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THEME

Croissance et oscillation des solutions des équations différentielles linéaires au voisinage d'un point singulier

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Abstract

This thesis is devoted to the study of the growth and oscillation of solutions to certain classes of linear differential equations whose coefficients are analytic in the closed complex plane except a finite singular point. For that, we use the Nevanlinna theory with an adapted notions by a conformal mapping. We point out that there are several similarities between the results of the complex plane and the results obtained in this work.

Key words : Linear differential equation, finite singular point, growth of solutions, exponent of convergence, Nevanlinna theory, order of growth, type of growth, oscillation, entire function, meromorphic function, analytic function.

ملخص

هذه الاطروحة مخصصة لدراسة تزايد و تذبذب حلول بعض الاصناف من المعادلات التفاضلية الخطية ذات عوامل تحليلية في الستوى المركب المغلق باستثناء نقطة شاذة منتهية. لهذا الغرض نستعمل نظرية نيفانلينا مع تكييف بعض المفاهيم والتعاريف و هذا باستعمال تحويل نقطي حافض للزوايا. نشير إلى أن هناك عدة أوجه تشابه بين نتائج المستوى المركب والنتائج المتحصل عليها في هذا العمل.

الكلمات المفتاحية: المعادلات التفاضلية الخطية، نقطة شذة منتهية، تزايد الحلول، أس التقارب، نظرية نيفانلينا، رتبة التزايد، نوع التزايد، التذبذب، الدوال الصحيحة، الدوال الميرومورفية، الدوال التحليلية.

Résumé

Cette thèse est consacré à l'étude de la croissance et l'oscillation des solutions de certaines classes d'équations différentielles linéaires dont les coefficients sont des fonctions analytiques dans le plan complexe fermé privé d'un point singulier fini. Pour celà, nous utilisons la théorie de Nevanlinna avec des notions adaptées par une tranformation conforme. Nous soulignons qu'ils existent plusieurs similitudes entre les résultats du plan complexe et les résultats obtenus dans ce travail.

Mots clés: Equations différentielles linéaires, point singulier fini, croissance des solutions, exposant de convergence, théorie de Nevanlinna, ordre de croissance, type de croissance, oscillation, fonction entière, fonction méromorphe, fonction analytique
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Contents

Introduction

Since 1925, the year when R. Nevanlinna published the results of his work on the theory of the distribution of the values of meromorphic functions, researchers have not ceased to publish in the same thematic and several problems have been studied and resolved. Close links with other fields are highlighted, in particular with the analytical theory of differential equations. Nevanlinna theory is an essential tool in the study of the properties of solutions of complex differential equations notably the growth and oscillation of solutions. For an introduction to the theory of differential equations in the complex plane with the theory of Nevanlinna see [49, 55]. Active research in this field was launched by H. Wittich and his students in the 1950s and 1960s. One of the important results due to Wittich concerning the growth of solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f^{'} + A_0(z) f = 0, \qquad (0.0.1)$$

is as follows: The coefficients $A_0, \ldots, A_{k-1}$ are polynomials if and only if all the solutions of the previous equation are entire functions of finite order of growth. Mr. Frei extended the above result, assuming that A_j, the last coefficient which is transcendent while all the coefficients $A_{j+1}, \ldots, A_{k-1}$ are polynomials, and he demonstrated that the equation has at most j linearly independent solutions of finite order.

In 1982, the theory of complex oscillation of solutions of linear differential equations in the complex plane $\mathbb{C}$ was first introduced by Bank and Laine [2]; they studied the oscillation of differential equations of the form $f^{''} + Af = 0$ where A is an entire function, then in 1983, they studied the zeros of meromorphic solutions of second order linear differential equations.

The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [8, 45, 46, 47, 53]. Recently in [25, 33], Fettouch and Hamouda investigated the growth of solutions of certain linear differential equations near a finite singular point by using an adapted definitions and properties of Nevanlinna theory by introducing a conformal mapping. In this work, we continue this investigation to study the behavior of solutions near a finite singular point to other types of linear differential equations.

This thesis contains an introduction and five chapters. In the first chapter, we will cite some notations, definitions and results that we need in our work. In the second chapter, we investigate the growth of solutions of the differential equation

$$f'' + A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z) \exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \qquad (0.0.2)$$

where $A(z)$ and $B(z)$ are analytic functions in the closed complex plane except at z_0 and a, b are complex constants such that $ab \neq 0$ and $a = cb$ $(c > 1)$. Another case has been studied for higher order linear differential equations with analytic coefficients having the same order near a finite singular point. In the third chapter, we will study the growth of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\} f^{(k-1)} + ... + A_0(z) \exp\left\{\frac{a_0}{(z_0 - z)^n}\right\} f = 0,$$

where $A_j(z)$ are analytic functions in the closed complex plane except at z_0 of order near z_0 less than n and a_j $(j = 0, \ldots, k-1)$ are distinct complex numbers. In the fourth chapter, firstly we will prove that all non trivial solutions are of infinite order when $A_0(z)$ dominates the other coefficients on a curve near the singular point z_0. In the other hand, we study the existence of a finite order solution in the case when a coefficient $A_s(z), s \neq 0$ dominates the others in a sector near z_0. In the last chapter, we investigate the exponent of convergence of $f^{(i)} - \varphi$ where $f \not\equiv 0$ is a solution to certain class of linear differential equations with analytic and meromorphic coefficients in the closed complex plane except a finite singular point and φ is a small function of f.

Chapter 1

Preliminaries

1.1 Some elements of the theory of R. Nevanlinna

1.1.1 Jensen's formula

Theorem 1.1.1 *[37, 49] Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let a_1, a_2, ... (resp. b_1, b_2, ...) denote its zeros (resp. poles), each taken into account according to its multiplicity. Then*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|d\varphi + \sum_{|b_j|<r} \log\frac{r}{|b_j|} - \sum_{|a_i|<r} \log\frac{r}{|a_i|}.$$

Definition 1.1.1 *For any real number $x > 0$, we define $\log^+ x$ by*

$$\log^+ x = \max\{0,\ \log x\}.$$

The properties of the truncated logarithm are contained in the following lemma.

Lemma 1.1.1 *[49] Let α, β, α_i strictly positive real numbers. So we have*

a) $\log\alpha \leq \log^+\alpha$,

b) $\log^+\alpha \leq \log^+\beta$ *for* $\alpha \leq \beta$,

c) $\log\alpha = \log^+\alpha - \log^+\frac{1}{\alpha}$,

d) $|\log\alpha| = \log^+\alpha + \log^+\frac{1}{\alpha}$,

e) $\log^+(\prod_{i=1}^{n}\alpha_i) \leq \sum_{i=1}^{n}\log^+\alpha_i$,

f) $\log^+\left(\sum_{i=1}^{n}\alpha_i\right) \leq \sum_{i=1}^{n}\log^+\alpha_i + \log n$,

Definition 1.1.2 *[37, 49] Let f be a meromorphic function, not being identically equal to $a \in \mathbb{C}$. Let $i(z, a, f)$ denotes the multiplicity of an a -point of f at z. Then we define*

$$n(r, a, f) = n\left(r, \frac{1}{f-a}\right) = \sum_{\substack{|z| \le r \\ f(z)=a}} i(z, a, f),$$

i.e, $n(r, a, f)$ counts the number of the roots of $f(z) = a$ in $|z| \le r$, each root according to its multiplicity. for the poles of f, we define similarly

$$n(r, \infty, f) = n(r, f) = \sum_{\substack{|z| \le r \\ f(z)=\infty}} i(z, \infty, f).$$

1.1.2 Counting function

Definition 1.1.3 *[37, 49] For a meromorphic function f, we define*

$$N(r, a, f) = N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f)\log r$$

for $a \ne \infty$ and

$$N(r, \infty, f) = N(r, f) = \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f)\log r$$

$N(r, a, f)$ is called the counting function or the function a-points of f in the disc $|z| \le r$.

1.1.3 Proximity function

Definition 1.1.4 *[37, 49] For a meromorphic function f, we define*

$$m(r, a, f) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\varphi}) - a}\right| d\varphi, \ a \ne \infty$$

and

$$m(r, \infty, f) = m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi,$$

$m(r, a, f)$ is called the proximity function of the function f at point a.

1.1.4 Characteristic function

Definition 1.1.5 *[42] The characteristic function of a meromorphic function f will be defined as*

$$T(r,f) = N(r,f) + m(r,f).$$

Example 1.1.1 *For the function $f(z) = e^{az}$ ($a \in \mathbb{C}^*$) , we have $m(r,f) = \dfrac{|a|\,r}{\pi}$, $N(r,f) = 0$, so $T(r,f) = \dfrac{|a|\,r}{\pi}$.*

Lemma 1.1.2 *[49] Let $f, f_1, ..., f_n$ be meromorphic functions and $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Then*

(A) $\quad m\left(r, \sum\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} m(r, f_i) + \log n$

(B) $\quad m\left(r, \prod\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} m(r, f_i)$

(C) $\quad N\left(r, \sum\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} N(r, f_i)$

(D) $\quad N\left(r, \prod\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} N(r, f_i)$

(E) $\quad T\left(r, \sum\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} T(r, f_i) + \log n \quad$ *for* $n \geq 1$

(F) $\quad T\left(r, \prod\limits_{i=1}^{n} f_i\right) \leq \sum\limits_{i=1}^{n} T(r, f_i) \quad$ *for* $n \geq 1$

(G) $\quad T(r, f^n) = nT(r, f), \quad$ *pour* $(n \in \mathbb{N}^*)$

(H) $\quad T\left(r, \frac{af+b}{cf+d}\right) = T(r,f) + O(1), \ f \not\equiv -\dfrac{d}{c}.$

Proof. We will prove only some properties.
(E) We have

$$\begin{aligned} T\left(r, \sum_{i=1}^{n} f_i\right) &= m\left(r, \sum_{i=1}^{n} f_i\right) + N\left(r, \sum_{i=1}^{n} f_i\right) \\ &\leq \sum_{i=1}^{n} m(r, f_i) + \log n + \sum_{i=1}^{n} N(r, f_i) \\ &= \sum_{i=1}^{n} T(r, f_i) + \log n. \end{aligned}$$

(F) We have

$$m\left(r, \prod_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} m(r, f_i)$$

and

$$N\left(r,\prod_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} N\left(r,f_i\right)$$

therefore

$$T\left(r,\prod_{i=1}^{p} f_i\right) = m\left(r,\prod_{i=1}^{n} f_i\right) + N\left(r,\prod_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} T\left(r,f_i\right).$$

(G) We have $|f^n| = |f|^n \leq 1 \Leftrightarrow |f| \leq 1$
If $|f| \leq 1$, so

$$T\left(r,f^n\right) = N\left(r,f^n\right) = nN\left(r,f\right).$$

If $|f| > 1$, so

$$\begin{aligned} T\left(r,f^n\right) &= m\left(r,f^n\right) + N\left(r,f^n\right) \\ &= nm\left(r,f\right) + nN\left(r,f\right) = nT\left(r,f\right). \end{aligned}$$

(H) Set $f_0 = f,\ f_1 = f_0 + \dfrac{d}{c},\ f_2 = cf_1,\ f_3 = \dfrac{1}{f_2}$,
$f_4 = \dfrac{bc-ad}{c} f_3,\ f_5 = f_4 + \dfrac{a}{c}$, if $c \neq 0, T(r,f_{k+1}) = T(r,f_k) + O\left(1\right)$.
Hence the result.

Proposition 1.1.1 *[37, 49] Let f be a meromorphic function with the development of Laurent expansion*

$$f\left(z\right) = \sum_{i=m}^{\infty} c_i z^i, \quad c_m \neq 0,\ m \in \mathbb{Z},$$

at the origin. Then

$$\log\left|c_m\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + N\left(r,f\right) - N\left(r,\frac{1}{f}\right).$$

1.1.5 First fundamental theorem of R. Nevanlinna

Theorem 1.1.2 *[37, 49] Let f be a meromorphic function , $a \in \mathbb{C}$ and*

$$f\left(z\right) - a = \sum_{i=m}^{\infty} c_i z^i, \quad c_m \neq 0,\ m \in \mathbb{Z},$$

be the Laurent expansion of $f - a$ at the origin. Then ,

$$T(r,a,f) = T(r,\frac{1}{f-a}) = T(r,f) - \log\left|c_m\right| + \varphi\left(r,a\right),$$

where

$$\left|\varphi\left(r,a\right)\right| \leq \log^{+}\left|a\right| + \log 2.$$

Proof. Assume first $a=0$. By Lemma 1.1.1 (c) and Proposition 1.1.1, we obtain

$$\begin{aligned}
\log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f\left(re^{i\theta}\right)}\right|d\theta+N\left(r,f\right)-N\left(r,\frac{1}{f}\right)\\
&= m\left(r,f\right)-m\left(r,\frac{1}{f}\right)+N\left(r,f\right)-N\left(r,\frac{1}{f}\right)\\
&= T\left(r,f\right)-T\left(r,\frac{1}{f}\right).
\end{aligned}$$

therefore

$$T\left(r,\frac{1}{f}\right)=T\left(r,f\right)-\log|c_m|, \qquad (1.1.1)$$

with $\varphi\left(r,0\right)\equiv 0$.
Proceeding now to the general case $a\neq 0$. we define $h=f-a$. Then

$$N\left(r,\frac{1}{h}\right)=N\left(r,\frac{1}{f-a}\right),\quad N\left(r,h\right)=N\left(r,f-a\right)=N\left(r,f\right),$$

and

$$m\left(r,\frac{1}{h}\right)=m\left(r,\frac{1}{f-a}\right).$$

Moreover,

$$\begin{aligned}
\log^+|h| &= \log^+|f-a|\leq\log^+|f|+\log^+|a|+\log 2.\\
\log^+|f| &= \log^+|f-a+a|=\log^+|h+a|\\
&\leq \log^+|h|+\log^+|a|+\log 2.
\end{aligned}$$

By integrating the two members from 0 to 2π ,we find

$$\begin{aligned}
m\left(r,h\right) &\leq m\left(r,f\right)+\log^+|a|+\log 2,\\
m\left(r,f\right) &\leq m\left(r,h\right)+\log^+|a|+\log 2,
\end{aligned}$$

Set

$$\varphi\left(r,a\right)=m\left(r,h\right)-m\left(r,f\right),$$

Then

$$|\varphi\left(r,a\right)|\leq\ln^+|a|+\ln 2.$$

Applying (1.1.1) for h we obtain

$$\begin{aligned}
T\left(r,\frac{1}{h}\right) &= m\left(r,\frac{1}{h}\right)+N\left(r,\frac{1}{h}\right)\\
&= T\left(r,h\right)-\log|c_m|\\
&= m\left(r,h\right)+N\left(r,h\right)-\log|c_m|\\
&= m\left(r,f\right)+\varphi\left(r,a\right)+N\left(r,f\right)-\log|c_m|\\
&= T\left(r,f\right)-\log|c_m|+\varphi\left(r,a\right).
\end{aligned}$$

So

$$T(r,a,f)=T(r,\frac{1}{f-a})=T(r,f)-\log|c_m|+\varphi(r,a),$$

where

$$|\varphi(r,a)|\leq\log^+|a|+\log 2.$$

Remark 1.1.1 *The first main theorem may be expressed as:*

$$T(r,\frac{1}{f-a})=T(r,f)+O(1)$$

for all $a\in\mathbb{C}$.

1.1.6 The growth of a meromorphic and entire function in the complex plane

Definition 1.1.6 *[37, 49] Let f be an entire function. Then the order and the hyper-order of f are defined respectively by*

$$\sigma(f)=\limsup_{r\longrightarrow+\infty}\frac{\log\log M(r,f)}{\log r}$$

and

$$\sigma_2(f)=\limsup_{r\longrightarrow+\infty}\frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f)=\max\limits_{|z|=r}f(z)$. If f is a meromorphic function, then the order and the hyper-order of f are defined respectively by

$$\sigma(f)=\limsup_{r\longrightarrow+\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\sigma_2(f)=\limsup_{r\longrightarrow+\infty}\frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of f.

Definition 1.1.7 *[37, 49] Let f be a meromorphic function , we define the n-iterated order of f by*

$$\sigma_n(f)=\limsup_{r\to+\infty}\frac{\log_n T(r,f)}{\log r},$$

and if f is an entire function , we define the n-iterated order of f by

$$\sigma_n(f)=\limsup_{r\to+\infty}\frac{\log_{n+1} M(r,f)}{\log r}.$$

where $\log_{n+1}(x)=\log\log_n(x)$ *($n\geq 1$ is an integer)*

Theorem 1.1.3 *[63] Suppose $h(z)$ is a non-constant entire function and $f(z) = e^{h(z)}$. Then*

$$\sigma_2(f) = \sigma(h).$$

Example 1.1.2 *[5] Let $f(z) = \exp\left\{\frac{\sin\sqrt{z}}{\sqrt{z}}\right\}$. Put*

$$h(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{+\infty} (-1)^n \frac{(\sqrt{z})^{2n}}{(2n+1)!} = \sum_{n=0}^{+\infty} (-1)^n \frac{z^n}{(2n+1)!}.$$

Since

$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = \limsup_{n\to+\infty} \sqrt[n]{\left|\frac{(-1)^n}{(2n+1)!}\right|} = \limsup_{n\to+\infty} \left(\frac{1}{(2n+1)!}\right)^{\frac{1}{n}} = 0,$$

then h is an entire function. Then by Theorem 1.1.3, we have

$$\sigma_2(f) = \sigma(h).$$

Let us calculate $\sigma(h)$. We have

$$M(r,h) = \max_{|z|=r} \left|\frac{\sin\sqrt{z}}{\sqrt{z}}\right| = \max_{|z|=r} \left|\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} z^n\right|$$

$$\leq \sum_{n=0}^{+\infty} \frac{1}{(2n+1)!} r^n = \frac{\sinh\sqrt{r}}{\sqrt{r}}$$

and for $z_0 = -r$ $(|z_0| = r)$, we get $|h(z_0)| = \left|\frac{\sin\sqrt{-r}}{\sqrt{-r}}\right| = \left|\frac{\sin(\pm i\sqrt{r})}{\pm i\sqrt{r}}\right| = \frac{\sinh\sqrt{r}}{\sqrt{r}}$,
so

$$M(r,h) = \frac{\sinh\sqrt{r}}{\sqrt{r}} = \frac{e^{\sqrt{r}} - e^{-\sqrt{r}}}{2\sqrt{r}} \sim \frac{e^{\sqrt{r}}}{2\sqrt{r}} \quad (r \to +\infty).$$

Then

$$\sigma(h) = \limsup_{r\to+\infty} \frac{\log\log M(r,h)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\left(\frac{e^{\sqrt{r}}}{2\sqrt{r}}\right)}{\log r}$$

$$= \limsup_{r\to+\infty} \frac{\log\left(\sqrt{r} - \log(2\sqrt{r})\right)}{\log r} = \limsup_{r\to+\infty} \frac{\log\sqrt{r}\left(1 - \frac{\log(2\sqrt{r})}{\sqrt{r}}\right)}{\log r}$$

$$= \limsup_{r\to+\infty} \frac{\frac{1}{2}\log r + \log\left(1 - \frac{\log(2\sqrt{r})}{\sqrt{r}}\right)}{\log r} = \frac{1}{2},$$

and

$$\sigma_2(f) = \sigma(h) = \frac{1}{2}.$$

Remark 1.1.2 *If f is of finite order, then the hyper order of this function is zero.*

Lemma 1.1.3 *[49] If f is a non-constant meromorphic function in $\mathbb{C}$, then $\sigma(f^{(k)}) = \sigma(f)$.*

Definition 1.1.8 *[39, 51] Let f be a meromorphic function in the complex plane of order $\sigma(f) = \sigma$ $(0 < \sigma < \infty)$. We define the type of f by*

$$\tau(f) = \limsup_{r \longrightarrow +\infty} \frac{T(r, f)}{r^{\sigma}}.$$

If f is an entire function of order $\sigma_M(f) = \sigma_M$ $(0 < \sigma_M < \infty)$, we define the type of f by

$$\tau_M(f) = \limsup_{r \longrightarrow +\infty} \frac{\log M(r, f)}{r^{\sigma_M}}.$$

Remark 1.1.3 *In general $\tau(f)$ does not equal $\tau_M(f)$, as the following example shows.*

Example 1.1.3 *For $f(z) = \exp\{z\}$, we have $T(r, f) = \frac{r}{\pi}$ and $M(r, f) = \exp\{r\}$. Therefore $\tau(f) = \frac{1}{\pi}$ and $\tau_M(f) = 1$.*

1.1.7 Exponent of convergence of zeros of a meromorphic function

Definition 1.1.9 *[50, 55, 56] Let f be a meromorphic function. We define the exponent of convergence of zeros of the function f by*

$$\lambda(f) = \limsup_{r \longrightarrow +\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}$$

and the exponent of convergence of distinct zeros of the function f by

$$\overline{\lambda}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}$$

where

$$\overline{N}\left(r, \frac{1}{f}\right) = \int_0^r \frac{\overline{n}\left(t, \frac{1}{f}\right) - \overline{n}\left(0, \frac{1}{f}\right)}{t} dt + \overline{n}\left(0, \frac{1}{f}\right) \log r,$$

$\overline{n}\left(t, \frac{1}{f}\right)$ denotes the number of distinct zeros of the function f located in the disc $|z| \leq t$.

Similarly, we define the n-iterative exponent of convergence of zeros of the function f by

$$\lambda_n(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_n N\left(r, \frac{1}{f}\right)}{\log r},$$

and distinct zeros by

$$\overline{\lambda}_n(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_n \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

Example 1.1.4 *Let* $f(z) = e^z - a$, $a \neq 0, \infty$.
We have $e^z = a \Leftrightarrow z = \ln a = \ln|a| + i(\arg a + 2k\pi)$, $k \in \mathbb{Z}$

$$\begin{aligned}
|z| &\leq t \Rightarrow \sqrt{(\ln|a|)^2 + (\arg a + 2k\pi)^2} \leq t \\
&\Rightarrow \frac{-\sqrt{t^2 - (\ln a)^2} - \arg a}{2\pi} \leq k \leq \frac{\sqrt{t^2 - (\ln a)^2} - \arg a}{2\pi} \\
&\Rightarrow n\left(t, \frac{1}{f}\right) \sim \frac{\sqrt{t^2 - (\ln a)^2}}{\pi} \sim \frac{t}{\pi},\ t \to +\infty \\
&\Rightarrow N\left(r, \frac{1}{f}\right) = \frac{r}{\pi} + o(1) \\
&\Rightarrow \lambda(f) = 1.
\end{aligned}$$

1.2 Linear and logarithmic measure

Definition 1.2.1 *[37, 49] The linear measure of a set* $E \subset [0, +\infty)$ *is defined by*

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt,$$

where $\chi_E(t)$ *is the characteristic function of the set* E *and the logarithmic measure of a set* $F \subset [1, +\infty)$ *is defined by*

$$lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt.$$

Example 1.2.1 *The linear measure of the set* $E = [2; e] \subset [0; +\infty[$ *is*

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt = \int_2^e dt = e - 2.$$

The logarithmic measure of the set $F = [1; e^4] \subset [1; +\infty[$ *is*

$$lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt = \int_1^{e^4} \frac{dt}{t} = 4.$$

Lemma 1.2.1 *[37] Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(rT\left(r, f\right)\right)\right),$$

and if f is of finite order, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log r\right).$$

1.3 Maximum term and Central index

Definition 1.3.1 *[38, 49] Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function. It is clear that if for all $r > 0$, the series $\sum_{n=0}^{+\infty} a_n z^n$ converges, then for all given $r > 0$:*

$$\lim_{n\to\infty} |a_n| r^n = 0,$$

and the maximum term

$$\mu(r) = \mu(r, f) = \max_{n\geq 0} |a_n| r^n$$

is well defined.

Definition 1.3.2 *[38, 49] We define the central index by*

$$V(r) = V(r, f) = \max_{m\geq 0} \{m : |a_m| r^m = \mu(r, f)\}.$$

Example 1.3.1 1. *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $a_n \neq 0$ be a polynomial. For r sufficiently large, we have*

$$\mu(r, p) = |a_n| r^n,$$

and consequently

$$V(r) = V(r, p) = n.$$

2. *Let $f(z) = e^z$. The Taylor expansion of f is $f(z) = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n$. Set $a_n = \frac{1}{n!}$. we have*

$$\mu(r) = \mu(r, f) = \max_{n\geq 0} |a_n| r^n = \max_{n\geq 0} \frac{1}{n!} r^n.$$

Set $u_n = |a_n| r^n = \frac{1}{n!} r^n$. *Let's study the monotony of the sequence* u_n. *we have*

$$\frac{u_{n+1}}{u_n} = \frac{r}{n+1}.$$

So u_n *is decreasing if* $\frac{u_{n+1}}{u_n} < 1$, *that is to say* $n > [r] - 1$, *where the hook* $[r]$ *designates the entire part of* r. *the sequence* (u_n) *is increasing if* $\frac{u_{n+1}}{u_n} > 1$, *that is to say* $n < [r] - 1$, *hence*

$$\mu(r, f) = \frac{1}{[r]!} r^{[r]},$$

and consequently

$$V(r) = V(r, f) = [r].$$

1.4 Wiman-Valiron theorem

We recall the Wiman-Valiron theorem of entire functions.

Theorem 1.4.1 *[38, 49] Let* f *be an entire transcendental function. Then there exists a set* $E \subset (1, +\infty)$ *that has finite logarithmic measure, such that for all* $j = 0, 1, ...$ *we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V(r)}{z_r} \right)^j,$$

as $r \to \infty$, $r \notin E$, *where* z_r *is a point on the circle* $|z| = r$ *that satisfies* $|f(z_r)| = M(r, f) = \max_{|z|=r} |f(z)|$, $0 < r < +\infty$.

1.5 The growth and distribution of the values of an analytic and meromorphic function near an isolated finite singular point

Definition 1.5.1 *[25] Let* f *be a meromorphic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *where* $z_0 \in \mathbb{C}$, *we define the counting function near* z_0 *by*

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r,$$

where $n(t, f)$ *counts the number of poles of* $f(z)$ *in the region* $\{z \in \mathbb{C} : t \le |z - z_0|\} \cup \{\infty\}$ *each pole according to its multiplicity*

Definition 1.5.2 *[25] Let f be a meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ where $z_0 \in \mathbb{C}$, we define the proximity function near z_0 by*

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(z_0 - re^{i\varphi}\right)\right| d\varphi.$$

Definition 1.5.3 *[25] Let f be a meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ where $z_0 \in \mathbb{C}$. We define the characteristic function near z_0 by*

$$T_{z_0}(r,f) = N_{z_0}(r,f) + m_{z_0}(r,f).$$

Definition 1.5.4 *[25] Let f be an analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Then the order and the hyper-order of f near z_0 are defined respectively by*

$$\begin{aligned} \sigma_M(f,z_0) &= \limsup_{r\to 0}\frac{\log^+\log^+ M_{z_0}(r,f)}{-\log r}, \\ \sigma_{2,M}(f,z_0) &= \limsup_{r\to 0}\frac{\log^+\log^+\log^+ M_{z_0}(r,f)}{-\log r}, \end{aligned}$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z-z_0| = r\}$. If f is a meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$, then the order and the hyper-order of f are defined by

$$\begin{aligned} \sigma_T(f,z_0) &= \limsup_{r\to 0}\frac{\log^+ T_{z_0}(r,f)}{-\log r}, \\ \sigma_{2,T}(f,z_0) &= \limsup_{r\to 0}\frac{\log^+\log^+ T_{z_0}(r,f)}{-\log r}. \end{aligned}$$

Definition 1.5.5 *[25] Let f be a meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite order $(0 < \sigma_T(f,z_0) = \sigma < \infty)$, we define the type of f near z_0 by*

$$\tau_T(f,z_0) = \limsup_{r\to 0} r^{\sigma} T_{z_0}(r,f).$$

If $f(z)$ is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite order $(0 < \sigma_M(f,z_0) = \sigma < \infty)$, we define the type of f by

$$\tau_M(f,z_0) = \limsup_{r\to 0} r^{\sigma}\log^+ M_{z_0}(r,f).$$

Definition 1.5.6 *[22] Let f be a meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$, we define the n-iterated order of f near z_0 by*

$$\sigma_{n,T}(f,z_0) = \limsup_{r\to 0}\frac{\log_n^+ T_{z_0}(r,f)}{-\log r},$$

If $f(z)$ is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, we define the n-iterated order of f near z_0 by

$$\sigma_{n,M}(f,z_0)=\limsup_{r\rightarrow 0}\frac{\log^+_{n+1}M_{z_0}(r,f)}{-\log r}.$$

where $\log^+_{n+1}(x)=\log^+\log^+_n(x)$ ($n\geq 1$ is an integer)

Remark 1.5.1 *It is shown in [25] that if f is a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ and $g(w)=f\left(z_0-\frac{1}{w}\right)$ then $g(w)$ is a meromorphic in $\mathbb{C}$ and we have*

$$m(R,g)=m_{z_0}(r,f),\ T(R,g)=T_{z_0}(r,f),$$

where $|w|=R=\frac{1}{r}$, therefore $\sigma(f,z_0)=\sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, then $g(w)$ is entire and thus $\sigma_T(f,z_0)=\sigma_M(f,z_0)$ and in general $\sigma_{n,T}(f,z_0)=\sigma_{n,M}(f,z_0)$ $n\geq 1$. So, we can use the notation $\sigma_n(f,z_0)$ without any ambiguity. But concerning the type, as in the complex plane, $\tau_T(f,z_0)$ does not equal to $\tau_M(f,z_0)$.

Example 1.5.1 *For the function $f(z)=\exp\left\{\frac{1}{(z_0-z)^n}\right\}$, where $n\in\mathbb{N}^*$, we have $M_{z_0}(r,f)=\exp\left\{\frac{1}{r^n}\right\}$, then $\sigma_M(f,z_0)=n$ and $\tau_M(f,z_0)=1$. In the other side, we have $T_{z_0}(r,f)=m_{z_0}(r,f)=\frac{1}{\pi r^n}$, so $\sigma_T(f,z_0)=n$ and $\tau_T(f,z_0)=\dfrac{1}{\pi}$.*

Definition 1.5.7 *[22] Let f be a meromorphic function. We define the exponent of convergence of zeros of the function f near z_0 by*

$$\lambda(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log N_{z_0}\left(r,\frac{1}{f}\right)}{-\log r}$$

where

$$N_{z_0}\left(r,\frac{1}{f}\right)=-\int_{\infty}^{r}\frac{n\left(t,\frac{1}{f}\right)-n\left(\infty,\frac{1}{f}\right)}{t}dt-n\left(\infty,\frac{1}{f}\right)\log r,$$

$n\left(t,\frac{1}{f}\right)$ denotes the number of poles of f in the region $\{z\in\mathbb{C}:t\leq|z-z_0|\}\cup\{\infty\}$ and the exponent of convergence of distinct zeros of the function f near z_0 by

$$\overline{\lambda}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log\overline{N}_{z_0}\left(r,\frac{1}{f}\right)}{-\log r}$$

where

$$\overline{N}_{z_0}\left(r,\frac{1}{f}\right)=-\int_{\infty}^{r}\frac{\overline{n}\left(t,\frac{1}{f}\right)-\overline{n}\left(\infty,\frac{1}{f}\right)}{t}dt-\overline{n}\left(\infty,\frac{1}{f}\right)\log r,$$

$\overline{n}\left(t,\frac{1}{f}\right)$ denotes the number of distinct poles of f in the region $\{z\in\mathbb{C}:t\leq|z-z_0|\}\cup\{\infty\}$.

Similarly, we define the n-iterated exponent of convergence of zeros of the function f near z_0 by

$$\lambda_n(f, z_0) = \limsup_{r \longrightarrow +0} \frac{\log_n N_{z_0}\left(r, \frac{1}{f}\right)}{-\log r},$$

and the n-iterated exponent of convergence of distinct zeros by

$$\overline{\lambda}_n(f) = \limsup_{r \longrightarrow 0} \frac{\log_n \overline{N}_{z_0}\left(r, \frac{1}{f}\right)}{-\log r}.$$

Definition 1.5.8 *[22] The linear measure of a set $E \subset (0, \infty)$ is defined as $\int_0^\infty \chi_E(t)\, dt$ and the logarithmic measure of E is defined by $\int_0^1 \frac{\chi_E(t)}{t} dt$ where $\chi_E(t)$ is the characteristic function of the set E.*

Definition 1.5.9 *[33] Let $f(z) = \sum_{n=0}^{+\infty} a_n \frac{1}{(z-z_0)^n}$ be an analytic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. Then for all given $|z - z_0| = r > 0$:*

$$\lim_{n \to \infty} |a_n| \frac{1}{r^n} = 0,$$

and the maximum term

$$\mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| \frac{1}{r^n}$$

is well defined and we define the central index of $f(z)$ by

$$V_{z_0}(r) = V_{z_0}(r, f) = \max_{m \geq 0} \left\{ m : |a_m| \frac{1}{r^m} = \mu(r) \right\}.$$

Theorem 1.5.1 *[33] Let f be non-constant analytic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. Then, there exists a set $E \subset (0, 1)$ that has finite logarithmic measure, that is $\int_0^1 \frac{\chi_E(t)}{t} dt < \infty$, such that for all $j = 0, 1, ...$ we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_0 - z_r} \right)^j,$$

as $r \to 0$, $r \notin E$, where z_r is a point on the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = M(r, f) = \max_{|z_0 - z| = r} |f(z)|$.

Chapter 2

Growth of solutions of a class of linear differential equations near a singular point

2.1 Introduction and results

The linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{2.1.1}$$

where $A(z)$ and $B(z)$ are entire functions, was investigated by many authors; see for example[1, 12, 15, 30, 48]. In [48], Kwon proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ with $0 < c < 1$, then every solution $f(z) \not\equiv 0$ of (2.1.1) is of infinite order; after, Chen completed the case $c > 1$ in [12]. In 2012, Hamouda proved results in the unit disc concerning the differential equation

$$f'' + A(z)e^{\frac{a}{(z_0 - z)^{\mu}}}f' + B(z)e^{\frac{b}{(z_0 - z)^{\mu}}}f = 0, \tag{2.1.2}$$

where $\mu > 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$, see [34]. After that, Fettouch and Hamouda proved the following two results.

Theorem 2.1.1 *[25] Let z_0, a, b be complex constants such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$ and n be a positive integer. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\}f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\}f = 0,$$

satisfies $\sigma(f,z_0)=\infty$, *with* $\sigma_2(f,z_0)=n$.

Theorem 2.1.2 *[25] Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *satisfying* $\max\{\sigma(A_j,z_0): j\neq 0\} < \sigma(A_0,z_0)$. *Then, every solution* $f(z)\not\equiv 0$ *of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_1(z)f'+A_0(z)f=0. \tag{2.1.3}$$

satisfies $\sigma(f,z_0)=\infty$, *with* $\sigma_2(f,z_0)=\sigma(A_0,z_0)$.

In this chapter, we will investigate the case $c>1$ to complete the remaining case in Theorem 2.1.1, in the following two results.

Theorem 2.1.3 *[20] Let* $n\in\mathbb{N}\setminus\{0\}$, $A(z)\not\equiv 0$, $B(z)\not\equiv 0$ *be analytic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *such that* $\max\{\sigma(A,z_0),\sigma(B,z_0)\}<n$. *Let* a,b *be complex constants such that* $ab\neq 0$ *and* $a=cb$, $c>1$. *Then every solution* $f(z)\not\equiv 0$ *of the differential equation*

$$f''+A(z)\exp\left\{\frac{a}{(z_0-z)^n}\right\}f'+B(z)\exp\left\{\frac{b}{(z_0-z)^n}\right\}f=0, \tag{2.1.4}$$

that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfies* $\sigma(f,z_0)=\infty$.

Theorem 2.1.4 *[20] Let* $n\in\mathbb{N}\setminus\{0\}$, $A(z)\not\equiv 0$, $B(z)\not\equiv 0$ *be polynomials. Let* a,b *be complex constants such that* $a=cb$, $c>1$. *Then every solution* $f(z)\not\equiv 0$ *of the differential equation*

$$f''+A\left(\frac{1}{z_0-z}\right)\exp\left\{\frac{a}{(z_0-z)^n}\right\}f'+B\left(\frac{1}{z_0-z}\right)\exp\left\{\frac{b}{(z_0-z)^n}\right\}f=0 \tag{2.1.5}$$

that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfies* $\sigma(f,z_0)=\infty$, *with* $\sigma_2(f,z_0)=n$.

In the following result, we will improve Theorem 2.1.2 by studying the case when $\max\{\sigma(A_j,z_0): j\neq 0\}\leq\sigma(A_0,z_0)$.

Theorem 2.1.5 *[20] Let* $A_0(z)\not\equiv 0$, $A_1(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfying the following conditions*

i) $0<\sigma(A_j,z_0)\leq\sigma(A_0,z_0)<\infty, j=1,...,k-1;$

ii) $\max\{\tau_M(A_j,z_0):\sigma(A_j,z_0)=\sigma(A_0,z_0)\}<\tau_M(A_0,z_0)$.

Then, every solution $f(z)\not\equiv 0$ *of (2.1.3) that is analytic in* $\overline{\mathbb{C}}\setminus\{z_0\}$, *satisfies* $\sigma(f,z_0)=\infty$, *with* $\sigma_2(f,z_0)=\sigma(A_0,z_0)$.

Remark 2.1.1 *If we replace* τ_M *by* τ_T *in the condition* $ii)$ *in Theorem 2.1.5 we get the same result.*

We can find the analogous of Theorem 2.1.5 in the complex plane and in the unit disc in ([59, Theorem 1], [35, Theorem 3]).

We signal here that when the coefficients $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$, it may happen that the solution f of (2.1.3) is not analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$. For example, $f(z) = z$ is a solution of the differential equation

$$f'' - \exp\left\{\frac{1}{z}\right\} f' + \frac{1}{z} \exp\left\{\frac{1}{z}\right\} f = 0, \tag{2.1.6}$$

where the coefficients of (2.1.6) are analytic in $\overline{\mathbb{C}} \setminus \{0\}$, but the solution $f(z) = z$ is not analytic in $\overline{\mathbb{C}} \setminus \{0\}$. That's why we wrote in our results (every solution $f(z) \not\equiv 0$ of (2.1.3), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}, ...$). So, it is a priori assumed that f is analytic in Theorem 2.1.1 and Theorem 2.1.2. It is similar to the case when the coefficients are meromorphic in $\mathbb{C}$, it is well known that the solutions of (2.1.3) may be non meromorphic in $\mathbb{C}$.

2.2 Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 2.2.1 *[25] Let* $A(z) \not\equiv 0$ *be analytic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *with* $\sigma(A, z_0) < n$, n *is a positive integer. Set* $g(z) = A(z) \exp\left\{\frac{a}{(z_0 - z)^n}\right\}$ *where* $a = \alpha + i\beta \neq 0$ *is complex number,* $z_0 - z = re^{i\varphi}$, $\delta_a(\varphi) = \alpha \cos(n\varphi) + \beta \sin(n\varphi)$, *and* $H = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$, *(obviously,* H *is of linear measure zero). Then for any given* $\varepsilon > 0$ *and for any* $\varphi \in [0, 2\pi) \setminus H$, *there exists* $r_0 > 0$ *such that for* $0 < r < r_0$, *we have*

(i) if $\delta_a(\varphi) > 0$, *then*

$$\exp\left\{(1 - \varepsilon) \delta_a(\varphi) \frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1 + \varepsilon) \delta_a(\varphi) \frac{1}{r^n}\right\}, \tag{2.2.1}$$

(ii) if $\delta_a(\varphi) < 0$, *then*

$$\exp\left\{(1 + \varepsilon) \delta_a(\varphi) \frac{1}{r^n}\right\} \leq |g(z)| \leq \exp\left\{(1 - \varepsilon) \delta_a(\varphi) \frac{1}{r^n}\right\}. \tag{2.2.2}$$

Lemma 2.2.2 *[25] Let* f *be a non constant meromorphic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$. *Let* $\alpha > 0$, $\varepsilon > 0$ *be given real constants and* $j \in \mathbb{N}$. *Then* *i) there exists a set* $E_1 \subset (0,1)$ *that has finite logarithmic measure and a constant* $A > 0$ *that depends on* α *and* j *such that for all* $r = |z - z_0|$ *satisfying* $r \in (0,1) \backslash E_1$ *we have*

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq A \left[\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f) \right]^j \tag{2.2.3}$$

ii) there exists a set $E_2 \subset [0, 2\pi)$ *that has a linear measure zero and a constant* $A > 0$ *that depends on* α *and* j *such that for all* $\theta \in [0, 2\pi) \backslash E_2$ *there exists a constant* $r_0 = r_0(\theta) > 0$ *such that (2.2.3) holds for all* z *satisfying* $\arg(z - z_0) \in [0, 2\pi) \backslash E_2$ *and* $r = |z - z_0| < r_0$.

Lemma 2.2.3 *[33] Let* f *be a non-constant meromorphic function in* $\overline{\mathbb{C}} \backslash \{z_0\}$ *of finite order* $\sigma(f, z_0) < \infty$. *Let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_1 \subset (0,1)$ *that has finite logarithmic measure such that for all* $r = |z - z_0| \in (0,1) \backslash E_1$, *we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \frac{1}{r^{k(\sigma+1)+\varepsilon}}, \ (k \in \mathbb{N}). \tag{2.2.4}$$

Lemma 2.2.4 *[22] Let* $f(z)$ *be a non constant meromorphic function in* $\overline{\mathbb{C}} \backslash \{z_0\}$. *Then*

$$\sigma\left(f^{(j)}, z_0\right) = \sigma(f, z_0), \quad j \in \mathbb{N}.$$

Proof. It is sufficient to prove that $\sigma(f', z_0) = \sigma(f, z_0)$. By Remark 1.5.1, $g(w) = f\left(z_0 - \frac{1}{w}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma(g) = \sigma(f, z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g') = \sigma(g)$, (see [60], [57]). We have $f'(z) = \frac{1}{w^2} g'(w)$. Set $h(w) = \frac{1}{w^2} g'(w)$. Obviously, we have $\sigma(h) = \sigma(g')$. In the other hand, by Remark 1.5.1, we have $\sigma(h) = \sigma(f', z_0)$. So, we conclude that $\sigma(f', z_0) = \sigma(f, z_0)$.

Lemma 2.2.5 *[20] Let* f *be a non constant analytic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded on some ray* $\arg(z_0 - z) = \theta$. *Then there exists an infinite sequence of points* $z_m = z_0 - r_m e^{i\theta}$, $m = 1, 2, ...,$ *where* $r_m \to 0$, *such that* $f^{(k)}(z_m) \to \infty$ *and*

$$\left| \frac{f^{(j)}(z_m)}{f^{(k)}(z_m)} \right| \leq M, \quad M > 0, \ j = 0, 1, ..., k-1. \tag{2.2.5}$$

Proof. Let $M\left(r, \theta, f^{(k)}\right)$ denotes the maximum modulus of $f^{(k)}(z)$ on the line segment $\left[z_0 - r_1 e^{i\theta}, \ z_0 - r e^{i\theta}\right]$. Clearly, we may construct a sequence of points $z_m =$

$z_0 - r_m e^{i\theta},\ m \geq 1,\ r_m \to 0$, such that $M\left(r, \theta, f^{(k)}\right) = \left|f^{(k)}\left(z_m\right)\right| \to \infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1, z_m]$ we have

$$\begin{aligned} f^{(j)}\left(z_m\right) &= f^{(j)}\left(z_1\right) + f^{(j+1)}\left(z_1\right)\left(z_m - z_1\right) \\ &+ ... + \frac{1}{(k-j-1)!} f^{(k-1)}\left(z_1\right)\left(z_m - z_1\right)^{k-j-1} \\ &+ \int_{z_1}^{z_m} ... \int_{z_1}^{y} f^{(k)}\left(x\right) dx dy ... dt, \end{aligned}$$

and by an elementary triangle inequality estimate we obtain

$$\begin{aligned} \left|f^{(j)}\left(z_m\right)\right| &\leq \left|f^{(j)}\left(z_1\right)\right| + \left|f^{(j+1)}\left(z_1\right)\right| \left|\left(z_m - z_1\right)\right| \\ &+ .... + \frac{1}{(k-j-1)!} \left|f^{(k-1)}\left(z_1\right)\right| \left|\left(z_m - z_1\right)\right|^{k-j-1} \\ &+ \frac{1}{(k-j)!} \left|f^{(k)}\left(z_m\right)\right| \left|\left(z_m - z_1\right)\right|^{k-j}. \end{aligned} \quad (2.2.6)$$

From (2.2.6) and taking account that when $m \to \infty,\ f^{(k)}\left(z_m\right) \to \infty,\ z_m \to z_0$, we obtain:

$$\left|\frac{f^{(j)}\left(z_m\right)}{f^{(k)}\left(z_m\right)}\right| \leq M, \quad (M > 0).$$

Lemma 2.2.6 *[20] Let f be a non constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma\left(f, z_0\right) = \sigma > 0$ and a finite type $\tau_M\left(f, z_0\right) = \tau > 0$. Then for any given $0 < \beta < \tau$ there exists a set $F \subset (0,1)$ of infinite logarithmic measure such that for all $r \in F$ we have*

$$\log M_{z_0}\left(r, f\right) > \frac{\beta}{r^{\sigma}}.$$

Proof. By the definition of $\tau_M\left(f, z_0\right)$, there exists a decreasing sequence $\{r_m\} \to 0$ satisfying $\frac{m}{m+1} r_m > r_{m+1}$ and

$$\lim_{m \to \infty} r_m^{\sigma} \log M_{z_0}\left(r_m, f\right) = \tau.$$

Then there exists m_0 such that for all $m > m_0$ and for a given $\varepsilon > 0$ we have

$$\log M_{z_0}\left(r_m, f\right) > \frac{\tau - \varepsilon}{r_m^{\sigma}}. \quad (2.2.7)$$

There exists m_1 such that for all $m > m_1$ and for a given $0 < \varepsilon < \tau - \beta$, we have

$$\left(\frac{m}{m+1}\right)^{\sigma} > \frac{\beta}{\tau - \varepsilon}. \quad (2.2.8)$$

By (2.2.7) and (2.2.8), for all $m > m_2 = \max\{m_0, m_1\}$ and for any $r \in \left[\frac{m}{m+1} r_m, r_m\right]$, we have

$$\log M_{z_0}(r,f) > \log M_{z_0}(r_m,f) > \frac{\tau-\varepsilon}{r_m^\sigma} > \frac{\tau-\varepsilon}{r^\sigma}\left(\frac{m}{m+1}\right)^\sigma > \frac{\beta}{r^\sigma}.$$

Set $F = \bigcup_{m=m_2}^{\infty} \left[\frac{m}{m+1} r_m, r_m\right]$. Then we have

$$\sum_{m=m_2}^{\infty} \int_{\frac{m}{m+1} r_m}^{r_m} \frac{dt}{t} = \sum_{m>m_2} \log \frac{m+1}{m} = \infty.$$

Lemma 2.2.7 *[20] Let f be a non constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with the hyper-order $\sigma_2(f, z_0) = \sigma$ and let $V_{z_0}(r)$ be the central index of f (see [33]). Then*

$$\limsup_{r\to 0} \frac{\log^+ \log^+ V_{z_0}(r)}{-\log r} = \sigma. \tag{2.2.9}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is an entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f, z_0) = \sigma$, and if $V(R)$ denotes the central index of g, then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [18, Lemma 2], we have

$$\limsup_{R\to+\infty} \frac{\log^+ \log^+ V(R)}{\log R} = \sigma. \tag{2.2.10}$$

Substituting R by $\frac{1}{r}$ in (2.2.10), we get (2.2.9).

Lemma 2.2.8 *[20] Let $A_j(z)$, $j = 0, ..., k-1$, be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that $\sigma(A_j, z_0) \le \alpha < \infty$. If f is a solution of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{2.2.11}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then $\sigma_2(f, z_0) \le \alpha$.

Proof. For any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $0 < r = |z_0 - z| < r_0$, we have

$$|A_j(z)| \le \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.2.12}$$

By the Wiman-Valiron near a finite singular point (see [33]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1+o(1))\left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j, \quad j = 0, ..., k-1, \tag{2.2.13}$$

where $V_{z_0}(r)$ is the central index of f and $|f(z_r)| = M(r,f) = \max_{|z_0-z|=r} |f(z)|$. From (2.2.11), we can write

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)|. \qquad (2.2.14)$$

Substituting (2.2.12) and (2.2.13) into (2.2.14), we obtain

$$|1+o(1)| \frac{(V_{z_0}(r))^k}{r^k} \leq k \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\} \frac{(V_{z_0}(r))^{k-1}}{r^{k-1}} |1+o(1)|,$$

and so

$$V_{z_0}(r) \leq kr \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\} |1+o(1)|. \qquad (2.2.15)$$

By (2.2.15) and Lemma 2.2.7 , we get

$$\sigma_2(f, z_0) \leq \alpha.$$

Lemma 2.2.9 *[33] Let $P(z) = a_n z^n + ... + a_0$ with $a_n \neq 0$ be a polynomial and $A(z) = P\left(\frac{1}{z_0-z}\right)$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $0 < r = |z_0 - z| \leq r_0$, the inequalities*

$$(1-\varepsilon)\frac{|a_n|}{r^n} \leq |A(z)| \leq (1+\varepsilon)\frac{|a_n|}{r^n}$$

hold.

Lemma 2.2.10 *[20] Let f be a non constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with the hyper-order $\sigma_2(f, z_0) = \alpha$, and let $V_{z_0}(r)$ be the central index of f. Let $E \subset (0, 1]$ be a set of finite logarithmic measure. Then, there exists a sequence of points $\{z_m = z_0 - r_m e^{i\theta_m}\}$, $m \geq 1$, such that $|f(z_m)| = M_{z_0}(r_m, f)$, $\lim_{m\to\infty} \theta_m = \theta^* \in [0, 2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have*

$$\limsup_{r\to 0} \frac{\log^+ V_{z_0}(r)}{-\log r} = \infty, \qquad (2.2.16)$$

$$\exp\left\{\frac{1}{r^{\alpha-\varepsilon}}\right\} \leq V_{z_0}(r) \leq \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \qquad (2.2.17)$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is an entire function of infinite order with the hyper-order $\sigma_2(g) = \sigma_2(f, z_0) = \alpha$ and if $V(R)$ denotes the central index of g then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [12, Remark 1], we have

$$\limsup_{R\to\infty} \frac{\log^+ V_{z_0}(R)}{\log R} = \infty, \qquad (2.2.18)$$

$$\exp\left\{R^{\alpha-\varepsilon}\right\} \leq V(R) \leq \exp\left\{R^{\alpha+\varepsilon}\right\}. \qquad (2.2.19)$$

Substituting R by $\frac{1}{r}$ in (2.2.18) and (2.2.19), we get (2.2.16) and (2.2.17).

2.3 Proof of Theorem 2.1.3

We assume that $\sigma(f, z_0) = \alpha < \infty$, and we prove that is failing. By Lemma 2.2.3, for any given $\varepsilon > 0$ there exists a set $E \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \setminus E$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \setminus E$ and $r = |z - z_0| < r_0$, we have

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{r^{\sigma+1+\varepsilon}}. \tag{2.3.1}$$

Set $a = \alpha + i\beta,\ z_0 - z = re^{i\theta},\ \delta = \delta_a(\theta) = \alpha \cos(n\theta) + \beta \sin(n\theta)$,

$$H = \{\theta \in [0, 2\pi) : \delta_a(\theta) = 0\}, \tag{2.3.2}$$

(obviously, H is of linear measure zero). By Lemma 2.2.1, for any given $0 < \varepsilon < 1$ and for any $\theta \in [0, 2\pi) \setminus E \cup H$, there exists $r_0 > 0$ such that for $0 < r < r_0$, (2.2.1) and (2.2.2) hold.

Now we take $\theta \in [0, 2\pi) \setminus E \cup H$ (obviously, $E \cup H$ is of linear measure zero). Then we have two cases: $\delta_a(\theta) < 0$ or $\delta_a(\theta) > 0$.

Case (i) $\delta_a = \delta < 0$. By $a = cb,\ c > 1,\ \delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta$. By (2.1.4), we get

$$1 \leq \left| A(z) \exp\left\{ \frac{a}{(z_0 - z)^n} \right\} \right| \left| \frac{f'}{f''} \right| + \left| B(z) \exp\left\{ \frac{b}{(z_0 - z)^n} \right\} \right| \left| \frac{f}{f''} \right|. \tag{2.3.3}$$

If $|f''(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5 there exists an infinite sequence of points $\left\{ z_m = z_0 - r_m e^{i\theta} \right\},\ m \geq 1$, where $r_m \to 0$ such that $f''(z_m) \to \infty$ and

$$\left| \frac{f(z_m)}{f''(z_m)} \right| \leq M_1, \quad \left| \frac{f'(z_m)}{f''(z_m)} \right| \leq M_2. \tag{2.3.4}$$

Using Lemma 2.2.1 and (2.3.4) into (2.3.3), we get as $m \to \infty$

$$1 \leq M_1 \exp\left\{ (1 - \varepsilon) \frac{\delta}{r_m^n} \right\} + M_2 \exp\left\{ (1 - \varepsilon) \frac{1}{c} \frac{\delta}{r_m^n} \right\} \to 0,$$

a contradiction. Hence

$$|f''(z)| \leq C_1, \tag{2.3.5}$$

holds on $\arg(z_0 - z) = \theta$, where C_1 is a constant. By integration along the line segment $\left[z_0 - r_1 e^{i\theta}, z_0 - re^{i\theta} \right]$, from (2.3.5) and the equality

$$f'(z) = f'(z_1) + \int_{z_1}^{z} f''(t)\, dt,$$

we obtain

$$|f'(z)| \leq C_2 + C_1 |z - z_1| \leq C_3, \tag{2.3.6}$$

as $z \to z_0$. Analogously, by (2.3.6), we can obtain

$$|f(z)| \leq C_4, \tag{2.3.7}$$

holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Case (ii) $\delta > 0$. We have $\delta_b(\theta) = \frac{1}{c}\delta_a(\theta) = \frac{1}{c}\delta > 0$. By (2.1.4), we have

$$\left|A(z)\exp\left\{\frac{a}{(z_0 - z_k)^n}\right\}\right| \leq \left|\frac{f''(z)}{f'(z)}\right| + \left|B(z)\exp\left\{\frac{b}{(z_0 - z_k)^n}\right\}\right|\left|\frac{f(z)}{f'(z)}\right|. \tag{2.3.8}$$

If $|f'(z)|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5, there exists an infinite sequence of points $\{z_m = z_0 - r_m e^{i\theta}\}$, $m \geq 1$ where $r_m \to 0$ such that $f'(z_m) \to \infty$ and

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leq M_3. \tag{2.3.9}$$

Substituting (2.3.1) and (2.3.9) into (2.3.8) and by Lemma 2.2.1, we obtain

$$\begin{aligned} \exp\left\{(1-\varepsilon)\frac{\delta}{r_m^n}\right\} &\leq \frac{1}{r_m^{\sigma+1+\varepsilon}} + M_3 \exp\left\{(1+\varepsilon)\frac{1}{c}\frac{\delta}{r_m^n}\right\} \\ &\leq \frac{M_4}{r_m^{\sigma+1+\varepsilon}} \exp\left\{(1+\varepsilon)\frac{1}{c}\frac{\delta}{r_m^n}\right\}, \end{aligned}$$

which implies that

$$1 \leq \frac{M_4}{r_m^{\sigma+1+\varepsilon}} \exp\left\{\left[(1+\varepsilon)\frac{1}{c} - (1-\varepsilon)\right]\frac{\delta}{r_m^n}\right\}. \tag{2.3.10}$$

By taking $0 < \varepsilon < \frac{c-1}{1+c}$, a contradiction follows in (2.3.10) as $m \to \infty$. So

$$|f'(z)| \leq C_5.$$

As above, we obtain that

$$|f(z)| \leq C_6,$$

holds on $\arg(z_0 - z) = \theta$ as $z \to z_0$.

Now, we proved that $|f(z)| \leq C$ on any ray $\arg(z_0 - z) = \theta \in [0, 2\pi) \setminus (E \cup H)$. Set $g(w) = f(z)$ such that $w = \frac{1}{z_0 - z}$. $g(w)$ is an entire function in $\mathbb{C}$ and $|g(w)| \leq C'$ $(C' > 0)$ on any ray $\arg(w) = -\theta$ such that $\theta \in [0, 2\pi) \setminus (E \cup H)$. By Phragmen-Lindelof theorem in sectors, we get that $|g(w)| \leq C'$ in $\mathbb{C}$ and By Liouville theorem we conclude that $g(w)$ is a constant. So, $f(z)$ is constant. We know that the only constant solution of (2.1.4) is $f \equiv 0$. Hence, every solution $f(z) \not\equiv 0$ of (2.1.4) is of infinite order.

2.4 Proof of Theorem 2.1.4

Assume that $f \not\equiv 0$ is a analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (2.1.5). By Theorem 2.1.3 and Lemma 2.2.8, we have $\sigma(f, z_0) = \infty$ and $\sigma_2(f, z_0) = \alpha \le n$. We assume that $\sigma_2(f, z_0) = \alpha < n$, and we prove that is failing. Since the Wiman-Valiron near a finite singular point (see [33]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_0 - z_r} \right)^j, \ j = 1, 2, \tag{2.4.1}$$

where $|f(z_r)| = M_{z_0}(r, f) = \max_{|z_0 - z| = r} |f(z)|$. By Lemma 2.2.10, there is a sequence $\{z_m = z_0 - r_m e^{i\theta_m}\}$, $m \ge 1$, such that $|f(z_m)| = M_{z_0}(r_m, f)$, $\lim_{m\to\infty} \theta_m = \theta^* \in [0, 2\pi)$, $r_m \notin E$, $r_m \to 0$ and for any given $\varepsilon > 0$, we have

$$\limsup_{m\to\infty} \frac{\log V_{z_0}(r_m)}{-\log r_m} = \infty,$$

$$\exp\left\{ \frac{1}{r_m^{\alpha-\varepsilon}} \right\} \le V_{z_0}(r_m) \le \exp\left\{ \frac{1}{r_m^{\alpha+\varepsilon}} \right\}. \tag{2.4.2}$$

Set $a = \alpha + i\beta$, $z_0 - z = re^{i\theta_0}$, $\delta = \delta_a(\theta^*) = \alpha\cos(n\theta^*) + \beta\sin(n\theta^*)$. Since $a = cb$, $c > 1$, we have $\delta_b(\theta^*) = \frac{1}{c}\delta_a(\theta^*) = \frac{1}{c}\delta$. There is three cases: (i) $\delta < 0$, (ii) $\delta > 0$, (iii) $\delta = 0$.

Case (i). $\delta < 0$. By $\lim_{m\to\infty} \theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m < 0$, $\delta_a(\theta_m) = c\delta_m < 0$. From (2.1.5), we can write

$$\left| \exp\left\{ \frac{-b}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f''}{f} \right| \le \left| A\left(\frac{1}{z_0 - z_m} \right) \exp\left\{ \frac{a - b}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f'}{f} \right| + \left| B\left(\frac{1}{z_0 - z_m} \right) \right|. \tag{2.4.3}$$

Substituting (2.4.1) -(2.4.2) into (2.4.3) and by Lemma 2.2.1 and Lemma 2.2.9, for any given ε $(0 < \varepsilon < n - \alpha)$ as m is sufficiently large, we have

$$\begin{aligned}
& \exp\left\{ (1-\varepsilon) \frac{-\delta_m}{r_m^n} \right\} \exp\left\{ \frac{2}{r_m^{\alpha-\varepsilon}} \right\} \frac{1}{r_m^2} |1 + o(1)| \\
\le\ & \left| \exp\left\{ \frac{-b}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f''}{f} \right| \\
\le\ & \exp\left\{ (1-\varepsilon)(c-1) \frac{\delta_m}{r_m^n} \right\} \exp\left\{ \frac{1}{r_m^{\alpha+\varepsilon}} \right\} \frac{|1 + o(1)|}{r_m} + \frac{1}{r_m^{d+1}} \\
\le\ & M_1 \exp\left\{ (1-\varepsilon)(c-1) \frac{\delta_m}{r_m^n} \right\} \exp\left\{ \frac{1}{r_m^{\alpha+\varepsilon}} \right\} \frac{1}{r_m^{d+2}},
\end{aligned}$$

where $M_1 > 0$ is a constant and $d = \deg B$; which implies

$$\exp\left\{ \frac{2}{r_m^{\alpha-\varepsilon}} \right\} (1 + o(1)) \le M_1 \exp\left\{ (1-\varepsilon) c \frac{\delta_m}{r_m^n} \right\} \exp\left\{ \frac{1}{r_m^{\alpha+\varepsilon}} \right\} \frac{1}{r_m^d}. \tag{2.4.4}$$

By taking $0 < \varepsilon < \max\{1, n-\alpha\}$, the right side of inequality (2.4.4) tend to zero as $m \to \infty$. This is a contradiction.

Case (ii). $\delta > 0$. By $\lim_{m\to\infty} \theta_m = \theta^*$, as m is sufficiently large, we have $\delta_b(\theta_m) = \delta_m > 0,\ \delta_a(\theta_m) = c\delta_m > 0$. By (2.1.5), we can write

$$\left|A\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\left|\frac{f'}{f}\right| \le \left|\frac{f''}{f}\right| + \left|B\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}\right|. \tag{2.4.5}$$

Substituting (2.4.1)-(2.4.2) into (2.4.5) and by Lemma 2.2.1, as m is sufficiently large, we have

$$\begin{aligned} &\exp\left\{(1-\varepsilon)\frac{c\delta_m}{r_m^n}\right\}\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}\frac{|1+o(1)|}{r_m} \\ \le\ & \left|A\left(\frac{1}{z_0 - z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right|\left|\frac{f'}{f}\right| \\ \le\ & \exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\frac{|1+o(1)|}{r_m^2} + \exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\} \\ \le\ & \frac{M_2}{r_m^2}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{(1+\varepsilon)\frac{\delta_m}{r_m^n}\right\}; \end{aligned}$$

where $M_2 > 0$ is a constant, which implies the following inequality

$$\exp\left\{\frac{1}{r_m^{\alpha-\varepsilon}}\right\}(1+o(1)) \le \frac{1}{r_m}\exp\left\{\frac{2}{r_m^{\alpha+\varepsilon}}\right\}\exp\left\{[(1+\varepsilon)-(1-\varepsilon)c]\frac{\delta_m}{r_m^n}\right\}. \tag{2.4.6}$$

By taking $0 < \varepsilon < \max\left\{\frac{c-1}{c+1}, n-\alpha\right\}$, the right side of inequality (2.4.6) tend to zero as $m \to \infty$ therefore a contradiction follows.

Case (iii). $\delta = 0$. Since $\arg(z_0 - z) = \theta^*$ is an asymptotic line of $\frac{a}{(z_0-z_m)^n}$, there is $m_0 > 0$ such that as $m > m_0$ we have

$$e^{-1} \le \left|\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\right| \le e, \tag{2.4.7}$$

$$e^{\frac{-1}{c}} \le \left|\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}\right| \le e^{\frac{1}{c}}. \tag{2.4.8}$$

By (2.1.5), (2.4.1) and (2.4.7)-(2.4.8), we obtain

$$\begin{aligned} -\left(\frac{V_{z_0}(r_m)}{z_0 - z_m}\right)^2(1+o(1)) &= A\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{a}{(z_0-z_m)^n}\right\}\left(\frac{V_{z_0}(r)}{z_0-z_m}\right)(1+o(1)) \\ &+ B\left(\frac{1}{z_0-z_m}\right)\exp\left\{\frac{b}{(z_0-z_m)^n}\right\}. \end{aligned} \tag{2.4.9}$$

By (2.4.7)-(2.4.9) and Lemma 2.2.1, for m large enough, we have

$$\left(\frac{V_{z_0}(r_m)}{r_m}\right)^2|1+o(1)| \le \frac{M_3}{r_m^{d+1}}\left(\frac{V_{z_0}(r_m)}{r_m}\right)|1+o(1)|,$$

where $M_3 > 0$ is a constant. So

$$V_{z_0}(r_m) \leq \frac{M_3}{r_m^d} |1 + o(1)|, \tag{2.4.10}$$

where $d = \max\{\deg A, \deg B\}$. (2.4.10) contradicts (2.4.2). Thus $\sigma_2(f, z_0) \geq n$, and by Lemma 2.2.8, we obtain $\sigma_2(f, z_0) = n$.

2.5 Proof of Theorem 2.1.5

We assume that $f \not\equiv 0$ is analytic solution of (2.1.3) in $\overline{\mathbb{C}} \setminus \{z_0\}$. From (2.1.3), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{2.5.1}$$

By Lemma 2.2.2, for any given $\alpha > 0$ there exists a set $E_1 \subset (0, 1)$ that has finite logarithmic measure and a constant $\lambda > 0$ such that for all $r = |z - z_0|$ satisfying $r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j}, \quad j = 1, ..., k. \tag{2.5.2}$$

There exist β_1, β_2 such that $\max\{\tau_T(A_j, z_0) : \sigma(A_j, z_0) = \sigma(A_0, z_0)\} < \beta_1 < \beta_2 < \tau_T(A_0, z_0)$. There exists a set $E_2 \subset (0, 1)$ that has finite logarithmic measure such that for all $r = |z - z_0|$ satisfying $r \notin E_2$, we have

$$|A_j(z)| \leq \exp\left\{\frac{\beta_1}{r^\sigma}\right\}, \quad j = 1, ..., k. \tag{2.5.3}$$

By Lemma 2.2.6, there exists a set $F \subset (0, 1)$ of infinite logarirhmic measure such that for all $r \in F$ we have

$$M_{z_0}(r, A_0) > \exp\left\{\frac{\beta_2}{r^\sigma}\right\}. \tag{2.5.4}$$

From (2.5.1)-(2.5.4), for all z satisfying $r = |z - z_0| \in F \setminus (E_1 \cup E_2)$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we obtain

$$\exp\left\{\frac{\beta_2}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k} \exp\left\{\frac{\beta_1}{r^\sigma}\right\};$$

and thus

$$\exp\left\{\frac{\beta_2 - \beta_1}{r^\sigma}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k}. \tag{2.5.5}$$

From (2.5.5), it is easy to obtain that $\sigma_2(f, z_0) \geq \sigma$ and combining this with Lemma 2.2.8, we get the equality $\sigma_2(f, z_0) = \sigma = \sigma(A_0, z_0)$.

Chapter 3

Linear differential equations with analytic coefficients having the same order near a singular point

3.1 Introduction and results

The linear differential equation

$$f'' + A(z)\, e^{az} f' + B(z)\, e^{bz} f = 0, \tag{3.1.1}$$

where $A(z)$ and $B(z)$ are entire functions was investigated by many authors; see for example [1, 12, 15, 30, 48]. In [48], Kwon proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ with $0 < c < 1$, then every solution $f(z) \not\equiv 0$ of (3.1.1) is of infinite order; after, Chen complete the case $c > 1$ in [12]. These results have been generalized in different ways to the higher order differential equation

$$f^{(k)} + A_{k-1}(z)\, e^{a_{k-1}z^n} f^{(k-1)} + ... + A_0(z)\, e^{a_0 z^n} f = 0, \tag{3.1.2}$$

where $A_j(z)$ $(j = 0, ..., k-1)$ are entire functions, by many authors, see [13, 14, 59]. In [34], Hamouda proved results similar to (3.1.1) in the unit disc concerning the differential equation

$$f'' + A(z)\, e^{\frac{a}{(z_0 - z)^{\mu}}} f' + B(z)\, e^{\frac{b}{(z_0 - z)^{\mu}}} f = 0, \tag{3.1.3}$$

where $\mu > 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. To investigate the counterpart of these results near a finite singular point, Fettouch and Hamouda proved the following result.

Theorem 3.1.1 *[25] Let* z_0, a, b *be complex constants such that* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$ *and* n *be a positive integer. Let* $A(z), B(z) \not\equiv 0$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *with* $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. *Then, every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\} f = 0, \qquad (3.1.4)$$

satisfies $\sigma(f, z_0) = \infty$ *with* $\sigma_2(f, z_0) = n$.

In chapter 2, we have completed the case $c > 1$ of (3.1.4). A natural question is: can we generalize (3.1.4) to higher order differential equations as in the complex plane? The following results give the affirmative answer.

Theorem 3.1.2 *[21] Let* $n \in \mathbb{N}\setminus\{0\}$; $A_j(z)$ $(j = 0, ..., k-1)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *such that* $\sigma(A_j, z_0) < n$ *and* $A_0(z) \not\equiv 0$. *If* a_j $(j = 0, ..., k-1)$ *are distinct complex numbers, then every solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f^{(k)} + A_{k-1}(z)\exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\} f^{(k-1)} + ... + A_0(z)\exp\left\{\frac{a_0}{(z_0 - z)^n}\right\} f = 0, \quad (3.1.5)$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, *satisfies* $\sigma(f, z_0) = \infty$.

Theorem 3.1.3 *[21] Let* $n \in \mathbb{N}\setminus\{0\}$; $A_j(z)$ $(j = 0, ..., k-1)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *such that* $\sigma(A_j, z_0) < n$ *and* $A_0(z) \not\equiv 0$. *Let* a_j $(j = 0, ..., k-1)$ *be complex constants. Suppose that there exist nonzero complex numbers* a_s *and* a_l, *such that* $0 < s < l \leq k-1$, $a_s = |a_s| e^{i\theta_s}$, $a_l = |a_l| e^{i\theta_l}$, $\theta_s, \theta_l \in [0, 2\pi)$, $\theta_s \neq \theta_l$, $A_s A_l \not\equiv 0$; *for* $j \neq s, l$; a_j *satisfies either* $a_j = d_j a_s$ $(0 < d_j < 1)$ *or* $a_j = d_j a_l$ $(0 < d_j < 1)$. *Then, every solution* $f(z) \not\equiv 0$ *of (3.1.5), that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *satisfies* $\sigma(f, z_0) = \infty$.

Now, we will study the case $s = 0$ of Theorem 3.1.3 in which we can also express the hyper-order of the solutions.

Theorem 3.1.4 *[21] Let* $n \in \mathbb{N}\setminus\{0\}$; $A_j(z)$ $(j = 0, ..., k-1)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *such that* $\sigma(A_j, z_0) < n$, *and let* a_j $(j = 0, ..., k-1)$ *be complex numbers such that* $a_0 = |a_0| e^{i\theta_0}$, $a_s = |a_s| e^{i\theta_s}$, $a_0 a_s \neq 0$ $(0 < s \leq k-1)$, $\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0 A_s \not\equiv 0$; *for* $j \neq 0, s$, a_j *satisfies either* $a_j = d_j a_0$ $(d_j < 1)$ *or* $\arg a_j = \arg a_s$. *Then, every solution* $f \not\equiv 0$ *of (3.1.5), that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *satisfies* $\sigma(f, z_0) = \infty$ *and* $\sigma_2(f, z_0) = n$.

Clearly, Theorem 3.1.1 is a particular case of Theorem 3.1.4.

3.2 Preliminary lemmas

By Lemma 2.2.3 and Lemma 2.2.4, we get the following lemma.

Lemma 3.2.1 *[33] Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ of finite order $\sigma(f, z_0) < \infty$; let $\varepsilon > 0$ be a given constant. Then:*
i) there exists a set $E_1^* \subset (0,1)$ that has finite logarithmic measure, such that for all $r = |z - z_0| \in (0,\ 1)\setminus E_1^*$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq \frac{1}{r^{(j-i)(\sigma+1)+\varepsilon}},\ (0 \leq i < j); \qquad (3.2.1)$$

ii) there exists a set $E_2^* \subset [0, 2\pi)$ that has a linear measure zero, such that for all $\theta \in [0,2\pi)\setminus E_2^*$, there exists a constant $r_0 = r_0(\theta) > 0$ such that (3.2.1) holds for all z satisfying $\arg(z - z_0) \in [0, 2\pi)\setminus E_2^*$ *and $r = |z - z_0| < r_0$.*

In addition to this lemma, we need also to Lemma 2.2.1, Lemma 2.2.2, Lemma 2.2.4, Lemma 2.2.5 and Lemma 2.2.8 cited in the second chapter.

3.3 Proof of Theorem 3.1.2

We assume that $f \not\equiv 0$ is a analytic solution of (3.1.5) in $\overline{\mathbb{C}}\setminus\{z_0\}$ with $\sigma(f, z_0) = \sigma < \infty$, and we prove that is failing. By Lemma 3.2.1, for any given $\varepsilon > 0$ there exists a set $E_1 \subset [0, 2\pi)$ that has a linear measure zero, such that for all $\theta \in [0,2\pi)\setminus E_1$, there exists a constant $r_0 = r_0(\theta) > 0$, such that for all z satisfying $\arg(z - z_0) = \theta \in [0,2\pi)\setminus E_1$ and $r = |z - z_0| < r_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq \frac{1}{r^{(j-i)(\sigma+1)+\varepsilon}} \quad (k \geq j > i \geq 0). \qquad (3.3.1)$$

Set $a_j = \alpha_j + i\beta_j,\ \delta_{a_j}(\theta) = \alpha_j \cos(n\theta) + \beta_j \sin(n\theta),\ z_0 - z = re^{i\theta}$,

$$E_2 = \left\{\theta \in [0, 2\pi) : \delta_{a_j}(\theta) = 0, j = 0, 1, ..., k-1\right\},$$

$$E_3 = \left\{\theta \in [0, 2\pi) : \delta_{a_j - a_i}(\theta) = 0,\ 0 \leq i < j \leq k-1\right\}.$$

By Lemma 2.2.1, for each function $A_j(z)\exp\left\{\frac{a_j}{(z_0-z)^n}\right\}$ $(j = 0, ..., k-1)$, there exists a set $H_j \subset [0, 2\pi)$ with linear measure zero such that for all $\theta \in [0,2\pi)\setminus H_j$, (2.2.1) and (2.2.2) hold. Set $E_4 = \bigcup_{j=0}^{k-1} H_j$, and then E_4 is also a set of linear measure zero. For any given $\theta \in [0, 2\pi)\setminus(E_1 \cup E_2 \cup E_3 \cup E_4)$, we have $\delta_{a_j}(\theta) \neq 0$, $\delta_{a_i}(\theta) \neq \delta_{a_j}(\theta)$ $(0 \leq i < j \leq k)$. Since a_j are distinct complex numbers, for any given $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$, there exists only one $s \in \{0, ..., k-1\}$ such that $\delta_{a_s}(\theta) = \max\left\{\delta_{a_j}(\theta) : j = 0, ..., k-1\right\}$. Set $\delta = \delta_{a_s}(\theta)$, $\delta' = \max\left\{\delta_{a_j}(\theta) : j \neq s\right\}$; then $\delta' < \delta$.

We divide the proof into two cases: (i) $\delta > 0$, (ii) $\delta < 0$.

Case (i) $\delta > 0$. By Lemma 2.2.1, for any given ε $(\varepsilon > 0)$, we obtain

$$\left|A_s(z)\exp\left\{\frac{a_s}{(z_0 - z)^n}\right\}\right| \geq \exp\left\{(1-\varepsilon)\frac{\delta}{r^n}\right\}, \tag{3.3.2}$$

if $\delta' > 0$, then

$$\left|A_j(z)\exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1+\varepsilon)\frac{\delta'}{r^n}\right\} \quad (j \neq s);$$

and If $\delta' < 0$, we have

$$\left|A_j(z)\exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1-\varepsilon)\frac{\delta'}{r^n}\right\} \quad (j \neq s).$$

Set $\delta' < \delta_1 < \delta$ such that $\delta_1 > 0$. In both cases $\delta' > 0$, $\delta' < 0$, we have

$$\left|A_j(z)\exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1+\varepsilon)\frac{\delta_1}{r^n}\right\} \quad (j \neq s). \tag{3.3.3}$$

Now we prove that $\left|f^{(s)}(z)\right|$ is bounded on the ray $\arg(z_0 - z) = \theta$.

If $\left|f^{(s)}(z)\right|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5, there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$ $(m \geq 1)$ where $r_m \to 0$ such that $f^{(s)}(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \leq M, \ (M > 0) \ \ (j = 0, ..., s-1). \tag{3.3.4}$$

By (3.1.5), we have

$$\begin{aligned}
\left|A_s(z_m)\exp\left\{\frac{a_s}{(z_0 - z_m)^n}\right\}\right| \leq\ & \left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right| + ... + \left|A_{s+1}(z_m)\exp\left\{\frac{a_{s+1}}{(z_0 - z_m)^n}\right\}\right| \\
& + \left|\frac{f^{(s+1)}(z_m)}{f^{(s)}(z_m)}\right| + \left|A_{s-1}(z_m)\exp\left\{\frac{a_{s-1}}{(z_0 - z_m)^n}\right\}\right| \\
& + \left|\frac{f^{(s-1)}(z_m)}{f^{(s)}(z_m)}\right| + ... + \left|A_0(z_m)\exp\left\{\frac{a_0}{(z_0 - z_m)^n}\right\}\right| \\
& + \left|\frac{f(z_m)}{f^{(s)}(z_m)}\right|.
\end{aligned} \tag{3.3.5}$$

Substituting (3.3.1)-(3.3.4) into (3.3.5), we obtain

$$\begin{aligned}
\exp\left\{(1-\varepsilon)\frac{\delta}{r_m^n}\right\} &\leq \frac{K_1}{r^{(k-s)(\sigma+1)+\varepsilon}}\exp\left\{(1+\varepsilon)\frac{\delta_1}{r_m^n}\right\} + K_2\exp\left\{(1+\varepsilon)\frac{\delta_1}{r_m^n}\right\} \\
&\leq \frac{K_3}{r^{(k-s)(\sigma+1+\varepsilon)}}\exp\left\{(1+\varepsilon)\frac{\delta_1}{r_m^n}\right\},
\end{aligned}$$

which implies that:

$$1 \leq \frac{K_3}{r^{(k-s)(\sigma+1)+\varepsilon}} \exp\left\{ \left[(1+\varepsilon)\,\delta_1 - (1-\varepsilon)\,\delta\right] \frac{1}{r_m^n} \right\}. \tag{3.3.6}$$

By taking $0 < \varepsilon < \frac{\delta-\delta_1}{\delta+\delta_1}$, a contradiction follows in (3.3.6) as $m \to \infty$; so $\left|f^{(s)}(z)\right| \leq C_1$. We can easily obtain $|f(z)| \leq C_2$, on $\arg(z_0 - z) = \theta$, as $z \to z_0$.

Case (ii) $\delta < 0$. By (3.1.5), we have

$$1 \leq \left|A_{k-1}(z)\exp\left\{\frac{a_{k-1}}{(z_0-z)^n}\right\}\right| \left|\frac{f^{(k-1)}(z)}{f^{(k)}(z)}\right| + ... + \left|A_0(z)\exp\left\{\frac{a_0}{(z_0-z)^n}\right\}\right| \left|\frac{f(z)}{f^{(k)}(z)}\right|. \tag{3.3.7}$$

If $\left|f^{(k)}(z)\right|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5, there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$ $(m \geq 1)$ where $r_m \to 0$ such that $f^{(k)}(z_m) \to \infty$ and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leq M', \ \left(M' > 0\right) \ (j = 0, ..., k-1). \tag{3.3.8}$$

Substituting (3.3.8) into (3.3.7) and by Lemma 2.2.1, we obtain

$$1 \leq kM' \exp\left\{(1-\varepsilon)\frac{\delta}{r_m^n}\right\} \to 0,$$

a contradiction. Hence $\left|f^{(k)}(z)\right| \leq C_3$, and so $|f(z)| \leq C_4$, on $\arg(z_0 - z) = \theta$, as $z \to z_0$.

Now, we proved that $|f(z)| \leq C$ on any ray $\arg(z_0 - z) = \theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$. Set $g(w) = f(z)$ such that $w = \frac{1}{z_0 - z}$. $g(w)$ is entire function in $\mathbb{C}$ and $|g(w)| \leq C'$ $(C' > 0)$ on any ray $\arg(w) = -\theta$, such that $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$. By Remark 1.5.1, $\sigma(g) = \sigma(f, z_0) = \sigma < \infty$. There exists a finite family of rays $\arg(w) = -\theta_i$ where $\theta_i \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$, such that the angle between every pair of adjacent rays does not exceed $\alpha\pi$, such that $\alpha < \frac{1}{\sigma}$ and the sum of all angles equals 2π. By Phragmen-Lindelof theorem in sectors (see [54]), we get that $|g(w)| \leq C'$ in $\mathbb{C}$, and By Liouville theorem, we conclude that $g(w)$ is a constant. Therefore, $f(z)$ is constant. We know that the only constant. It is clear that the only constant solution of (3.1.5) is $f \equiv 0$. Hence, every solution $f(z) \not\equiv 0$ of (3.1.5), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ is of infinite order.

3.4 Proof of Theorem 3.1.3

We assume that $f \not\equiv 0$ is analytic solution of (3.1.5) in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $\sigma(f, z_0) = \sigma < \infty$, and we prove that is failing. By Lemma 3.2.1, for any given $\varepsilon > 0$, there exists a set $E_5 \subset [0, 2\pi)$ with linear measure zero, such that for all $\theta \in [0, 2\pi) \setminus E_5$ there exists a constant $r_0 = r_0(\theta) > 0$, such that for all z satisfying $\arg(z - z_0) = \theta \in [0, 2\pi) \backslash E_5$ and $r = |z - z_0| < r_0$, we have:

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq \frac{1}{r^{(j-i)(\sigma+1)+\varepsilon}} \quad (k \geq j > i \geq 0). \tag{3.4.1}$$

Set $E_6 = \{\theta \in [0, 2\pi) : \delta_{a_s}(\theta) = \delta_{a_l}(\theta)\}$; since $\theta_s \neq \theta_l$, E_6 is of linear measure zero. By taking $\theta \in [0, 2\pi) \backslash (E_5 \cup E_6)$, (obviously, $E_5 \cup E_6$ is of linear measure zero), we have $\delta_{a_s}(\theta) > \delta_{a_l}(\theta)$ or $\delta_{a_s}(\theta) < \delta_{a_l}(\theta)$. Set $c_1 = \delta_{a_s}(\theta)$, $c_2 = \delta_{a_l}(\theta)$, we divide the proof into two cases:

(i) $c_1 > c_2$;

(ii) $c_1 < c_2$.

Case (i) $c_1 > c_2$. Here we divide also (i) into three cases:

(a) $c_1 > c_2 > 0$;

(b) $c_1 > 0 > c_2$;

(c) $0 > c_1 > c_2$.

Case(a) $c_1 > c_2 > 0$. Set $c_3 = \max\{\delta_{a_j}(\theta),\ j \neq s\}$, and then $c_3 < c_1$.

By Lemma 2.2.1, for any given ε ($\varepsilon > 0$) there exists $r_0 > 0$, such that for $0 < r < r_0$, we obtain:

$$\left| A_s(z) \exp\left\{ \frac{a_s}{(z_0 - z)^n} \right\} \right| \geq \exp\left\{ (1 - \varepsilon) \frac{c_1}{r^n} \right\}, \tag{3.4.2}$$

and

$$\left| A_j(z) \exp\left\{ \frac{a_j}{(z_0 - z)^n} \right\} \right| \leq \exp\left\{ (1 + \varepsilon) \frac{c_3}{r^n} \right\} \quad (j \neq s). \tag{3.4.3}$$

Now we prove that $\left| f^{(s)}(z) \right|$ is bounded on the ray $\arg(z_0 - z) = \theta$.

If $\left| f^{(s)}(z) \right|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5, there exists an infinite sequence of points $\{z_m = z_0 - r_m e^{i\theta}\}$ $(m \geq 1)$ where $r_m \to 0$, such that $f^{(s)}(z_m) \to \infty$ and

$$\left| \frac{f^{(j)}(z_m)}{f^{(s)}(z_m)} \right| \leq M_1,\ (M_1 > 0) \quad (j = 0, ..., s-1). \tag{3.4.4}$$

Substituting (3.4.1)-(3.4.4) into (3.1.5), we obtain

$$\begin{aligned}
\exp\left\{ (1 - \varepsilon) \frac{c_1}{r_m^n} \right\} &\leq \left| A_s(z_m) \exp\left\{ \frac{a_s}{(z_0 - z_m)^n} \right\} \right| \\
&\leq \left| \frac{f^{(k)}(z_m)}{f^{(s)}(z_m)} \right| + ... + \left| A_{s+1}(z_m) \exp\left\{ \frac{a_{s+1}}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f^{(s+1)}(z_m)}{f^{(s)}(z_m)} \right| \\
&\quad + \left| A_{s-1}(z_m) \exp\left\{ \frac{a_{s-1}}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f^{(s-1)}(z_m)}{f^{(s)}(z_m)} \right| \\
&\quad +... + \left| A_0(z_m) \exp\left\{ \frac{a_0}{(z_0 - z_m)^n} \right\} \right| \left| \frac{f(z_m)}{f^{(s)}(z_m)} \right| \\
&\leq \frac{M_2}{r^{(k-s)(\sigma+1+\varepsilon)}} \exp\left\{ (1 + \varepsilon) \frac{c_3}{r_m^n} \right\},
\end{aligned}$$

where $M_2 > 0$ is a constant, which implies that

$$1 \leq \frac{M_2}{r^{(k-s)(\sigma+1+\varepsilon)}} \exp\left\{\left[(1+\varepsilon)\, c_3 - (1-\varepsilon)\, c_1\right] \frac{1}{r_m^n}\right\}. \tag{3.4.5}$$

By taking $0 < \varepsilon < \frac{c_1 - c_3}{c_1 + c_3}$, a contradiction follows in (3.4.5) as $m \to \infty$. Therefore, $\left|f^{(s)}(z)\right| \leq K_1$ on $\arg(z_0 - z) = \theta$. We can easily obtain $|f(z)| \leq K_2$ on $\arg(z_0 - z) = \theta$.

Case(b) $c_1 > 0 > c_2$. Using the same reasoning as in the case (a), we can also obtain $|f(z)| \leq K_3$ on $\arg(z_0 - z) = \theta$.

Case(c) $0 > c_1 > c_2$. By (3.1.5), we obtain

$$\begin{aligned} 1 \leq\ & \left|A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\}\right| \left|\frac{f^{(k-1)}(z)}{f^{(k)}(z)}\right| \\ & + \ldots + \left|A_l(z) \exp\left\{\frac{a_l}{(z_0 - z)^n}\right\}\right| \left|\frac{f^{(l)}(z)}{f^{(k)}(z)}\right| \\ & + \ldots + \left|A_s(z) \exp\left\{\frac{a_s}{(z_0 - z)^n}\right\}\right| \left|\frac{f^{(s)}(z)}{f^{(k)}(z)}\right| \\ & + \ldots + \left|A_0(z) \exp\left\{\frac{a_0}{(z_0 - z)^n}\right\}\right| \left|\frac{f(z)}{f^{(k)}(z)}\right|. \end{aligned} \tag{3.4.6}$$

By Lemma 2.2.1, we have

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \leq \exp\left\{(1-\varepsilon) \frac{c_1}{r^n}\right\} \quad (j = 0, \ldots, k-1). \tag{3.4.7}$$

If $\left|f^{(k)}(z)\right|$ is unbounded on the ray $\arg(z_0 - z) = \theta$, then by Lemma 2.2.5, there exists an infinite sequence of points $\left\{z_m = z_0 - r_m e^{i\theta}\right\}$ $(m \geq 1)$ where $r_m \to 0$, such that $f^{(k)}(z_m) \to \infty$ and:

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leq M_3, \ (M_3 > 0) \quad (j = 0, \ldots, k-1). \tag{3.4.8}$$

Substituting (3.4.7) and (3.4.8) into (3.4.6), we obtain:

$$1 \leq M_4 \exp\left\{(1-\varepsilon) \frac{c_1}{r^n}\right\} \to 0,$$

a contradiction. Hence, $\left|f^{(k)}(z)\right| \leq K_4$, and then $|f(z)| \leq K_5$, on $\arg(z_0 - z) = \theta$, as $z \to z_0$.

Now, we proved that $|f(z)| \leq K$ on any ray $\arg(z_0 - z) = \theta \in [0, 2\pi) \setminus (E_5 \cup E_6)$. Set $g(w) = f(z)$, such that $w = \frac{1}{z_0 - z}$. $g(w)$ is an entire function in $\mathbb{C}$ and $|g(w)| \leq K'$ $(K' > 0)$ on any ray $\arg(w) = -\theta$, such that $\theta \in [0, 2\pi) \setminus (E_5 \cup E_6)$. By Phragmen-Lindelof theorem in sectors, we get that $|g(w)| \leq K'$ in $\mathbb{C}$, and By Liouville theorem, we conclude that $g(w)$ is a constant. So, $f(z)$ is constant. We know that the only constant solution of (3.1.5) is $f \equiv 0$. Hence, every solution $f(z) \not\equiv 0$ of (3.1.5), that is analytic in $\overline{\mathbb{C}} \backslash \{z_0\}$, is of infinite order.

Case(ii) $c_1 < c_2$.

Using the same reasoning as in the case(i), we can also obtain that $f(z)$ is constant, which contradicts our assumption. Therefore, every solution $f(z) \not\equiv 0$ of (3.1.5), that is analytic in $\overline{\mathbb{C}} \backslash \{z_0\}$, is of infinite order.

3.5 Proof of Theorem 3.1.4

Assume that $f \not\equiv 0$ is analytic solution of (3.1.5) in $\overline{\mathbb{C}} \backslash \{z_0\}$. By (3.1.5), we obtain

$$\left|A_0(z)\exp\left\{\frac{a_0}{(z_0-z)^n}\right\}\right| \leq \left|\frac{f^{(k)}}{f}\right| + \left|A_{k-1}(z)\exp\left\{\frac{a_{k-1}}{(z_0-z)^n}\right\}\right|\left|\frac{f^{(k-1)}}{f}\right| + ... + \left|A_1(z)\exp\left\{\frac{a_1}{(z_0-z)^n}\right\}\right|\left|\frac{f'}{f}\right|. \quad (3.5.1)$$

Since $\theta_0 \neq \theta_s$, there exist $(\theta_1, \theta_2) \subset [0, 2\pi)$ such that for $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$, we have $\delta_{a_0}(\theta) > 0$, $\delta_{a_s}(\theta) < 0$. Set $\delta_0 = \delta_{a_0}(\theta)$, $\delta' = \max\left\{\delta_{a_j}(\theta) : \arg a_j = \arg a_0\right\}$, $\delta' < \delta_1 < \delta_0$ such that $\delta_1 > 0$. By Lemma 2.2.1, for any given $0 < \varepsilon < 1$, we have:

$$\left|A_0(z)\exp\left\{\frac{a_0}{(z_0-z)^n}\right\}\right| \geq \exp\left\{(1-\varepsilon)\frac{\delta_0}{r^n}\right\} \quad (3.5.2)$$

and

$$\left|A_j(z)\exp\left\{\frac{a_j}{(z_0-z)^n}\right\}\right| \leq \exp\left\{(1+\varepsilon)\frac{\delta_1}{r^n}\right\} \quad (j = 1, ..., k-1). \quad (3.5.3)$$

By Lemma 2.2.2, there exist a set $E^* \subset [0, 2\pi)$ that has a linear measure zero, such that for all $\theta \subset [0, 2\pi) \backslash E^*$, there exists a constant $r_0 = r_0(\theta) > 0$, such that for all z satisfying $\arg(z - z_0) = \theta \in [0, 2\pi) \backslash E^*$ and $r = |z - z_0| < r_0$, we have:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \frac{\lambda}{r^{2j}}\left[T_{z_0}(\alpha r, f)\right]^{2j} \quad (j = 1, ..., k). \quad (3.5.4)$$

Substituting (3.5.2)-(3.5.4) into (3.5.1), we obtain:

$$\exp\left\{(1-\varepsilon)\frac{\delta_0}{r^n}\right\} \leq \frac{\lambda k}{r^{2k}}\left[T_{z_0}(\alpha r, f)\right]^{2k}\exp\left\{(1+\varepsilon)\frac{\delta_1}{r^n}\right\},$$

which implies that:

$$\exp\left\{((1-\varepsilon)\delta_0 - (1+\varepsilon)\delta_1)\frac{1}{r^n}\right\} \leq \frac{\lambda k}{r^{2k}}\left[T_{z_0}(\alpha r, f)\right]^{2k}, \quad (3.5.5)$$

where $0 < \varepsilon < \frac{\delta_0 - \delta_1}{\delta_0 + \delta_1}$. By (3.5.5), we obtain $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 2.2.8, we have $\sigma_2(f, z_0) \leq n$. Therefore $\sigma_2(f, z_0) = n$.

Chapter 4

Finite and infinite order of growth of solutions to linear differential equations near a singular point

4.1 Introduction and results

For $k \geq 2$, The linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \qquad (4.1.1)$$

where $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ are entire and meromorphic functions in the complex plane, is investigated by many authors with some conditions; see for example [31, 6, 50, 59, 36]. In [35], Hamouda studied the differential equation (4.1.1) in the unit disc. In 2016, Fettouch and Hamouda proved the following result.

Theorem 4.1.1 *[25] Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0$$

satisfies $\sigma(f, z_0) = \infty$ with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In the following two results, we will base our study on the domination of A_0 on only a curve tending to z_0. In this case, it may hapen that

$$\sigma(A_0, z_0) \leq \max\{\sigma(A_j, z_0) : j \neq 0\}.$$

Theorem 4.1.2 *[22] Let $A_0(z) \not\equiv 0,\ A_1(z), \ldots,\ A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0 = \{|z_0 - z| : z \in \gamma\} \cap (0,1)$ is of infinite logarithmic measure, such that for $z \in \gamma,\ r = |z_0 - z| \in \gamma_0$ and for any fixed $\mu > 0$, we have*

$$\lim_{r \to 0} \frac{1}{|A_0(z)| r^{\mu}} \left(\sum_{j=1}^{k-1} |A_j(z)| + 1 \right) = 0, \tag{4.1.2}$$

then every solution $f(z) \not\equiv 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f' + A_0(z) f = 0, \tag{4.1.3}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ is of infinite order.

Corollary 4.1.1 *Let $P_j(z),\ j = 1, 2, \ldots, k-1$ be polynomials and $P_0(z)$ be a transcendental entire function; let $A_j(z) = P_j(1/(z_0 - z))$; then every solution $f(z) \not\equiv 0$ of (4.1.3), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, is of infinite order.*

Example 4.1.1 *The differential equation*

$$f''' + \frac{1}{z^3} f'' + \frac{1}{z^2} f' + \sum_{n=1}^{\infty} \frac{1}{n^{n^2} z^n} f = 0, \tag{4.1.4}$$

fulfills the assumptions of Theorem 4.1.2 as z tend to $z_0 = 0$ on the ray $\arg\theta = 0$. So, every solution $f(z) \not\equiv 0$ of (4.1.4) is of infinite order. We signal here that $\sigma(A_0, 0) = \sigma(A_1, 0) = \sigma(A_2, 0) = 0$.

Theorem 4.1.3 *[22] Let $A_0(z) \not\equiv 0,\ A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0 = \{|z_0 - z| : z \in \gamma\} \cap (0,1)$ is of infinite logarithmic measure, such that for $z \in \gamma$ and $r = |z_0 - z| \in \gamma_0$, we have*

$$\lim_{r \to 0} \frac{1}{|A_0(z)|} \left(\sum_{j=1}^{k-1} |A_j(z)| + 1 \right) \exp_n \frac{\lambda}{r^{\mu}} = 0 \tag{4.1.5}$$

where $n \geq 1$ is an integer, $\lambda > 0$, $\mu > 0$ are real constant, then every solution $f(z) \not\equiv 0$ of (4.1.3), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, satisfies $\sigma_n(f, z_0) = \infty$ and furthermore $\sigma_{n+1}(f, z_0) \geq \mu$.

Example 4.1.2 *The differential equation*

$$f''' + \exp\left\{\frac{1}{z}\right\} f'' + \exp_2\left\{\frac{1}{z^3}\right\} f' + \exp_2\left\{\frac{1}{z^2}\right\} f = 0, \tag{4.1.6}$$

fulfills the assumptions of Theorem 4.1.3 as z *tend to* $z_0 = 0$ *on the ray* $\arg\theta = \frac{\pi}{5}$. *So, every solution* $f(z) \not\equiv 0$ *of (4.1.6) is of infinite order with* $\sigma_3(f, 0) \geq 2$

Now, we will investigate the case when A_s, $s \neq 0$ dominates the other coefficients in a sector. Let $I(\varepsilon) = (\theta_1 + \varepsilon, \theta_2 - \varepsilon) \subset [0, 2\pi)$ and $S(\varepsilon)$ denote the sector $\{z : \arg(z_0 - z) \in I(\varepsilon)\}$, $\varepsilon \geq 0$.

Theorem 4.1.4 *[22] Let* $A_0(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *Satisfying that there exist real constants* $0 \leq \theta_1 < \theta_2 \leq 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| : \arg(z - z_0) = \theta\} \subset (0, 1)$ *of infinite logarithmic measure, and for every fixed* $\mu > 0$, *we have*

$$\lim_{z \to z_0} \frac{1}{|A_s(z)| r^\mu} \left(\sum_{j=0, j \neq s}^{k-1} |A_j(z)| + 1 \right) = 0, \quad s \neq 0 \tag{4.1.7}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$. *Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of (4.1.3) that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) There exists $j \in \{0, ..., s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$, *we have*

$$\lim_{z \to z_0} \frac{\left| f^{(j)}(z) - b_j \right|}{r^\mu} = 0 \tag{4.1.8}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) For each integer $m \geq j + 1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$ *we have*

$$\lim_{z \to z_0} \frac{\left| f^{(m)}(z) \right|}{r^\mu} = 0, \tag{4.1.9}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

Example 4.1.3 *The function* $f(z) = e^{\frac{1}{z}} - 1$ *satisfies the differential equation*

$$f''' + e^{-\frac{1}{z}} f'' + \left(\frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4} \right) f' + \left(\frac{2}{z^3} + \frac{1}{z^4} \right) f = 0. \tag{4.1.10}$$

The differential equation(4.1.10) fulfills the assumptions of Theorem 4.1.4 in any sector $(\theta_1, \theta_2) \subset \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ *with* $z_0 = 0$. *In this example,* $A_2(z) = e^{-\frac{1}{z}}$ *is the dominating coefficient, while we have* $j = 0$ *and* $b_j = -1$.

Theorem 4.1.5 *[22] Let* $A_0(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exist real constants* $0 \le \theta_1 < \theta_2 \le 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| : \arg(z - z_0) = \theta\} \subset (0, 1)$ *of infinite logarithmic measure, such that we have*

$$\lim_{z \to z_0} \frac{1}{|A_s(z)|} \left(\sum_{j=,0 j \neq s}^{k-1} |A_j(z)| + 1 \right) \exp \frac{\lambda}{r^\alpha} = 0, \quad s \neq 0 \tag{4.1.11}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\lambda > 0$, $\alpha > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of (4.1.3), analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*
(i) There exists $j \in \{0, ..., s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\lambda > \lambda' > 0$ *we have*

$$\left| f^{(j)}(z) - b_j \right| < \exp\left(-\frac{\lambda'}{r^\alpha} \right)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.
(ii) For each integer $m \ge j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\lambda' > 0$ *we have*

$$\left| f^{(m)}(z) \right| < \exp\left(-\frac{\lambda'}{r^\alpha} \right)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Corollary 4.1.2 *[22] Let* $A_0(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exists real constants* $0 \le \theta_1 < \theta_2 \le 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| : \arg(z - z_0) = \theta\} \subset (0, 1)$ *of infinite logarithmic measure, we have*

$$|A_s(z)| \ge \exp \frac{\alpha}{r^\mu}, \quad s \neq 0,$$

$$|A_j(z)| \le \exp \frac{\beta}{r^\mu}$$

where $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\alpha > \beta \ge 0$, $\mu > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of (4.1.3) that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*
(i) There exists $j \in \{0, ..., s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$\left| f^{(j)}(z) - b_j \right| < \exp\left(-\frac{\lambda'}{r^\mu} \right) \tag{4.1.12}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.
(ii) For each integer $m \geq j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$\left|f^{(m)}(z)\right| < \exp\left(-\frac{\lambda'}{r^\mu}\right) \tag{4.1.13}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Indeed, by taking $\alpha - \beta > \lambda > 0$, the condition (4.1.11) holds; and then the assertions (4.1.12)-(4.1.13) hold by taking $\lambda > \lambda' > 0$.
We can see similar results of these theorems in the complex plane and in the unit disc in [36, 35, 50].

4.2 Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 4.2.1 *[25] Let* f *be a non constant meromorphic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of finite order* $\sigma(f, z_0) < \infty$*; let* $\varepsilon > 0$ *be a given constant. Then,*
i) there exists a set $E_1 \subset (0,1)$ *that has finite logarithmic measure such that for all* $r = |z - z_0| \in (0,1) \setminus E_1$, *we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{r^{k(\sigma+1)+\varepsilon}}, \quad k \in \mathbb{N}; \tag{4.2.1}$$

ii) and there exists a set $E_2 \subset [0, 2\pi)$ *that has a linear measure zero such that for all* $\theta \in [0, 2\pi) \setminus E_2$ *there exists a constant* $r_0 = r_0(\theta) > 0$ *such that for all* z *satisfying* $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and* $r = |z - z_0| < r_0$, *the inequality (4.2.1) holds.*

Lemma 4.2.2 *[22] Let* f *be a non constant meromorphic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of finite order* $\sigma_n(f, z_0) = \sigma_n < \infty$ $(n \geq 1)$ *and let* $\varepsilon > 0$ *be a given constant. Then, there exists a set* $E_1 \subset (0,1)$ *that has finite logarithmic measure such that for all* $r = |z - z_0| \in (0,1) \setminus E_1$, *we have*
i) if $n = 1$, *(4.2.1) holds,*
ii) and if $n \geq 2$

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \left(\exp_{n-1} \frac{1}{r^{\sigma_n + \varepsilon}}\right)^k, \quad k \in \mathbb{N}. \tag{4.2.2}$$

Proof. By the definition

$$\sigma_n(f, z_0) = \limsup_{r \to 0} \frac{\log_n T_{z_0}(r, f)}{-\log r} = \sigma_n,$$

for given $\varepsilon' > 0$ there exists r_0 such that for $0 < r < r_0$, we have

$$\frac{\log_n T_{z_0}(r, f)}{-\log r} < \sigma_n + \varepsilon';$$

which implies

$$T_{z_0}(r, f) < \exp_{n-1} \frac{1}{r^{\sigma_n + \varepsilon'}}. \tag{4.2.3}$$

Combining (4.2.3) with Lemma 2.2.2, for $\alpha > 0$, there exists a set $E_1 \subset (0, 1)$ that has finite logarithmic measure and a constant $A > 0$ such that for all $r = |z - z_0|$ satisfying $r \notin (0, 1) \setminus E_1$, we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq A \left[\frac{1}{r^2} \exp_{n-1} \left(\frac{\alpha}{r} \right)^{\sigma_n + \varepsilon'} \exp_{n-2} \left(\frac{\alpha}{r} \right)^{\sigma_n + \varepsilon'} \right]^k.$$

Then, for $\varepsilon > \varepsilon^{'} > 0$ and r near enough to 0, we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \left(\exp_{n-1} \frac{1}{r^{\sigma_n + \varepsilon}} \right)^k.$$

Lemma 4.2.3 *[22] Let f be an analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Let $a \geq \frac{1}{2}$ and*

$$G = \left\{ z : |\arg(z_0 - z)| < \frac{\pi}{2a} \right\}.$$

Suppose that $\limsup_{z \to \varsigma} |f(z)| \leq M$ for all $\varsigma \in \partial G$, where M is a fixed constant. Suppose further that there exist constants K, $b < a$ such that

$$|f(z)| \leq K \exp \frac{1}{r^b} \quad \text{as } r \to 0,$$

where $r = |z_0 - z|$ and $z \in G$. Then, $|f(z)| \leq M$ for all $z \in G$.

Proof. The change of variable $w = \frac{1}{z_0 - z}$ maps G into $H = \left\{ w : |\arg(w)| < \frac{\pi}{2a} \right\}$ and the function $g(w) = f(z)$ is an entire on $w \in \mathbb{C}$ and we have $|\arg(z_0 - z)| = \frac{\pi}{2a} \Leftrightarrow |\arg(w)| = \frac{\pi}{2a}$ and $\limsup_{w \to \xi} |g(w)| = \limsup_{z \to \varsigma} |f(z)| \leq M$ for all $\xi \in \partial H$. Further, we have

$$|g(w)| = |f(z)| \leq K \exp \frac{1}{r^b} = K \exp R^b \quad \text{as } R \to \infty,$$

where $R = |w| = \frac{1}{r}$. Then, by Phragmen-Lindelof theorem we get $|g(w)| \leq M$ for all $w \in H$. Therfore, $|f(z)| \leq M$ for all $z \in G$.

Lemma 4.2.4 *[22] If f is analytic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ such that for any $\mu > 0$, we have*

$$\left|f\left(z_0 - re^{i\theta}\right)\right| \leq r^{\mu} \quad as\ r \to 0$$

then $\int\limits_0^r \left|f\left(z_0 - te^{i\theta}\right)\right| dt$ converges and for every $\alpha > 0$, we have

$$\int\limits_0^r \left|f\left(z_0 - te^{i\theta}\right)\right| dt \leq r^{\alpha} \quad as\ r \to 0.$$

Proof. It is easy to show that $\int\limits_0^r \left|f\left(z_0 - te^{i\theta}\right)\right| dt$ converges; and we have

$$\int\limits_0^r \left|f\left(z_0 - te^{i\theta}\right)\right| dt \leq \int\limits_0^r t^{\mu} dt = \frac{r^{\mu+1}}{\mu+1}.$$

Let $\alpha > 0$. By taking $\mu + 1 > \alpha$, we have

$$\int\limits_0^r \left|f\left(z_0 - te^{i\theta}\right)\right| dt \leq \frac{r^{\mu+1}}{\mu+1} \leq r^{\alpha} \quad \text{as } r \to 0.$$

Lemma 4.2.5 *[22] Let f be an analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$. The two following assertions are equivalent:*
i) for any $\mu > 0$, $\left|f\left(z_0 - re^{i\theta}\right)\right| \leq r^{\mu}$ as $r \to 0$,
ii) for any $\alpha > 0$, $\lim\limits_{r\to 0} \frac{\left|f\left(z_0 - re^{i\theta}\right)\right|}{r^{\alpha}} = 0$.

Proof. (ii) $\Rightarrow$ (i). Suppose that for any $\alpha > 0$, $\lim\limits_{r\to 0} \frac{\left|f\left(z_0 - re^{i\theta}\right)\right|}{r^{\alpha}} = 0$. For any $\alpha > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < r < \delta$ we have

$$\left|f\left(z_0 - re^{i\theta}\right)\right| \leq \varepsilon r^{\alpha}.$$

By taking $\varepsilon = 1$ we get the assertion (i).
(i) $\Rightarrow$ (ii). Suppose that for any $\mu > 0$, $\left|f\left(z_0 - re^{i\theta}\right)\right| \leq r^{\mu}$ as $r \to 0$. Let $\alpha > 0$. We have

$$\frac{\left|f\left(z_0 - re^{i\theta}\right)\right|}{r^{\alpha}} \leq \frac{r^{\mu}}{r^{\alpha}}.$$

By taking $\mu > \alpha$, we obtain

$$\lim_{r\to 0} \frac{\left|f\left(z_0 - re^{i\theta}\right)\right|}{r^{\alpha}} = 0.$$

Lemma 4.2.6 *[22] If f is analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ such that*

$$\left|f\left(z_0-te^{i\theta}\right)\right|\leq\exp\left(-\frac{\lambda}{t^{\alpha}}\right),$$

where $\alpha>0$, $\lambda>0$, then $\int\limits_0^r\left|f\left(z_0-te^{i\theta}\right)\right|dt$ converges and we have

$$\int\limits_0^r\left|f\left(z_0-te^{i\theta}\right)\right|dt\leq\exp\left(-\frac{\lambda}{r^{\alpha}}\right)\quad as\ r\to 0.$$

Proof. It is easy to show that $\int\limits_0^r\left|f\left(z_0-te^{i\theta}\right)\right|dt$ converges; and we have

$$\begin{aligned}\int\limits_0^r\left|f\left(z_0-te^{i\theta}\right)\right|dt &\leq \int\limits_0^r\exp\left(-\frac{\lambda}{r^{\alpha}}\right)dt\leq\exp\left(-\frac{\lambda}{r^{\alpha}}\right)\int\limits_0^r dt\\ &= r\exp\left(-\frac{\lambda}{r^{\alpha}}\right)\leq\exp\left(-\frac{\lambda}{r^{\alpha}}\right)\quad\text{as } r\to 0.\end{aligned}$$

4.3 Proof of Theorem 4.1.2

Suppose that $f\not\equiv 0$ is a solution of (4.1.3) of finite order $\sigma(f,z_0)=\sigma<\infty$. By Lemma 4.2.1, for any given $\varepsilon>0$ there exists a set $E\subset(0,1)$ that has finite logarithmic measure such that for all $r=|z_0-z|\in(0,1)\backslash E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\frac{1}{r^{j(\sigma+2+\varepsilon)}},\quad j=1,...,k. \tag{4.3.1}$$

From (4.1.3) we can write

$$1\leq\frac{1}{|A_0(z)|}\left|\frac{f^{(k)}}{f}\right|+\frac{|A_{k-1}(z)|}{|A_0(z)|}\left|\frac{f^{(k-1)}}{f}\right|+...+\frac{|A_1(z)|}{|A_0(z)|}\left|\frac{f'}{f}\right|. \tag{4.3.2}$$

By the assumption (4.1.2), for $r\in F$ and any fixed $\mu>0$, we have

$$\lim_{r\to 0}\frac{|A_j(z)|}{|A_0(z)|\,r^{\mu}}=0,\quad j=1,...,k \tag{4.3.3}$$

and

$$\lim_{r\to 0}\frac{1}{|A_0(z)|\,r^{\mu}}=0. \tag{4.3.4}$$

Using (4.3.1), (4.3.3) and (4.3.4) in (4.3.2), a contradiction follows as $r\to 0$ with $r=|z_0-z|\in F\backslash E$.

4.4 Proof of Theorem 4.1.3

Suppose that $f \not\equiv 0$ is a solution of (4.1.3) with $\sigma_n(f, z_0) = \sigma_n < \infty$, $n \geq 1$. If $n = 1$ we have (4.3.1) and if $n \geq 2$, by Lemma 4.2.2, for any given $\varepsilon > 0$ there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z_0 - z| \in (0,1) \setminus E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon}}\right)^j, \quad j = 1, ..., k. \tag{4.4.1}$$

By the assumption (4.1.5), for $r \in F$, we have

$$\lim_{r\to 0}\frac{|A_j(z)|}{|A_0(z)|}\exp_n\frac{\lambda}{r^\mu} = 0, \quad j = 1, ..., k \tag{4.4.2}$$

and

$$\lim_{r\to 0}\frac{1}{|A_0(z)|}\exp_n\frac{\lambda}{r^\mu} = 0. \tag{4.4.3}$$

Using (4.3.1) or (4.4.1), (4.4.2) and (4.4.3) in (4.3.2), a contradiction follows as $r \to 0$ on γ with $r = |z_0 - z| \in F\backslash E$. So, $\sigma_n(f, z_0) = \infty$ for $n \geq 1$. Now by lemma 2.2.2 and since $\sigma_n(f, z_0) = \infty$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq A\left(\frac{1}{r}T_{z_0}(\alpha r, f)\right)^{2k}, \quad j = 1, ..., k. \tag{4.4.4}$$

By the assumption (4.1.5), for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we have

$$\frac{|A_j(z)|}{|A_0(z)|} \leq \frac{\varepsilon_1}{\exp_n\left(\frac{\lambda}{r^\mu}\right)}, \quad j = 1, ..., k \tag{4.4.5}$$

and

$$\frac{1}{|A_0(z)|} \leq \frac{\varepsilon_2}{\exp_n\left(\frac{\lambda}{r^\mu}\right)}, \tag{4.4.6}$$

as $r \to 0$ on γ with $r = |z_0 - z| \in F$. Using (4.4.4)-(4.4.6) in (4.3.2), we obtain, for $r = |z_0 - z| \in F\backslash E$,

$$1 \leq \frac{M}{\exp_n\left(\frac{\lambda}{r^\mu}\right)}\left(\frac{1}{r}T_{z_0}(\alpha r, f)\right)^{2k}, \tag{4.4.7}$$

where $M > 0$ is a real constant. Set $R = \alpha r$. We signal here that E is of finite logarithmic measure if and only if αE is of finite logarithmic measure. So, from (4.4.7), we get

$$\exp_n\frac{\lambda\alpha^\mu}{R^\mu} \leq M\left(\frac{\alpha}{R}T_{z_0}(R, f)\right)^{2k}, \quad R \in F\backslash E. \tag{4.4.8}$$

From (4.4.8) we obtain

$$\sigma_{n+1}(f, z_0) = \limsup_{r\to 0}\frac{\log^+_{n+1} T_{z_0}(r, f)}{-\log R} \geq \mu.$$

4.5 Proof of Theorem 4.1.4

First, we have to prove that $f(z)$ is bounded in $S(\varepsilon)$, for $\varepsilon > 0$ small enough and for that we prove that $f^{(s)}(z)$ is also bounded in $S(\varepsilon)$. From Lemma 3.2.1, it follows that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for all $j \in \{s+1, ..., k\}$

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq \frac{1}{r^{(j-s)(\sigma+1)+\varepsilon}}, \tag{4.5.1}$$

where $\arg(z_0 - z) \in I(0) \backslash E$ and $r = |z_0 - z| \in \Gamma_\theta$. If we suppose that $f^{(s)}(z)$ is unbounded on some ray $\arg(z_0 - z) = \varphi \in I(0) \backslash E$, then by lemma 2.2.5 there exists an infinite sequence of points $z_m = z_0 - r_m e^{i\varphi}$, $m = 1, 2, ...$, whith $r_m \to 0$, such that $f^{(k)}(z_m) \to \infty$ and

$$\left| \frac{f^{(q)}(z_m)}{f^{(s)}(z_m)} \right| \leq M_1, \tag{4.5.2}$$

where $M_1 > 0$, $q \in \{0, 1, ..., s-1\}$ and m large enough. From (4.1.3) we can write

$$\begin{aligned} 1 \leq\ & \frac{1}{|A_s(z)|} \left| \frac{f^{(k)}}{f^{(s)}} \right| + \frac{|A_{k-1}(z)|}{|A_s(z)|} \left| \frac{f^{(k-1)}}{f^{(s)}} \right| + ... + \frac{|A_{s+1}(z)|}{|A_s(z)|} \left| \frac{f^{(s+1)}}{f^{(s)}} \right| \\ & + \frac{|A_{s-1}(z)|}{|A_s(z)|} \left| \frac{f^{(s-1)}}{f^{(s)}} \right| + ... + \frac{|A_0(z)|}{|A_s(z)|} \left| \frac{f}{f^{(s)}} \right|. \end{aligned} \tag{4.5.3}$$

Combining now (4.1.7), (4.5.1)- (4.5.3) and letting $m \to \infty$ we obtain a contradiction. Therefore, $f^{(s)}(z)$ remains bounded on all rays $\arg(z_0 - z) = \varphi \in I(0) \backslash E$. By Lemma 4.2.3, we conclude that $f^{(s)}(z)$ is bounded, say $\left|f^{(s)}(z)\right| \leq M_2$, in the whole sector $S\left(\frac{\varepsilon}{2}\right)$, for $\varepsilon > 0$ small enough.

By integrating s times along the line segment $[z_1, z]$ in $S\left(\frac{\varepsilon}{2}\right)$, we have

$$\begin{aligned} f(z) &= f(z_1) + f'(z_1)(z - z_1) + ... + \frac{1}{(s-1)!} f^{(s-1)}(z_1)(z - z_1)^{s-1} \\ & + \int_{z_1}^{z} ... \int_{z_1}^{z} f^{(s)}(t)\, dt ... dt; \end{aligned}$$

and by an elementary triangle inequality estimate, we obtain

$$|f(z)| \leq |f(z_1)| + |f'(z_1)| |z - z_1| + ... + \frac{1}{(s-1)!} \left|f^{(s-1)}(z_1)\right| |z - z_1|^{s-1} + \frac{1}{(s)!} M |z - z_1|^s$$

and therfore, as $z \to z_0$, we get

$$|f(z)| \leq M_3 \tag{4.5.4}$$

for a certain constant $M_3 > 0$.
Now, we begin to prove (4.1.9) for $m = s$. Using (4.1.3), we can write

$$\begin{aligned} \left|f^{(s)}(z)\right| &\leq |f| \left(\frac{1}{|A_s(z)|} \left|\frac{f^{(k)}}{f}\right| + \frac{|A_{k-1}(z)|}{|A_s(z)|} \left|\frac{f^{(k-1)}}{f}\right| + ... + \frac{|A_{s+1}(z)|}{|A_s(z)|} \left|\frac{f^{(s+1)}}{f}\right| \right. \\ &\left. + \frac{|A_{s-1}(z)|}{|A_s(z)|} \left|\frac{f^{(s-1)}}{f}\right| + ... + \frac{|A_1(z)|}{|A_s(z)|} \left|\frac{f'}{f}\right| + \frac{|A_0(z)|}{|A_s(z)|} \right). \end{aligned} \quad (4.5.5)$$

By the assumption (4.1.7), for any $\mu > 0$, for every $j \in \{0, 1, ..., s-1, s+1, ..., k-1\}$ and for $\varepsilon > 0$, there exist δ such that for $|z_0 - z| < \delta$ we have

$$\frac{|A_j(z)|}{|A_s(z)|} \leq \varepsilon |z_0 - z|^{\mu}, \quad (4.5.6)$$

$$\frac{1}{|A_s(z)|} \leq \varepsilon |z_0 - z|^{\mu}, \quad (4.5.7)$$

where $\arg(z_0 - z) = \theta \in I(0)$ and $|z_0 - z| = r \in \Gamma_\theta$. Substituting (4.5.1), (4.5.4), (4.5.6) and (4.5.7) into (4.5.5), we obtain that for any $\mu > 0$, we have

$$\left|f^{(s)}(z)\right| \leq M_4 \frac{|z_0 - z|^{\mu}}{r^{k(\sigma+1)+\varepsilon}} \quad \text{as } r \to 0.$$

We conclude that for any fixed $\alpha > 0$

$$\lim_{z \to z_0} \frac{\left|f^{(s)}(z)\right|}{r^{\alpha}} = 0, \quad (4.5.8)$$

with $r = |z_0 - z| \in \Gamma_\theta$ and $\arg(z_0 - z) = \varphi \in I\left(\frac{\varepsilon}{2}\right) \backslash E$.

Proof of equality (4.1.9) for $m > s$. Consider $z = z_0 - re^{i\theta} \in S(\varepsilon)$ and $C(z)$ the circle centered at z of radius ρ small enough such that $C(z)$ is contained in $S\left(\frac{\varepsilon}{2}\right)$, we may take $\rho = r \sin\left(\frac{\varepsilon}{2}\right)$. By cauchy formula applied to the function $f^{(s)}(z)$ we have

$$f^{(m)}(z) = \frac{(m-s)!}{2\pi} \int_{C(z)} \frac{f^{(s)}(\zeta)}{(z-\zeta)^{m-s+1}} d\zeta, \quad (4.5.9)$$

and using (4.5.9), we get

$$\begin{aligned} \left|f^{(m)}(z)\right| &\leq \frac{(m-s)!}{2\pi} \int_0^{2\pi} \frac{|z_0 - z|^{\mu}}{\rho^{m-s+1}} \rho d\theta \\ &\leq \frac{(m-s)!}{\sin^{m-s}\left(\frac{\varepsilon}{2}\right)} \frac{|z_0 - z|^{\mu}}{r^{m-s}}. \end{aligned}$$

We conclude that, for any fixed $\alpha > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, we have

$$\lim_{z \to z_0} \frac{\left|f^{(m)}(z)\right|}{|z_0 - z|^\alpha} = 0.$$

Until now, we have proved the second assertion for $m \geq s$.
We start to prove the first assertion for $j = s - 1$. Set

$$a_s = \int_0^\infty f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt.$$

By (4.5.8), it is easy to see that $\int_0^\infty f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt$ converges. Moreover, a_s is independent of θ, because by (4.5.8), the integral of $f^{(s)}(\zeta)$ over the arc $z_0 - re^{i\theta}$, $\theta \in (\varphi, \varphi') \subset I\left(\frac{\varepsilon}{2}\right)$, we get

$$\left|\int_\varphi^{\varphi'} f^{(s)}\left(z_0 - re^{i\theta}\right) ire^{i\theta} d\theta\right| \leq Mr^{\alpha+1} |\varphi' - \varphi| \to 0, \quad r \to 0, \ M > 0.$$

Define now $b_{s-1} = f^{(s-1)}(\infty) + a_s$, and suppose that $b_{s-1} \neq 0$. Let $z = z_0 - re^{i\theta}$ be an arbitrary point in $S(\varepsilon)$. Then, since

$$f^{(s-1)}(z) - b_{s-1} = \int_\infty^z f^{(s)}(\zeta) d\zeta - \int_0^\infty f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt,$$

we may apply (4.5.8) and Lemma 4.2.4, we get

$$\begin{aligned}
\left|f^{(s-1)}(z) - b_{s-1}\right| &= \left|\int_\infty^z f^{(s)}(\zeta) d\zeta - \int_0^\infty f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt\right| \\
&= \left|\int_r^\infty f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt + \int_\infty^0 f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt\right| \\
&= \left|\int_r^0 f^{(s)}\left(z_0 - te^{i\theta}\right) e^{i\theta} dt\right| \\
&\leq \int_0^r \left|f^{(s)}\left(z_0 - te^{i\theta}\right)\right| dt \leq r^\mu \ \text{ as } r \to 0, \qquad (4.5.10)
\end{aligned}$$

for any $\mu > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By Lemma 4.2.5, We have completed the proof in the case $b_{s-1} \neq 0$. If $b_{s-1} = 0$, we define $a_{s-1} =$

$\int\limits_0^\infty f^{(s-1)}\left(z_0 - te^{i\theta}\right)e^{i\theta}dt$ and $b_{s-2} = f^{(s-2)}(\infty) + a_{s-1}$ and by applying Lemma 4.2.4 with (4.5.10) we obtain that, for every fixed $\mu > 0$,

$$\left|f^{(s-2)}(z) - b_{s-2}\right| \leq r^{\mu} \ \text{ as } r \to 0,$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By the same method, if $b_{s-1} = b_{s-2} = ... = b_{j+1} = 0$ and $b_j \neq 0$, $j \in \{0, ..., s-1\}$, then for any fixed $\mu > 0$

$$\left|f^{(j)}(z) - b_j\right| \leq r^{\mu} \text{ as } r \to 0,$$

and

$$\left|f^{(m)}(z)\right| \leq r^{\mu} \ \text{ as } r \to 0 \text{ for all } m \geq j+1 \tag{4.5.11}$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. Now it remains to show that the case $b_{s-1} = b_{s-2} = ... = b_0 = 0$ is not possible. In this case, we have, for any fixed $\mu > 0$

$$\left|f^{(m)}(z)\right| \leq r^{\mu} \ \text{ as } r \to 0, \tag{4.5.12}$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, for every $m \geq 0$ and any $\mu > 0$, there exists $r_0(\mu, m) > 0$ such that if $|z_0 - z| = r < r_0$ then $\left|f^{(m)}(z)\right| \leq |z_0 - z|^{\mu}$. Now we take $z \in S(\varepsilon)$ such that $r = |z_0 - z| < r_1 = \min\limits_{m=0,...,s} r_0(\mu, m)$; we remark here that if z is fixed then (4.5.12) is valid for only some $\mu > 0$ and not for any $\mu > 0$. From (4.1.3) we can write

$$\begin{aligned}\frac{\left|f^{(s)}(z)\right|}{|f(z)|} \leq\ & \frac{1}{|A_s(z)|}\left|\frac{f^{(k)}}{f}\right| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\left|\frac{f^{(k-1)}}{f}\right| + ... + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}}{f}\right| \\ & + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}}{f}\right| + ... + \frac{|A_1(z)|}{|A_s(z)|}\left|\frac{f'}{f}\right| + \frac{|A_0(z)|}{|A_s(z)|},\end{aligned} \tag{4.5.13}$$

and by using (4.1.7) and Lemma 4.2.1 in (4.5.13), we obtain

$$\frac{\left|f^{(s)}(z)\right|}{|f(z)|} \leq |z_0 - z|^{\mu}, \tag{4.5.14}$$

and by (4.5.12) for $m = 0$ in (4.5.13), we get

$$\left|f^{(s)}(z)\right| \leq |z_0 - z|^{2\mu}, \tag{4.5.15}$$

for $|z_0 - z| < r_1$ and $\arg(z_0 - z) \in I(\varepsilon) \backslash E$, hence in $S\left(\varepsilon + \frac{\varepsilon}{2}\right)$ by Lemma 4.2.3. Repeating the reasoning of (4.5.10)-(4.5.12) with (4.5.15), we obtain

$$|f(z)| \leq |z_0 - z|^{2\mu},$$

and by combining with (4.5.14), we get

$$\left|f^{(s)}(z)\right| \leq |z_0 - z|^{3\mu},$$

in $S\left(\varepsilon+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}\right)$. Inductively, by the same reasoning, after $(T-1)$ steps, we obtain

$$\left|f^{(s)}(z)\right| \leq |z_0-z|^{T\mu} \tag{4.5.16}$$

in

$$S\left(\varepsilon+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\ldots+\frac{\varepsilon}{2^{T-1}}\right)=S\left(2\varepsilon\left(1-\frac{1}{2^{T-1}}\right)\right)$$

with $|z_0-z|<r_1$. Thus, we have proved, in this special case $b_{s-1}=b_{s-2}=\ldots=b_0=0$, that (4.5.16) is valid in $S(2\varepsilon)$ for all $T\in\mathbb{N}$, provided $|z_0-z|<r_1$. Fix now a finite line segment $L\subset S(2\varepsilon)$ with $|z_0-z|<\min(1,r_1)$. By taking $T\to\infty$ in (4.5.16), $f^{(s)}(z)$ vanishes identically on such a line segment. Therefore, f must be a polynomial. Since f is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, f has to be a constant. It is easy to see that the only constant solution of (4.1.3) is $f\equiv 0$, a contradiction.

4.6 Proof of Theorem 4.1.5

We will use the same method of the proof of Theorem 4.1.4. The assumption (4.1.11) implies that for any $\varepsilon>0$ there exists $\delta>0$ such that for $r=|z_0-z|<\delta$, we have

$$\frac{|A_j(z)|}{|A_s(z)|} \leq \varepsilon \exp\left(-\frac{\lambda}{r^\alpha}\right), \tag{4.6.1}$$

$$\frac{1}{|A_s(z)|} \leq \varepsilon \exp\left(-\frac{\lambda}{r^\alpha}\right) \tag{4.6.2}$$

By the same steps (4.5.1)- (4.5.3) with (4.6.1) and (4.6.2), we can prove that $f^{(s)}(z)$ is bounded in $S(\varepsilon)$, say

$$\left|f^{(s)}(z)\right| \leq M_1,$$

in the whole sector $S\left(\frac{\varepsilon}{2}\right)$, for some $\varepsilon>0$ small enough. As above, we can prove also that

$$|f(z)| \leq M_2.$$

By using (4.6.1)- (4.6.2) in (4.5.5), for $r=|z_0-z|\in\Gamma_\theta$ and $\arg(z_0-z)=\varphi\in I\left(\frac{\varepsilon}{2}\right)\setminus E$, we get

$$\left|f^{(s)}(z)\right| \leq \exp\left(\frac{-\lambda+\tau}{r^\alpha}\right),$$

where $0<\tau<\lambda$. For $m>s$, as above, by (4.5.9) we obtain

$$\left|f^{(m)}(z)\right| \leq \exp\left(\frac{-\lambda+\tau}{r^\alpha}\right),$$

for all $z\in S(\varepsilon)$ with $r=|z_0-z|\in\Gamma_\theta$, $0<\tau<\lambda$. Puting a_s and b_{s-1} as above and by Lemma 4.2.6, we get

$$\left|f^{(s-1)}(z)-b_{s-1}\right| \leq \exp\left(\frac{-\lambda+\tau}{r^\alpha}\right),$$

as $r = |z_0 - z| \to 0$, where $0 < \tau < \lambda$. By the same method used in the proof of Theorem 4.1.4, we can prove the impossibility of the case $b_{s-1} = b_{s-2} = ... = b_0 = 0$.

Chapter 5

Exponent of convergence of solutions to linear differential equations near a singular point

5.1 Introduction and results

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \qquad (5.1.1)$$

where $A_j(z)$ are entire and meromorphic functions in the complex plane.

In 2012, Xu, Tu and Zheng studied the relation between the small functions and the derivatives of the solutions of differential equation (5.1.1) and obtained the following two results.

Theorem 5.1.1 *[62] Let $A_j(z)$, $j = 0, 1, ..., k-1$ be entire functions with finite order and satisfy one of the following conditions:*
i) $\max\{\sigma(A_j) : j = 1, ..., k-1\} < \sigma(A_0) < \infty$;
ii) $0 < \sigma(A_{k-1}) = ... = \sigma(A_1) = \sigma(A_0) < \infty$ *and* $\max\{\tau(A_j) : j = 1, ..., k-1\} = \tau_1 < \tau(A_0) = \tau$,
Then for every solution $f \not\equiv 0$ of (5.1.1) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have

$$\overline{\lambda}_2(f-\varphi) = \overline{\lambda}_2\left(f^{(i)} - \varphi\right) = \lambda_2\left(f^{(i)} - \varphi\right) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

Theorem 5.1.2 *[62] Let $A_j(z)$, $j = 0, 1, ..., k-1$ be meromorphic functions satisfying* $\max\{\sigma(A_j) : j = 1, ..., k-1\} < \sigma(A_0)$ *and $\delta(\infty, A_0) > 0$. Then, for every* *meromorphic solution* $f \not\equiv 0$ *of (5.1.1) and for any meromorphic function* $\varphi(z) \not\equiv 0$ *satisfying* $\sigma_2(\varphi) < \sigma(A_0)$, *we have*

$$\overline{\lambda}_2(f-\varphi) = \overline{\lambda}_2\left(f^{(i)}-\varphi\right) = \lambda_2\left(f^{(i)}-\varphi\right) = \sigma_2(f) \geq \sigma(A_0) \quad (i \in \mathbb{N}).$$

There has been an extension of these results to the unit disc, see [7, 26, 65]. In this chapter, we will extend this investigation to linear differential equations whose coefficients are analytic and meromorphic in the closed complex plane except a finite point. In fact, we will prove the following results.

Theorem 5.1.3 *Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfying* $0 < \sigma(A_0, z_0) = \sigma < \infty$, $0 < \tau(A_0, z_0) = \tau < \infty$, *and* $\sigma(A_j, z_0) \leq \sigma(A_0, z_0)$, $\tau(A_j, z_0) < \tau$ *if* $\sigma(A_j, z_0) = \sigma(A_0, z_0)$ $(j = 1, ..., k-1)$. *Then, every analytic solution* $f(z) \not\equiv 0$, *in* $\overline{\mathbb{C}}\setminus\{z_0\}$, *of (5.1.1), and for any analytic function* $\varphi(z) \not\equiv 0$ *in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfying* $\sigma_2(\varphi, z_0) < \sigma(A_0, z_0)$, *we have*

$$\overline{\lambda}_2(f-\varphi, z_0) = \overline{\lambda}_2\left(f^{(i)}-\varphi, z_0\right) = \lambda_2\left(f^{(i)}-\varphi, z_0\right) = \sigma_2(f, z_0) = \sigma(A_0, z_0) \quad (i \in \mathbb{N}). \tag{5.1.2}$$

Corollary 5.1.1 *Let* $A_0(z) \not\equiv 0, A_1(z)$ *be analytic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfying* $\sigma(A_j, z_0) < n \ (j = 1, 2)$; *suppose that* a, b *are complex numbers that satisfy* $0 < |a| < |b|$. *If* $\varphi(z) \not\equiv 0$ *is an analytic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *satisfying* $\sigma_2(\varphi, z_0) < n$, *then every analytic solution* $f(z) \not\equiv 0$, *in* $\overline{\mathbb{C}}\setminus\{z_0\}$, *of the differential equation*

$$f'' + A_1(z)\exp\left\{\frac{a}{(z_0-z)^n}\right\}f' + A_0(z)\exp\left\{\frac{b}{(z_0-z)^n}\right\}f = 0,$$

satisfies

$$\overline{\lambda}_2(f-\varphi, z_0) = \overline{\lambda}_2\left(f^{(i)}-\varphi, z_0\right) = \lambda_2\left(f^{(i)}-\varphi, z_0\right) = \sigma_2(f, z_0) = n \ (i \in \mathbb{N}).$$

Theorem 5.1.4 *Let* $A_j(z)$ $j = 0, 1, ..., k-1$ *be meromorphic functions in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *with* $\sigma(A_j, z_0) = 0$ *and such that there exist a constant* $\beta > 0$ *and a set* F *of infinite logarithmic measure such that for all* $|z - z_0| = r \in F$ *and for any* $\alpha > 0$, *we have* $\max\{|A_j(z)| : j = 1, ..., k-1\} \leq \frac{1}{r^{\beta}}$ *and* $|A_0(z)| \geq \frac{1}{r^{\alpha}}$ *where* $|A_0(z)| = M(r, A_0)$. *Then, for every meromorphic solution* $f \not\equiv 0$ *of (5.1.1), and for any meromorphic function* $\varphi(z) \not\equiv 0$ *in* $\overline{\mathbb{C}}\setminus\{z_0\}$ *of finite order* $\sigma(\varphi, z_0) < \infty$, *we have*

$$\overline{\lambda}(f-\varphi, z_0) = \overline{\lambda}\left(f^{(i)}-\varphi, z_0\right) = \lambda\left(f^{(i)}-\varphi, z_0\right) = \sigma(f, z_0) = \infty \ (i \in \mathbb{N}). \tag{5.1.3}$$

Corollary 5.1.2 *Let* $P_j(z)$, $j = 1, 2, ..., k-1$ *be polynomials and* $P_0(z)$ *be a transcendental entire function with* $\sigma(P_0, z_0) = 0$, *let* $A_j(z) = P_j\left(\frac{1}{z_0 - z}\right)$; *then for every analytic solution* $f(z) \not\equiv 0$, *in* $\overline{\mathbb{C}} \setminus \{z_0\}$, *of (5.1.1), and for any analytic function* $\varphi(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of finite order* $\sigma(\varphi, z_0) < \infty$, *the equalities (5.1.3) hold.*

Theorem 5.1.5 *Let* $A_j(z)$ $j = 0, 1, ..., k-1$ *be meromorphic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying* $\max\{\sigma(A_j, z_0) : j = 1, ..., k-1\} < \sigma(A_0, z_0) = \sigma$ *with* $\liminf\limits_{r \to 0} \frac{m_{z_0}(r, A_0)}{T_{z_0}(r, A_0)} > 0$. *Then, for every solution* $f(z) \not\equiv 0$ *of (5.1.1), and for any meromorphic function* $\varphi(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying* $\sigma_2(\varphi, z_0) < \sigma$, *we have*

$$\overline{\lambda}_2(f - \varphi, z_0) = \overline{\lambda}_2\left(f^{(i)} - \varphi, z_0\right) = \lambda_2\left(f^{(i)} - \varphi, z_0\right) = \sigma_2(f, z_0) \geq \sigma(A_0, z_0) \quad (i \in \mathbb{N}).$$

5.2 Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 5.2.1 *[62] Assume* $f \not\equiv 0$ *is a solution of equation (5.1.1), set* $g = f - \varphi$, *then* g *satisfies the equation*

$$g^{(k)} + A_{k-1}g^{(k-1)} + ... + A_1 g' + A_0 g = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_1\varphi' + A_0\varphi\right] \tag{5.2.1}$$

Lemma 5.2.2 *[62] Assume* $f \not\equiv 0$ *is a solution of equation (5.1.1), set* $g_i = f^{(i)} - \varphi$, *then* g_i *satisfies the equation*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + ... + U_0^i g_i = -\left[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + ... + U_0^i \varphi\right] \tag{5.2.2}$$

where

$$U_j^0 = A_j, \quad j = 0, 1, ..., k-1,$$
$$U_j^{i+1} = U_{j-1}^i + U_j^{i\prime} - \frac{U_0^{i\prime}}{U_0^i} U_{j+1}^i, \ j = 0, 1, ..., k-1, \ i \in \mathbb{N}, \tag{5.2.3}$$

and $U_k^i \equiv 1$, $A_k \equiv 1$.

Lemma 5.2.3 *[25] Let* f *be a non constant analytic function in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of order* $\sigma(f, z_0) > \alpha > 0$. *Then, there exists a set* $F \subset (0, 1)$ *of infinite logarithmic measure such that for all* $r \in F$, *we have*

$$\log M_{z_0}(r, f) > \frac{1}{r^\alpha}. \tag{5.2.4}$$

Lemma 5.2.4 *Let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $0 < \sigma(A_0, z_0) = \sigma < \infty$, $0 < \tau(A_0, z_0) = \sigma < \infty$, and $\sigma(A_j, z_0) < \sigma(A_0, z_0)$ or $\sigma(A_j, z_0) = \sigma(A_0, z_0)$ with $\tau(A_j, z_0) < \tau_1 < \tau$, $(j = 1, ..., k-1)$ and U_j^i $(j = 0, 1, ..., k)$ $(i \in \mathbb{N})$ be stated as in (5.2.3). Then, for any given ε $(0 < 2\varepsilon < \tau - \tau_1)$, there exists a set F of infinite logarithmic measure such that for $r \in F$, we have*

$$\left|U_0^i\right| \geq \exp\left\{\frac{\tau - \varepsilon}{r^\sigma}\right\} \text{ and } \left|U_j^i\right| \leq \exp\left\{\frac{\tau_1 + \varepsilon}{r^\sigma}\right\} \tag{5.2.5}$$

where $|U_0^i(z)| = M(r, U_0^i)$, $i \in \mathbb{N}$ and $j = 1, 2, ..., k-1$.

Proof. *The inductive method will be used to prove it.*

We first prove (5.2.5) for $i = 1$. From (5.2.3), we have $U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0} A_{j+1} = A_j + A_{j+1}\left(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0}\right)$, $j = 0, 1, ..., k-1$ and $A_k \equiv 1$. So

$$\left|U_0^1\right| \geq |A_0| - |A_1|\left(\left|\frac{A'_1}{A_1}\right| + \left|\frac{A'_0}{A_0}\right|\right), \tag{5.2.6}$$

$$\left|U_j^1\right| \leq |A_j| + |A_{j+1}|\left(\left|\frac{A'_{j+1}}{A_{j+1}}\right| + \left|\frac{A'_0}{A_0}\right|\right). \tag{5.2.7}$$

By Lemma 2.2.3 and (5.2.6)-(5.2.7), there exists a set F with infinite logarithmic measure such that

$$\begin{aligned}\left|U_0^1\right| &\geq \exp\left\{\frac{\tau - \frac{\varepsilon}{2^m}}{r^\sigma}\right\} - 2\exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^m}}{r^\sigma}\right\}\frac{1}{r^M} \\ &\geq \exp\left\{\frac{\tau - \frac{\varepsilon}{2^{m-1}}}{r^\sigma}\right\},\end{aligned} \tag{5.2.8}$$

$$\begin{aligned}\left|U_j^1\right| &\leq \exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^m}}{r^\sigma}\right\} + 2\exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^m}}{r^\sigma}\right\}\frac{1}{r^M} \\ &\leq \exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^{m-1}}}{r^\sigma}\right\},\ j \neq 0,\end{aligned} \tag{5.2.9}$$

where $M > 0$ is a constant. Now for $i = 2$ in (5.2.3), we have

$$\left|U_0^2\right| \geq \left|U_0^1\right| - \left|U_1^1\right|\left(\left|\frac{(U_1^1)'}{U_1^1}\right| + \left|\frac{(U_0^1)'}{U_0^1}\right|\right), \tag{5.2.10}$$

$$\left|U_j^2\right| \leq \left|U_j^2\right| + \left|U_{j+1}^2\right|\left(\left|\frac{(U_{j+1}^2)'}{U_{j+1}^2}\right| + \left|\frac{(U_0^2)'}{U_0^2}\right|\right),\ j \neq 0. \tag{5.2.11}$$

From (5.2.8)-(5.2.11), we obtain

$$\left|U_0^2\right| \geq \exp\left\{\frac{\tau - \frac{\varepsilon}{2^{m-2}}}{r^\sigma}\right\} \text{ and } \left|U_j^2\right| \leq \exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^{m-2}}}{r^\sigma}\right\}. \tag{5.2.12}$$

By (5.2.12) and for $i = 3$ in (5.2.3), we have

$$\left|U_0^3\right| \geq \exp\left\{\frac{\tau - \frac{\varepsilon}{2^{m-3}}}{r^\sigma}\right\} \text{ and } \left|U_j^3\right| \leq \exp\left\{\frac{\tau_1 + \frac{\varepsilon}{2^{m-3}}}{r^\sigma}\right\}.$$

By the same method until $i = m$, we obtain

$$\left|U_0^i\right| \geq \exp\left\{\frac{\tau - \varepsilon}{r^\sigma}\right\} \text{ and } \left|U_j^i\right| \leq \exp\left\{\frac{\tau_1 + \varepsilon}{r^\sigma}\right\}.$$

Remark 5.2.1 *By the properties of the characteristic function and Remark 1.5.1, we can obtain*

$$\sigma\left(U_j^i, z_0\right) \leq \max\left\{\sigma\left(A_0, z_0\right), ..., \sigma\left(A_{k-1}, z_0\right)\right\} = \sigma\left(A_0, z_0\right),$$

where $i \in \mathbb{N}$ and $j = 0, 1, ..., k-1$.

Lemma 5.2.5 *Let $H_j(z)$ $j = 0, 1, ..., k-1$ be meromorphic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order satisfying $\max\{|H_j(z)| : j = 1, ..., k-1\} \leq \exp\left\{\frac{\beta_1}{r^\sigma}\right\}$ and $|H_0(z)| \geq \exp\left\{\frac{\beta}{r^\sigma}\right\}$ where $|H_0(z)| = M(r, H_0)$, $0 < \beta_1 < \beta$, $\sigma > 0$ and $|z - z_0| = r \in F \subset (0,\ 1)$ with F is of infinite logarithmic measure. Then, every meromorphic solution $f \not\equiv 0$ of the differential equation*

$$f^{(k)} + H_{k-1}(z) f^{(k-1)} + ... + H_1(z) f' + H_0(z) f = 0. \tag{5.2.13}$$

Satisfies $\sigma_2(f, z_0) \geq \sigma$.

Proof. Let $f \not\equiv 0$ be a meromorphic solution of (5.2.13) of finite order $\sigma(f, z_0) = \sigma < \infty$. From (5.2.13), we obtain

$$|H_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1} |H_j(z)| \left|\frac{f^{(j)}}{f}\right|. \tag{5.2.14}$$

By lemma 2.2.3, for a given $\varepsilon > 0$ there exists a set $E \subset (0, 1)$ of finite logarithmic measure such that for all $|z - z_0| = r \notin E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \frac{1}{r^{j(\sigma+1)+\varepsilon}},\ (j = 1, ..., k). \tag{5.2.15}$$

From (5.2.14)-(5.2.15) and the assumptions of lemma 5.2.4, we obtain

$$\exp\left\{\frac{\beta}{r^{\sigma}}\right\} \leq \frac{M_1}{r^{k(\sigma+1)+\varepsilon}} \exp\left\{\frac{\beta_1}{r^{\sigma}}\right\}, \tag{5.2.16}$$

where $M_1 > 0$ is a constant. Since $\beta_1 < \beta$, a contradiction follows From (5.2.16) as $r \to 0$. So, $\sigma(f, z_0) = \infty$; and by lemma 2.2.2, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j}, \ (j = 1, ..., k). \tag{5.2.17}$$

From (5.2.14)-(5.2.17) and the assumptions of lemma 5.2.4, we obtain

$$\exp\left\{\frac{\beta}{r^{\sigma}}\right\} \leq \frac{M_2}{r^{2k}} \left[T_{z_0}(\alpha r, f)\right]^{2k} \exp\left\{\frac{\beta_1}{r^{\sigma}}\right\}, \tag{5.2.18}$$

and thus

$$\exp\left\{\frac{\beta - \beta_1}{r^{\sigma}}\right\} \leq \frac{M_2}{r^{2k}} \left[T_{z_0}(\alpha r, f)\right]^{2k}. \tag{5.2.19}$$

From (5.2.19), it is easy to obtain that $\sigma_2(f, z_0) \geq \sigma$.

By the same reasoning of lemma 5.2.4 and using lemma 5.2.3, we obtain the following lemma.

Lemma 5.2.6 *Let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ with finite order and satisfy* $\max\{\sigma(A_j, z_0) : j \neq 0\} = \sigma_1 < \sigma(A_0, z_0) = \sigma < \infty$, *and U_j^i $(j = 0, 1, ..., k)$ $(i \in \mathbb{N})$ be stated as in (5.2.3). Then, for any given ε $(0 < 2\varepsilon < \sigma - \sigma_1)$, there exists a set F of infinite logarithmic measure such that for $|z - z_0| = r \in F$, we have*

$$\left|U_0^i\right| \geq \exp\left\{\frac{1}{r^{\sigma-\varepsilon}}\right\} \ \text{and} \ \left|U_j^i\right| \leq \exp\left\{\frac{1}{r^{\sigma_1+\varepsilon}}\right\} \tag{5.2.20}$$

where $|U_0^i(z)| = M(r, U_0^i)$, $i \in \mathbb{N}$ and $j = 1, 2, ..., k-1$.

By using the same method of the proof of lemma 5.2.5, we obtain the following lemma.

Lemma 5.2.7 *Let $H_j(z)$ $j = 0, 1, ..., k-1$ be meromorphic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order satisfying* $\max\{|H_j(z)| : j = 1, ..., k-1\} \leq \exp\left\{\frac{1}{r^{\sigma_1}}\right\}$ *and* $|H_0(z)| \geq \exp\left\{\frac{1}{r^{\sigma}}\right\}$ *where $|H_0(z)| = M(r, H_0)$, $0 < \sigma_1 < \sigma$, and $|z - z_0| = r \in F \subset (0, 1)$ with F is of infinite logarithmic measure. Then, every meromorphic solution f of (5.2.13) Satisfies $\sigma_2(f, z_0) \geq \sigma$.*

By the well known logarithmic derivative lemma of meromorphic functions in $\mathbb{C}$ we can prove its new version near a singular point as the following.

Lemma 5.2.8 *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$, and let $k\in\mathbb{N}$. Then*

$$m_{z_0}\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right),$$

for all $r\in(0,1)-E$, where $\int_E\frac{dr}{r}<\infty$. If f is of finite order, then

$$m_{z_0}\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log\frac{1}{r}\right),\ r\notin E.$$

Proof. Set $g(w)=f\left(z_0-\frac{1}{w}\right)$, $z=z_0-\frac{1}{w}$. from Remark 1.5.1, $g(w)$ is meromorphic function in $\mathbb{C}$ and $m(R,f)=m_{z_0}(r,f)$, $T(R,f)=T_{z_0}(r,f)$, where $r=\frac{1}{R}$. By differentiating $g(w)=f\left(z_0-\frac{1}{w}\right)$, we obtain $f'(z)=w^2g'(w)$ and so

$$\frac{f'(z)}{f(z)}=w^2\frac{g'(w)}{g(w)},$$

from which, we get

$$\begin{aligned} m_{z_0}\left(r,\frac{f'}{f}\right) &\leq 2\log R+m\left(r,\frac{g'}{g}\right)\\ &= O(\log T(R,f)+\log R). \end{aligned}$$

By differentiating $f'(z)=w^2g'(w)$, we have

$$f''(z)=w^4g''(w)+2w^3g'(w),$$

and so

$$\frac{f''(z)}{f(z)}=w^4\frac{g''(w)}{g(w)}+2w^3\frac{g'(w)}{g(w)}. \tag{5.2.21}$$

By (5.2.21), properties of the proximity function and logarithmic derivative lemma, we obtain

$$m_{z_0}\left(r,\frac{f''}{f}\right)=O(\log T(R,f)+\log R).$$

In general, we can obtain

$$\frac{f^{(k)}(z)}{f(z)}=w^{2k}\frac{g^{(k)}(w)}{g(w)}+a_{k-1}w^{2k-1}\frac{g^{(k-1)}(w)}{g(w)}+...+a_1w^{k+1}\frac{g'(w)}{g(w)}.$$

where a_j $(j=1,...,k-1)$ are positive integers; and similarly as above, we get

$$m_{z_0}\left(r,\frac{f^{(k)}}{f}\right)=O(\log T(R,f)+\log R).$$

Lemma 5.2.9 *Let* $F \not\equiv 0, A_j(z)$ $j = 0, 1, ..., k-1$ *be meromorphic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying* $\max\{\sigma_n(F, z_0), \sigma_n(A_j, z_0), j = 0, 1, ..., k-1\} < \sigma_n(f, z_0)$. *If* f *is a meromorphic solution in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_0 f = F, \quad (5.2.22)$$

then $\lambda_n(f, z_0) = \overline{\lambda}_n(f, z_0) = \sigma_n(f, z_0)$ $(n = 1, 2, ...)$.

Proof. From (5.2.22), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + ... + A_0\right) \quad (5.2.23)$$

if f has a zero at $z_1 \in \overline{\mathbb{C}} \setminus \{z_0\}$ of order $\alpha > k$, then F has a zero at z_1 of order $\alpha - k$. Hence,

$$n_{z_0}\left(r, \frac{1}{f}\right) \leq k\overline{n}_{z_0}\left(r, \frac{1}{f}\right) + n_{z_0}\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n_{z_0}(r, A_j)$$

and then

$$N_{z_0}\left(r, \frac{1}{f}\right) \leq k\overline{N}_{z_0}\left(r, \frac{1}{f}\right) + N_{z_0}\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N_{z_0}(r, A_j). \quad (5.2.24)$$

By (5.2.23), we have

$$m_{z_0}\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m_{z_0}(r, A_j) + m_{z_0}\left(r, \frac{1}{F}\right) + O(1). \quad (5.2.25)$$

By Lemma 5.2.9, we have

$$m_{z_0}\left(r, \frac{f^{(j)}}{f}\right) = O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right) \quad (j = 1, ..., k-1) \quad (5.2.26)$$

holds for all $r \in (0, 1) - E$ where E is of finite logarithmic measure. By (5.2.24), (5.2.25) and (5.2.26), we get

$$\begin{aligned} T_{z_0}(r, f) &= T_{z_0}\left(r, \frac{1}{f}\right) + O(1) \\ &\leq k\overline{N}_{z_0}\left(r, \frac{1}{f}\right) + \sum_{j=0}^{k-1} T_{z_0}(r, A_j) + T_{z_0}(r, F) + \\ &\quad + O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right), \ r \notin E \end{aligned} \quad (5.2.27)$$

By (5.2.27) and by taking account that $O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right)\leq\frac{1}{2}T_{z_0}(r,f)$, we obtain

$$\frac{1}{2}T_{z_0}(r,f)\leq k\overline{N}_{z_0}\left(r,\frac{1}{f}\right)+\sum_{j=0}^{k-1}T_{z_0}(r,A_j)+T_{z_0}(r,F). \tag{5.2.28}$$

By (5.2.28), we have

$$\sigma_n(f,z_0)\leq\max\left\{\overline{\lambda}_n(f,z_0),\sigma_n(A_j,z_0),\sigma_n(F,z_0)\right\}.$$

Since

$$\max\{\sigma_n(F,z_0),\sigma_n(A_j,z_0);j=0,1,...,k-1\}<\sigma_n(f,z_0),$$

we get $\sigma_n(f,z_0)\leq\overline{\lambda}_n(f,z_0)$. Therefore $\sigma_n(f,z_0)=\overline{\lambda}_n(f,z_0)=\lambda_n(f,z_0)$.

Lemma 5.2.10 *Let $A_j(z)$ $j=0,1,...,k-1$ be meromorphic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ with $\sigma(A_j,z_0)=0$, such that there exists a constant $\beta_1>0$, there exists a set F of infinite logarithmic measure such that for all $|z-z_0|=r\in F$ and for any $\alpha>0$, we have $\max\{|A_j(z)|:j=1,...,k-1\}\leq\frac{1}{r^{\beta_1}}$ and $|A_0(z)|\geq\frac{1}{r^{\alpha}}$ where $|A_0(z)|=M(r,A_0)$, and $|z-z_0|=r\in F$ and U_j^i $(j=0,1,...,k)$ $(i\in\mathbb{N})$ be stated as in (5.2.3). Then, there exists a constant $\beta>0$, for all $|z-z_0|=r\in F$ and for any $\alpha>0$, we have*

$$\left|U_0^i(z)\right|\geq\frac{1}{r^{\alpha}}\text{ and }\left|U_j^i(z)\right|\leq\frac{1}{r^{\beta}}, \tag{5.2.29}$$

where $|U_0^i(z)|=M(r,U_0^i)$, $i\in\mathbb{N}$ and $j=1,2,...,k-1$.

Proof. We use the same method of the proof of Lemma 5.2.4. By Lemma 2.2.3, (5.2.6) and (5.2.7), there exist $\beta_2>0$ and for any $\alpha>0$, we have

$$\left|U_j^1(z)\right|\leq\frac{1}{r^{\beta_2}}\ (j=1,2,...,k-1);$$

$$\left|U_0^1(z)\right|\geq\frac{1}{r^{\alpha}},\ r\in F.$$

By the same method, for any integer $i\geq1$, there exist $\beta>0$ and for any $\alpha>0$, we obtain

$$\left|U_j^i(z)\right|\leq\frac{1}{r^{\beta_2}}\ (j=1,2,...,k-1);$$

$$\left|U_0^i(z)\right|\geq\frac{1}{r^{\alpha}}.$$

Lemma 5.2.11 *Let $H_j(z)$ $j=0,1,...,k-1$ be meromorphic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ such that there exists a constant $\beta>0$, there exists a set F of infinite logarithmic measure such that for all $|z-z_0|=r\in F$ and for any $\alpha>0$, we have*

$\max\{|H_j(z)| : j = 1, ..., k-1\} \leq \frac{1}{r^\beta}$ *and* $|H_0(z)| \geq \frac{1}{r^\alpha}$ *where* $|H_0(z)| = M(r, H_0)$, *and* $|z - z_0| = r \in F$. *Then, every meromorphic solution* $f \not\equiv 0$ *of (5.2.13) Satisfies* $\sigma(f, z_0) = \infty$.

Proof. Suppose that $f \not\equiv 0$ is a solution of (5.2.13) of finite order $\sigma(f, z_0) = \sigma < \infty$. By Lemma 5.2.2, for any given $\varepsilon > 0$ there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z_0 - z| \in (0,1) - E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \frac{1}{r^{j(\sigma+2+\varepsilon)}}, \ (j = 1, ..., k). \tag{5.2.30}$$

From (5.2.13) we can write

$$1 \leq \frac{1}{|H_0(z)|}\left|\frac{f^{(k)}}{f}\right| + \frac{|H_{k-1}(z)|}{|H_0(z)|}\left|\frac{f^{(k-1)}}{f}\right| + ... + \frac{|H_1(z)|}{|H_0(z)|}\left|\frac{f'}{f}\right|. \tag{5.2.31}$$

By the assumptions, for $r \in F - E$ and any $\alpha > 0$, we have

$$1 \leq k\frac{r^\alpha}{r^{k(\sigma+2+\varepsilon)+\beta}}. \tag{5.2.32}$$

By taking $\alpha > k(\sigma + 2 + \varepsilon) + \beta$ in (5.2.13), a contradiction follows as $r \to 0$.

Lemma 5.2.12 *Let* $A_j(z)$ $j = 0, 1, ..., k-1$ *be meromorphic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying* $\max\{\sigma(A_j, z_0) : j = 1, ..., k-1\} < \sigma_1 < \sigma(A_0, z_0) = \sigma$ *with*

$$\liminf_{r\to 0}\frac{m_{z_0}(r, A_0)}{T_{z_0}(r, A_0)} > 0.$$

Then, every meromorphic solution $f \not\equiv 0$ *of (5.1.1) Satisfies* $\sigma_2(f, z_0) \geq \sigma$.

Proof. From (5.1.1), we can write

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + ... + A_1\frac{f'}{f};$$

and then

$$m_{z_0}(r, A_0) \leq \sum_{j=1}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k-1} m_{z_0}(r, A_j) + O(1). \tag{5.2.33}$$

By the condition $\liminf_{r\to 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)} > 0$, there exist $0 < \alpha < 1$ and $r_0 > 0$ such that for all $0 < r < r_0$, we have

$$m_{z_0}(r, A_0) > \alpha T_{z_0}(r, A_0). \tag{5.2.34}$$

By the same argument of Lemma 5.2.3, for any given $0 < \varepsilon < \sigma$, there exists a set $F \subset (0,1)$ of infinite logarithmic measure such that for all $r \in F$, we have

$$T_{z_0}(r, A_0) > \frac{1}{r^{\sigma-\varepsilon}}. \tag{5.2.35}$$

By Lemma 5.2.8 and (5.2.33)-(5.2.35), for $0 < \varepsilon < \sigma - \sigma_1$, we have

$$\frac{\alpha}{r^{\sigma-\varepsilon}} \leq \alpha T_{z_0}(r, A_0) \leq c \log T_{z_0}(r, f) + c \log \frac{1}{r} + \frac{1}{r^{\sigma_1}},\ c > 0. \tag{5.2.36}$$

From (5.2.36), it is easy to get $\sigma_2(f, z_0) \geq \sigma$.

Lemma 5.2.13 *Let* $A_j(z)$ $j = 0, 1, ..., k-1$ *be meromorphic functions in* $\overline{\mathbb{C}} \backslash \{z_0\}$ *satisfying* $\max\{\sigma(A_j, z_0) : j = 1, ..., k-1\} < \sigma(A_0, z_0) = \sigma$ *with* $\liminf\limits_{r\to 0} \frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)} > 0$ *and* U_j^i $(j = 0, 1, ..., k)$ $(i \in \mathbb{N})$ *be stated as in (5.2.3). Then, there exists a set* $F \subset (0,1)$ *of infinite logarithmic measure such that for* $|z - z_0| = r \in F$, *we have*

$$\lim_{r\to 0} \frac{\log m_{z_0}(r, U_0^i)}{\log r} = \sigma, \tag{5.2.37}$$

$$\limsup_{r\to 0} \frac{\log m_{z_0}\left(r, U_j^i\right)}{\log r} < \sigma,\ (j = 1, ..., k-1) \tag{5.2.38}$$

Proof. We will use the inductive method. For $i = 1$, we have

$$U_j^1 = A_j + A_{j+1}\left(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0}\right),\ j = 0, 1, ..., k-1, \tag{5.2.39}$$

where $A_k \equiv 1$; and then

$$m_{z_0}\left(r, U_0^1\right) \leq m_{z_0}(r, A_0) + m_{z_0}(r, A_1) + m_{z_0}\left(r, \frac{A'_1}{A_1}\right) + m_{z_0}\left(r, \frac{A'_0}{A_0}\right) + O(1); \tag{5.2.40}$$

$$m_{z_0}\left(r, U_j^1\right) \leq m_{z_0}(r, A_j) + m_{z_0}(r, A_{j+1}) + m_{z_0}\left(r, \frac{A'_{j+1}}{A_{j+1}}\right) + m_{z_0}\left(r, \frac{A'_0}{A_0}\right) + O(1), \tag{5.2.41}$$

where $j = 1, ..., k-1$. From (5.2.39), we can write $A_0 = U_0^1 + A_1\left(\frac{A'_1}{A_1} - \frac{A'_0}{A_0}\right)$, and then

$$m_{z_0}(r, A_0) \leq m_{z_0}\left(r, U_0^1\right) + m_{z_0}(r, A_1) + m_{z_0}\left(r, \frac{A'_1}{A_1}\right) + m_{z_0}\left(r, \frac{A'_0}{A_0}\right) + O(1). \tag{5.2.42}$$

By the assumptions, Lemma 5.2.8, and (5.2.40)-(5.2.42), we get

$$\lim_{r\to 0} \frac{\log m_{z_0}(r, U_0^i)}{\log r} = \lim_{r\to 0} \frac{\log m_{z_0}(r, A_0)}{\log r} = \sigma;$$

$$\limsup_{r\to 0}\frac{\log m_{z_0}\left(r,U_j^i\right)}{\log r}\leq \limsup_{r\to 0}\frac{\max\limits_{1\leq j\leq k-1}\log m_{z_0}\left(r,A_j\right)}{\log r}<\sigma.$$

Now, we suppose that (5.2.40)-(5.2.42) hold for i and prove it for $i+1$. By using the similar argument with (5.2.3) we can easily prove.(5.2.40)-(5.2.42) for $i+1$.

5.3 Proof of Theorem 5.1.3

We will consider two cases as follows

Case 1. $\max\{\sigma(A_j,z_0):j=1,2,...,k-1\}<\sigma(A_0,z_0)<\infty$. Suppose that $f\not\equiv 0$ is a solution of (5.1.1) and $\varphi(z)\not\equiv 0$ is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)$. We start to prove $\overline{\lambda}_2(f-\varphi,z_0)=\lambda_2(f-\varphi,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. From Theorem 2.1.2, we have $\sigma_2(f,z_0)=\sigma(A_0,z_0)$. Set $g=f-\varphi$. Since $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)=\sigma_2(f,z_0)$, we have $\sigma_2(g,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. By Lemma 5.2.1, g satisfies (5.2.1). Set $F=\varphi^{(k)}+A_{k-1}\varphi^{(k-1)}+...+A_0\varphi$. If $F\equiv 0$, then by Theorem 2.1.2, we have $\sigma_2(\varphi,z_0)=\sigma(A_0,z_0)$, a contradiction; thus $F\not\equiv 0$. Now, since $\sigma_2(g,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)>\max\{\sigma_2(F,z_0),\sigma_2(A_j,z_0)\}$, the assumption of Lemma 5.2.9 holds, and then $\overline{\lambda}_2(g,z_0)=\lambda_2(g,z_0)=\sigma_2(g,z_0)$. So, we conclude that $\overline{\lambda}_2(f-\varphi,z_0)=\lambda_2(f-\varphi,z_0)=\sigma(A_0,z_0)$. Now we prove (5.1.2) for $i\geq 1$. Set $g_i=f^{(i)}-\varphi$. By Lemma 2.2.4 and the assumptions we have $\sigma_2\left(f^{(i)},z_0\right)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$ and $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)$, then we have $\sigma_2(g_i,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. By Lemma 5.2.2, g_i satisfies (5.2.2). Set $F_i=\varphi^{(k)}+U_{k-1}^i\varphi^{(k-1)}+...+U_0^i\varphi$. If $F_i\equiv 0$, by Lemma 5.2.6 and Lemma 5.2.7, $\sigma_2(\varphi,z_0)\geq\sigma(A_0,z_0)$; a contradiction with $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)$; so $F_i\not\equiv 0$. From Remerk 5.2.1, we have $\sigma\left(U_j^i,z_0\right)\leq\sigma(A_0,z_0)<\infty$; therefore $\sigma_2\left(U_j^i,z_0\right)=0$. We have $\sigma_2(g_i,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)>\max\left\{\sigma_2(F_i,z_0),\sigma_2\left(U_j^i,z_0\right)\right\}$, and by Lemma 5.2.9, we obtain $\overline{\lambda}_2(g_i,z_0)=\lambda_2(g_i,z_0)=\sigma_2(g_i,z_0)$, i.e.

$$\overline{\lambda}_2\left(f^{(i)}-\varphi,z_0\right)=\lambda_2\left(f^{(i)}-\varphi,z_0\right)=\sigma_2(f,z_0)=\sigma(A_0,z_0).$$

Case 2. $\sigma(Aj,z_0)\leq(A_0,z_0)<\infty$ and $\tau(A_j,z_0)<\tau(A_0,z_0)$ if $\sigma(Aj,z_0)=(A_0,z_0)$, $(j=1,2,...,k-1)$. Assume that $f\not\equiv 0$, is a solution of (5.1.1) and $\varphi(z)\not\equiv 0$ is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)$. As above, we start to prove $\overline{\lambda}_2(f-\varphi,z_0)=\lambda_2(f-\varphi,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. From Theorem 2.1.5, we have $\sigma_2(f,z_0)=\sigma(A_0,z_0)$. Set $g=f-\varphi$, we have $\sigma_2(g,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. As above, g satisfies (5.2.1). If $F\equiv 0$, then by Theorem 2.1.5, we have $\sigma_2(\varphi,z_0)=\sigma(A_0,z_0)$, a contradiction; hence $F\not\equiv 0$. From the assumptions of Theorem 5.1.3, we get $\max\{\sigma_2(F,z_0),\sigma_2(A_j,z_0):j=0,1,...,k-1\}<\sigma_2(g,z_0)=\sigma(A_0,z_0)$. From Lemma 5.2.9, we have $\overline{\lambda}_2(f-\varphi,z_0)=\lambda_2(f-\varphi,z_0)=\sigma_2(f,z_0)=\sigma(A_0,z_0)$. Now we prove (5.1.2) for $i\geq 1$. Set $g_i=f^{(i)}-\varphi$. We have $\sigma_2(g_i,z_0)=\sigma_2(f,z_0)$, and g_i satisfies (5.2.2). If $F_i\equiv 0$, by lemma 5.2.4 and lemma 5.2.5, we obtain $\sigma_2(\varphi,z_0)\geq\sigma(A_0,z_0)$. a contradiction with $\sigma_2(\varphi,z_0)<\sigma(A_0,z_0)$; so $F_i\not\equiv 0$.

As above by lemma 5.2.9, we obtain $\overline{\lambda}_2(g_i, z_0) = \lambda_2(g_i, z_0) = \sigma_2(g_i, z_0)$. i.e.

$$\overline{\lambda}_2\left(f^{(i)} - \varphi, z_0\right) = \lambda_2\left(f^{(i)} - \varphi, z_0\right) = \sigma_2(f, z_0) = \sigma(A_0, z_0).$$

5.4 Proof of Theorem 5.1.4

We use the same method of the proof of Theorem 5.1.3 and the same notations. From Lemma 5.2.11, we have $\sigma(f, z_0) = \infty$. Since $\sigma(\varphi, z_0) < \infty$ we have $\sigma(g, z_0) = \sigma(f, z_0) = \infty$. If $F \equiv 0$, then by Lemma 5.2.11, we have $\sigma(\varphi, z_0) = \infty$, a contradiction; thus $F \not\equiv 0$. Since $\sigma(g, z_0) = \sigma(f, z_0) = \infty > \max\{\sigma(F, z_0), \sigma(A_j, z_0)\}$, the assumption of lemma 5.2.9 holds, and then $\overline{\lambda}(g, z_0) = \lambda(g, z_0) = \sigma(g, z_0)$, i.e. $\overline{\lambda}(f - \varphi, z_0) = \lambda(f - \varphi, z_0) = \sigma(f, z_0) = \infty$. By Lemma 2.2.4 and the assumptions of the theorem, we have $\sigma\left(f^{(i)}, z_0\right) = \sigma(f, z_0) = \infty$ and $\sigma(g_i, z_0) = \infty$. If $F_i \equiv 0$, then by Lemma 5.2.10 and lemma 5.2.11, we have $\sigma(\varphi, z_0) = \infty$, a contradiction; thus $F_i \not\equiv 0$. As above, by lemma 5.2.9, we obtain

$$\overline{\lambda}\left(f^{(i)} - \varphi, z_0\right) = \lambda\left(f^{(i)} - \varphi, z_0\right) = \sigma(f, z_0) = \infty.$$

5.5 Proof of Theorem 5.1.5

By using the similar argument as in the proof of Theorem 5.1.3 and by Lemma 5.2.12, Lemma 5.2.13 and Lemma 5.2.9, we can get the conclusion

$$\overline{\lambda}_2\left(f^{(i)} - \varphi, z_0\right) = \lambda_2\left(f^{(i)} - \varphi, z_0\right) = \sigma_2(f, z_0) \geq \sigma(A_0, z_0) \ \ (i \in \mathbb{N}).$$

Conclusion

Through this work, we have seen that there are several similarities between our results and those of the complex plane case. However, there are also some differences: We know that all solutions of LDE with entire coefficients are entire functions but when the coefficients are analytic in $\overline{\mathbb{C}}-\{z_0\}$, the solutions may not be analytic in $\overline{\mathbb{C}}-\{z_0\}$; also as indicated in ([33]) that $f_1(z) = \frac{1}{z}$ and $f_2(z) = \frac{1}{z^2}$ constitute the fundamental system of solutions of the differential equation

$$f'' + \frac{4}{z}f' + \frac{2}{z^2}f = 0;$$

and we have $\sigma(f_1, 0) = \sigma(f_2, 0) = 0$. Contrary in the complex plane, for the case of polynomial coefficients, it is not possible to have only polynomials for solutions, see [32]. So, to extend the results from the complex plane case to a neighborhood of a finite singular point, the problem in its generality remains open. In the other hand, there is another interesting question: How about the case when the differential equation contains more than one singular point? Hamouda has investigated this question in [33] and this is just the biginning and there is still a lot to do in this direction.

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