the case* $\alpha = 0$, *The fractional integral* I^0 *is interpreted as an identity operator.*

ii) If $\alpha = n \in \mathbb{N}$, *the definition* (1.3) *coincide with the nth integrals of the form*

$$\begin{aligned} \left(I_a^n f\right)(t) &= \int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_n} f\left(t_n\right) dt_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt \end{aligned}$$

Exampel 1.3.1 *Let* $f(\xi) = (\xi - a)^{\lambda}, \xi \in (\Omega), \lambda > -1$, *then for* $\alpha > 0$, *we have*

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^{\xi} (\xi - \tau)^{\alpha - 1} (\xi - a)^{\lambda}\, d\tau. \tag{1.4}$$

We put $\xi = a + \rho(\xi - \tau)$, $0 \le \rho \le 1$. *Then tthe formula* (1.4), *is written in the form*

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{(\xi - a)^{\alpha+\lambda}}{\Gamma(\alpha)} \int_0^1 \rho^{\lambda} (1-\rho)^{\alpha-1}\, d\rho$$

Thanks to (1.1), *we get*

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} (\xi - a)^{\alpha+\lambda}.$$

If $\lambda = 0$, *then* $\left({}^{RL}I_a^{\alpha} 1\right)(\xi) = \dfrac{1}{\Gamma(\alpha)} (\xi - a)^{\alpha}$.

Proposition 1.3.1 *Let* $\alpha \in \mathbb{R}$ *such as* $\alpha > 0$, *then the operator* ${}^{RL}I_a^{\alpha}$ *is well defined.*

Proof. Let $f \in L^1(\Omega)$ and $\alpha \in \mathbb{R}$ $(\alpha > 0)$. According to Fubini's theorem, we have

$$\begin{aligned} \int_a^t \left|I_a^{\alpha} f(t)\right| dt &\le \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (t-s)^{\alpha-1} |f(s)|\, ds dt \\ &\le \frac{1}{\Gamma(\alpha)} \int_a^b |f(s)| \left(\int_s^b (t-s)^{\alpha-1} dt\right) ds \\ &\le \frac{1}{\alpha\Gamma(\alpha)} \int_a^b |f(s)| (b-s)^{\alpha}\, ds \\ &\le \frac{b^{\alpha}}{\Gamma(\alpha+1)} \int_a^b |f(s)|\, ds < \infty. \end{aligned}$$

Proposition 1.3.2 *Let $f \in L^1(\Omega)$. Then.*

$$^{RL}I_a^{\alpha}\left(I_a^{\beta} f(x)\right) =^{RL} I_a^{\alpha+\beta} f(t),\ \forall \alpha, \beta \in \mathbb{R}\ \ (\alpha, \beta > 0).$$

Proof. Using the Dirichlet formula, we have

$$\begin{aligned} ^{RL}I_a^{\alpha} I_a^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} du \frac{1}{\Gamma(\beta)} \int_a^u (u-t)^{\beta-1} f(t)\, dt \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(t)\, dt \int_t^x (x-u)^{\alpha-1} (u-t)^{\beta-1} du. \end{aligned}$$

We put $y = \dfrac{u-t}{x-t}$, so we have

$$\begin{aligned} ^{RL}I_a^{\alpha} I_a^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(t)\, dt\, (x-t)^{\alpha+\beta-1} \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy \\ &= \frac{B(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha+\beta-1} f(t)\, dt =^{RL} I_a^{\alpha+\beta} f(t). \end{aligned}$$

1.3.2 Approache of Hadamard

Définition 1.3.2 *The Hadamard fractional integral of order $\alpha > 0$, for a continuous function f on Ω is defined as:*

$$^{H}I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{\log t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau},\ \ \alpha > 0,\ \ a \leq t \leq b$$

In particulair if $\alpha = 0$, we have $^{H}I^{0} f(t) = f(t)$.

Proposition 1.3.3 *Let $f \in L^1(\Omega)$. Then $\forall \alpha, \beta \in \mathbb{R}\ (\alpha, \beta > 0)$, we have*

$$^{H}I_a^{\alpha}\left(^{H}I_a^{\beta} f(x)\right) =^{H} I_a^{\alpha+\beta} f(t),$$

$$^{H}I_a^{\alpha}\left(^{H}I_a^{\beta} f(x)\right) =^{H} I_a^{\beta}\left(^{H}I_a^{\alpha} f(x)\right)$$

1.4 Some approache of fractional derivation

There are several approximations of fractional derivation, in this section we will present the approaches of Riemann-Liouville and Caputo which are very useful in our work, for more detail, see [29], [31], [35]

1.4.1 Approache of Riemann-Liouville

Définition 1.4.1 *Let $f \in L^1(\Omega)$. The fractional derivative of Riemann-Liouville of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is given by*

$$\begin{aligned} \left({}^{RL}D_a^\alpha f\right)(t) \quad &: = \left(\frac{d}{dt}\right)^n \left(I_a^{n-\alpha} f\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \;\; n \in \mathbb{N}^*, n-1 < \alpha \leq n, t > a. \end{aligned} \tag{1.5}$$

Remark 1.4.1 *If $\alpha = n \in \mathbb{Z}_+$, Then*

$$\begin{aligned} \left({}^{RL}D_a^0 f\right)(t) &= f(t); \\ \left({}^{RL}D_a^\alpha f\right)(t) &= f^{(n)}(t), \end{aligned}$$

Where $f^{(n)}$ is the usual derivative of order n of the function f.

Exampel 1.4.1 *Let f the function defined by $f(t) = t^\lambda, \lambda > -1$ and $n-1 < \alpha \leq n, n \in \mathbb{N}^*$, so we have*

$$\begin{aligned} \left({}^{RL}D_0^\alpha f\right)(x) \quad &: = \left(\frac{d}{dx}\right)^n \left(\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{n-\alpha-1} s^\lambda ds\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^n \left(x^{n-\alpha+\lambda} \int_0^1 (1-u)^{n-\alpha-1} u^\lambda du\right) \\ &= \frac{1}{\Gamma(m-\alpha)} B(\lambda+1, n-\alpha) \left(\frac{d}{dx}\right)^n x^{n-\alpha+\lambda}. \end{aligned}$$

Knowing that, $\forall p \in \mathbb{R} \backslash \{-1, -2, -3, ...\}$, we have

$$\left(\frac{d}{dx}\right)^n x^p = p(p-1)\ldots(p-n+1)\, x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}.$$

From where

$$\begin{aligned} \left({}^{RL}D_0^\alpha f\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \times \frac{\Gamma(\lambda+1)\Gamma(n-\alpha)}{\Gamma(\lambda+1+n-\alpha)} \times \frac{\Gamma(n-\alpha+\lambda+1)}{\Gamma(n-\alpha+\lambda-m+1)} x^{\lambda-\alpha} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} x^{\lambda-\alpha} \end{aligned}$$

Remark 1.4.2 *As a special case, if $\lambda = 0$, then we have*

$$\begin{aligned} {}^{RL}D_0^{\alpha}1 &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \ \forall \alpha \in \mathbb{R}^{+} \setminus \{0,1,2,3,...\} \\ {}^{RL}D_0^{\alpha}1 &= 0, \forall \alpha \in \mathbb{Z}_{+}. \end{aligned}$$

Remark 1.4.3 *The fractional derivative of a constant function in the sense of Riemann-Liouville is not zero.*

Proposition 1.4.1 *Let $\alpha, \beta > 0$ such as $n-1 < \alpha \leq n$ and $m-1 < \beta \leq m, n, m \in \mathbb{N}^*$. If $\alpha > \beta > 0$,then for $f \in L^1(\Omega)$, we have*

$$\left({}^{RL}D^{\beta} I_a^{\alpha} f\right)(t) = I_a^{\alpha-\beta} f(t)$$

Proposition 1.4.2 *Let $\alpha > 0$ such as $n-1 < \alpha \leq n, n \in \mathbb{N}^*$. For $f \in L^1(\Omega)$, we have*

$$\left({}^{RL}D^{\alpha} I_a^{\alpha} f\right)(t) = f(t)$$

Proposition 1.4.3 *Let $n-1 < \alpha \leq n, n \in \mathbb{N}^*, m \in \mathbb{N}^*$ and $f \in L^1(\Omega)$. If the fractional derivatives $\left({}^{RL}D^{\alpha} f\right)(t)$ and $(D^{\alpha+m} f)(t)$ exist, then*

$$(D^m D^{\alpha} f)(t) = \left(D^{\alpha+m} f\right)(t)$$

1.4.2 Approache of Caputo

In this part we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties. for more detail, see [19], [29], [31]

Définition 1.4.2 *The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ of a function $f \in C^n(\Omega)$ is defined by*

$${}^{c}D_a^{\alpha} f(t) := {}^{RL} I_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \ n \in \mathbb{N}^*, n-1 < \alpha < n, t > a.$$

Remark 1.4.4

i) In particular, when $0 < \alpha < 1$ and $f \in C(\Omega)$, then

$${}^{c}D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s)\, ds = I_a^{1-\alpha} f'(t)$$

ii) If $\alpha \in \mathbb{N}$, then we have

$$^{c}D_{a}^{\alpha}f(t) = f^{(n)}(t).$$

Exampel 1.4.2 *Let $f(t) = C$ the constant function, then we have*

$$^{c}D^{\alpha}f(t) = 0 \text{ but } {}^{RL}D^{\alpha}f(t) \neq 0$$

Proposition 1.4.4 *let f and g be two functions such that $^{c}D^{\alpha}f(t),^{c}D^{\alpha}g(t)$ exist. Then the Caputo fractional derivation is a linear operator:*

$$^{c}D_{a}^{\alpha}(\lambda f + \gamma g)(t) = \lambda^{c}D_{a}^{\alpha}f(t) + \gamma^{c}D_{a}^{\alpha}g(t),\ \lambda, \gamma \in \mathbb{R}.$$

Proposition 1.4.5 *Let $n-1 < \alpha < n, n \in \mathbb{N}^{*}, m \in \mathbb{N}$ and let the function f such that $^{c}D^{\alpha}f(t)$ exists, then:*

$$^{c}D^{\alpha}D^{m}f(t) =^{c} D^{\alpha+m}f(t) \neq D^{mc}D^{\alpha}f(t).$$

The following theorem establishes the relation between the fractional derivative in the sense of caputo and that in the sense of Riemann-Liouville.

Théorème 1.2 *Let $\alpha > 0$ with $n-1 < \alpha < n, n \in \mathbb{N}^{*}$, and let f a function such as $^{c}D_{a}^{\alpha}f(t)$ et $^{RL}D_{a}^{\alpha}f(t)$ exist, then :*

$$^{c}D_{a}^{\alpha}f(t) =^{RL} D_{a}^{\alpha}f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}.$$

1.4.3 Approache of Caputo-Hadamard

In this section we are interested in the fractional derivative in the sense of Caputo-type Hadamard and its properties.

Définition 1.4.3 *The Caputo-type Hadamard fractional derivative of order $\alpha > 0$ or an at least $n-$times delta differentiable function f is defined as*

$$^{H}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \left(\frac{\log t}{\tau}\right)^{n-\alpha-1} \delta^{n}f(\tau)\frac{d\tau}{\tau}, \quad n-1 < \alpha < n,\ n = [\alpha]+1, t > a.$$

where, $\delta = t\dfrac{d}{dt}$ *and* $\log(.) = \log_{e}(.)$.

Proposition 1.4.6 *Let f and g be two fonctions whose fractional derivatives in the sense of Hadamard exist. Then* (1). *For all λ, $\mu \in \mathbb{R}$, ${}^{H}D_{a}^{\alpha}\left[\lambda f\left(t\right)+\mu g\left(t\right)\right]$ exist and*

$$ {}^{H}D_{a}^{\alpha}\left[\lambda f\left(t\right)+\mu g\left(t\right)\right]=\lambda {}^{H}D_{a}^{\alpha}\left[f\left(t\right)\right]+\mu {}^{H}D_{a}^{\alpha}\left[g\left(t\right)\right] \tag{1.6}$$

(2)

$$ {}^{H}D_{a}^{\alpha}\left[{}^{H}D_{a}^{\beta}\left[f\left(t\right)\right]\right]\neq {}^{H}D_{a}^{\alpha+\beta}\left[f\left(t\right)\right]. \tag{1.7}$$

(3)

$$ {}^{H}D_{a}^{\alpha}\left[{}^{H}D_{a}^{\beta}\left[f\left(t\right)\right]\right]\neq {}^{H}D_{a}^{\beta}\left[{}^{H}D_{a}^{\alpha}\left[f\left(t\right)\right]\right]. \tag{1.8}$$

1.5 Fundamental Lemmas

In this section, we provide some lemmas of fractional derivatives, witch will play major roles in our analysis, for more details, see [1, 6, 8, 18]

Lemma 1.2 *Let $\alpha>0$. Then the general solution of the differential equation ${}^{c}D_{a}^{\alpha}x\left(t\right)=0$ can be given by:*

$$x\left(t\right)=\sum_{i=0}^{n-1}c_{i}\left(t-a\right)^{i},$$

such that $c_{i}\in\mathbb{R}, i=0,1,2,..,n-1,\ n=\left[\alpha\right]+1$.

Lemma 1.3 *We consider an $\alpha>0$. Then, it yields that*

$$ {}^{RL}I^{\alpha}D^{\alpha}x\left(t\right)=x\left(t\right)+\sum_{i=0}^{n-1}c_{i}\left(t-a\right)^{i},$$

for $c_{i}\in\mathbb{R}, i=0,1,2,...,n-1,\ n=\left[\alpha\right]+1$.

Lemma 1.4 *Let $\alpha>0$ and $x\in AC_{\delta}^{n}\left[a,b\right]$. Then we have*

$$ {}^{H}I^{\alpha}\left({}^{H}D^{\alpha}\right)x\left(t\right)=x\left(t\right)-\sum_{i=0}^{n-1}c_{i}\left(\log\left(\frac{t}{a}\right)\right)^{i},$$

where, $c_{i}\in\mathbb{R}, i=1,2,...,n-1,\left(n=\left[\alpha\right]+1\right)$.

1.6 Fixed Point Theorems

In this section we present some fixed point theorems and definitions that are used in our work to prove the existence and uniqueness of solutions of the problems studied. For more details on these theorems you can see [21], [26], [37], [45].
In the following, X denotes a Banach space.

Définition 1.6.1 *An operator $\mathcal{T} : X \longrightarrow X$ is said to be contraction if there is a constant $k \in]0,1[$, such that, for any $x, y \in X$, we have*

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leq k \|x - y\|.$$

Définition 1.6.2 *The operator $\mathcal{T} : X \longrightarrow X$ is completely continuous if it transforms all bounded of X in a relatively compact part.*

Définition 1.6.3 *Let $\mathcal{T} : X \longrightarrow X$. We call a fixed point of $\mathcal{T}$ any point $x \in X$ such that*

$$\mathcal{T}(x) = x.$$

Théorème 1.3 (*Banach Fixed Point Theorem*) *Let $\mathcal{T} : X \longrightarrow X$ a contraction operator. Then, T has a unique fixed point in X.*

Théorème 1.4 (*Arzelà-Ascoli Theorem*) *Let Ω be a set of X. Then Ω is relatively compact in X if and only if the following conditions are valid :*

i) Ω is uniformly bounded.
ii) Ω is equicontinuous.

Théorème 1.5 (*Schaefer's Fixed Point Theorem*) *Let $\mathcal{T} : X \longrightarrow X$ a completely continuous operator. If the set*

$$\Omega = \{x \in X : x = \lambda x, 0 < \lambda < 1\}.$$

is bounded, then T has at least one fixed point.

Chapter 2

Some Fractional Integral Estimates

2.1 Motivation

This chapter deals with tow important lemmas that are needed to elaborate the chapter three. We begin by proving a first main result in chapter three. Then we prove another auxiliary lemma involving the Hadamard operator.

2.2 Estimations Using Riemann-Liouville Operator

In this section, we present some estimates related to fractional integral of Riemann-Liouville type which have an important role in the chapter three.
Before starting to give the results, we need some quantities and definitions : Let χ, ψ are tow functions, $1 < \alpha \leq 2, 0 < \beta \leq 1, \gamma > 0, 0 < \mu, \rho \leq T$. We put

$$\begin{aligned}
\Delta \quad &: \quad = \int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} \chi(\xi) \frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta)} d\xi - \frac{b_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+ \quad c_1 \int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} \chi(\xi) d\xi - (a_1+b_1) \\
\Lambda \quad &: \quad = c_2 \int_0^{\rho} \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} \psi(\xi) \frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} d\xi - b_2 \frac{T^{\alpha-2\beta}}{\Gamma(\alpha-\beta+1)}
\end{aligned}$$

We define the space

$$\Pi := \left\{ u, u \in C[0;T], \ {}^{c}D^{\beta} u(t) \in C[0;T] \right\}$$

For computational convenience, with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $N_\chi := \sup_{0\leq t\leq T} |\chi(.)|$, $N_\psi := \sup_{0\leq t\leq T} |\psi(.)|$, we set

$$\begin{aligned}
\mu_1 &= \Bigg(\frac{T^\alpha}{\Gamma(\beta)\Gamma(\alpha-\beta)} + \frac{\rho^{\alpha+\gamma} N_\psi}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)\left|I^\gamma \psi(\rho)\right|} + \frac{|b_2|\Gamma(\alpha-2\beta) T^{\alpha-\beta}}{\left|I^\gamma\psi(\rho)\right| |c_2|} \\
&\quad + \frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\left|I^\gamma\psi(\rho)\right||c_2|\Gamma(\beta)} + \frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_\chi}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|} \\
&\quad + \frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)} \\
&\quad + \frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\Gamma(\alpha-\beta)} \Bigg) \\
\mu_2 &= \Bigg(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)} + \frac{|\Lambda_2||c_1|\rho^{\alpha-\beta+\gamma}N_\psi}{|\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda_2||b_1|T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2|\Delta|} \\
&\quad + \frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1|\right)|c_2|\rho^{\alpha-\beta+\gamma}N_\psi T^{\alpha-\beta}}{\Gamma(\gamma)\Gamma(\alpha-\beta)|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{\left|I^\gamma\psi(\rho)\right|(\Gamma(\alpha-\beta))^2|\Delta|} \\
&\quad + \frac{|c_2|\rho^{\alpha-\beta+\gamma}N_\psi T^{\alpha-\beta}}{|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)\Gamma(\gamma)\Gamma(\alpha-\beta)} \\
&\quad + \frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}T^{\alpha-\beta}}{|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)} \Bigg), \\
\mu_3 &= \Bigg(\frac{T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-2\beta)} + \frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_\chi}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|} \\
&\quad + \frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1|\right)|c_2|\rho^{\alpha+\gamma}N_\psi T^{\alpha-\beta}}{|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} \\
&\quad + \frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{2(\alpha-\beta)}}{|\Delta|\left|I^\gamma\psi(\rho)\right|\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta)} \Bigg)
\end{aligned}$$

$$\mu_4 = \left(+ \frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)} + \frac{|\Lambda| |c_1| \rho^{\alpha-\beta+\gamma} N_\psi}{|\Delta| \Gamma(\gamma) \Gamma(\alpha-\beta)} + \frac{|\Lambda| |b_1| T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2 |\Delta|} \right.$$
$$+ \frac{\left(\frac{|c_1| \rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1| \right) |c_2| \rho^{\alpha-\beta+\gamma} N_\psi T^{\alpha-\beta}}{\Gamma(\gamma) \Gamma(\alpha-\beta) |\Delta| |I^\gamma \psi(\rho)| \Gamma(\alpha-\beta+1)}$$
$$\left. + \frac{\left(\frac{|c_1| \rho^\gamma N_\psi}{\Gamma(\gamma)} + |a_1+b_1| \right) T^{\alpha-\beta} |b_2| \Gamma(\alpha-2\beta) T^{\alpha-2\beta}}{|\Delta| |I^\gamma \psi(\rho)| \Gamma(\alpha-\beta+1)} \right)$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $N_\chi := \sup\limits_{0 \le t \le T} |\chi(.)|, N_\psi := \sup\limits_{0 \le t \le T} |\psi(.)|$

Lemma 2.1 *Let φ be a function and let $\Psi : [0;T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a bounded function. Then operator defined on $C([0;T], \mathbb{R})$ by*

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) = \left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-u)^{\alpha-\beta-1} \right. \\
&\times \int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) du \\
&+ \frac{1}{|I^\gamma \psi(\rho)|} \left\{ \int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} \chi(\xi) \left(\frac{1}{\Gamma(\alpha-\beta)} \int_0^\xi (\xi-u)^{\alpha-\beta-1} \right. \right. \\
&\left. \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) \right) du \right) d\xi \\
&+ \frac{|b_2| \Gamma(\alpha-2\beta)}{|c_2|} \int_0^T (T-u)^{\alpha-\beta-1} \\
&\times \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) \right) du \\
&+ \Lambda \left[\frac{|c_1|}{|\Delta|} \int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} \chi(\xi) \frac{1}{\Gamma(\alpha-\beta)} \int_0^\xi (\xi-u)^{\alpha-\beta-1} \right. \\
&\times \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Psi\left(s, \varphi(s), D^\beta \varphi(s)\right) ds + k\varphi(u) \right) du \Bigg) d\xi
\end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&+\frac{|b_1|}{|\Delta|\,\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1} \\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^\beta\varphi(s)\right)ds+k\varphi(u)\right)du\Bigg] \\
&+\frac{1}{\Delta I^\gamma h(\rho)}\Bigg[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\psi(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right. \\
&\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^\beta\varphi(s)\right)ds+k\varphi(u)\right)du\right)d\xi \\
&+|b_2|\,\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1} \\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left(s,\varphi(s),D^\beta\varphi(s)\right)ds+k\varphi(u)\right)du\Bigg] \\
&\times\left(|c_1|\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\,d\xi+(a_1+b_1)\right)\Bigg\}\Bigg)
\end{aligned}
$$

maps bounded sets into bounded sets in $C([0;T],\mathbb{R})$.

Proof. Now we consider the ball B_r defined by $B_r=\{\varphi\in\Pi:\|\varphi\|_\Pi\le r\}$ in Π. We have

$$
\begin{aligned}
&|(\mathcal{A}\varphi)(t)|\le\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-u)^{\alpha-\beta-1}\right. \\
&\times\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\left(\left|\Psi\left(s,\varphi(s),D^\beta\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du \\
&+\frac{1}{|I^\gamma\psi(\rho)|}\left\{\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Psi\left|\left(s,\varphi(s),D^\beta\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du\right)d\xi \\
&+\frac{|b_2|\,\Gamma(\alpha-2\beta)}{|c_2|}\int_0^T (T-u)^{\alpha-\beta-1} \\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|\Psi\left(s,\varphi(s),D^\beta\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\\
&\times\left(\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du\right)d\xi\\
&+\frac{|b_1|}{|\Delta|\Gamma(\alpha-\beta)}\int_0^T(T-u)^{\alpha-\beta-1}\\
&\left.\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du\right]\\
&+\frac{1}{\Delta I^{\gamma}\psi(\rho)}\left[|c_2|\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\psi(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right.\\
&\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du\right)d\xi\\
&+|b_2|\Gamma(\alpha-2\beta)\int_0^T(T-u)^{\alpha-2\beta-1}\\
&\left.\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|\Psi\left(s,\varphi(s),D^{\beta}\varphi(s)\right)\right|ds+k|\varphi(u)|\right)du\right]\\
&\times\left.\left.\left(|c_1|\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)d\xi+|a_1+b_1|\right)\right\}\right).
\end{aligned}
$$

Using the fact that functions Ψ is bounded, then there exists $M>0$, such that $|\Psi(.,.,.)|\leq M$. Then from (2.2), we get

$$
\begin{aligned}
&|(\mathcal{A}\varphi)(t)|\leq\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-u)^{\alpha-\beta-1}\right.\\
&\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}Mds+krdu\right)\\
&+\frac{1}{|I^{\gamma}\psi(\rho)|}\left\{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right.\\
&\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du\right)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|b_2|\,\Gamma(\alpha-2\beta)}{|c_2|}\int_0^T (T-u)^{\alpha-\beta-1} \\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du \\
&+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right. \\
&\times\left(\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du\right)d\xi \\
&+\frac{|b_1|}{|\Delta|\,\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1} \\
&\left.\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du\right] \\
&+\frac{1}{\Delta I^\gamma\psi(\rho)}\left[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du\right)d\xi \\
&+|b_2|\,\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1} \\
&\left.\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}Mds+kr\right)du\right] \\
&\left.\left.\times\left(|c_1|\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\,d\xi+|(a_1+b_1)|\right)\right\}\right).
\end{aligned}
$$

Hence, we have the following inequality

$$
\begin{aligned}
|(\mathcal{A}\varphi)(t)| \;\leq\; & M\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-u)^{\alpha-\beta-1}\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}dsdu\right. \\
&+\frac{1}{|I^\gamma\psi(\rho)|}\left\{\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdud\xi\right. \\
&+\frac{|b_2|\,\Gamma(\alpha-2\beta)}{|c_2|}\int_0^T (T-u)^{\alpha-\beta-1}\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdu
\end{aligned}
$$

$$
+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\int_0^{u}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdud\xi\right.
$$

$$
\left.\frac{|b_1|}{|\Delta|\Gamma(\alpha-\beta)}\int_0^{T}(T-u)^{\alpha-\beta-1}\int_0^{u}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdu\right]
$$

$$
+\left[\frac{|c_2|}{\Delta I^{\gamma}\psi(\rho)}\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\int_0^{u}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdud\xi\right.
$$

$$
\left.\left.\left.|b_2|\Gamma(\alpha-2\beta)\int_0^{T}(T-u)^{\alpha-2\beta-1}\int_0^{u}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}dsdu\right]\right\}\right)
$$

$$
\times\left(|c_1|\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)d\xi+|(a_1+b_1)|\right)
$$

$$
+kr\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t}(t-u)^{\alpha-\beta-1}du+\frac{1}{|I^{\gamma}\psi(\rho)|}\left\{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\right.\right.
$$

$$
\times\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}dud\xi+\frac{|b_2|\Gamma(\alpha-2\beta)}{|c_2|}\int_0^{T}(T-u)^{\alpha-\beta-1}du
$$

$$
+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}dud\xi\right.
$$

$$
\left.+\frac{|b_1|}{|\Delta|\Gamma(\alpha-\beta)}\int_0^{T}(T-u)^{\alpha-\beta-1}du\right]+\frac{|c_2|}{\Delta I^{\gamma}\psi(\rho)}\left[\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\right.
$$

$$
\left.\left.\left.\times\ \frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}dud\xi+|b_2|\Gamma(\alpha-2\beta)\int_0^{T}(T-u)^{\alpha-2\beta-1}du\right)\right]\right\}\right)
$$

$$
\times\left(|c_1|\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}\chi(\xi)d\xi+|a_1+b_1|\right)
$$

By computation, we find that $|\mathcal{A}\varphi(t)|$ can be estimate as follows :

$$
\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\leq M\left(\frac{T^{\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}+\frac{\rho^{\alpha+\gamma}N_{\psi}}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)|I^{\gamma}\psi(\rho)|}+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}\psi(\rho)||c_2|}\right. \\
&\quad+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}\psi(\rho)||c_2|\Gamma(\beta)}+\frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_{\chi}}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|}
\end{aligned}
$$

$$+\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta|\,|I^\gamma h(\rho)|\,\Gamma(\alpha-\beta+1)}+\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\Bigg)$$

$$+kr\left(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)}+\frac{|\Lambda_2|\,|c_1|\,\rho^{\alpha-\beta+\gamma}N_\psi}{|\Delta|\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}+\frac{|\Lambda_2|\,|b_1|\,T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2\,|\Delta|}\right.$$

$$+\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\,\rho^{\alpha-\beta+\gamma}N_\psi T^{\alpha-\beta}}{\Gamma(\gamma)\,\Gamma(\alpha-\beta)\,|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)}+\frac{|\Lambda|\,|b_1|\,T^{\alpha-\beta}}{|I^\gamma\psi(\rho)|\,(\Gamma(\alpha-\beta))^2\,|\Delta|}$$

$$\left.+\frac{|c_2|\,\rho^{\alpha-\beta+\gamma}N_\psi T^{\alpha-\beta}}{|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}+\frac{|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-2\beta}T^{\alpha-\beta}}{|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)}\right)$$

Hence, we obtain

$$|(\mathcal{T}x)(t)|\leq M\mu_1+kr\mu_2 \tag{2.3}$$

On the other hand, by application D^β to (2.1), we side

$$\begin{aligned}\left|D^\beta\mathcal{A}(x)(t)\right| &\leq M\left(\frac{T^{\alpha-\beta}}{\Gamma(\beta)\,\Gamma(\alpha-2\beta)}+\frac{|\Lambda|\,|c_1|\,\mu^{\alpha+\gamma}N_\chi}{|\Delta|\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}\right.\\ &+\frac{|\Lambda|\,|b_1|\,T^{\alpha-\beta}}{\Gamma(\beta)\,\Gamma(\alpha-\beta)\,|\Delta|}+\frac{\left(\frac{|c_1|\rho^\gamma N\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\,\rho^{\alpha+\gamma}N_\psi T^{\alpha-\beta}}{|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}\\ &\left.+\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\,\Gamma(\alpha-2\beta)\,T^{2(\alpha-\beta)}}{|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\beta)}\right)\end{aligned}$$

$$\begin{aligned}&+kr\left(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)}+\frac{|\Lambda|\,|c_1|\,\rho^{\alpha-\beta+\gamma}N_\psi}{|\Delta|\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}+\frac{|\Lambda_2|\,|b_1|\,T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2\,|\Delta|}\right.\\ &+\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\,\rho^{\alpha-\beta+\gamma}N_\psi T^{\alpha-\beta}}{\Gamma(\gamma)\,\Gamma(\alpha-\beta)\,|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)}+\\ &\left.\frac{\left(\frac{|c_1|\rho^\gamma N_\psi}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}\,|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-2\beta}}{|\Delta|\,|I^\gamma\psi(\rho)|\,\Gamma(\alpha-\beta+1)}\right)\end{aligned}$$

As a result of the above inequality, we find

$$\left|D^{\beta}\mathcal{A}\left(x\right)\left(t\right)\right| \leq M\mu_3 + kr\mu_4 \tag{2.4}$$

Based on the two results, (2.3) and (2.4), we deduce that

$$\left\|\mathcal{T}\left(x\right)\right\| < \infty,\ \forall t \in \left[0,T\right].$$

Hence, the operator $\mathcal{T}$ maps bounded sets into bounded sets in Π.

2.3 Other Estimations Using Hadamard Operator

To prove our result, we need some notations and quantities :

$$\begin{aligned}
\Lambda_1 &: = \lambda_1 \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}, \quad \Lambda_2 := \lambda_1 - \lambda_2 \frac{\gamma_1}{\gamma_2}, \\
\Lambda_3 &: = \lambda_4 - \lambda_3 \frac{\gamma_4}{\gamma_3}, \quad \Lambda_4 := \lambda_3 \frac{\left(b-a\right)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)} \left(\frac{\theta_1}{\gamma_2} - \frac{\gamma_1}{\gamma_2}\right), \\
\Sigma &: = \Lambda_4\Lambda_1 - \Lambda_3\Lambda_2.
\end{aligned}$$

and

$$\begin{aligned}
Q_1 &: = \frac{\left|\lambda_1\right| l_1}{\left|\Sigma\right|} \left(\left|\Lambda_3\right| + \left|\Lambda_4\right| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}\left(1\right)\right)\left(b\right) \\
&+ \frac{\left|\lambda_4\right| l_1}{\left|\Sigma\right|} \left(\left|\Lambda_1\right| + \left|\Lambda_2\right| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)^{RL} I^{\alpha_1}\left(1\right)\left(b\right) + l_1^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}\left(1\right)\right)\left(b\right), \\
Q_2 &: = \frac{\left|\lambda_2\right| l_2}{\left|\Sigma\right|} \left(\left|\Lambda_3\right| + \left|\Lambda_4\right| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)^{H} I^{\beta_2}\left(1\right)\left(b\right) \\
&+ \frac{\left|\lambda_3\right| l_2}{\left|\Sigma\right|} \left(\left|\Lambda_1\right| + \left|\Lambda_2\right| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)^{RL} I^{\alpha_2}\left({}^{H}I^{\beta_2}\left(1\right)\right)\left(b\right),
\end{aligned}$$

$$
\begin{aligned}
Q_3 \ : \ &= \frac{|\lambda_1| \, l_1}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^H I^{\beta_1} \left({}^{RL} I^{\alpha_1} (1) \right) (b) \\
&+ \frac{|\lambda_4| \, l_1}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^{RL} I^{\alpha_1} (1) (b), \\
Q_4 \ : \ &= \frac{|\lambda_2| \, l_2}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^H I^{\beta_2} (1) (b) \\
&+ \frac{|\lambda_3 l_2|}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^{RL} I^{\alpha_2} \left({}^H I^{\beta_2} (1) \right) (b) + l_2^{RL} I^{\alpha_2} \left({}^H I^{\beta_2} (1) \right) (b), \\
M \ : \ &= \frac{\left(\log \dfrac{b}{a} \right)^{1-\beta_1}}{\Gamma(1-\beta_1)},
\end{aligned}
$$

where $0 < \alpha_i, \beta_i \leq 1$, $i = \overline{1,2}$ and γ_i, λ_i, θ_i, $\left(i = \overline{1,4}\right)$ are real numbers such that γ_2, γ_2 are no zero numbers, $a, b \in \mathbb{R}$ with $a > 0$. We denote by E_1, E_2 the Banach spaces defined as follows

$$
\begin{aligned}
X \ : \ &= \left\{ u \in C([a,b], \mathbb{R}), {}^H D^{\beta_1} u(t) \in C([a,b], \mathbb{R}) \right\}, \\
Y \ : \ &= \left\{ v \in C([a,b], \mathbb{R}), {}^H D^{\alpha_2} v(t) \in C([a,b], \mathbb{R}) \right\}.
\end{aligned}
$$

The two spaces can be equipped by the norms

$$
\|u\|_X := \max\left(\|x\|, \left\| {}^H D^{\beta_1} x \right\| \right), \|x\| = \sup_{a \leq t \leq b} |x(t)|, \left\| {}^H D^{\beta_1} x \right\| = \sup_{a \leq t \leq b} \left| {}^H D^{\beta_1} x(t) \right|.
$$

We consider the operator

$$
\begin{aligned}
\mathcal{F} : \quad X \times Y &\longrightarrow X \\
(u, v)(t) &\longmapsto \mathcal{T}(u, v)(t)
\end{aligned}
$$

defined by

$$
\begin{aligned}
\mathcal{F}(u, v)(t) \ : \ &= \frac{1}{\Sigma} \Bigg[-\Lambda_3 \theta_2 + \left(\Lambda_3 - \Lambda_4 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} \right) \\
&\times \left(\lambda_1^H I^{\beta_1} \left({}^{RL} I^{\alpha_1} \phi \left(b, u(b), v(b), {}^H D^{\alpha_2} v(b) \right) \right) \right.
\end{aligned} \tag{2.5}
$$

$$+ \lambda_2^H I^{\beta_2} \omega \left(b, u(b), v(b), ^H D^{\beta_1} u(b)\right)\right) + \lambda_2 \frac{\theta_1}{\gamma_2}\Bigg]$$

$$+\frac{1}{\Sigma}\left[\left(\Lambda_1+\Lambda_4\right)\theta_4 - \left(\Lambda_1+\Lambda_2 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right.$$

$$\times \left(\lambda_3^{RL} I^{\alpha_2}\left(^H I^{\beta_2} \omega \left(b, u(b), v(b), ^H D^{\beta_1} u(b)\right)\right)\right.$$

$$\left. + \lambda_4 I^{\alpha_1} \phi \left(b, u(b), v(b), ^H D^{\alpha_2} v(b)\right)\right) + \lambda_3 \frac{\theta_3}{\gamma_3}\Bigg]$$

$$+^H I^{\beta_1}\left(^{RL} I^{\alpha_1} \phi \left(t, u(t), v(t), ^H D^{\alpha_2} v(t)\right)\right),$$

where $f, g : [a; b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ are two given functions.

Lemma 2.2 *If ϕ and ω are two bounded functions, then the operator* (2.5) *maps bounded sets into bounded sets in X.*

Proof. Let Ω bounded in $X \times Y$. For each $t \in [a, b]$ and $(u, v) \in \Omega$, we have

$$\left|F(u,v)(t)\right| \leq \frac{1}{|\Sigma|}\left[\left|\Lambda_3 \theta_2\right| + \left(\left|\Lambda_3\right| + \left|\Lambda_4\right| \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right. \tag{2.6}$$

$$\times \left(\lambda_1^H I^{\beta_1}\left(^{RL} I^{\alpha_1}\left|\phi\left(b, u(b), v(b), ^H D^{\alpha_2} v(b)\right)\right|\right)\right.$$

$$\left. + \left|\lambda_2\right|^H I^{\beta_2}\left|\omega\left(b, u(b), v(b), ^H D^{\beta_1} u(b)\right)\right|\right) + \left|\lambda_2\right| \left|\frac{\theta_1}{\gamma_2}\right|\Bigg]$$

$$+\frac{1}{|\Sigma|}\left[\left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left(\left|\Lambda_1\right| + \left|\Lambda_2\right| \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right.$$

$$\times \left(\left|\lambda_3\right|^{RL} I^{\alpha_2}\left(^H I^{\beta_2}\left|\omega\left(b, u(b), v(b), ^H D^{\beta_1} u(b)\right)\right|\right)\right.$$

$$\left. + \left|\lambda_4\right| I^{\alpha_1}\left|\phi\left(b, u(b), v(b), ^H D^{\alpha_2} v(b)\right)\right|\right) + \left|\lambda_3\right| \left|\frac{\theta_3}{\gamma_3}\right|\Bigg]$$

$$+^H I^{\beta_1}\left(^{RL} I^{\alpha_1}\left|\phi\left(t, u(t), v(t), ^H D^{\alpha_2} v(t)\right)\right|\right).$$

Thanks to the conditions of ϕ and ω , there exist two constants N_1, N_2 such that

$$|\phi(.,.,.)| \leq N_1, \text{ and } |\omega(.,.,.)| \leq N_2 \tag{2.7}$$

Using(2.7) in (2.6), we obtain

$$\begin{aligned} |\digamma(u,v)(t)| &\leq \frac{1}{|\Sigma|}\left[|\Lambda_3\theta_2| + \left(|\Lambda_3| + |\Lambda_4|\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\left(N_1|\lambda_1| I^{\beta_1} I^{\alpha_1}(b)\right.\right. \qquad (2.8) \\ &\quad \left.\left. + N_2|\lambda_2|^H I^{\beta_2}(b)\right) + |\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|\right] \\ &\quad + \frac{1}{|\Sigma|}\left[(|\Lambda_1| + |\Lambda_4|)|\theta_4| + \left(|\Lambda_1| + |\Lambda_2|\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right. \\ &\quad \left.\times\left(N_2|\lambda_3| I^{\alpha_2 H} I^{\beta_2}(b) + |\lambda_4| N_1 I^{\alpha_1}(b)\right) + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right] + N_1^H I^{\beta_1} I^{\alpha_1}(b) \\ &\leq N_1 Q_1 + N_2 Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right) \end{aligned}$$

On the other hand, we can see that

$$\begin{aligned} \left|{}^H D^{\beta_1}\digamma(u,v)(t)\right| &\leq \frac{1}{\Gamma(1-\alpha_2)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha_2}|\digamma(u,v)(t)|\frac{ds}{s} \qquad (2.9) \\ &\leq M\left(K_1 Q_1 + K_2 Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\right) \end{aligned}$$

So, (2.8) and (2.9) yield to

$$\|\digamma\|_{E_1} < \infty.$$

Then $\digamma$ maps bounded sets into bounded sets in X.

Chapter 3

A Class of Fractional Differential Equations With Sequential Derivatives

3.1 Introduction

In recent years, the theory of fractional derivatives has attracted the attention of many authors, especially the fractional differential equations, where several studies and research have been prepared related to the existence and uniqueness of the solution on Banach spaces, This is because of the importance of these equations in solving some important problems in numerous areas, such as physics, biomathematics, control theory, etc. For more information, we refer the reader to [31], [38], [24].We shall note also that the existence and uniqueness studies of the above new theory have been published in many research papers, we cite [3], [4], [6], [41]. Related to these studies, we need to cite some papers that have motivated the present research work, and we can begin by [7] where their authors have studied the existence of solution of the problem :

$$\begin{cases} \left({}^cD^{\alpha}+k{}^cD^{\alpha-1}\right)u\left(t\right)=f\left(t,u\left(t\right)\right),\ 1<\alpha\leq 2,\ 0<t<T,\ T>0, \\ \\ \alpha_1 u\left(0\right)+\rho_1 u\left(T\right)=\beta_1\ ,\ \alpha_2 u'\left(0\right)+\rho_2 u'\left(T\right)=\beta_2, \\ \\ \alpha_1 u\left(0\right)+\rho_1 u\left(T\right)=\lambda_1\int_0^{\eta}u(s)ds+\lambda_2, \\ \\ \alpha_2 u'\left(0\right)+\rho_2 u'\left(T\right)=\mu_1\int_{\xi}^{T}u(s)ds+\mu_2, \end{cases}$$

taking into account that ${}^cD^{\alpha}_{0^+}$ denotes the Caputo derivative of order α, $k\in\mathbb{R}^+, 0<\eta<\xi<T$ and $\alpha_1,\alpha_2,\beta_1,\beta_2,\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$. In an interesting paper, Mahmudov et al. [33] have studied the existence of solutions for:

$$\begin{cases} \left({}^cD^{\alpha}+k{}^cD^{\alpha-1}\right)u\left(t\right)=f\left(t,u\left(t\right),D^{\alpha-1}u\left(t\right)\right),\ 1<\alpha\leq 2,\ 0\leq t\leq T, \\ \\ \alpha_1 u\left(\eta\right)+\beta_1 u\left(T\right)=\gamma_1\int_0^{\xi}u(s)ds+\epsilon_1, \\ \\ \alpha_2 D^{\alpha-1}u\left(\eta\right)+\beta_2 D^{\alpha-1}u\left(T\right)=\gamma_2\int_{\zeta}^{T}u(s)ds+\epsilon_2, \end{cases}$$

under the conditions that ${}^cD^{\alpha}$ is Caputo derivative of order $\alpha, 0\leq\eta\leq T, 0<\xi<\zeta<T$ and $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\epsilon_1,\epsilon_2\in\mathbb{R}$.
In this chapter we are interested with the study the existence and uniqueness of solutions of the following problem of sequential type with initial conditions:

$$\begin{cases} \left({}^cD^{\alpha}+k{}^cD^{\beta}\right)x\left(t\right)=f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right),\ 1<\alpha\leq 2,\ 0<\beta\leq 1,\ 0\leq t\leq T, \\ a_1 x\left(0\right)+b_1 x\left(T\right)=c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)g(\xi)d\xi, \\ a_2 D^{\beta}x\left(0\right)+b_2 D^{\beta}x\left(T\right)=c_2\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)h(\xi)d\xi, \end{cases} \tag{3.1}$$

by taking into account that ${}^cD^{\alpha},{}^cD^{\beta}$ denote the Caputo derivatives, with, $1<\alpha\leq 2$, $0<\beta\leq 1$, $0<\mu,\rho\leq T$ and $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$, $f:[0;T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ and $g,h:[0,T]\longrightarrow\mathbb{R}$ are continuous and k,γ are positive real constants. This chapter is structured as follows :

In the first section, we study the existence of solutions for the linear system of fractional differential equations. In the second section we present the first result using the Banach contraction principle, then we establish the second existence result using Schauder's fixed point theorem. At the end of this chapter we provide some examples.

3.2 A Sequential System With Integral Representation

Let us prove the result:

Lemma 3.1 *We take* $\varphi \in C([0,T],\mathbb{R})$. *The solution of*

$$\begin{cases} \left({}^{c}D^{\alpha}+k{}^{c}D^{\beta}\right)x(t)=\varphi(t),\ 1<\alpha\leq 2,\ 0<\beta\leq 1,\ 0\leq t\leq T, \\ a_1x(0)+b_1x(T)=c_1\int_0^{\mu}\dfrac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)g(\xi)d\xi, \\ a_2D^{\beta}x(0)+b_2D^{\beta}x(T)=c_2\int_0^{\rho}\dfrac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}x(\xi)h(\xi)d\xi \end{cases}$$

is:

$$\begin{aligned} x(t) &:= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-u)^{\alpha-\beta-1}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du \\ &-\frac{1}{I^{\gamma}h(\rho)}\left\{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right. \\ &\times\left.\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi\right. \\ &-\frac{b_2\Gamma(\alpha-2\beta)}{c_2}\int_0^T(T-u)^{\alpha-\beta-1}\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du \\ &-\Lambda\left[\frac{c_1}{\Delta}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right. \\ &\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\bigg)d\xi \end{aligned}$$

$$
\begin{aligned}
&- \frac{b_1}{\Delta\Gamma(\alpha-\beta)} \int_0^T (T-u)^{\alpha-\beta-1} \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)ds - kx(u) \right) du \\
&- \frac{1}{\Delta I^{\gamma} h(\rho)} \left[c_2 \int_0^{\rho} \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} \left(h(\xi) \frac{1}{\Gamma(\alpha-\beta)} \int_0^{\xi} (\xi-u)^{\alpha-\beta-1} \right.\right. \\
&\times \left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)ds - kx(u) \right) du \right) d\xi \\
&-b_2\Gamma(\alpha-2\beta) \int_0^T (T-u)^{\alpha-2\beta-1} \left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)ds - kx(u) \right) du \right] \\
&\times \left(c_1 \int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\, d\xi - (a_1+b_1) \right) \Bigg\} \times \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)},
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta \ : \ &= \int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi) \frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta)} d\xi - \frac{b_1 T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&+c_1 \int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\, d\xi - (a_1+b_1) \\
\Lambda \ : \ &= c_2 \int_0^{\rho} \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} h(\xi) \frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} d\xi - b_2 \frac{T^{\alpha-2\beta}}{\Gamma(\alpha-\beta+1)}
\end{aligned}
$$

Proof. We see that

$$
\left({}^cD^{\alpha} + k\,{}^cD^{\beta}\right) x(t) = \varphi(t) \tag{3.2}
$$

We see also that (4.7) can be written as follows:

$$
{}^cD^{\beta}\left({}^cD^{\alpha-\beta} x(t) + kx(t)\right) = \varphi(t). \tag{3.3}
$$

The solution of (3.3) can be written as

$$
x(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-u)^{\alpha-\beta-1} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)ds - kx(u) \right) du - A_0 - \frac{A_1 t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. \tag{3.4}
$$

Therefore,

$$
D^{\beta}x(t) = \frac{1}{\Gamma(\alpha-2\beta)} \int_0^t (t-u)^{\alpha-2\beta-1} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)ds - kx(u) \right) du - \frac{A_1 t^{\alpha-2\beta}}{\Gamma(\alpha-2\beta+1)}
$$

Thanks to the second condition of (3.1) , we have

$$
\begin{aligned}
a_2D^{\beta}x(0)+b_2D^{\beta}x(T) &= \frac{b_2}{\Gamma(\alpha-2\beta)}\int_0^T (T-u)^{\alpha-2\beta-1} \qquad (3.5)\\
&\times\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)dsdu - kx(u)\right) - \frac{A_1b_2\ T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta+1)}\\
&= c_2\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\\
&\left.\times\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds\ -kx(u)\right)du\right)d\xi\\
&= -c_2A_0\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)d\xi\\
&\quad -c_2A_1\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi
\end{aligned}
$$

Using the first condition of (3.1) , we can write

$$
\begin{aligned}
a_1x(0)+b_1x(T) &= -A_0(a_1+b_1)+\frac{b_1}{\Gamma(\alpha-\beta)}\int_0^T(T-u)^{\alpha-\beta-1} \qquad (3.6)\\
&\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du-\frac{A_1b_1T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\
&= c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\\
&\left.\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi-c_1A_0\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)d\xi\\
&\quad -c_1A_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi
\end{aligned}
$$

Thanks to (3.5), it yields then that

$$-\frac{A_1 b_2\, T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta+1)}+c_2 A_0\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)d\xi$$

$$+\,c_2 A_1\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi$$

$$=c_2\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi$$

$$-\frac{b_2}{\Gamma(\alpha-2\beta)}\int_0^{T}(T-u)^{\alpha-2\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du$$

Thanks to (3.6), we observe that

$$A_0\left(c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)d\xi-(a_1+b_1)\right)$$

$$+\,A_1\left(c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi-\frac{b_1\, T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)$$

$$=c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.$$

$$\left.\times\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi-\frac{b_2}{\Gamma(\alpha-2\beta)}\int_0^{T}(T-u)^{\alpha-2\beta-1}$$

$$\times\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du.$$

$$-\frac{b_1}{\Gamma(\alpha-\beta)}\int_0^{T}(T-u)^{\alpha-\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du.$$

Hence,

$$A_1\left(c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi-\frac{b_1\ T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)$$

$$=c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)$$
$$\times\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi$$

$$-\frac{b_1}{\Gamma(\alpha-\beta)}\int_0^{T}(T-u)^{\alpha-\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du$$

$$-\left[\frac{c_2}{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)d\xi}\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right.$$

$$\left.\left.\times\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du\right)d\xi\right.$$

$$-\frac{b_2}{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)d\xi}\Gamma(\alpha-2\beta)\int_0^{T}(T-u)^{\alpha-2\beta-1}\left(\int_0^{u}\frac{(u-s)^{\alpha-\beta-1}}{\Gamma(\beta)}\varphi(s)ds-kx(u)\right)du$$

$$\left.-\frac{A_1}{\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)d\xi}\left(c_2\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma((\gamma)}h(\xi)\frac{\xi^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}d\xi-\frac{b_2\ T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta+1)}\right)\right]$$

$$\times\left(c_1\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma((\gamma)}g(\xi)d\xi-(a_1+b_1)\right). \tag{3.7}$$

So,

$$
\begin{aligned}
A_1 \;=\; & \frac{c_1}{\Delta}\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right. \\
& \left.\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)\,ds - kx(u)\right)du\right)d\xi \\
& -\frac{b_1}{\Delta\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1}\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)\,ds - kx(u)\right)du \\
& -\frac{1}{\Delta I^\gamma h(\rho)}\left[c_2\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right. \\
& \left.\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)\,ds - kx(u)\right)du\right)d\xi \\
& \left.- b_2\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1}\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)\,ds - kx(u)\right)du\right] \\
& \times\left(c_1\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\,d\xi - (a_1+b_1)\right)
\end{aligned}
$$

Therefore, we can state that

$$A_0 = \frac{1}{I^{\gamma} h(\rho)} \left\{ \int_0^{\rho} \frac{(\rho - \xi)^{\gamma - 1}}{\Gamma(\gamma)} h(\xi) \left(\frac{1}{\Gamma(\alpha - \beta)} \int_0^{\xi} (\xi - u)^{\alpha - \beta - 1} \right. \right.$$

$$\times \left. \left. \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du \right) d\xi \right.$$

$$- \frac{b_2 \Gamma(\alpha - 2\beta)}{c_2} \int_0^T (T - u)^{\alpha - \beta - 1} \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du$$

$$- \Lambda \left[\frac{c_1}{\Delta} \int_0^{\mu} \frac{(\mu - \xi)^{\gamma - 1}}{\Gamma(\gamma)} g(\xi) \left(\frac{1}{\Gamma(\alpha - \beta)} \int_0^{\xi} (\xi - u)^{\alpha - \beta - 1} \right. \right.$$

$$\times \left. \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du \right) d\xi$$

$$- \frac{b_1}{\Delta \Gamma(\alpha - \beta)} \int_0^T (T - u)^{\alpha - \beta - 1} \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du$$

$$- \frac{1}{\Delta I^{\gamma} h(\rho)} \left[c_2 \int_0^{\rho} \frac{(\rho - \xi)^{\gamma - 1}}{\Gamma(\gamma)} h(\xi) \left(\frac{1}{\Gamma(\alpha - \beta)} \int_0^{\xi} (\xi - u)^{\alpha - \beta - 1} \right. \right.$$

$$\times \left. \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du \right) d\xi$$

$$\times \left(\int_0^u \frac{(\mu - s)^{\beta - 1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du \right) d\xi$$

$$- \frac{b_1}{\Delta\Gamma(\alpha-\beta)} \int_0^T (T-u)^{\alpha-\beta-1} \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du$$

$$-b_2 \Gamma(\alpha-2\beta) \int_0^T (T-u)^{\alpha-2\beta-1} \left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \varphi(s)\, ds - kx(u) \right) du \Bigg]$$

$$\times \left(c_1 \int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\, d\xi - (a_1+b_1) \right) \Bigg] \Bigg\}$$

Substituting A_0 and A_1, we end our proof.

3.3 Criteria of Existence and Uniqueness

We begin this section by introducing the Banach space

$$E := \left\{ u, u \in C[0;T],\ {}^cD^\beta u(t) \in C[0;T] \right\}$$

that can be equipped with

$$\|u\|_E := \max\left(\|u\|_\infty, \left\| {}^cD^\beta u \right\|_\infty \right),$$

where

$$\|u\|_\infty := \sup_{0\le t\le T} |u(t)| \quad \text{and} \quad \left\| {}^cD^\beta u \right\|_\infty := \sup_{0\le t\le T} \left| {}^cD^\beta u(t) \right|.$$

Then, we need to transform (3.1) to the following equivalent fixed point problem :

$$x = \mathcal{T}x, \tag{3.8}$$

where $\mathcal{T}:E\to E$ is defined by:

$$
\begin{aligned}
(\mathcal{T}x)(t) &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-u)^{\alpha-\beta-1}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du \\
&- \frac{1}{I^{\gamma}h(\rho)}\left\{\int_0^{\rho} \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\times \left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du\right) d\xi \\
&- \frac{b_2\Gamma(\alpha-2\beta)}{c_2}\int_0^T (T-u)^{\alpha-\beta-1}\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du \\
&- \Lambda\left[\frac{c_1}{\Delta}\int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\times \left.\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du\right) d\xi\right] \\
&- \frac{b_1}{\Delta\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1}\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du \\
&- \frac{1}{\Delta I^{\gamma}h(\rho)}\left[c_2\int_0^{\rho} \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}\left(h(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\xi}(\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\times \left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du\right) d\xi \\
&-b_2\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1}\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} f\left(s, x(s), D^{\beta}x(s)\right) ds - kx(u)\right) du\right] \\
&\times \left.\left(c_1\int_0^{\mu} \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\, d\xi - (a_1+b_1)\right)\right\} \times \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.
\end{aligned}
$$

The following quantities are introduced :

$$\begin{aligned}
\Omega_1 &= L\left(\frac{T^{\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} + \frac{\rho^{\alpha+\gamma}N_h}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)|I^{\gamma}h(\rho)|} + \frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|}\right. \\
&\quad + \frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|\Gamma(\beta)} + \frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|} \\
&\quad \left. + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)} + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\right) \\
&\quad + k\left(\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta)} + \frac{\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)|\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|I^{\gamma}h(\rho)||c_2|}\right. \\
&\quad \left. + \frac{|\Lambda||c_1|\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)||\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)T^{\alpha-\beta}|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\right)
\end{aligned}$$

$$\begin{aligned}
\Omega_2 &= L\left(\frac{T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-2\beta)} + \frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|}\right. \\
&\quad + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)|c_2|\rho^{\alpha+\gamma}N_hT^{\alpha-\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)} \\
&\quad \left. + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{2(\alpha-\beta)}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta)}\right) \\
&\quad + k\left(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)} + \frac{|\Lambda_2||c_1|\rho^{\alpha-\beta+\gamma}N_h}{|\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{|\Lambda||b_1|T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2|\Delta|}\right. \\
&\quad + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)|c_2|\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{\Gamma(\gamma)\Gamma(\alpha-\beta)|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)} \\
&\quad \left. + \frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)} + |a_1+b_1|\right)T^{\alpha-\beta}|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\right), \\
L &= \max(L_1, L_2).
\end{aligned}$$

Before establishing the existence of solutions for the problem (3.1), we impose the following assumptions :

(H_1) : There are two positive reals numbers L_1 and L_2, such as :

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq L_1|x_1-x_2|+L_2|y_1-y_2|,\quad \forall t\in[0,T],x_1,y_1,x_2,y_2\in\mathbb{R}.$$

(H_2) : The function $f:[0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a continuous.
(H_3) : There is $\delta>0$ such that $|f(t,x,y)|\leq\delta$ $\forall t\in[0,T]$ and $x,y\in\mathbb{R}$

Our first result in this work is based on the Banach's contraction principle and reads as follows.

Théorème 3.1 *suppose that the assumption (H_1) is verified. If*

$$\max(\Omega_1,\Omega_2)<1$$

Then the problem (3.1) has a unique solution on $[0,T]$.

Proof. We prove that $\mathcal{T}$ is contractive over the space E.
We have :

$$\begin{aligned}
&|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)|\leq\sup_{t\in[0,T]}\Bigg(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-u)^{\alpha-\beta-1}\\
&\times\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\left(\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k|x(u)-y(u)|\right)du\\
&+\frac{1}{|I^\gamma h(\rho)|}\left\{\int_0^\rho\frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi(\xi-u)^{\alpha-\beta-1}\right.\right.\\
&\left.\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k|x(u)-y(u)|\right)du\right)d\xi\right.\\
&+\frac{|b_2|\Gamma(\alpha-2\beta)}{|c_2|}\int_0^T(T-u)^{\alpha-\beta-1}\\
&\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k|x(u)-y(u)|\right)du\\
&+\Lambda\left[\frac{|c_1|}{|\Delta|}\int_0^\mu\frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi(\xi-u)^{\alpha-\beta-1}\right.\\
&\left.\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k|x(u)-y(u)|\right)du\right)d\xi\right]
\end{aligned}$$

$$+\frac{|b_1|}{|\Delta|\,\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1}$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du$$
$$+\frac{1}{\Delta I^\gamma h(\rho)}\left[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right.$$
$$\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du\Bigg)d\xi$$
$$+|b_2|\,\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1}$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du\Bigg]$$
$$\times\left(|c_1|\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\,d\xi+|(a_1+b_1)|\right)\Bigg\}\Bigg)$$
$$+\sup_{t\in[0,T]}\left(\left\{\frac{|c_1|}{|\Delta|}\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right.\right.$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du\Bigg)d\xi$$
$$+\frac{|b_1|}{|\Delta|\,\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1}$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du$$
$$+\frac{1}{|\Delta|\,I^\gamma h(\rho)}\left[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right.$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du\Bigg)d\xi$$
$$+|b_2|\,\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1}$$
$$\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,x(s),D^\beta x(s)\right)-f\left(s,y(s),D^\beta y(s)\right)\right|ds+k\,|x(u)-y(u)|\right)du\Bigg]$$
$$\times\left(|c_1|\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\,d\xi+|(a_1+b_1)|\right)\Bigg\}\times\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\Bigg).$$

Consequently, we have

$$\begin{aligned}
|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \leq\ & \left(L_1|x-y|+L_2\left|D^{\beta}(x-y)\right|\right)\\
&\times\left(\frac{T^{\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}+\frac{\rho^{\alpha+\gamma}N_h}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)|I^{\gamma}h(\rho)|}\right.\\
&+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|}\\
&+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|\Gamma(\beta)}+\frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}\\
&+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|}\\
&+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\\
&\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\right)\\
&+\|x-y\|\left(\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta)}+\frac{k\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)|\Gamma(\gamma)\Gamma(\alpha-\beta)}\right.\\
&+\frac{k|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|I^{\gamma}h(\rho)||c_2|}+\frac{k|\Lambda||c_1|\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)||\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)}\\
&\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\right).
\end{aligned}$$

Using the right hand side of the obove inequality, we can write :

$$\begin{aligned}
|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| \leq\ & \left[L\times\left(\frac{T^{\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}+\frac{\rho^{\alpha+\gamma}N_h}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)|I^{\gamma}h(\rho)|}+\right.\right.\\
&+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|}+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{|I^{\gamma}h(\rho)||c_2|\Gamma(\beta)}\\
&+\frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda_2||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|}\\
&+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)}\\
&\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\right)\\
&+\left(\frac{kT^{\alpha-\beta}}{\Gamma(\alpha-\beta)}+\frac{k\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)|\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}+\frac{k\,|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-2\beta}}{|I^{\gamma}h(\rho)|\,|c_2|}\right.\\
&+\frac{k\,|\Lambda|\,|c_1|\,\rho^{\alpha-\beta+\gamma}N_h}{|I^{\gamma}h(\rho)|\,|\Delta|\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}\\
&\left.\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}\,|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-2\beta}}{|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)}\right)\right]\|x-y\|.
\end{aligned}
$$

Now we pass to work with the D^{β}, we have

$$
\begin{aligned}
\left|D^{\beta}\mathcal{T}(x)(t)-D^{\beta}\mathcal{T}(y)(t)\right| \;\leq\; &\|x-y\|\left(\frac{LT^{\alpha-\beta}}{\Gamma(\beta)\,\Gamma(\alpha-2\beta)}+\frac{L\,|\Lambda|\,|c_1|\,\mu^{\alpha+\gamma}N_g}{|\Delta|\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}\right.\\
&+\frac{L\,|\Lambda|\,|b_1|\,T^{\alpha-\beta}}{\Gamma(\beta)\,\Gamma(\alpha-\beta)\,|\Delta|}\\
&+\frac{L\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\,\rho^{\alpha+\gamma}N_hT^{\alpha-\beta}}{|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}\\
&+\frac{L\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\,\Gamma(\alpha-2\beta)\,T^{2(\alpha-\beta)}}{|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)\,\Gamma(\alpha-\beta)}\\
&+\frac{kT^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)}+\frac{k\,|\Lambda_2|\,|c_1|\,\rho^{\alpha-\beta+\gamma}N_h}{|\Delta|\,\Gamma(\gamma)\,\Gamma(\alpha-\beta)}+\frac{k\,|\Lambda|\,|b_1|\,T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2\,|\Delta|}\\
&+\frac{k\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\,\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{\Gamma(\gamma)\,\Gamma(\alpha-\beta)\,|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)}\\
&\left.+\frac{k\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}\,|b_2|\,\Gamma(\alpha-2\beta)\,T^{\alpha-2\beta}}{|\Delta|\,|I^{\gamma}h(\rho)|\,\Gamma(\alpha-\beta+1)}\right).
\end{aligned}
$$

From the last two inequalities, we have:

$$
\|\mathcal{T}(x)-\mathcal{T}(y)\|_E\leq\max(\Omega_1,\Omega_2)\,\|x-y\|_E\leq\|x-y\|_E\,.
$$

Hence, we deduce that $\mathcal{T}$ is a contraction. As a consequence of Banach contraction principle, (3.1) has exactly one solution defined on $[0,T]$.
The following theorem studies the existence of at least one solution of the system (3.1)

3.4 Criteria of Existence

The next main result is based on Scheafer's fixed point theorem :

For this section we need the following quantities :

$$
\begin{aligned}
\mu_1 &= \delta\Bigg(\frac{T^{\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)}+\frac{\rho^{\alpha+\gamma}N_h}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)\left|I^{\gamma}h(\rho)\right|}+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\left|I^{\gamma}h(\rho)\right||c_2|} \\
&\quad+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\left|I^{\gamma}h(\rho)\right||c_2|\Gamma(\beta)}+\frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|} \\
&\quad+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}}{|\Delta|\left|I^{\gamma}h(\rho)\right|\Gamma(\alpha-\beta+1)} \\
&\quad+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\Bigg) \\
&\quad+kr\Bigg(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)}+\frac{|\Lambda_2||c_1|\rho^{\alpha-\beta+\gamma}N_h}{|\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda_2||b_1|T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2|\Delta|} \\
&\quad+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{\Gamma(\gamma)\Gamma(\alpha-\beta)|\Delta|\left|I^{\gamma}h(\rho)\right|\Gamma(\alpha-\beta+1)}+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{\left|I^{\gamma}h(\rho)\right|(\Gamma(\alpha-\beta))^2|\Delta|} \\
&\quad+\frac{|c_2|\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{|\Delta|\left|I^{\gamma}h(\rho)\right|\Gamma(\alpha-\beta+1)\Gamma(\gamma)\Gamma(\alpha-\beta)} \\
&\quad+\frac{|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}T^{\alpha-\beta}}{|\Delta|\left|I^{\gamma}h(\rho)\right|\Gamma(\alpha-\beta+1)}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= \delta\left(\frac{T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-2\beta)}+\frac{|\Lambda||c_1|\mu^{\alpha+\gamma}N_g}{|\Delta|\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{\Gamma(\beta)\Gamma(\alpha-\beta)|\Delta|}\right.\\
&+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\rho^{\alpha+\gamma}N_hT^{\alpha-\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha-\beta)}\\
&\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|b_2|\Gamma(\alpha-2\beta)T^{2(\alpha-\beta)}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta)}\right)\\
&+kr\left(\frac{T^{\alpha-2\beta}}{\Gamma(\alpha-2\beta)}+\frac{|\Lambda||c_1|\rho^{\alpha-\beta+\gamma}N_h}{|\Delta|\Gamma(\gamma)\Gamma(\alpha-\beta)}+\frac{|\Lambda||b_1|T^{\alpha-\beta}}{(\Gamma(\alpha-\beta))^2|\Delta|}\right.\\
&+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)|c_2|\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{\Gamma(\gamma)\Gamma(\alpha-\beta)|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\\
&\left.+\frac{\left(\frac{|c_1|\rho^{\gamma}N_h}{\Gamma(\gamma)}+|a_1+b_1|\right)T^{\alpha-\beta}|b_2|\Gamma(\alpha-2\beta)T^{\alpha-2\beta}}{|\Delta||I^{\gamma}h(\rho)|\Gamma(\alpha-\beta+1)}\right).
\end{aligned}
$$

At the moment, we are ready to present to the reader the following main result.

Théorème 3.2 *Assume that the assumptions* (H_2) *and* (H_3) *hold. Then, the problem* (3.1) *has at least one solution on* $[0,T]$.

Proof. The proof of this theorem is done in four steps :

Step1 : Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \longrightarrow u$ in $C([0,T],\mathbb{R})$. Then for $t \in$

$[0,T]$, we have

$$
\begin{aligned}
&\left|\mathcal{T}u_n(t)-\mathcal{T}u(t)\right| \leq \sup_{t\in[0,T]}\Bigg(\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-u)^{\alpha-\beta-1}\\
&\times\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\left(\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\\
&+\frac{1}{\left|I^\gamma h(\rho)\right|}\Bigg\{\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\Bigg(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\\
&\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\Bigg)d\xi\\
&+\frac{|b_2|\,\Gamma(\alpha-2\beta)}{|c_2|}\int_0^T (T-u)^{\alpha-\beta-1}\\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\\
&+\Lambda\Bigg[\frac{|c_1|}{|\Delta|}\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)}g(\xi)\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|x(u)-u(u)\right|\right)du\Bigg)d\xi\Bigg]\\
&+\frac{|b_1|}{|\Delta|\,\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1}\\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\\
&+\frac{1}{\Delta I^\gamma h(\rho)}\Bigg[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)}h(\xi)\Bigg(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\\
&\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s,u_n(s),D^\beta u_n(s)\right)-f\left(s,u(s),D^\beta u(s)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\Bigg)d\xi
\end{aligned}
\tag{3.9}
$$

$$+\left|b_2\right|\Gamma\left(\alpha-2\beta\right)\int_0^T\left(T-u\right)^{\alpha-2\beta-1}$$
$$\times\left(\int_0^u\frac{\left(\mu-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,u_n\left(s\right),D^\beta u_n\left(s\right)\right)-f\left(s,u\left(s\right),D^\beta u\left(s\right)\right)\right|ds+k\left|u_n(u)-u(u)\right|\right)du\Bigg]$$
$$\times\left(\left|c_1\right|\int_0^\mu\frac{\left(\mu-\xi\right)^{\gamma-1}}{\Gamma\left(\gamma\right)}g\left(\xi\right)d\xi+\left|\left(a_1+b_1\right)\right|\right)\Bigg\}\Bigg)$$

On the other side, we can say that

$$\begin{aligned}\left|D^\beta\mathcal{T}u_n\left(t\right)-D^\beta\mathcal{T}u\left(t\right)\right| \leq\ & \left|u_n\left(t\right)-u\left(t\right)\right|\Bigg(\frac{LT^{\alpha-\beta}}{\Gamma\left(\beta\right)\Gamma\left(\alpha-2\beta\right)}+\frac{L\left|\Lambda\right|\left|c_1\right|\mu^{\alpha+\gamma}N_g}{\left|\Delta\right|\Gamma\left(\beta\right)\Gamma\left(\gamma\right)\Gamma\left(\alpha-\beta\right)}\\
&+\frac{L\left|\Lambda\right|\left|b_1\right|T^{\alpha-\beta}}{\Gamma\left(\beta\right)\Gamma\left(\alpha-\beta\right)\left|\Delta\right|}\\
&+\frac{L\left(\frac{\left|c_1\right|\rho^\gamma N_h}{\Gamma(\gamma)}+\left|a_1+b_1\right|\right)\left|c_2\right|\rho^{\alpha+\gamma}N_hT^{\alpha-\beta}}{\left|\Delta\right|\left|I^\gamma h\left(\rho\right)\right|\Gamma\left(\alpha-\beta+1\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)\Gamma\left(\alpha-\beta\right)}\\
&+\frac{L\left(\frac{\left|c_1\right|\rho^\gamma N_h}{\Gamma(\gamma)}+\left|a_1+b_1\right|\right)\left|b_2\right|\Gamma\left(\alpha-2\beta\right)T^{2(\alpha-\beta)}}{\left|\Delta\right|\left|I^\gamma h\left(\rho\right)\right|\Gamma\left(\alpha-\beta+1\right)\Gamma\left(\alpha-\beta\right)}\\
&+\frac{kT^{\alpha-2\beta}}{\Gamma\left(\alpha-2\beta\right)}+\frac{k\left|\Lambda_2\right|\left|c_1\right|\rho^{\alpha-\beta+\gamma}N_h}{\left|\Delta\right|\Gamma\left(\gamma\right)\Gamma\left(\alpha-\beta\right)}+\frac{k\left|\Lambda\right|\left|b_1\right|T^{\alpha-\beta}}{\left(\Gamma\left(\alpha-\beta\right)\right)^2\left|\Delta\right|}\\
&+\frac{k\left(\frac{\left|c_1\right|\rho^\gamma N_h}{\Gamma(\gamma)}+\left|a_1+b_1\right|\right)\left|c_2\right|\rho^{\alpha-\beta+\gamma}N_hT^{\alpha-\beta}}{\Gamma\left(\gamma\right)\Gamma\left(\alpha-\beta\right)\left|\Delta\right|\left|I^\gamma h\left(\rho\right)\right|\Gamma\left(\alpha-\beta+1\right)}\\
&+\frac{k\left(\frac{\left|c_1\right|\rho^\gamma N_h}{\Gamma(\gamma)}+\left|a_1+b_1\right|\right)T^{\alpha-\beta}\left|b_2\right|\Gamma\left(\alpha-2\beta\right)T^{\alpha-2\beta}}{\left|\Delta\right|\left|I^\gamma h\left(\rho\right)\right|\Gamma\left(\alpha-\beta+1\right)}\Bigg).\end{aligned} \tag{3.10}$$

Thanks to the assumptions (3.9) *and* (3.10), and using $(H2)$, we can find

$$\|\mathcal{T}u_n-\mathcal{T}u\|\longrightarrow 0 \text{ when } n\longrightarrow+\infty.$$

Then T is continuous on $C\left([0,T],\mathbb{R}\right)$.

Step2 : Let $r>0$. Now we consider the ball $\mathfrak{B}_r$ defined by $\mathfrak{B}_r=\{x\in E:\|x\|_E\leq r\}$ in E. To show that the operator $\mathcal{T}$ is uniformly bounded, it suffices to replace $\mathcal{T}$ by $\mathcal{A}$ and $f=\Psi$ in lemma 2.1.

Step3 : In this step, we show that $\mathcal{T}$ is equicontinuous. Suppose that $0<t_1<t_2<T$

and $x \in B_r$. Then,

$$
\begin{aligned}
&\left|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)\right| \leq \frac{1}{\Gamma(\alpha-\beta)}\left[\int_0^{t_2}(t_2-u)^{\alpha-\beta-1} - \int_0^{t_1}(t_1-u)^{\alpha-\beta-1}\right] \\
&\times \int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\left|\left(f\left(s, x(s), D^\beta x(s)\right) ds\right)\right| du \\
&+\left\{\frac{|c_1|}{|\Delta|}\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\times\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds + k|x(u)|\right) du\right) d\xi \\
&+\frac{|b_1|}{|\Delta|\Gamma(\alpha-\beta)}\int_0^T (T-u)^{\alpha-\beta-1} \\
&\times\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds + k|x(u))|\right) du \\
&+\frac{1}{|\Delta I^\gamma h(\rho)|}\left[|c_2|\int_0^\rho \frac{(\rho-\xi)^{\gamma-1}}{\Gamma(\gamma)} h(\xi)\left(\frac{1}{\Gamma(\alpha-\beta)}\int_0^\xi (\xi-u)^{\alpha-\beta-1}\right.\right. \\
&\times\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds + k|x(u)|\right) du\right) d\xi \\
&+|b_2|\Gamma(\alpha-2\beta)\int_0^T (T-u)^{\alpha-2\beta-1} \\
&\times\left.\left(\int_0^u \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds + k|x(u)|\right) du\right] \\
&\times\left.\left(|c_1|\int_0^\mu \frac{(\mu-\xi)^{\gamma-1}}{\Gamma(\gamma)} g(\xi)\, d\xi + |(a_1+b_1)|\right)\right\} \times \frac{\left|t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right|}{\Gamma(\alpha-\beta+1)}
\end{aligned}
$$

It is to note that the right hand side of the previous inequality tends to zero independently of the variable $x \in B_r$, when $t_2 - t_1 \longrightarrow 0$.

The same remark for the following inequality :

$$
\begin{aligned}
&\left|D^{\beta}\left(\mathcal{T}x\right)(t_2)-D^{\beta}\left(\mathcal{T}x\right)(t_1)\right| \leq \frac{1}{\Gamma(\alpha-2\beta)}\left[\int_0^{t_2}(t_1-u)^{\alpha-2\beta-1}-\int_0^{t_1}(t_2-u)^{\alpha-2\beta-1}\right]\\
&\times\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}\left|\left(f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)ds\right)\right|du\\
&+\left\{\frac{|c_1|}{|\Delta|}\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma\left(\gamma\right)}g\left(\xi\right)\left(\frac{1}{\Gamma\left(\alpha-\beta\right)}\int_0^{\xi}\left(\xi-u\right)^{\alpha-\beta-1}\right.\right.\\
&\times\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds+k\left|x(u)\right|\right)du\right)d\xi\\
&+\frac{b_1}{\Delta\Gamma\left(\alpha-\beta\right)}\int_0^T\left(T-u\right)^{\alpha-\beta-1}\\
&\times\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds+k\left|x(u))\right|\right)du\\
&+\frac{1}{\left|\Delta I^{\gamma}h\left(\rho\right)\right|}\left[\left|c_2\right|\int_0^{\rho}\frac{(\rho-\xi)^{\gamma-1}}{\Gamma\left(\gamma\right)}h\left(\xi\right)\left(\frac{1}{\Gamma\left(\alpha-\beta\right)}\int_0^{\xi}\left(\xi-u\right)^{\alpha-\beta-1}\right.\right.\\
&\times\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds+k\left|x(u)\right|\right)du\right)d\xi\\
&+\left|b_2\right|\Gamma\left(\alpha-2\beta\right)\int_0^T\left(T-u\right)^{\alpha-2\beta-1}\\
&\times\left.\left(\int_0^u\frac{(\mu-s)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds+k\left|x(u)\right|\right)du\right]\\
&\times\left.\left(\left|c_1\right|\int_0^{\mu}\frac{(\mu-\xi)^{\gamma-1}}{\Gamma\left(\gamma\right)}g\left(\xi\right)d\xi+\left|\left(a_1+b_1\right)\right|\right)\right\}\times\frac{\left|t_2^{\alpha-2\beta}-t_1^{\alpha-2\beta}\right|}{\Gamma(\alpha-2\beta+1)}
\end{aligned}
$$

So, $\mathcal{T}:E\longrightarrow E$ is equicontinuous. Then, by Arzela-Ascoli theorem, $\mathcal{T}$ is completely continuous.

Let consider finally:

$$\mathcal{G}=\left\{x\in E: x=\lambda\mathcal{T}x,\ \text{for some } 0<\lambda<1\right\}.$$

For $x \in \mathcal{G}$ and $t \in [0,T]$, we have $x(t) = \lambda \mathcal{T}(x)$, for some $0 < \lambda < 1$. Then

$$\begin{aligned} \|x\| &\leq \|\mathcal{T}(x)\| \leq \mu_1, \\ \left\|D^{\beta} x\right\| &\leq \left\|D^{\beta} \mathcal{T}(x)\right\| \leq \mu_2. \end{aligned}$$

Consequently,

$$\|x\| \leq \max(\mu_1, \mu_2) < \infty$$

We deduce that $\mathcal{G}$ is bounded. Finally, (3.1) has at least one solution defined over $[0,T]$. So, $\mathcal{T} : E \longrightarrow E$ is equi continuous. Then, by Arzela-Ascoli theorem, $\mathcal{T}$ is completely continuous.

3.5 Examples

Problem 1

To give an illustration of Theorem 3.1, we take the following problem

$$\begin{cases} \left({}^{c}D^{\frac{7}{5}} + 2\,{}^{c}D^{\frac{3}{4}}\right) x(t) = e^{-t} + 2\frac{x(t)}{10^{12}} + 7\frac{\tan^{-1}\left({}^{c}D^{\frac{3}{4}} x(t)\right)}{10^{14}} \\ D^{\beta} x(0) + D^{\beta} x(4) = \int_0^1 x(\xi) \sin \xi d\xi \\ x(0) + x(4) = \int_0^2 x(\xi) \cos \xi d\xi \end{cases} \tag{3.11}$$

In this problem, we have

$$\begin{cases} \alpha = \frac{7}{5},\ \beta = \frac{3}{4},\ a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1, \\ T = 4,\ k = \frac{1}{10^{12}},\ \mu = 2,\ \rho = \gamma = 1, \\ f(t,x,y) = e^{-t} + 2\dfrac{x(t)}{10^{12}} + 7\dfrac{\tan^{-1}(y(t))}{10^{14}}, \\ h(t) = \sin t, \quad g(t) = \cos t. \end{cases}$$

Also, $\forall t \in [0,4]$, and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, so we have

$$|f(t, x_2, y_2) - f(t, x_1, y_1)| \leq \frac{2}{10^{12}} |x_2 - x_1| + \frac{7}{10^{14}} |y_2 - y_1|$$

It is clair that $L_1 = \dfrac{2}{10^{12}}$, $L_2 = \dfrac{7}{10^{14}}$, $L = \max(L_1, L_2) = \dfrac{2}{10^{12}}$.

Moreover, the previous values give us

$$\begin{aligned} \Omega_1 &\simeq 2.9488 \times 10^{-2}, \\ \Omega_2 &\simeq 5.1709 \times 10^{-12}. \end{aligned}$$

Then, $\max(\Omega_1, \Omega_2) \leq 1$, thus the assumptions of Theorem 6 are satisfied, hence, the BVP (3.11) has a solution in $[0, 4]$.

Problem 2

For Theorem 3.2, we consider the problem (3.1) with : $f(t, x, y) = \dfrac{\sqrt{t}e^{-2t^2}}{2 + x + ty^2}$

Clearly, f is continuous. We have also

$$|f(t, x, y)| = \left| \frac{\sqrt{t}e^{-2t^2}}{2 + x + ty^2} \right| \leq 1 = \delta$$

This means that f is bounded, thus, the BVP (3.11) has at least one solution in $[0, 4]$.

Chapter 4

A System of Differential Equations With Sequential Fractional Derivatives

4.1 Introduction

Differential equations theory is considered as one of the most important fields of mathematics. It has many applications in physics, electrochemistry, biomathematics, aerodynamics, dynamics, electromagnetic, control theory of dynamical systems, etc. For more details, we refer the reader to [24], [38], [39]. In particular, the existence and uniqueness problems of differential equations of fractional order have been investigated by many authors. For instance, we cite the papers [8], [42], [43]. Recently, in [43] some existence and the uniqueness results are given for the following system of sequential Caputo and Hadamard fractional differential equations

$$\begin{cases} {}^{C}D^{\alpha H}D^{\beta}x\left(t\right)=f\left(t,x\left(t\right)\right), \qquad a\leq t\leq b, \\ \gamma_1 x\left(a\right)+\gamma_2^{H}D^{\beta}x\left(a\right)=0,\ \ \lambda_1 x\left(b\right)+\lambda_2^{H}D^{\beta}x\left(b\right)=0, \end{cases}$$

where ${}^{C}D^{\alpha}, {}^{H}D^{\beta}$ denote the Caputo and Hadamard fractional derivatives of orders α and β, respectively with, $0<\alpha,\beta\leq 1$ and $\gamma_i,\lambda_i\in\mathbb{R}\left(i=\overline{1,2}\right), f:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function. Very recently, S. Asawasamrit et al. [8] studied the existence and uniqueness of solutions for the coupled system of nonlinear sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions defined as

$$\begin{cases} {}^{C}D^{p_1 H}D^{q_1}x(t) = f(t, x(t), y(t)), \ a \le t \le b, \\ {}^{H}D^{q_2 C}D^{p_2}y(t) = g(t, x(t), y(t)), \ \ a \le t \le b, \\ \alpha_1 x(a) + \alpha_2^{C}D^{p_2}y(a) = 0, \ \beta_1 x(b) + \beta_2^{C}D^{p_2}y(b) = 0, \\ \alpha_3 y(a) + \alpha_4^{H}D^{q_1}x(a) = 0, \ \beta_3 y(b) + \beta_4^{H}D^{\beta_1}x(b) = 0 \\ \end{cases} \tag{4.1}$$

where ${}^{C}D^{p_i}, {}^{H}D^{q_i}$ are the Caputo and Hadamard fractional derivatives of orders p_i and q_i, respectively with, $0 < \alpha_i, \beta_i \le 1$, $i = \overline{1,2}$ and $\alpha_i, \beta_i \left(i = \overline{1,4}\right)$ are real constants and $f, g : [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ are continuous functions.

Our objective in this chapter is to study the following problem [10] :

$$\begin{cases} {}^{C}D^{\alpha_1 H}D^{\beta_1}x(t) = f\left(t, x(t), y(t), {}^{H}D^{\alpha_2}y(t)\right), \ a \le t \le b, \\ {}^{H}D^{\beta_2 C}D^{\alpha_2}y(t) = g\left(t, x(t), {}^{H}D^{\beta_1}x(t), y(t)\right), \ \ a \le t \le b, \\ \gamma_1 x(a) + \gamma_2^{C}D^{\alpha_2}y(a) = \theta_1, \ \lambda_1 x(b) + \lambda_2^{C}D^{\alpha_2}y(b) = \theta_2, \\ \gamma_3 y(a) + \gamma_4^{H}D^{\beta_1}x(a) = \theta_3, \ \lambda_3 y(b) + \lambda_4^{H}D^{\beta_1}x(b) = \theta_4, \\ \end{cases} \tag{4.2}$$

where ${}^{C}D^{\alpha_i}, {}^{H}D^{\beta_i}$ denote the Caputo and Hadamard fractional derivatives of orders α_i and β_i, respectively with, $0 < \alpha_i, \beta_i \le 1$, $i = \overline{1,2}$ and γ_i, λ_i, θ_i, $\left(i = \overline{1,4}\right)$ are real numbers such that $\gamma_i \left(i = \overline{1,4}\right)$ are no zero numbers, $a, b \in \mathbb{R}$ with $a > 0$, and $f, g : [a; b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ are two given functions.

This chapter is structured as follows : In the first section, we study the existence of solutions for the linear system of fractional differential equations. In the second section, we present the first result using the Banach contraction principle. Then we establish the second existence based on Schaefer fixed point theorem. We conclude this chapter by an application.

Sequential System and Integral Representation

For computational convenience, we set

$$\begin{aligned}
\Lambda_1 &: = \lambda_1 \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}, \quad \Lambda_2 := \lambda_1 - \lambda_2 \frac{\gamma_1}{\gamma_2}, \\
\Lambda_3 &: = \lambda_4 - \lambda_3 \frac{\gamma_4}{\gamma_3}, \quad \Lambda_4 := \lambda_3 \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \left(\frac{\theta_1}{\gamma_2} - \frac{\gamma_1}{\gamma_2}\right), \\
\Sigma &: = \Lambda_4 \Lambda_1 - \Lambda_3 \Lambda_2.
\end{aligned}$$

In the following lemma, we prove a first auxiliary main result.

Lemma 4.1 *Let the functions* $\varphi, \psi \in C([a,b], \mathbb{R})$. *Then, the solution of the problem*

$$\begin{cases}
{}^{C}D^{\alpha_1 H}D^{\beta_1} x(t) = \varphi(t), \quad a \leq t \leq b, \\
{}^{H}D^{\beta_2 C}D^{\alpha_2} y(t) = \psi(t), \quad a \leq t \leq b, \\
\gamma_1 x(a) + \gamma_2^{C} D^{\alpha_2} y(a) = \theta_1, \ \lambda_1 x(b) + \lambda_2^{C} D^{\alpha_2} y(b) = \theta_2, \\
\gamma_3 y(a) + \gamma_4^{H} D^{\beta_1} x(a) = \theta_3, \ \lambda_3 y(b) + \lambda_4^{H} D^{\beta_1} x(b) = \theta_4
\end{cases} \tag{4.3}$$

is given by $(x(t), y(t))$, $t \in [a,b]$, *where*

$$\begin{aligned}
x(t) &= \frac{1}{\Sigma}\left[-\Lambda_3\theta_2 + \left(\Lambda_3 - \Lambda_4 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right. \\
&\quad \left. \times \left(\lambda_1^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(b) + \lambda_2^{H} I^{\beta_2}\psi(b) + \lambda_2 \frac{\theta_1}{\gamma_2}\right)\right] \\
&\quad + \frac{1}{\Sigma}\left[(\Lambda_1 + \Lambda_4)\theta_4 - \left(\Lambda_1 + \Lambda_2 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right. \\
&\quad \left. \times \left(\lambda_3 I^{\alpha_2}\left({}^{H}I^{\beta_2}\psi\right)(b) + \lambda_4^{RL} I^{\alpha_1}\varphi(b) + \lambda_3 \frac{\theta_3}{\gamma_3}\right)\right] \\
&\quad + {}^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(t),
\end{aligned}$$

and

$$
\begin{aligned}
y(t) &= \frac{\theta_3}{\gamma_3}+\frac{\theta_1}{\gamma_2}\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&-\frac{1}{\Sigma}\left(\frac{\gamma_4}{\gamma_3}\Lambda_4+\frac{\gamma_1}{\gamma_2}\Lambda_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\
&\times\left(\theta_2-\left(\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(b)+\lambda_2^H I^{\beta_2}\psi(b)\right)-\lambda_2\frac{\theta_1}{\gamma_2}\right) \\
&-\frac{1}{\Sigma}\left[\frac{\gamma_4}{\gamma_3}\Lambda_4\theta_4-\left(\frac{\gamma_4}{\gamma_3}\Lambda_4+\frac{\gamma_1}{\gamma_2}\Lambda_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\right. \\
&\times\left(\lambda_3^{RL}I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(b)+\lambda_4 I^{\alpha_1}\varphi(b)-\lambda_3\frac{\theta_3}{\gamma_3}\right) \\
&\left.+\frac{\gamma_1}{\gamma_2}\theta_1\Lambda_1\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]+{}^{RL}I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(t)
\end{aligned}
$$

Proof. We apply lemmas 1.3 to the first equation of (4.3), we can write

$$ {}^H D^{\beta_1}x(t)=c_1+I^{\alpha_1}\varphi(t),\quad c_1\in\mathbb{R} \tag{4.4}$$

We apply lemma 1.4 to (4.4), we get

$$ x(t)=c_2+c_1\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}+{}^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(t),\quad c_2\in\mathbb{R} \tag{4.5}$$

By using the Hadamard fractional integral of order β_2 to the second equation of (4.3), it yields that

$$ {}^C D^{\alpha_2}y(t)=c_3+{}^H I^{\beta_2}\psi(t),\quad c_3\in\mathbb{R}. \tag{4.6}$$

Thanks to lemma 1.3 to (4.6), yields the following formula

$$ y(t)=c_4+c_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+{}^{RL}I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(t),\quad c_4\in\mathbb{R}. \tag{4.7}$$

Thanks to the initial conditions of (4.3), we obtain

$$\begin{cases} \gamma_1 c_2 + \gamma_2 c_3 = \theta_1, \\ \lambda_1 \left(c_2 + c_1 \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} +^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}\varphi \right)(b) \right) + \lambda_2 \left(c_3 +^H I^{\beta_2}\psi(b) \right) = \theta_2, \\ \gamma_3 c_4 + \gamma_4 c_1 = \theta_3, \\ \lambda_3 \left(c_4 + c_3 \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} +^{RL} I^{\alpha_2} \left({}^H I^{\beta_2}\psi \right)(b) \right) + \lambda_4 \left(c_1 + I^{\alpha_1}\varphi(b) \right) = \theta_4 \ , \end{cases} \tag{4.8}$$

so, we have

$$\begin{cases} \lambda_1 \dfrac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} c_1 + \left(\lambda_1 - \lambda_2 \dfrac{\gamma_1}{\gamma_2} \right) c_2 = \theta_2 - \lambda_1^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}\varphi \right)(b) - \lambda_2^H I^{\beta_2}\psi(b) - \lambda_2 \dfrac{\theta_1}{\gamma_2}, \\ \left(\lambda_4 - \lambda_3 \dfrac{\gamma_4}{\gamma_3} \right) c_1 + \lambda_3 \dfrac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \left(\dfrac{\theta_1}{\gamma_2} - \dfrac{\gamma_1}{\gamma_2} \right) c_2 = \theta_4 - \lambda_3^{RL} I^{\alpha_2} \left({}^H I^{\beta_2}\psi \right)(b) - \lambda_4 I^{\alpha_1}\varphi(b) - \lambda_3 \dfrac{\theta_3}{\gamma_3}, \\ c_3 = \dfrac{\theta_1}{\gamma_2} - \dfrac{\gamma_1}{\gamma_2} c_2, \\ c_4 = \dfrac{\theta_3}{\gamma_3} - \dfrac{\gamma_4}{\gamma_3} c_1. \end{cases} \tag{4.9}$$

By solving the system, we obtain

$$\begin{cases} \Lambda_1 c_1 + \Lambda_2 c_2 = \Delta_1 \\ \Lambda_3 c_1 + \Lambda_4 c_2 = \Delta_2, \end{cases}$$

where

$$\begin{aligned} \Delta_1 \ &: = \theta_2 - \lambda_1^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}\varphi \right)(b) - \lambda_2^H I^{\beta_2}\psi(b) - \lambda_2 \frac{\theta_1}{\gamma_2}, \\ \Delta_2 \ &: = \theta_4 - \lambda_3^{RL} I^{\alpha_2} \left({}^H I^{\beta_2}\psi \right)(b) - \lambda_4 I^{\alpha_1}\varphi(b) - \lambda_3 \frac{\theta_3}{\gamma_3}, \end{aligned}$$

we obtain

$$\begin{aligned} c_1 &= \frac{\Lambda_4}{\Sigma}\Delta_1 + \frac{\Lambda_2}{\Sigma}\Delta_2, \\ c_2 &= \frac{\Lambda_1}{\Sigma}\Delta_2 - \frac{\Lambda_3}{\Sigma}\Delta_1. \end{aligned}$$

Using (4.9) , we get the quantities

$$\begin{aligned} c_3 &= \frac{\theta_1}{\gamma_2} - \frac{\gamma_1\Lambda_1}{\gamma_2\Sigma}\Delta_2 + \frac{\gamma_1\Lambda_3}{\gamma_2\Sigma}\Delta_1, \\ c_4 &= \frac{\theta_3}{\gamma_3} - \frac{\gamma_4\Lambda_2}{\gamma_3\Sigma}\Delta_2 - \frac{\gamma_4\Lambda_4}{\gamma_3\Sigma}\Delta_1. \end{aligned}$$

Substitute the values of c_1, c_2 in (4.5) and c_3, c_4 in (4.7) , then lemma 9 is thus proved.

4.2 Solvability: Existence and Uniqueness

We introduce the spaces

$$\begin{aligned} X &: = \left\{x \in C\left([a,b],\mathbb{R}\right), {}^{H}D^{\beta_1}x\left(t\right) \in C\left([a,b],\mathbb{R}\right)\right\}, \\ Y &: = \left\{y \in C\left([a,b],\mathbb{R}\right), {}^{H}D^{\alpha_2}y\left(t\right) \in C\left([a,b],\mathbb{R}\right)\right\}. \end{aligned}$$

We endowed the space X by the norm

$$\|u\|_X := \max\left(\|x\|, \left\|{}^{H}D^{\beta_1}x\right\|\right), \quad \|x\| = \sup_{a\leq t\leq b}|x(t)|, \quad \left\|{}^{H}D^{\beta_1}x\right\| = \sup_{a\leq t\leq b}\left|{}^{H}D^{\beta_1}x(t)\right|.$$

In the same manner with Y, we consider

$$\|y\|_Y : \max\left(\|y\|, \left\|{}^{H}D^{\alpha_2}y\right\|\right), \quad \|y\| = \sup_{a\leq t\leq b}|y(t)|, \quad \left\|{}^{H}D^{\alpha_2}y\right\| = \sup_{a\leq t\leq b}\left|{}^{H}D^{\alpha_2}y(t)\right|.$$

Thus, $\left(X \times Y, \|.\|_{X\times Y}\right)$ is a Banach space with norm

$$\|(x,y)\|_{X\times Y} := \max\left(\|x\|_X, \|x\|_Y\right).$$

The Riemann-Liouville and Hadamard fractional integrals of a function with three variables are given by

$$^{H}I^{q}\left(^{RL}I^{p}\left(f_{x,y,z}\right)\right)(\zeta)=\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{\zeta}\int_{a}^{s}\left(\log\frac{\zeta}{s}\right)^{q-1}(s-r)^{p-1}f\left(r,x\left(r\right),y(r),z\left(r\right)\right)dr\frac{ds}{s},$$

and

$$^{RL}I^{p}\left(^{H}I^{q}\left(f_{x,y,z}\right)\right)(\zeta)=\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{\zeta}\int_{a}^{s}(\zeta-r)^{p-1}\left(\log\frac{s}{r}\right)^{q-1}f\left(r,x\left(r\right),y(r),z\left(r\right)\right)\frac{dr}{r}ds.$$

where $0<p,q\leq 1$ and $\zeta\in\{t,b\}$.

As a special case that will be needed in this paper, we consider the following two quantities:

$$^{H}I^{q}\left(^{RL}I^{p}(1)\right)(\zeta)=\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{\zeta}\int_{a}^{s}\left(\log\frac{\zeta}{s}\right)^{q-1}(s-r)^{p-1}dr\frac{ds}{s},$$

$$^{RL}I^{p}\left(^{H}I^{q}(1)\right)(\zeta)=\frac{1}{\Gamma(p)\Gamma(q)}\int_{a}^{\zeta}\int_{a}^{s}(\zeta-r)^{p-1}\left(\log\frac{s}{r}\right)^{q-1}\frac{dr}{r}ds.$$

We consider the operator $\mathcal{T}$ defined as follows:

$$\begin{array}{rccc}\mathcal{T}: & X\times Y & \longrightarrow & X\times Y\\ & (x,y)(t) & \longmapsto & \left(\mathcal{T}_{1}(x,y)(t),\mathcal{T}_{2}(x,y)(t)\right),\end{array}$$

where, $\forall t\in[a,b]$,

$$\begin{aligned}\mathcal{T}_{1}(x,y)(t)\quad:&=\frac{1}{\Sigma}\Bigg[-\Lambda_{3}\theta_{2}+\left(\Lambda_{3}-\Lambda_{4}\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_{1}}}{\Gamma(\beta_{1}+1)}\right)\\ &\times\left(\lambda_{1}^{H}I^{\beta_{1}}\left(^{RL}I^{\alpha_{1}}f\left(b,x(b),y(b),^{H}D^{\alpha_{2}}y(b)\right)\right)\right.\\ &\left.+\lambda_{2}^{H}I^{\beta_{2}}g\left(b,x(b),y(b),^{H}D^{\beta_{1}}x(b)\right)\right)+\lambda_{2}\frac{\theta_{1}}{\gamma_{2}}\Bigg]\\ &+\frac{1}{\Sigma}\Bigg[(\Lambda_{1}+\Lambda_{4})\theta_{4}-\left(\Lambda_{1}+\Lambda_{2}\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_{1}}}{\Gamma(\beta_{1}+1)}\right)\end{aligned}$$

$$\times\left(\lambda_3^{RL}I^{\alpha_2}\left({}^HI^{\beta_2}g\left(b,x\left(b\right),y\left(b\right),{}^HD^{\beta_1}x\left(b\right)\right)\right)\right.$$

$$\left.+\lambda_4I^{\alpha_1}f\left(b,x\left(b\right),y\left(b\right),{}^HD^{\alpha_2}y\left(b\right)\right)\right)+\lambda_3\frac{\theta_3}{\gamma_3}\Bigg]$$

$$+{}^HI^{\beta_1}\left({}^{RL}I^{\alpha_1}f\left(t,x\left(t\right),y\left(t\right),{}^HD^{\alpha_2}y\left(t\right)\right)\right)$$

and

$$\begin{aligned}
\mathcal{T}_2\left(x,y\right)\left(t\right) \; : \; &= \frac{\theta_3}{\gamma_3}+\frac{\theta_1}{\gamma_2}\frac{\left(t-a\right)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)} \\
&-\frac{1}{\Sigma}\left(\frac{\gamma_4}{\gamma_3}\Lambda_4+\frac{\gamma_1}{\gamma_2}\Lambda_3\frac{\left(t-a\right)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right) \\
&\times\left(\theta_2-\left(\lambda_1^HI^{\beta_1}\left({}^{RL}I^{\alpha_1}f\left(b,x\left(b\right),y\left(b\right),{}^HD^{\alpha_2}y\left(b\right)\right)\right)\right.\right. \\
&\left.\left.+\lambda_2^HI^{\beta_2}g\left(b,x\left(b\right),y\left(b\right),{}^HD^{\beta_1}x\left(b\right)\right)\right)-\lambda_2\frac{\theta_1}{\gamma_2}\right) \\
&-\frac{1}{\Sigma}\left[\frac{\gamma_4}{\gamma_3}\Lambda_4\theta_4-\left(\frac{\gamma_4}{\gamma_3}\Lambda_4+\frac{\gamma_1}{\gamma_2}\Lambda_3\frac{\left(t-a\right)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right)\right. \\
&\times\left(\lambda_3^{RL}I^{\alpha_2}\left({}^HI^{\beta_2}g\left(b,x\left(b\right),y\left(b\right),{}^HD^{\beta_1}x\left(b\right)\right)\right)\right. \\
&\left.\left.+\lambda_4^{RL}I^{\alpha_1}f\left(b,x\left(b\right),y\left(b\right),{}^HD^{\alpha_2}y\left(b\right)\right)-\lambda_3\frac{\theta_3}{\gamma_3}\right)+\frac{\gamma_1}{\gamma_2}\theta_1\Lambda_1\frac{\left(t-a\right)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right] \\
&+{}^{RL}I^{\alpha_2}\left({}^HI^{\beta_2}g\left(t,x\left(t\right),y\left(t\right),{}^HD^{\beta_1}x\left(t\right)\right)\right)
\end{aligned}$$

Before proving the existence of the solution to the problem (4.2), the following hypotheses are proposed :

$(H1)$: Suppose that there exists constants $l_{ij}>0, i=\overline{1,2}, j=\overline{1,3}$ such that

$$\begin{aligned}
\left|f\left(t,x_2,y_2,z_2\right)-f\left(t,x_1,y_1,z_1\right)\right| &\leq l_{11}\left|x_2-x_1\right|+l_{12}\left|y_2-y_1\right|+l_{13}\left|z_2-z_1\right|, \\
\left|g\left(t,x_2,y_2,z_2\right)-g\left(t,x_1,y_1,z_1\right)\right| &\leq l_{21}\left|x_2-x_1\right|+l_{22}\left|y_2-y_1\right|+l_{23}\left|z_2-z_1\right|,
\end{aligned}$$

for each $t\in[a,b]$ and all $x_i,y_i,z_i\in\mathbb{R}$.

(H_2) : The functions $f, g : [a,b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ are continuous

(H_3) : There exist two constants $K_1, K_2, > 0$ such that, $\forall t \in [a,b], x, y, z \in \mathbb{R}$

$$|f(t,x,y,z)| \leq K_1, \quad |g(t,x,y,z)| \leq K_2.$$

Then, we introduce the quantities:

$$\begin{aligned}
Q_1 &: = \frac{|\lambda_1| \, l_1}{|\Sigma|} \left(|\Lambda_3| + |\Lambda_4| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} \right)^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}(1) \right)(b) \\
&+ \frac{|\lambda_4| \, l_1}{|\Sigma|} \left(|\Lambda_1| + |\Lambda_2| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} \right)^{RL} I^{\alpha_1}(1)(b) + l_1^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}(1) \right)(b), \\
Q_2 &: = \frac{|\lambda_2| \, l_2}{|\Sigma|} \left(|\Lambda_3| + |\Lambda_4| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} \right)^H I^{\beta_2}(1)(b) \\
&+ \frac{|\lambda_3| \, l_2}{|\Sigma|} \left(|\Lambda_1| + |\Lambda_2| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)} \right)^{RL} I^{\alpha_2} \left({}^{H}I^{\beta_2}(1) \right)(b), \\
Q_3 &: = \frac{|\lambda_1| \, l_1}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}(1) \right)(b) \\
&+ \frac{|\lambda_4| \, l_1}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^{RL} I^{\alpha_1}(1)(b), \\
Q_4 &: = \frac{|\lambda_2| \, l_2}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^H I^{\beta_2}(1)(b) \\
&+ \frac{|\lambda_3 l_2|}{|\Sigma|} \left(\frac{|\gamma_4|}{|\gamma_3|} |\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|} |\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \right)^{RL} I^{\alpha_2} \left({}^{H}I^{\beta_2}(1) \right)(b) + l_2^{RL} I^{\alpha_2} \left({}^{H}I^{\beta_2}(1) \right)(b), \\
M_1 &: = \frac{\left(\log \frac{b}{a} \right)^{1-\beta_1}}{\Gamma(2-\beta_1)}, \\
M_2 &: = \frac{\left(\log \frac{b}{a} \right)^{1-\alpha_2}}{\Gamma(2-\alpha_2)},
\end{aligned}$$

where,

$$l_1 = \max(l_{11}, l_{12}, l_{13}), \quad l_2 = \max(l_{21}, l_{22}, l_{23}).$$

Now, we are able to prove the following result.

Théorème 4.1 *Assume that* $(H1)$ *is satisfied. Then, the problem* (4.2) *has a unique solution* *on* $[a,b]$, *provided that* $Q<1$, *where*

$$Q:=\max\left\{\max\left(\left(Q_1+Q_2\right),M_1\left(Q_1+Q_2\right)\right),\max\left(\left(Q_3+Q_4\right),M_2\left(Q_3+Q_4\right)\right)\right\}.$$

Proof. We show that the operator $\mathcal{T}$ is contractive. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then, for each $t\in[a,b]$, we have

$$\begin{aligned}
\left|\mathcal{T}_1(x_2,y_2)(t)-\mathcal{T}_1(x_1,y_1)(t)\right| \leq\; & \frac{1}{|\Sigma|}\left[\left(|\Lambda_3|+|\Lambda_4|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\left(|\lambda_1|\,l_1\left(\|x_2-x_1\|\right.\right.\right.\\
& +\|y_2-y_1\|+\left\|{}^{H}D^{\alpha_2}(y_2-y_1)\right\|\Big)^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b)\\
& +|\lambda_2|\,l_2\left(\|x_2-x_1\|+\|y_2-y_1\|+\left\|{}^{H}D^{\beta_1}(x_2-x_1)\right\|\right)\\
& \times{}^{H}I^{\beta_2}(1)(b)\Big)\Big]+\frac{1}{|\Sigma|}\left[\left(|\Lambda_1|+|\Lambda_2|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\right.\\
& \times\left(|\lambda_3|\,l_2\,\|x_2-x_1\|+\|y_2-y_1\|+\left\|{}^{H}D^{\beta_1}(x_2-x_1)\right\|\right)\\
& \times{}^{RL}I^{\alpha_2}\left({}^{H}I^{\beta_2}(1)\right)(b)+|\lambda_4|\,l_1\left(\|x_2-x_1\|+\|y_2-y_1\|\right.\\
& \left.\left.+{}^{H}D^{\alpha_2}\|y_2-y_1\|\right)^{RL}I^{\alpha_1}(1)(b)\right]\\
& +l_1\left(\|x_2-x_1\|+\|y_2-y_1\|\right.\\
& \left.+{}^{H}D^{\alpha_2}\|y_2-y_1\|\right)^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b).
\end{aligned}$$

Consequently, the following estimate is valid

$$\begin{aligned}
\left|\mathcal{T}_1(x_2,y_2)(t)-\mathcal{T}_1(x_1,y_1)(t)\right| &\leq \frac{|\lambda_1| l_1}{|\Sigma|}\left(|\Lambda_3|+|\Lambda_4|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \qquad (4.10)\\
&\times\left(\|x_2-x_1\|_X+\|y_2-y_1\|_Y\right)^H I^{\beta_1} I^{\alpha_1}(1)(b)\\
&+\frac{|\lambda_2| l_2}{|\Sigma|}\left(|\Lambda_3|+|\Lambda_4|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\\
&\times\left(\|x_2-x_1\|_X+\|y_2-y_1\|_Y\right)^H I^{\beta_2}(1)(b)\\
&+\frac{|\lambda_3| l_2}{|\Sigma|}\left(|\Lambda_1|+|\Lambda_2|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\\
&\times\left(\|x_2-x_1\|_X+\|y_2-y_1\|_Y\right) I^{\alpha_2 H} I^{\beta_2}(1)(b)\\
&+\frac{|\lambda_4| l_1}{|\Sigma|}\left(|\Lambda_1|+|\Lambda_2|\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)\\
&+l_1\left(\|x_2-x_1\|_X+\|y_2-y_1\|_Y\right)^H I^{\beta_1} I^{\alpha_1}(1)(b)\\
&\leq (Q_1+Q_2)\max\left(\|x_2-x_1\|_\infty,\|y_2-y_1\|_\infty\right).
\end{aligned}$$

On other hand, using the β_1-norm, we can write

$$\begin{aligned}
\left|{}^H D^{\beta_1}\mathcal{T}_1(x_2,y_2)(t)-{}^H D^{\beta_1}\mathcal{T}_1(x_1,y_1)(t)\right| &\leq \frac{1}{\Gamma(1-\beta_1)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{s}\right)^{-\beta_1} \qquad (4.11)\\
&\times\left|\mathcal{T}_1(x_2,y_2)(t)-\mathcal{T}_1(x_1,y_1)(s)\right|\frac{ds}{s}\\
&\leq M_1(Q_1+Q_2)\max\left(\|x_2-x_1\|_\infty,\|y_2-y_1\|_\infty\right)
\end{aligned}$$

Similarly, for $\mathcal{T}_2$, we remark that

$$\left|\mathcal{T}_2(x_2,y_2)(t)-\mathcal{T}_2(x_1,y_1)(t)\right|\leq (Q_3+Q_4)\max\left(\|x_2-x_1\|_\infty,\|y_2-y_1\|_\infty\right).$$

Also, using α_2-norm, we have

$$\begin{aligned}\left|{}^{H}D^{\alpha_2}\mathcal{T}_2(x_2,y_2)(t)-{}^{H}D^{\alpha_2}\mathcal{T}_2(x_1,y_1)(t)\right| &\leq \frac{1}{\Gamma(1-\alpha_2)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha_2} \\ &\quad\times\left|\mathcal{T}_2(x_2,y_2)(t)-\mathcal{T}_2(x_1,y_1)(s)\right|\frac{ds}{s} \\ &\leq M_2(Q_3+Q_4)\max\left(\|x_2-x_1\|_\infty,\|y_2-y_1\|_\infty\right)\end{aligned}$$

Thanks to $(4.11),(4.10)$, we obtain

$$\begin{aligned}\|\mathcal{T}_1(x_2,y_2)-\mathcal{T}_1(x_1,y_1)\|_X &= \max\left(\|\mathcal{T}_1(x_2,y_2)-\mathcal{T}_1(x_1,y_1)\|,\left\|{}^{H}D^{\beta_1}\mathcal{T}_1(x_2,y_2)-{}^{H}D^{\beta_1}\mathcal{T}_1(x_1,y_1)\right\|\right) \\ &\leq \max\left((Q_1+Q_2),M_1(Q_1+Q_2)\right)\max\left(\|x_2-x_1\|_X,\|y_2-y_1\|_Y\right)\end{aligned}$$

With the same arguments as before, for $\mathcal{T}_2$, we have

$$\|\mathcal{T}_2(x_2,y_2)-\mathcal{T}_2(x_1,y_1)\|_Y \leq \max\left((Q_3+Q_4),M_2(Q_3+Q_4)\right)\max\left(\|x_2-x_1\|_X,\|y_2-y_1\|_Y\right)$$

consequently, we obtain

$$\|\mathcal{T}(x_2,y_2)-\mathcal{T}(x_1,y_1)\|_{X\times Y}\leq Q\max\left(\|x_2-x_1\|_X,\|y_2-y_1\|_Y\right).$$

Using the fact that $Q<1$, we conclude that $\mathcal{T}$ is a contraction mapping.
As consequence of Banach's fixed point theorem, the problem (4.2) admits a unique solution over $[a,b]$.

4.3 Solvability: Existence

The second main result is based on Schaefer fixed point theorem. We prove the following existence result.

Théorème 4.2 *Assume that the following two hypotheses* (H_2) *and* (H_3) *are valid. Then, the problem* (4.2) *has at least one solution on* $[a,b]$.

Proof. First of all, it is to note that the operator is continuous since the given functions of our problem are also continuous.
Then, the following steps are needed to achieve the proof of this results.
Step1: We show that the operator $\mathcal{T}$ maps bounded sets into bounded sets in $X\times Y$.

we prove that $\mathcal{T}_1$ maps bounded sets into bounded set in X.Let Ω bounded in $X \times Y$. By applying lemma 2.2 for the operator $\mathcal{T}_1$ with $f = \phi, g = \omega$ and $M = M_1$, then for each $t \in [a,b]$ and $(x,y) \in \Omega$, we have

$$\left|\mathcal{T}_1(x,y)(t)\right| \leq K_1Q_1 + K_2Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right).$$

In other hand, we have

$$\left|{}^H D^{\beta_1}\mathcal{T}_1(x,y)(t)\right| \leq M_1\left(K_1Q_1 + K_2Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\right). \tag{4.12}$$

So, (2.8) and (4.12) yields

$$\|\mathcal{T}_1(x,y)(t)\|_X \leq \max\left(\begin{array}{c} K_1Q_1 + K_2Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right), \\ M_1\left(K_1Q_1 + K_2Q_2 + \frac{1}{|\Sigma|}\left(|\Lambda_3\theta_2| + (|\Lambda_1| + |\Lambda_4|)|\theta_4| + |\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\right) \end{array}\right) \tag{4.13}$$

By the same method followed in lemma 2.2, we prove that $\mathcal{T}_2$ maps bounded sets into bounded set in Y. We have

$$\begin{aligned} \left|\mathcal{T}_2(x,y)(t)\right| \leq\ & \left|\frac{\theta_3}{\gamma_3}\right| + \left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \\ & + \left|\frac{1}{\Sigma}\right|\left(\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4| + \left|\frac{\gamma_1}{\gamma_2}\Lambda_3\right|\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\ & \times\left(\theta_2 - \left(\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1} f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\right)\right.\right. \\ & \left.\left. + \lambda_2^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right) - \lambda_2\frac{\theta_1}{\gamma_2}\right) \\ & - \frac{1}{\Sigma}\left[\frac{\gamma_4}{\gamma_3}\Lambda_4\theta_4 - \left(\frac{\gamma_4}{\gamma_3}\Lambda_4 + \frac{\gamma_1}{\gamma_2}\Lambda_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\right. \\ & \times\left(\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right)\right) \end{aligned}$$

$$+\lambda_4^{RL}I^{\alpha_1}f\left(b,x\left(b\right),y\left(b\right),^{H}D^{\alpha_2}y\left(b\right)\right)-\lambda_3\frac{\theta_3}{\gamma_3}\bigg)+\frac{\gamma_1}{\gamma_2}\theta_1\Lambda_1\frac{(t-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\bigg]$$

$$+^{RL}I^{\alpha_2}\left(^{H}I^{\beta_2}g\left(t,x\left(t\right),y\left(t\right),^{H}D^{\beta_1}x\left(t\right)\right)\right)$$

then, we get the following estimate

$$\begin{aligned}\left|\mathcal{T}_2\left(x,y\right)(t)\right| &\leq \left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}+\frac{1}{|\Sigma|}\left(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4|\right.\\ &\quad\left.+\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)+K_1Q_3+K_2Q_4.\end{aligned}$$

Also, we have

$$\begin{aligned}\left|^{H}D^{\alpha_2}\mathcal{T}_2\left(x,y\right)(t)\right| &\leq M_2\left(K_1Q_3+K_2Q_4+\left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right.\\ &\quad+\frac{1}{|\Sigma|}\left(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4|+\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right.\\ &\quad\left.\left.+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\right).\end{aligned}$$

Then,

$$\left\|^{H}D^{\alpha_2}\mathcal{T}_2\left(x,y\right)(t)\right\|_Y\leq\max\begin{pmatrix}\left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{1}{|\Sigma|}\left(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4|\right.\\ \left.+\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)+K_1Q_3+K_2Q_4,\\ M_2\left(K_1Q_3+K_2Q_4+\left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}\right.\\ +\frac{1}{|\Sigma|}\left(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4|\right.\\ \left.\left.+\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\right)\end{pmatrix} \tag{4.14}$$

Hence, from (4.13) and(4.14), we deduce that $\mathcal{T}\Omega$ is a uniformly bounded set.

Step2: We prove that $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let $t_1,t_2\in[a,b]$

such that $t_1 < t_2$, and let $(x,y) \in \Omega$, then

$$\begin{aligned} \left|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)\right| &\leq \frac{|\Lambda_4|}{|\Sigma|\,\Gamma(\beta_1+1)} \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \\ &\times \Big(|\lambda_1|^H I^{\beta_1} \left({}^{RL}I^{\alpha_1} \left| f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\right|\right) \\ &+ |\lambda_2|^H I^{\beta_2} \left| g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right| \Big) \\ &+ \frac{|\Lambda_2|}{|\Sigma|\,\Gamma(\beta_1+1)} \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \\ &\times \Big(|\lambda_3|^{RL} I^{\alpha_2} \left({}^H I^{\beta_2} \left| g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right|\right) \\ &+ |\lambda_4|^{RL} I^{\alpha_1} \left| f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\right| \Big) \\ &+ {}^H I^{\beta_1 RL} I^{\alpha_1} \Big(\Big| f\left(t_2, x(t_2), y(t_2), {}^H D^{\alpha_2} y(t_2)\right) \\ &- f\left(t_1, x(t_1), y(t_1), {}^H D^{\alpha_2} y(t_1)\right) \Big| \Big). \end{aligned}$$

So, we can write

$$\begin{aligned} \left|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)\right| &\leq \frac{|\Lambda_4|}{|\Sigma|\,\Gamma(\beta_1+1)} \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \\ &\times \Big(K_1 |\lambda_1|^H I^{\beta_1} \left({}^{RL}I^{\alpha_1}(1)\right)(b) + K_2 |\lambda_2|^H I^{\beta_2}(1)(b) \Big) \\ &+ \frac{|\Lambda_2|}{|\Sigma|\,\Gamma(\beta_1+1)} \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \\ &\times \Big(K_2 |\lambda_3|^{RL} I^{\alpha_2} \left({}^H I^{\beta_2}(1)\right)(b) + |\lambda_4| K_1^{RL} I^{\alpha_1}(1)(b) \Big) \\ &+ \frac{K_1 (b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)\,\Gamma(\beta_1+1)} \left[2\left(\log\frac{t_2}{t_1}\right)^{\beta_1} + \left| \left(\log\frac{t_2}{a}\right)^{\beta_1} - \left(\log\frac{t_1}{a}\right)^{\beta_1} \right| \right] \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

On the other hand, we obtain

$$\left|{}^{H}D^{\beta_1}\mathcal{T}_1(x,y)(t_2) -{}^{H}D^{\beta_1}\mathcal{T}_1(x,y)(t_1)\right| \leq M_1 \left|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)\right|$$

Therefore, we obtain

$$\left|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)\right| \longrightarrow 0, \text{ as } t_1 \longrightarrow t_2.$$

With the same manner, we can show that

$$\left|\mathcal{T}_2(x,y)(t_2) - \mathcal{T}_2(x,y)(t_1)\right| \longrightarrow 0, \text{ as } t_1 \longrightarrow t_2.$$

Thanks to the Steps1,2 and using Arzela-Ascoli theorem, we conclude that the operator $\mathcal{T}$ completely continuous.

Stpe3: Now, we show that the set

$$\mathcal{E} = \{(x,y) \in X \times Y : (x,y) = \lambda \mathcal{T}(x,y),\ 0 < \lambda < 1\}$$

is bounded.
If $(x,y) \in \mathcal{E}$, this yields that

$$\begin{cases} x(t) = \lambda \mathcal{T}_1(x,y)(t) \\ y(t) = \lambda \mathcal{T}_2(x,y)(t) \end{cases}, \forall t \in [a,b]$$

Hence, we have

$$|x(t)| \leq \lambda \|\mathcal{T}_1(x,y)(t)\| \leq \|\mathcal{T}_1(x,y)\|$$

and

$$|y(t)| \leq \lambda \|\mathcal{T}_2(x,y)(t)\| \leq \|\mathcal{T}_2(x,y)\|$$

On the other hand,

$$\left|{}^{H}D^{\beta_1}x(t)\right| \leq \left|{}^{H}D^{\beta_1}\mathcal{T}_1(x,y)(t_2)\right|$$

$$\left|{}^{H}D^{\alpha_2}\mathcal{T}_2 y(t)\right| \leq \left|{}^{H}D^{\alpha_2}\mathcal{T}_2(x,y)(t)\right|$$

Thus, we get

$$\|(x,y)\|_{X\times Y} := \max\left(\|x(t)\|_X, \|y(t)\|_Y\right) \leq \max\left(\|\mathcal{T}_1(x,y)\|, \|\mathcal{T}_2(x,y)\|\right) < \infty$$

At the end, in view of Schaefer's fixed point theorem, we conclude that $\mathcal{T}$ has a fixed point which is a solution of the problem (4.2).

4.4 Illustration

Let us consider the example:

$$\begin{cases} {}^{C}D^{\frac{2}{3}\,H}D^{\frac{1}{4}}x(t) = f\left(t, x(t), y(t), {}^{H}D^{\frac{6}{7}}y(t)\right), \ 1 \leq t \leq 3, \\ {}^{H}D^{\frac{4}{5}\,C}D^{\frac{6}{7}}y(t) = g\left(t, x(t), {}^{H}D^{\frac{1}{4}}x(t), y(t)\right), \ \ 1 \leq t \leq 3, \\ 0.2x(1) + 1.2\,{}^{C}D^{\frac{6}{7}}y(1) = 1.3, \quad 0.6x(3) + 2.6\,{}^{C}D^{\frac{6}{7}}y(3) = 0.9, \\ 2.15y(1) + 1.6\,{}^{H}D^{\frac{1}{4}}x(1) = 1.7, \quad 0.3y(1) + 1.6\,{}^{H}D^{\frac{1}{4}}x(1) = 3.2. \end{cases} \tag{4.15}$$

Here, $\alpha_1 = \frac{2}{3}$; $\alpha_2 = \frac{6}{7}$; $\beta_1 = \frac{1}{4}$; $\beta_2 = \frac{4}{5}$; $\gamma_1 = 0.2$; $\gamma_2 = 1.2$; $\gamma_3 = 2.15$; $\gamma_4 = 1.6$; $\theta_1 = 1.3$; $\theta_2 = 0.9$; $\theta_3 = 1.7$; $\theta_4 = 3.2$; $\lambda_1 = 0.6$; $\lambda_2 = 2.6$; $\lambda_3 = 0.3$; $\lambda_4 = 1.6$, and the functions $f, g : [1;3] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by

$$\begin{cases} f\left(t, x(t), y(t), {}^{H}D^{\frac{6}{7}}y(t)\right) = \frac{1}{2}t + \frac{t^2 x(t)}{54(1+x(t))} + \frac{1}{6}\cos y(t) + \frac{t}{18}\tan^{-1}\left({}^{H}D^{\frac{6}{7}}y(t)\right) \\ g\left(t, x(t), {}^{H}D^{\frac{1}{4}}x(t), y(t)\right) = \frac{1}{7}\tan^{-1}(x(t)) + \frac{t y(t)}{21(1+y(t))} + \frac{t}{29}\frac{{}^{H}D^{\frac{1}{4}}x(t)}{\left(1 + {}^{H}D^{\frac{1}{4}}x(t)\right)} \end{cases}$$

It is clear that f, g are continuous functions and we have:

$$|f(t,x,y,z)| \leq \frac{13}{21} = K_1, |g(t,x,y,z)| \leq \frac{79}{203} = K_2$$

Thanks Theorem (4.2), the system (4.2) has at least one solution $(x(t), y(t)), t \in [1,3]$.

GENERAL CONCLUSION AND PERSPECTIVES

In our thesis project, we have presented some notions, some definitions, some auxiliary lemmas and some other integro- differential concepts and properties that we have used in the other chapters for proving the main results. Then, we have proved some integral lemmas that have relationships with the results of the other chapters. These integral estimates have allowed us to find the integral equations for our studied differential problems. We have also been concerned with some classes of differential equations that have the property of "sequential". This notion has not a sense in the classical cases of differential equations since the two important properties, regarding the commutativity and the semi group of the derivatives of integer orders, are satisfied. In this sense, some existence and uniqueness criteria have been established and proved. Some other results on the establishment of sufficient conditions "for the existence of at least" have also been discussed in this project. We have illustrated the main results with some examples to show to the reader the applicability of some of the main results.

At the end, we propose to the interested reader to investigate the studied problems using the important approach of fractional calculus in the sense of Khalil. Also the interested reader can use the Atangana Baleanu approach to study the above problems. Certainly, he will have the possibility to propose comparative studies in this sense. In our opinion, we suppose that these two approaches are important and they can be used for modelling many real phenomena.

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