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Applications of Integral Inequalities to Fractional Hybrid Differential Equations

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

Aknowledgement

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Chairperson Mr Mohand OULD ALI, Professor at University of Mostaganem,

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Abstract

The main focus of this thesis is applications of integral inequalities, in as many ways as possible, on hybrid differential equations of fractional order. For this purpose, generalizations of a certain type of integral inequalities are obtained. In addition to that, applications on a class of fractional hybrid differential equations using fixed point theory are established.

First, we present generalizations to some integral inequalities of Gronwall-Ballman type. This type of integral inequalities has many uses when it comes to differential equations. In that light, some applications to fractional hybrid differential equations with Hadamard derivative got included in this thesis.

Then, we present a different sense of applications of integral inequalities to a certain class of fractional hybrid differential equations. We study a boundary value problem which is a system of n-hybrid differential equations with Caputo derivative and nonlocal conditions. Accordingly, some results that address existence and uniqueness of the solution of the system are given. For the existence of at least one solution, two approaches are used: Shaefer fixed point theorem and another theorem developed by the mathematician Dhage. Illustrative examples will be presented as well to validate the results.

For stability of the system, we proceed through Ulam-Hyers stability as the main way to study it. We try to establish the necessary results that validate the stability of the system mentioned above.

Keywords : Hybrid differential equation, Caputo derivative, Hadamard derivative, integral inequalities, fixed point, existence, uniqueness, Ulam-Hyers stability,

Mathematical Subject Classification (2010) : 34A38, 26A33, 32A65, 39B05, 39A30

الملخص

ترتكز هذه الأطروحة في أساسها على تطبيق المتراجحات التكاملية, بمختلف الطرق الممكنة, على المعادلات التفاضلية الهجينة ذات الرتب الاختيارية (الغير صحيحة). من أجل تحقيق هذا الأمر, سنقوم بتعميم بعض المتراجحات التكاملية بالإضافة الى بعض التطبيقات على فئة معينة من المعادلات التفاضلية الهجينة باستعمال نظرية النقط الثابتة.

بداية, سنقوم بتقديم تعميمات لبعض المتراجحات التكاملية من نوع غرونوال بالمان. لهذا النوع من المتراجحات التكاملية استعمالات عديدة خصوصا عندما يتعلق الأمر بالمعادلات التفاضلية. لذلك, سنقوم بإرفاق بعض التطبيقات للنتائج المحصل عليها أنفا على معادلات تفاضلية هجينة ذات تفاضل من نوع هادامار.

بعد ذلك سنقدم بعض التطبيقات للمتراجحات التكاملية, و ذلك بطريقة مختلفة عما سبق, على فئة من المعادلات التفاضلية الهجينة ذات الرتب الغير صحيحة. سنقوم بدراسة مسألة ذات قيم حدية حيث أن هذه المسألة عبارة عن معادلات تفاضلية هجينة مترابطة ذات تفاضل من نوع كابوتو مع شروط حدية غير موضعية. سنهدف إلى الحصول على نتائج تدعم وجود حل وحيد للمسالة. أما فيما يتعلق بإمكانية وجود حل واحد على الأقل سنعتمد على طريقتين مختلفتين: طريقة النقط الثابتة لشايفر و نظرية أخرى للرياضي داج.

اما بالنسبة لموضوع استقرار جملة المعادلات التفاضلية المذكورة أعلاه, فسنقوم باستعمال نظرية الاستقرار ايلام هايرز كمنهج أساسي لدراسة هذا الأمر وذلك لتحصيل النتائج المرجوة من أجل اثبات أن جملة المعادلات مستقرة.

كلمات مفتاحية: معادلات تفاضلية هجينة, تفاضل من نوع كابوتو, تفاضل من نوع هادامار, متراجحات تكاملية, نقطة ثابتة, وجود, وحيد, استقرار ايلام هايرز.

تصنيف المواضيع الرياضية (2010):

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Résumé

L'objectif principal de cette thèse est les applications des inégalités intégrales, avec autant de manières que possible, aux équations différentielles hybrides fractionnaires. Des généralisations de quelques inégalités intégrales sont abouti pour ce but là. En plus, des applications sur une certaine classe d'équations différentielles hybrides fractionnaires, en utilisant la théorie des points fixes, sont obtenues.

D'abord, on présente des généralisations de quelques inégalités intégrales de type Gronwall-Ballman. Ces inégalités intégrales ont plusieurs utilisations quand les équations différentielles sont concernées. Quelques applications à des équations différentielles hybrides fractionnaires avec la dérivée de Hadamard sont aussi inclues dans cette thèse.

Ensuite, on présente des applications des inégalités intégrales dans un autre sens à une certaine classe d'équations différentielles hybrides fractionnaires. On fait l'étude à un problème aux limites qui est un système de n-équations différentielles hybrides avec la dérivée de Caputo et avec conditions non-locales. Des résultats qui adressent l'existence et l'unicité de solution du système sont bien donnés. Pour l'existence d'une solution au moins, deux approches sont utilisées: le théorème de point fixe de Shaefer et le théorème de point fixe développé par le mathématicien Dhage. Des exemples illustrant la validité des résultats sont aussi présentés.

Pour la stabilité du système, on a pris la stabilité au sens de Ulam-Hyers comme la méthode principale pour l'étudier. On cherche a établir les résultats nécessaires pour valider la stabilité du système mentionné ci dessus.

Mots clés : Équation différentielle hybride, dérivée de Caputo, dérivée de Hadamard, inégalités intégrales, point fixe, existence, unicité , stabilité Ulam-Hyers,

Mathematical Subject Classification (2010) : 34A38, 26A33, 32A65, 39B05, 39A30

Contents

Introduction

One of the beauties of science is that it is progressive in its nature. It builds up on what was found yesterday to create what is new today. What was once a scientific truth can turn to be a false hypothesis later in years just to be found useful in different ways as we go in time. Science grows through trial and error. No information is ever a waste. Every new perspective brings something new to the table and so did fractional calculus theory to many aspects of mathematics.

Fractional Calculus is a branch of mathematics that had seen huge development in the last few decades.

A conversation took place in 1695 between two great mathematicians Leibnitz and L'Hospital. It started with a notation and a question: Leibnitz gave the derivative of order n by $\frac{d^n y}{dx^n}$ and L'Hospital asked if n can take the value of $\frac{1}{2}$ "What if n be $\frac{1}{2}$?", asked L'Hopital. It will lead to a paradox","From this apparent paradox, one day useful consequences will be drawn." responded Leibnitz [50]. More than a century later, hints of a derivative of arbitrary order were mentioned by S. F. Lacroix, Euler, and Fourier. The year 1832 was marked in the historical development of fractional calculus by the works of N. H. Abel who used the concept in solving an integral equation that has a crucial part in what was called tautochrome problem (or isochrome problem)[50]. His work was considered the first official application of fractional calculus in physics even though it didn't have the proper representations back then, but the idea was there[50]. Between 1832 to 1855, Liouville was the first to make serious attempts to give this derivative a shape and form in a mathematical sense. He applied some of his works to address problems in potential theory. The works of Riemann followed through as well as O. Heaviside, P. A. Nekrassov, A. Krug, Laurent, ... The subject went dormant for few years between 1940s and 1960s to come back to the spotlight again around 1960s and 1970s. [50], [51].

A turning point this theory had seen was around the 1980s when physicists showed interest in the integral and derivative concept of arbitrary order. From this point on till now, fractional calculus knew huge development in the area of applied mathematics. It touched variety of subjects as it is mentioned in the book of [59], "from inverse mechanical problems to control theory and dynamical chaos, heat flow spreading, electrical and radio engineering to astrophysics and cosmology not forgetting biophysics and medicine" as we see in [8], [10], [11],[13], [22], [34], [42],

[49], [52], [53], [54], [56], [59], [60]. Mainly, it is fractional differential equations that are used as substitutes to old known models that represent the physical phenomena.

In this thesis, we address a certain class of hybrid differential equations with arbitrary order. Hybrid differential equations are results of perturbation techniques that were applied to unsolvable models. These techniques are not exactly specified but they are a common tool that is used as an approach to solve mathematical models. Since we are interested in the arbitrary order of these equations, and since these equations are not linear, we explore through both axes of fractional calculus : integration and derivation, many techniques in order to apply them to these equations as in Gronwall-Ballman type inequalities, Banach contraction principle, Shaefer fixed point theorem and some other tools to achieve results that help in solving and probably finding approximations to the solution of these equations. The stability of these equations is also addressed using Ulam-Hyers stability.

This thesis is ordered as follow:

☙. **Chapter one:** In this chapter, we present some of the fundamental notions of fractional calculus like special functions, integral operators and derivative operators and their properties as well. We also find it useful to mention some of the basics of analysis and topology as a helpful mean to understand functional analysis theorems and how they work.

☙. **Chapter two:** This chapter takes you first to a place where you can understand hybrid differential equations better. Then, generalizations of some integral inequalities of Gronwall-Ballman type are established to accommodate fractional hybrid differential equations. As a mean of application, the generalizations are applied on hybrid differential equation with Hadamard derivative.

☙. **Chapter three:** This chapter is dedicated to another type of applications of integral inequalities. It is through existence/ existence and uniqueness of solution for a system of n hybrid differential equations with fractional order by using fixed point theorems like Banach contraction principle, Shaefer fixed point theorem, and another fixed theorem developed specifically for hybrid differential equations.

☙. **Chapter four:** In this chapter, we cover specifically the stability of the system studied in chapter three. For this purpose, Ulam-Hyers stability is used to prove the desired results.

Finally, we conclude our work by summarizing the whole process while expressing some possibilities and aspects that can be addressed as future perspectives to new areas hoping that it will help in expending the research.

Notations index

For the sake of practicality, we found it important to first clarify some of the notations used in this thesis.

$\mathbb{R}$	:	The set of the real numbers,
$\mathbb{C}$	:	The set of the complex numbers,
$\mathbb{N}$	:	The set of natural numbers,
$\|.\|$	:	The infinity norm,
$\Gamma(.)$	:	Gamma function of Euler,
$\mathcal{B}(.,.)$	:	Beta function of Euler,
$log(.)$	:	the natural logarithm with base number e ($e \approx 2.718$),
I_a^α	:	The Riemann-Liouville integral of order α (noted I^α when $a = 0$).
${}_{RL}D_a^\alpha$	:	The Riemann-Liouville derivative of order α (noted ${}_{RL}D^\alpha$ when $a = 0$).
${}^cD_a^\alpha$	:	The Caputo derivative of order α (noted ${}^CD^\alpha$ when $a = 0$).
${}^HI_a^\alpha$	:	The Hadamard integral of order α (noted ${}^HI^\alpha$ when $a = 1$).
${}^HD_a^\alpha$	:	The Hadamard derivative of order α (noted ${}^HD^\alpha$ when $a = 1$).

Chapter 1

Basic and Important Notions in Fractional Calculus and Functional Analysis

1 Elementary Notions of Fractional Calculus

1.1 Special Functions in Fractional Calculus

One of the most basic aspects that the theory of fractional calculus was built on is the famous Gamma function of Euler. As it is considered a generalization for the factorial, it was used to develop what we now call fractional operators for integral and derivative.

Definition 1.1 *[46], [33] We call a Gamma function the following integration*

$$\Gamma(\chi) := \int_0^{+\infty} t^{\chi-1} \exp(-t) dt$$

where $\chi \in \mathbb{R}$ *and* $\chi \geq 0$.

Some of the properties of this function ([46], [33]) :

a). $\Gamma(\chi+1) = \chi\Gamma(\chi)$, $(\chi > 0)$.

b). $\Gamma(n+1) = n!$ and $\Gamma(1) = 1$ $(n \in \mathbb{N})$.

c). The gamma function has simple poles at the points $\chi = 0, -1, -2, ...$

d). For $\chi \in \mathbb{R}_+^*$, the following equality is valid: $\Gamma(\chi) = \dfrac{\Gamma(\chi-n)}{\chi(\chi+1)...(\chi+n-1)}$, $(n \in \mathbb{N})$.

Another special function is what is called the beta function of Euler

Definition 1.2 *[46], [33] The beta function of Euler is the function defined by*

$$\mathcal{B}(\chi, \lambda) := \int_0^1 u^{\chi-1}(1-u)^{\lambda-1}, \quad (\chi > 0, \lambda > 0). \tag{1.1}$$

- ⋆ Some of the properties of the beta function of Euler is that it can be written as follow ([46]):

$$\mathcal{B}(\chi, \lambda) = \frac{\Gamma(\chi)\Gamma(\lambda)}{\Gamma(\chi+\lambda)}, \quad (\chi > 0, \lambda > 0). \tag{1.2}$$

- ⋆ Another property to this function that is related to the previous property is that ([46])

$$\mathcal{B}(\chi, \lambda) = \mathcal{B}(\lambda, \chi), \quad (\chi > 0, \lambda > 0). \tag{1.3}$$

Next, we present some of the commonly used approaches of integration and differentiation in fractional calculus. But before that, let us introduce L_p spaces.

"Let $[a, b] \subset \mathbb{R}$. For $0 < p < +\infty$, the space $L_p([a, b])$ is the collection of all equivalence classes of measurable functions f for which the p-norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} < +\infty.$$

$$L^p([a, b]) = \{f : [a, b] \longrightarrow \mathbb{C}, \ \text{f measurable, and} \ \|f\|_p < +\infty\}.$$

". [5]

1.2 The Riemann-Liouville Approach

Definition 1.3 *[33] Let h be a continuous function on $[a, b]$ $(-\infty \leq a < b \leq +\infty)$. The integral of h of an arbitrary order α $(\alpha \in \mathbb{R}, \alpha > 0)$ with the approach of Riemann-Liouville is introduced as follow:*

$$(I_a^\alpha h)(t) := \int_a^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} h(\zeta) d\zeta, \tag{1.4}$$

where Γ is given in Definition 1.1.

Definition 1.4 *[33] Let h be a continuous function on $[a, b]$ $(-\infty \leq a < b \leq +\infty)$. The derivative of h of an arbitrary order α $(\alpha \in \mathbb{R}, \alpha > 0)$ with the approach of Riemann-Liouville is introduced as follow:*

$$\begin{aligned} ({}_{RL}D_a^\alpha h)(t) &:= \left(\frac{d}{dt}\right)^n (I_a^{n-\alpha} h)(t) \\ &= \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-\zeta)^{n-\alpha-1}}{\Gamma(n-\alpha)} h(\zeta) d\zeta \end{aligned} \tag{1.5}$$

with $n = [\alpha] + 1$.

Before the definitions above took form, specifically between 1832 and 1880, many attempts had been taken to give meaning to the fractional derivative of arbitrary order. Abel, Liouville, and Riemann are the most famous ones in that era to give something meaningful. However, there were obvious differences between the operators. The scientific committee of that period decided on some criteria that a fractional derivative should fulfill. One of these criteria is linearity. Accordingly we have the following property:

Property 1 *[33] Let* $\alpha > 0$, $f, g \in L_p(a,b)$ $(1 \leq p \leq \infty)$ *and let* $\sigma, \gamma \in \mathbb{R}$. *Then*

$$I_a^{\alpha}[(\sigma f(\chi) + \gamma g(\chi)] = \sigma I_a^{\alpha} f(\chi) + \gamma I_a^{\alpha} g(\chi). \tag{1.6}$$

Property 2 *[33] For* $\alpha > 0$, *and* $\beta \in \mathbb{R}$ $(\beta > 0)$, *we can obtain the following equalities:*

a). $(I_a^{\alpha}(\chi - a)^{\beta-1}) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\chi - a)^{\beta+\alpha-1}$ $(\alpha > 0)$,

b). $({}_{RL}D_a^{\alpha}(\chi - a)^{\beta-1}) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\chi - a)^{\beta-\alpha-1}$ $(\alpha > 0, \beta > \alpha)$.

Proof: Let us have $\alpha, \beta \in \mathbb{R}$ ($\alpha > 0$, $\beta > 0$).

a). Using the definition of Riemann-Liouville integral, we have

$$(I_a^{\alpha}(\chi - a)^{\beta-1}) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\chi - s)^{\alpha-1}(s-a)^{\beta-1} ds \tag{1.7}$$

Let us put $u = \frac{s-a}{\chi-a}$ Then, the equation becomes

$$\begin{aligned} (I_a^{\alpha}(\chi - a)^{\beta-1}) &= \frac{(\chi-a)^{\alpha+\beta-1}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du \\ &= \frac{(\chi-a)^{\alpha+\beta-1}}{\Gamma(\alpha)} \mathcal{B}(\alpha, \beta) \end{aligned} \tag{1.8}$$

Thanks to the property(1.2) of the function $\mathcal{B}$, we get

$$(I_a^{\alpha}(\chi - a)^{\beta-1}) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\chi - a)^{\alpha+\beta-1} \tag{1.9}$$

b). We have:

$$({}_{RL}D_a^{\alpha}(\chi - a)^{\beta-1}) = \left(\frac{d}{d\chi}\right)^n (I^{n-\alpha}(\chi - a)^{\beta-1}) \tag{1.10}$$

Thanks to the previous property, we have

$$({}_{RL}D_a^{\alpha}(\chi - a)^{\beta-1}) = \left(\frac{d}{d\chi}\right)^n \left(\frac{\Gamma(\beta)}{\Gamma(\beta+n-\alpha)}(\chi-a)^{n-\alpha+\beta-1}\right) \tag{1.11}$$

By calculating the term under the derivative $(\frac{d}{d\chi})^n$ and using the property (d) of the Gamma function, we get

$$({}_{RL}D_a^{\alpha}(\chi - a)^{\beta-1}) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\chi-a)^{\beta-\alpha-1}. \tag{1.12}$$

■

One of the special traits of Rieman-Liouville derivative is that applying it to a constant does not mean the derivative is equal to 0.

Property 3 *[33] Let $\beta = 1$ and with $\alpha \geq 0$, we have*

i/. $({}_{RL}D_a^{\alpha}(1)) = \dfrac{(\chi-a)^{-\alpha}}{\Gamma(1-\alpha)}$ $(0 < \alpha < 1)$,

ii/. Yet, for $j = 1, 2, ..., [\alpha] + 1$, we have $({}_{RL}D_a^{\alpha}(\chi-a)^{\alpha-j}) = 0$.

Property 4 *[33]"Semi-group property"*

For $\alpha > 0$ and $\beta > 0$, we have

$$(I_a^{\alpha} I_a^{\beta} h)(t) = (I_a^{\alpha+\beta} h)(t) \tag{1.13}$$

at almost every point $t \in [a, b]$ and $h \in L_p(a, b)$ $(1 \leq p \leq +\infty)$.

Proof

For $t \in [a, b]$, we have :

$$\begin{aligned}
(I_a^{\alpha}(I_a^{\beta} h))(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\zeta)^{\alpha-1} (I_a^{\beta} h)(\zeta) d\zeta \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^s (t-\zeta)^{\alpha-1} (\zeta-\chi)^{\beta-1} h(\chi) d\chi d\zeta \qquad (1.14) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t h(\chi) \int_{\chi}^t (t-\zeta)^{\alpha-1} (\zeta-\chi)^{\beta-1} d\zeta d\chi
\end{aligned}$$

We chose $u = \frac{\zeta-\chi}{t-\chi}$ as a change of variables. Then (1.14), becomes:

$$
\begin{aligned}
(I_a^{\alpha}(I_a^{\beta}h))(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(\chi)(t-\chi)^{\alpha+\beta-1}\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}dud\chi \\
&= \frac{\mathcal{B}(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t h(\chi)(t-\chi)^{\alpha+\beta-1}d\chi
\end{aligned} \tag{1.15}
$$

Thanks to the properties of beta function, we have :

$$
\begin{aligned}
(I_a^{\alpha}(I_a^{\beta}h))(t) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}\int_a^t h(\chi)(t-\chi)^{\alpha+\beta-1}d\chi \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-\chi)^{\alpha+\beta-1}h(\chi)d\chi \\
&= (I_a^{\alpha+\beta}h)(t)
\end{aligned} \tag{1.16}
$$

which is the desired result.

Lemma 1.1 *[33] If $\alpha > 0$, and $h \in L_p(a,b)$, $(1 \leq p < \infty)$, then the following equality*

$$({}_{RL}D_a^{\alpha}I_a^{\alpha}h)(\chi) = h(\chi) \tag{1.17}$$

holds almost everywhere on $[a,b]$.

Proof: Let us have for $\alpha > 0$ and $h \in L_p(a,b)$

$$({}_{RL}D_a^{\alpha}I_a^{\alpha}h)(\chi) = \left(\tfrac{d}{dt}\right)^n I_a^{n-\alpha}I_a^{\alpha}h(\chi) \tag{1.18}$$

Using the semi-group property, we get the desired results.

Property 5 *[33] If $\alpha > \beta > 0$, then for $h \in L_p(a,b)$, $(1 \leq p < \infty)$, we have*

$$({}_{RL}D_a^{\beta}I_a^{\alpha}h)(\chi) = I_a^{\alpha-\beta}h(\chi) \tag{1.19}$$

holds almost everywhere on $[a,b]$.

This property is proved similarly to Lemma 1.1 .

1.3 The Caputo Approach

Since Riemann-Liouville derivative has its own flaws when it comes to the use of it in practical matters as it is indicated in I. Podlubny's book [46], Caputo developed a derivative operator that covers the blind side of Riemann-Liouville

Definition 1.5 *[33] Let h be a function in $C^n([a,b])$. The derivative of h of an arbitrary order α $(n-1<\alpha<n)$ with the approach of Caputo is introduced as follow:*

$$\begin{aligned} {}^cD_a^\alpha h(t) &:= I_a^{n-\alpha}h^{(n)}(t) \\ &= \int_a^t \frac{(t-\zeta)^{n-\alpha-1}}{\Gamma(n-\alpha)}h^{(n)}(\zeta)d\zeta, \quad a<\zeta<t<b, \end{aligned} \tag{1.20}$$

with $n=[\alpha]+1$.

Property 6 *[33] Let $\alpha>0$, $\beta>0$, and n is given by $n=[\alpha]+1$.*

- $\left({}^cD_a^\alpha(\chi-a)^{\beta-1}\right)=\dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\chi-a)^{\beta-\alpha-1}$ *such that* $(\beta>n)$.
- $\left({}^cD_a^\alpha(\chi-a)^{k}\right)=0$, *where* $k=1,2,...,n-1$.
- $({}^cD_a^\alpha 1)=0$. *This is one of the main differences between Caputo derivative and Riemann-Liouville derivative.*

Lemma 1.2 *[33] Let $\alpha>0$, and $y\in C(a,b)$. If $\alpha\notin\mathbb{N}$, or $\alpha\in\mathbb{N}$ and $n=[\alpha]+1$, then*

$$({}^cD_a^\alpha I_a^\alpha y)(\chi)=y(\chi). \tag{1.21}$$

Lemma 1.3 *[33] Let $y\in C^n([a;b],\mathbb{R})$. For $\alpha>0$, the fractional differential equation ${}^cD_a^\alpha y(\chi)=0$ has a general solution given by:*

$$y(\chi)=\sum_{i=0}^{n-1}c_i(\chi-a)^i,$$

with $c_i\in\mathbb{R}, i=0,1,2,..,n-1$, and $n=[\alpha]+1$.

Proof: Let $\alpha>0$ and $y\in C^n([a;b],\mathbb{R})$.

$${}^cD_a^\alpha y(\chi)=0 \Rightarrow I_a^{n-\alpha}D_a^n y(\chi)=0. \tag{1.22}$$

We apply $D_a^{n-\alpha}$ and we get

$$D_a^n y(\chi)=0 \Rightarrow y(\chi)=\sum_{i=0}^{n-1}c_i(\chi-a)^i. \tag{1.23}$$

Lemma 1.4 *[33] Let $y\in C^n([a;b],\mathbb{R})$. For $\alpha>0$, we have:*

$$I_a^\alpha({}^cD_a^\alpha y)(\chi)=y(\chi)+\sum_{i=0}^{n-1}c_i(\chi-a)^i,$$

with $c_i\in\mathbb{R}, i=0,1,2,...,n-1$ and $n=[\alpha]+1$.

Proof: Let $\alpha > 0$ and $y \in C^n([a;b], \mathbb{R})$.

$$I_a^\alpha({}^cD_a^\alpha y(\chi)) = I_a^\alpha I_a^{n-\alpha} D_a^n y(\chi) \tag{1.24}$$

If we use the semi-group property, we find that

$$\begin{aligned} I_a^\alpha({}^cD_a^\alpha y(\chi)) &= I_a^n D_a^n y(\chi) \\ &= y(\chi) + \sum_{i=0}^{n-1} c_i(\chi - a)^i. \end{aligned} \tag{1.25}$$

The Link between Riemann-Liouville Derivative and Caputo Derivative

If we want to give Caputo-derivative a meaning through Riemann-Liouville derivative it would be given by the following representation: [33]

$$({}^cD_a^\alpha y)(\chi) := \left({}_{RL}D_a^\alpha \left[y(\chi) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(\chi - a)^k\right]\right) \tag{1.26}$$

where $y \in C^n(a, b)$, and $n = [\alpha] + 1$.

This representation can be simplified by applying the necessary tools that we presented and we can get the following formulation:[33]

$$({}^cD_a^\alpha y)(\chi) := ({}_{RL}D_a^\alpha y)(\chi) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k - \alpha + 1)}(\chi - a)^k \tag{1.27}$$

where $y \in C^n(a, b)$, and $n = [\alpha] + 1$.

This type of links serve as a good tool for the flexibility between derivative operators.

1.4 The Hadamard Approach

In 1892, the famous mathematician Hadamard, propose a new model of fractional operator that was later named after him. In their book [33], Kilbas et al. presented the definitions and properties of this operator.

Definition 1.6 *[33] For a continuous function h on the interval (a, b) $(0 \le a < b \le +\infty)$, the Hdamard integral of order α $(\alpha > 0)$ is given by the mathematical expression*

$$({}^HI_a^\alpha h)(\chi) := \frac{1}{\Gamma(\alpha)} \int_a^\chi \left(log\frac{\chi}{t}\right)^{\alpha-1} \frac{h(t)}{t} dt, \quad (a < \chi < b) \tag{1.28}$$

where Γ is given in Definition 1.1.

Definition 1.7 *[33] Let $[a,b]$ be a finite interval such that $-\infty < a < b < +\infty$ and let $AC[a,b]$ be a space that contains all absolutely continuous functions on $[a,b]$. Let us denote $\delta = t\frac{d}{dt}$ and define the space*

$$AC_\delta^n[a,b] = \{h : t \in [a,b] \to \mathbb{R} \text{ such that } (\delta^{n-1}h) \in AC[a,b]\}. \tag{1.29}$$

Clearly $AC_\delta^1[a,b] \equiv AC[a,b]$ for $n = 1$.

Definition 1.8 *[33] Let h be in the space $AC_\delta^n[a,b]$, with $0 \leq a < b < \infty$, $\delta = t\frac{d}{dt}$, and $n = [\alpha]+1$ ($\alpha > 0$). The derivative of the function h of an arbitrary order α with Hadamard approach is defined as*

$$\begin{aligned} {}^H D_a^\alpha h(t) &= \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n \int_a^t (log\frac{t}{\zeta})^{n-\alpha+1} h(\zeta)\frac{d\zeta}{\zeta} \\ &= \delta^n({}^H I_a^{n-\alpha} h)(t). \end{aligned} \tag{1.30}$$

Property 7 *[33]"Semi-group property" Let $\alpha > 0$, $\beta > 0$, and $1 \leq p \leq \infty$. If $0 < a < b < \infty$, then for $h \in L^p(a,b)$,*

a. ${}^H I_a^\alpha({}^H I_a^\beta h) = {}^H I_a^{\alpha+\beta} h$

b. ${}^H D_a^\beta({}^H I_a^\alpha h) = {}^H I_a^{\alpha-\beta} h$

c. ${}^H D_a^\alpha({}^H I_a^\alpha h) = h.$

Proof: a. Let h be in $L^p(a,b)$ and $\alpha > 0$. Then,

$${}^H I_a^\alpha(({}^H I_a^\beta h))(t)) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(log\frac{t}{s}\right)^{\alpha-1} \int_a^s \left(log\frac{s}{\chi}\right)^{\beta-1} h(\chi)\frac{d\chi}{\chi}\frac{ds}{s} \tag{1.31}$$

We notice that $a \leq \chi \leq s \leq t$. Accordingly, we get

$${}^H I_a^\alpha(({}^H I_a^\beta h))(t)) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_\chi^t \left(log\frac{t}{s}\right)^{\alpha-1} \left(log\frac{s}{\chi}\right)^{\beta-1} \frac{ds}{s} h(\chi)\frac{d\chi}{\chi} \tag{1.32}$$

Now, let's put $w = \frac{log\frac{s}{\chi}}{log\frac{t}{\chi}}$. If we accommodate (1.32) according to w, then, we get

$$\begin{aligned} {}^H I_a^\alpha(({}^H I_a^\beta h))(t)) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{\chi}\right)^{\alpha+\beta-1} \int_0^1 (1-w)^{\alpha-1} w^{\beta-1} dw f(\chi)\frac{d\chi}{\chi} \\ &= \frac{\mathcal{B}(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{\chi}\right)^{\alpha+\beta-1} h(\chi)\frac{d\chi}{\chi} \end{aligned} \tag{1.33}$$

Thanks to the property of the beta function of Euler, we get

$$
\begin{aligned}
{}^{H}I_a^{\alpha}(({}^{H}I_a^{\beta}h))(t)) &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t \left(\log\frac{t}{\chi}\right)^{\alpha+\beta-1} h(\chi)\frac{d\chi}{\chi} \\
&= ({}^{H}I_a^{\alpha+\beta}h)(t).
\end{aligned} \tag{1.34}
$$

which achieve the point. ■

Property 8 *[33] If $\alpha > 0$, and $\beta > 1$ and $0 < a < b < \infty$, then*

- $\left({}^{H}I_a^{\alpha}\left(\log\frac{\chi}{a}\right)^{\beta-1}\right) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(log\frac{\chi}{a}\right)^{\beta+\alpha-1},$
- $\left({}^{H}D_a^{\alpha}\left(\log\frac{\chi}{a}\right)^{\beta-1}\right) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(log\frac{\chi}{a}\right)^{\beta-\alpha-1}.$

Proof: Let $\alpha > 0$, $\alpha > 0$. We have

$$
\left({}^{H}I_a^{\alpha}\left(log\frac{\chi}{a}\right)^{\beta-1}\right) = \frac{1}{\Gamma(\alpha)}\int_a^{\chi}\left(log\frac{\chi}{s}\right)^{\alpha-1}\left(log\frac{s}{a}\right)^{\beta-1}\frac{ds}{s} \tag{1.35}
$$

If we put $u = \frac{log\frac{s}{a}}{log\frac{\chi}{a}}$, then we get

$$
\begin{aligned}
\left({}^{H}I_a^{\alpha}\left(log\frac{\chi}{a}\right)^{\beta-1}\right) &= \frac{1}{\Gamma(\alpha)}\left(log\frac{\chi}{a}\right)^{\beta+\alpha-1}\int_a^{\chi}(1-u)^{\alpha-1}(u)^{\beta-1}du \\
&= \frac{1}{\Gamma(\alpha)}\left(log\frac{\chi}{a}\right)^{\beta+\alpha-1}\mathcal{B}(\alpha,\beta)
\end{aligned} \tag{1.36}
$$

Using the property of the function beta of Euler, we get

$$
\left({}^{H}I_a^{\alpha}\left(log\frac{\chi}{a}\right)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(log\frac{\chi}{a}\right)^{\beta+\alpha-1} \tag{1.37}
$$

Similarily, we get the second equation.

Remark 1.1 *[33]*

- ⋆ *If $\beta = 1$, and $\alpha \geq 0$, then ${}^{H}D_a^{\alpha}C \neq 0$ where C is a real constant.*
- ⋆ *If $0 < \alpha < 1$, then ${}^{H}D_a^{\alpha}1 = \dfrac{1}{\Gamma(1-\alpha)}\left(log\frac{\chi}{a}\right)^{-\alpha}.$*
- ⋆ *For $j = [\alpha]+1$, we have $\left({}^{H}D_a^{\alpha}\left(\log\frac{\chi}{a}\right)^{\alpha-j}\right) = 0.$*

Corollary 1.1 *[33] Let $\alpha > 0$, $n = [\alpha]+1$, and $1 < a < b < \infty$. The equality $({}^{H}D_a^{\alpha} y)(\chi) = 0$ is valid if, and only if,*

$$y(\chi) = \sum_{j=1}^{n} c_j (log\frac{\chi}{a})^{\alpha-j} \tag{1.38}$$

and the following formula holds:

$${}^{H}I_a^{\alpha}({}^{H}D_a^{\alpha} y(\chi)) = y(\chi) + \sum_{j=1}^{n} c_j (log\frac{\chi}{a})^{\alpha-j}$$

where $c_j \in \mathbb{R}$, $j = 1, 2, ..., n$, and $n - 1 < \alpha < n$.

Remark 1.2 *[33] When $0 < \alpha \leq 1$, the relation $({}^{H}D_a^{\alpha} y)(\chi) = 0$ holds if and only if $y(\chi) = c\left(\log \frac{\chi}{a}\right)^{\alpha-1}$.*

2 Important Elements of Functional Analysis

Since everything is connected in mathematics, it is important to clear some of the fondamental concepts that keeps the process of work flowing. The following notions create a map for the reader to follow through.

2.1 Banach Space

Definition 1.9 *[58] Let B be a vector normed set and σ a metric on B. A metric pace (B, σ) is* **complete** *if every Cauchy sequence in B has a limit.*

Definition 1.10 *[39] We call a* **Banach space** *every normed vector space where the induced metric is complete.*

2.2 Completely Continuous Operators

Definition 1.11 *[39] A function $f : X \to Y$ between metric spaces is* **continuous** *when it preserves convergence,*

$$\chi_n \to \chi \in X \Rightarrow f(\chi_n) \to f(\chi) \in Y. \tag{1.39}$$

where $\{\chi_n\}_{n\in\mathbb{N}}$.

In this case, $f(\lim_{n\to+\infty} \chi_n) = \lim_{n\to+\infty} f(\chi_n)$.

Definition 1.12 *[39] A set B is* **bounded** *when the distance between any two points in the set has an upper bound,*

$$\exists r > 0, \quad \forall \chi, y \in B, \quad d(\chi, y) \leq r \tag{1.40}$$

Definition 1.13 *[47] Let us have the spaces X, Y that happens to be Banach spaces and let $T : D \subset X \to Y$.*

(a) We say that the operator T is **bounded** *if it maps any bounded subset of D into a bounded subset of Y.*

(b) We say that the operator T is **completely continuous** *if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.*

2.3 Ascoli-Arzela Theorem

Ascoli-Arzela theorem is one of the most used theorems in fixed point theory. It is a theorem that provided a simpler way to use fixed point theorems and add the factor of practicallity to them. It has created a bridge beween the concepts that are easily applied and those that are a bit harder to put to immediate use.

To be more clear, let (K, d) be a compact metric space and $C(K, \mathbb{R}^n)$ be the famous Banach space that encompasses all continuous functions from K to $\mathbb{R}^n$, under the sup-norm $|.|_\infty$

Theorem 1.1 *[47] A subset Y of $C(K, \mathbb{R}^n)$ is* **relatively compact** *if and only if the following conditions are satisfied:*

(i) Y is **bounded***, i. e., there exists a constant $c > 0$ such that*

$$|u(\chi)| \leq c \tag{1.41}$$

for all $\chi \in K$ and $u \in Y$.

(ii) Y is **equicontinuous***, i.e., for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $u \in Y$,*

$$|u(\chi) - u(\chi_1)| < \varepsilon \tag{1.42}$$

whenever $\chi, \chi_1 \in K$ and $d(\chi, \chi_1) < \delta$.

2.4 About Fixed Point Theorems

Banach Contraction Principle

Definition 1.14 *[39] A function $f : X \to Y$ is called a* **Liptschiz** *map when*

$$\exists c > 0, \forall \chi_1, \chi_2 \in X, \quad d_Y(f(\chi_1), f(\chi_2)) \leq c d_X(\chi_1, \chi_2). \tag{1.43}$$

where (X, d_X) *and* (Y, d_Y) *are metric spaces. Furthermore, it is called a* **contraction** *when it is Lipschitz with constant* $c < 1$.

Theorem 1.2 *[43], [35] Let* $(E, \|.\|)$ *be a Banach space, and let* $B \subseteq E$ *be nonempty and closed. If the function* $T : B \to B$ *satisfies*

$$\|T\chi - Ty\| \leq q\|\chi - y\|, \quad \text{for all} \quad \chi, y \in B \tag{1.44}$$

with $q < 1$*, then within* B *there exists a unique fixed point* χ^* *of* T.

Shaefer Fixed Point Theorem

Lemma 1.5 *[43],[35] In a Banach space named* Δ*, we define the completely continuous operator* $\varphi : \Delta \longrightarrow \Delta$*. If the set* $\digamma = \{\chi \in \Delta, \chi = \lambda\varphi(\chi), \lambda \in]0, 1[\}$ *is bounded, then, we can consider that* φ *has at least one solution.*

Dhage Fixed Point Approach

Lemma 1.6 *[15] Let* $\digamma$ *(*$\digamma \neq \varnothing$*) be a subset of the space* Δ *where* Δ *is a Banach space and* $\digamma$ *is bounded, closed, and convex. Now, let us have the following operators:* $\Lambda : \Delta \to \Delta$ *and* $\Theta : \digamma \to \Delta$*. These operators satisfy the following conditions:*

a) Λ *must be a Lipschitzian with Lipschitz constant noted* γ,

b) Θ *is completely continuous,*

c) *the equation* $\chi = \Lambda\chi\Theta y$ *amplies that* χ *is in* $\digamma$ *for all* $y \in \digamma$*, and*

d) *we have* $\gamma M < 1$*, where* $M = \|\Theta(\digamma)\| = sup\{\|\Theta(\chi)\| : \chi \in \digamma\}$

Therefore, we can say that the operator equation

$$\Lambda\chi\Theta\chi = \chi$$

possibly has a solution.

2.5 Additional Concepts and Tools

As another set of useful tools, here we present some notions that are important links to create a clear picture of the works presented in Chapter 2.

Theorem 1.3 *[21]* **"Cauchy-Schwartz Inequality"** *Let* $a_1, a_2, ..., a_n$ *and* $b_1, b_2, ..., b_n$ *be real numbers. Then, we have :*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}} \tag{1.45}$$

As an extension to the theorem, for $f, g \in L^2([a, b])$, we have

$$\left| \int_a^b f(\chi) g(\chi) d\chi \right| \leq \left(\int_a^b |f(\chi)|^2 d\chi \right)^{\frac{1}{2}} \left(\int_a^b |g(\chi)|^2 d\chi \right)^{\frac{1}{2}} \tag{1.46}$$

Theorem 1.4 *[21]* **" Hölder Inequality"** *Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers.If we have $\rho, \sigma \in [1, +\infty)$ such that $\frac{1}{\rho} + \frac{1}{\sigma} = 1$, then we have*

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left(\sum_{i=1}^{n} |a_i|^{\rho} \right)^{\frac{1}{\rho}} + \left(\sum_{i=1}^{n} |b_i|^{\sigma} \right)^{\frac{1}{\sigma}} \tag{1.47}$$

Let $f, g \in L^2([a, b])$. Then, Hölder's Inequality extension is given by

$$\left| \int_a^b f(\chi) g(\chi) d\chi \right| \leq \left(\int_a^b |f(\chi)|^{\rho} d\chi \right)^{\frac{1}{\rho}} \left(\int_a^b |g(\chi)|^{\sigma} d\chi \right)^{\frac{1}{\sigma}} \tag{1.48}$$

Definition 1.15 *[39] A finite (or infinite) inequality is* **positive** *if all variables $a, b, ...$ involved in it are real and non-negative.*

The following Jensen Lemma is also needed in chapter2.

Lemma 1.7 *[55] Let $n \in \mathbb{N}$, and let $a_1, ..., a_n$ be nonnegative real numbers. Then, for $r > 1$,*

$$\left(\sum_{i=1}^{n} a_i \right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r \tag{1.49}$$

Chapter 2

Applications of Integral Inequalities to Fractional Hybrid Differential Equations

Gronwall-Ballman type inequalities are a type of inequalities that can be used in many ways to study differential and integral equations. Whether it is for the effect of the order of the equation on the solution itself or on its uniqueness, or to find an approximation in someway to the said solution, in literature we find many works in that context as we see in the works of [1], [6], [7], [9], [14], [36], [45], [48], [55], [57], [62], [64], [65]. In this chapter, we present some of the results we obtained while addressing the heart of this thesis. To achieve our purpose of applying integral inequalities to fractional hybrid differential equations, first it is necessary to understand what are hybrid differential equations.

1 About Hybrid Differential Equations

Fractional hybrid differential equations are simply hybrid differential equations that have been generalized to their arbitrary order. In literature, specifically in the work of the mathematician B. C. Dhage [16], we understand that hybrid differential equations are the consequence of applying perturbations techniques on dynamical systems that are represented by nonlinear equations. These perturbation techniques are used due to the difficulty of solving the nonlinear differential equations for various reasons. He goes far in explaining : "For any closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$, consider the initial value problem of nonlinear first order ordinary differential equation

$$\begin{cases} \chi'(\zeta) &= h(\zeta, \chi(\zeta)), \quad \text{a.e. } \zeta \in J \\ \chi(0) &= \chi_0 \in \mathbb{R}. \end{cases} \tag{2.1}$$

where $h : J \times \mathbb{R} \to \mathbb{R}$."

If h is nonlinear and difficult to deal with, here the perturbation techniques become useful.

We usually notice two types of perturbed differential equations:

- ⋆ "Perturbation differential equation of first type: It is when the free unknown function is the part that has been perturbed in some way." [16]
- ⋆ "Perturbation differential equation of second type: It is when the unknown function under the derivative has gone under the process of perturbation." [16]

"Now, if the perturbation of second type involves multiplication or division, it is called *quadratic perturbation*. It has the following form

$$\begin{cases} \dfrac{d}{d\zeta}\left(\dfrac{\chi(\zeta)}{\phi_2(\zeta,\chi(\zeta))}\right) &= \phi_1(\zeta,\chi(\zeta)), \quad \text{a.e. } \zeta \in J \\ \chi(0) &= \chi_0 \in \mathbb{R}. \end{cases} \tag{2.2}$$

If the perturbation of second type involves addition or subtraction, it is called *linear perturbation*. It has the following form

$$\begin{cases} \dfrac{d}{d\zeta}\Big(\chi(\zeta) - \phi_2(\zeta,\chi(\zeta))\Big) &= \phi_1(\zeta,\chi(\zeta)), \quad \text{a.e. } \zeta \in J \\ \chi(0) &= \chi_0 \in \mathbb{R}. \end{cases} \tag{2.3}$$

where ϕ_1 and ϕ_2 has a direct relation to h.

In literature, these types of equations are named by *hybrid differential equations*."[16]

It is important to mention that in this dissertation, we focused on studying hybrid differential equations of second type with quadratic perturbation.

2 About Gronwall-Ballman Inequalities

In his book [12], C. Corduneanu states that a positive function χ for $\tau \in [t_0, T)$ $(T \leq +\infty)$ satisfies

$$\chi(\tau) \leq \phi(\tau) + \int_{t_0}^{\tau} \kappa(\zeta)\chi(\zeta)d\zeta, \tag{2.4}$$

where ϕ is a continuous function on $[t_0, T)$, and κ is a positive function on the same interval, then the Gronwall-Ballman inequality implies that

Lemma 2.1 *[12] For any function that satisfy the above inequality (2.4) and its assumptions, we have*

$$\chi(\tau) \leq \phi(\tau) + \int_{t_0}^{\tau} \phi(\zeta)\kappa(\zeta) \exp\left[\int_{\zeta}^{\tau} \kappa(\sigma)d\sigma\right] d\zeta, \quad \tau \in [t_0, T). \tag{2.5}$$

In 2013, J. Shao and F. Meng [55] considered a class of nonlinear Gronwall-Ballman inequalities that has generalized some results that was applied to fractional differential equations with Caputo derivative.

Lemma 2.2 *[55] Let $I = [t_0, T) \in \mathbb{R}$, $\kappa, \phi, \psi \in C(I, \mathbb{R}^+), (T \leq \infty)$. Suppose that $\chi \in C(I, \mathbb{R}^+)$, and*

$$\chi(\tau) \leq \kappa(\tau) + \int_{t_0}^{\tau} \phi(\zeta)\chi(\zeta)d\zeta + \int_{t_0}^{\tau} \psi(\zeta)\chi^{\gamma}(\zeta)d\zeta, \; \tau \in I, \tag{2.6}$$

where $0 \leq \gamma < 1$.

Then, for $\tau \in I$, we have

$$\chi(\tau) \leq \left[A^{1-\gamma}(\tau)+(1-\gamma) \int_{t_0}^{\tau} \exp\left((\gamma-1)\int_{t_0}^{\zeta} \phi(\sigma)d\sigma\right)\psi(\zeta)d\zeta\right]^{1/(1-\gamma)} \times \exp\left(\int_{t_0}^{\tau} \phi(\zeta)d\zeta\right), \tag{2.7}$$

where $A(\tau) = \max\limits_{t_0 \leq \zeta \leq \tau} \kappa(\zeta)$.

Theorem 2.1 *[9] Let $\chi, a, b, h_i, (i = 1, ..., n)$ be real valued nonnegative continuous functions and there exists positive real numbers $\rho_1, \rho_2, ..., \rho_n$ and $\chi(t)$ that satisfies the following inequality*

$$\chi^{\rho}(t) \leq a(t) + b(t) \int_0^t \sum_{i=1}^{i=n} h_i(s)\chi^{\rho_i}(s)ds, \tag{2.8}$$

with $t \in \mathbb{R}^+$.
Accordingly, we can get that

$$\chi(t) \leq \left\{a(t) + b(t)\int_0^t \sum_{i=1}^{n} h_i(s)\left(\frac{\rho_i}{\rho}a(s) + \frac{\rho-\rho_i}{\rho}\right) \times exp\left(\int_s^t b(\sigma)\sum_{i=1}^{n}\frac{\rho_i}{\rho}h_i(\sigma)d\sigma\right)ds\right\}^{1/\rho} \tag{2.9}$$

for $\rho \geq \rho^* = \max \rho_i, i = 1, ..., n$.

An attempt of extending results to inequalities of Gronwall-Ballman type to cover equations with Hadamard derivative has been established for problem with maxima by [57].

Inspired by the above works, we established some results that extend J. Shao et al. results in Lemma 2.2 and generalized some results of [9] mentioned in Theorem 2.1 to be applied to hybrid fractional differential equations with Hadamard derivative.

3 Main Results

We propose the following main result that generalizes Theorem 4 of [55]. We have

Theorem 2.2 *[19] Let $I = [t_0, T]$, $t_0 \geq 1$, $\alpha > 0, 0 < \gamma < 1$ and $a, b, p \in C(I, \mathbb{R}^+)$. For the case when $\chi \in C(I, \mathbb{R}^+)$ and it satisfies*

$$\chi(t) \leq a(t) + \int_{t_0}^{t} (log(\frac{t}{s}))^{\alpha-1} b(s)\chi(s)s^{-1}ds + \int_{t_0}^{t} (log(\frac{t}{s}))^{\alpha-1} p(s)\chi^{\gamma}(s)s^{-1}ds, \tag{2.10}$$

As a mean of approximation, the following two cases are valid:
(i) *If $\alpha > 1/2$, then*

$$\begin{aligned} \chi(t) &\leq \Big[A_1^{1-\gamma}(t) + (1-\gamma)G_1 \\ &\times \int_{t_0}^{t} \exp\left((\gamma-1)G_1 \int_{t_0}^{s} b^2(\sigma)\sigma^{-1}d\sigma\right) p^2(s)s^{3\gamma-4}ds\Big]^{\frac{1}{2(1-\gamma)}} \\ &\times t^{3/2} \exp\left((G_1/2)\int_{t_0}^{t} b^2(s)s^{-1}ds\right), \qquad t \in I, \end{aligned} \tag{2.11}$$

with $A_1(t) = \max\limits_{t_0 \leq s \leq t} 3s^{-3}a^2(s)$, and $G_1 = \Gamma(2\alpha - 1)/9^{\alpha-1}$.
(ii) *Suppose that $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1 + \alpha$. Then, we have*

$$\begin{aligned} \chi(t) &\leq \Big[A_2^{1-\gamma}(t) + (1-\gamma)G_2 \\ &\times \int_{t_0}^{t} p^q(s)s^{q(\gamma(\frac{p+1}{p})-2)} \exp\left((\gamma-1)G_2 \int_{t_0}^{s} b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right) ds\Big]^{\frac{1}{q(1-\gamma)}} \\ &\times t^{\frac{p+1}{p}} \exp\left(\frac{G_2}{q}\int_{t_0}^{t} b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right), \end{aligned} \tag{2.12}$$

where $A_2(t) = \max\limits_{t_0 \leq s \leq t} 3^{q-1}s^{-q(\frac{p+1}{p})}a^q(s)$, and $G_2 = 3^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{\frac{q}{p}}$.

Proof:
Let $t \in I$. We have:

$$\chi(t) \leq a(t) + \int_{t_0}^{t} (log(\frac{t}{s}))^{\alpha-1} b(s)s^{-2}s\chi(s)ds + \int_{t_0}^{t} (log(\frac{t}{s}))^{\alpha-1} p(s)s^{-2}s\chi^{\gamma}(s)ds \tag{2.13}$$

(i) With the help of the famous Cauchy-Schwartz inequality, we can get:

$$\begin{aligned}
\chi(t) &\leq a(t) + \Big(\int_{t_0}^{t} (log(\frac{t}{s}))^{2(\alpha-1)} s^2 ds\Big)^{1/2} \Big(\int_{t_0}^{t} b^2(s) s^{-4} \chi^2(s) ds\Big)^{1/2} \\
&+ \Big(\int_{t_0}^{t} (log(\frac{t}{s}))^{2(\alpha-1)} s^2 ds\Big)^{1/2} \Big(\int_{t_0}^{t} p^2(s) s^{-4} \chi^{2\gamma}(s) ds\Big)^{1/2} \\
&\leq a(t) + \Big(\frac{3t^3 \Gamma(2\alpha-1)}{9^{\alpha}}\Big)^{1/2} \Big(\int_{t_0}^{t} b^2(s) s^{-4} \chi^2(s) ds\Big)^{1/2} \\
&+ \Big(\frac{3t^3 \Gamma(2\alpha-1)}{9^{\alpha}}\Big)^{1/2} \Big(\int_{t_0}^{t} p^2(s) s^{-4} \chi^{2\gamma}(s) ds\Big)^{1/2}
\end{aligned} \tag{2.14}$$

where $\alpha > 1/2$.

Using Jensen Lemma (Lemma 1.7) for $r = 2$, the above inequality becomes

$$\begin{aligned}
\chi^2(t) &\leq 3a^2(t) + \left(\frac{t^3 \Gamma(2\alpha-1)}{9^{\alpha-1}}\right) \left(\int_{t_0}^{t} b^2(s) s^{-4} \chi^2(s) ds\right) \\
&+ \left(\frac{t^3 \Gamma(2\alpha-1)}{9^{\alpha-1}}\right) \left(\int_{t_0}^{t} p^2(s) s^{-4} \chi^{2\gamma}(s) ds\right)
\end{aligned} \tag{2.15}$$

As a transitional mean, let us introduce the function $w(t) := [\chi^2(t) t^{-3}]$. By adjusting (2.15) according to w, we get:

$$w(t) \leq A_1(t) + G_1 \left(\int_{t_0}^{t} b^2(s) s^{-1} w(s) ds\right) + G_1 \left(\int_{t_0}^{t} p^2(s) s^{3\gamma-4} w^{\gamma}(s) ds\right) \tag{2.16}$$

Since $A_1(t)$ is nondecreasing, then by Lemma 2.2, it yields that:

$$\begin{aligned}
w(t) &\leq \Big[A_1^{1-\gamma}(t) + (1-\gamma) G_1 \\
&\times \int_{t_0}^{t} p^2 s^{3\gamma-4}(s) \exp\left((\gamma-1) G_1 \int_{t_0}^{s} b^2(\sigma) \sigma^{-1} d\sigma\right) ds\Big]^{1/(1-\gamma)} \\
&\times \exp\left(G_1 \int_{t_0}^{t} b^2(s) s^{-1} ds\right),
\end{aligned} \tag{2.17}$$

Replacing w by its quantity, we get (3.2).

(ii) Taking $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1+\alpha$, then we get $(1/p)+(1/q) = 1$. Thanks to Hölder inequality, we obtain

$$\begin{aligned}\chi(t) \quad &\leq \quad a(t) + \left(\int_{t_0}^{t}\left(log\frac{t}{s}\right)^{p(\alpha-1)} s^p ds\right)^{\frac{1}{p}} \left(\int_{t_0}^{t} b^q(s)\chi^q(s)s^{-2q}ds\right)^{\frac{1}{q}} \\ &\quad + \left(\int_{t_0}^{t}\left(log\frac{t}{s}\right)^{p(\alpha-1)} s^p ds\right)^{\frac{1}{p}} \left(\int_{t_0}^{t} p^q(s)\chi^{q\gamma}(s)s^{-2q}ds\right)^{\frac{1}{q}}.\end{aligned} \tag{2.18}$$

As a direct consequence,

$$\begin{aligned}\chi(t) \quad &\leq \quad a(t) + \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{1}{p}} \left(\int_{t_0}^{t} b^q(s)\chi^q(s)s^{-2q}ds\right)^{\frac{1}{q}} \\ &\quad + \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{1}{p}} \left(\int_{t_0}^{t} p^q(s)\chi^{q\gamma}(s)s^{-2q}ds\right)^{\frac{1}{q}}\end{aligned} \tag{2.19}$$

By taking the aid of Jensen Lemma (Lemma 1.7) with $r = q$, we can write

$$\begin{aligned}\chi^q(t) \quad &\leq \quad 3^{q-1}a^q(t) + 3^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{\frac{q}{p}} \\ &\quad \times \left(\int_{t_0}^{t} b^q(s)\chi^q(s)s^{-2q}ds + \int_{t_0}^{t} p^q(s)\chi^{q\gamma}(s)s^{-2q}ds\right).\end{aligned} \tag{2.20}$$

Considering the function $w(t) := (\chi(t)t^{-(p+1)/p})^q$, it yields that

$$w(t) \quad \leq \quad A_2(t) + G_2\int_{t_0}^{t} b^q(s)s^{-q\left(\frac{p-1}{p}\right)}w(s)ds + G_2\int_{t_0}^{t} p^q(s)w^\gamma(s)s^{q(\gamma\left(\frac{p+1}{p}\right)-2)}ds. \tag{2.21}$$

Due to Lemma 2.2, we notice that

$$\begin{aligned}w(t) \quad &\leq \quad \Big[A_2^{1-\gamma}(t) + (1-\gamma)G_2 \\ &\quad \times \int_{t_0}^{t} p^q(s)s^{q(\gamma\left(\frac{p+1}{p}\right)-2)} \exp\left((\gamma-1)G_2\int_{t_0}^{t} b^q(\sigma)\sigma^{-q\left(\frac{p-1}{p}\right)}d\sigma\right)ds\Big]^{1/(1-\gamma)} \\ &\quad \times \exp\left(G_2\int_{t_0}^{t} b^q(\sigma)\sigma^{-q\left(\frac{p-1}{p}\right)}d\sigma\right).\end{aligned} \tag{2.22}$$

By replacing w with its value, we get the inequality (2.12). The proof is thus achieved. ■

Example: Let $t_0 = 1$, $T = e$, and $a(t) = \exp(t)$, $b(t) = \sqrt{3}t^{-2}$, $p(t) = t^{\frac{-3}{2}\gamma}$. It is obvious that a, b, and p are in $C(I, \mathbb{R}^+)$. So, for $\gamma = \frac{1}{2}$ and $\alpha = \frac{3}{4}$, we have:

$$\chi(t) \leq \exp(t) + \int_1^t (log(\frac{t}{s}))^{\frac{-1}{4}} \sqrt{3}s^{-2}\chi(s)s^{-1}ds + \int_1^t (log(\frac{t}{s}))^{\frac{-1}{4}} s^{\frac{-3}{4}} \chi^{\frac{1}{2}}(s)s^{-1}ds, \tag{2.23}$$

Since, $\alpha > 1/2$ and thanks to (3.2), we get:

$$\begin{aligned} \chi(t) \leq & \left[(3t^{-3}\exp(2t))^{1/2} + \frac{1}{3}\exp\left(-\frac{\sqrt{\pi}}{2\sqrt{3}}(t^{-3}-1)\right) - \frac{1}{3}\right]^{-1} \\ & \times t^{3/2}\exp\left(-\frac{\sqrt{\pi}}{2\sqrt{3}}(t^{-3}-1)\right). \end{aligned} \tag{2.24}$$

We propose this second main result that generalizes Theorem 2.1 ([9]).

Theorem 2.3 *[19] Let χ, a, k_i real nonnegative functions defined on $t \in [t_0, T]$ where $t_0 \geq 1$, $\delta_i < 1$ for $i = 1, ..., n$. If*

$$\chi(t) \leq a(t) + \int_{t_0}^t \left(\log\frac{t}{s}\right)^{\alpha-1} \sum_{i=1}^{i=n} k_i(s)\chi^{\delta_i}(s)s^{-1}ds, \tag{2.25}$$

as a consequence, we'd have these possible results:

(i) *If $\alpha > 1/2$, then*

$$\begin{aligned} \chi(t) \leq & \left\{2a^2(t) + \frac{6t^3}{9^\alpha}\Gamma(2\alpha-1)\int_{t_0}^t \sum_{i=1}^n nk_i^2(s)s^{-4}\left(\delta_i 2a^2(s) + 1 - \delta_i\right)\right. \\ & \left.\times exp\left(\int_s^t \frac{6\sigma^3}{9^\alpha}\Gamma(2\alpha-1)\sum_{i=1}^n n\delta_i k_i^2(\sigma)(\sigma)^{-4}d\sigma\right) ds\right\}^{1/2} \end{aligned} \tag{2.26}$$

(ii)*Suppose that $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1+\alpha$. Then, we have*

$$\begin{aligned} \chi(t) \leq & \left\{2^{q-1}a^q(t) + 2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}\right. \\ & \times \int_{t_0}^t \sum_{i=1}^n n^{q-1}k_i^q(s)s^{-2q}\left(\delta_i 2^{q-1}a^q(s) + (1-\delta_i)\right) \\ & \left.\times exp\left(\int_s^t 2^{q-1}\left(\frac{\sigma^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}\sum_{i=1}^n n^{q-1}\delta_i k_i^q(\sigma)\sigma^{-2q}\right) ds\right\}^{1/q} \end{aligned} \tag{2.27}$$

Proof:

For $t \in [t_0, T]$, we have

$$\chi(t) \leq a(t) + \int_{t_0}^t \left(\log\frac{t}{s}\right)^{\alpha-1} s\sum_{i=1}^{i=n} k_i(s)\chi^{\delta_i}(s)s^{-2}ds. \tag{2.28}$$

(i) Using Cauchy-Shwartz inequality and Lemma 1.7, we can write:

$$\chi(t) \leq a(t) + \left(\int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{2(\alpha-1)} s^2 ds \right)^{1/2} \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} n k_i^2(s) \chi^{2\delta_i}(s) s^{-4} ds \right)^{1/2}. \tag{2.29}$$

This leads to

$$\chi(t) \leq a(t) + \left(\frac{3t^3}{9^\alpha} \Gamma(2\alpha - 1) \right)^{1/2} \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} n k_i^2(s) \chi^{2\delta_i}(s) s^{-4} ds \right)^{1/2} \tag{2.30}$$

where $\alpha > 1/2$.

Thanks to the inequality (2.30) and Lemma 1.7, we get:

$$\chi^2(t) \leq 2a^2(t) + \left(\frac{6t^3}{9^\alpha} \Gamma(2\alpha - 1) \right) \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} n k_i^2(s) s^{-4} \chi^{2\delta_i}(s) ds \right) \tag{2.31}$$

Now, if we put $\tilde{p} = 2, \tilde{p}_i = 2\delta_i, \tilde{h}_i(t) = n k_i^2(t) t^{-4}, \tilde{a}(t) = 2a^2(t), \tilde{b}(t) = \frac{6t^3}{9^\alpha} \Gamma(2\alpha - 1)$, the inequality would take the following form:

$$\chi^{\tilde{p}}(t) \leq \tilde{a}(t) + \tilde{b}(t) \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} \tilde{h}_i(s) \chi^{\tilde{p}_i}(s) ds \right) \tag{2.32}$$

which, thanks to Theorem 2.1, gives

$$\chi(t) \leq \left\{ \tilde{a}(t) + \tilde{b}(t) \int_{0}^{t} \sum_{i=1}^{n} \tilde{h}_i(s) \left(\delta_i \tilde{a}(s) + 1 - \delta_i \right) \times exp \left(\int_{s}^{t} \tilde{b}(\sigma) \sum_{i=1}^{n} \delta_i \tilde{h}_i(\sigma) d\sigma \right) ds \right\}^{1/2}, \tag{2.33}$$

from which we conclude (2.26).

(ii) Let $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, and $p = 1 + \alpha$, then we get $(1/p) + (1/q) = 1$. Using Hölder inequality and Lemma 1.7, the inequality (2.28) becomes

$$\chi(t) \leq a(t) + \left(\int_{t_0}^{t} \left(\log \frac{t}{s} \right)^{p(\alpha-1)} s^p ds \right)^{1/p} \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} n^{q-1} k_i^q(s) \chi^{q\delta_i}(s) s^{-2q} ds \right)^{1/q} \tag{2.34}$$

Therefore, we get

$$\chi(t) \leq a(t) + \left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}} \Gamma(p(\alpha - 1) + 1) \right)^{1/p} \left(\int_{t_0}^{t} \sum_{i=1}^{i=n} n^{q-1} k_i^q(s) \chi^{q\delta_i}(s) s^{-2q} ds \right)^{1/q} \tag{2.35}$$

Thanks to (2.35) and using Lemma 1.7, we obtain

$$\chi^q(t) \leq 2^{q-1}a^q(t)+2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}\left(\int_{t_0}^{t}\sum_{i=1}^{i=n} n^{q-1}k_i^q(s)\chi^{q\delta_i}(s)s^{-2q}ds\right) \tag{2.36}$$

If we take the following notations $\hat{p} = q$, $\hat{p}_i = q\delta_i$, $\hat{a}(t) = 2^{q-1}a^q(t)$, $\hat{b}(t) = 2^{q-1}\left(\frac{t^{p+1}}{(p+1)^{p(\alpha-1)+1}}\Gamma(p(\alpha-1)+1)\right)^{q/p}$, $\hat{h}_i(t) = n^{q-1}k_i^q(t)t^{-2q}$, then,

$$\chi^{\hat{p}}(t) \leq \hat{a}(t) + \hat{b}(t)\left(\int_{t_0}^{t}\sum_{i=1}^{i=n}\hat{h}(s)\chi^{\hat{p}_i}(s)ds\right). \tag{2.37}$$

According to Theorem 2.1, we have

$$\chi(t) \leq \left\{\hat{a}(t) + \hat{b}(t)\int_0^t \sum_{i=1}^{n}\hat{h}_i(s)\left(\delta_i\hat{a}(s) + (1-\delta_i)\right) \times exp\left(\int_s^t \hat{b}(\sigma)\sum_{i=1}^{n}\delta_i\hat{h}_i(\sigma)\right)ds\right\}^{1/q} \tag{2.38}$$

Therefore, we have (2.27) which completes the proof. ∎

4 Applications on Fractional Hybrid Differential Equations

In this section, we are concerned with the following hybrid differential problem:

$$\begin{cases} {}^{H}D^{\alpha}\left(\dfrac{z(t)}{f(t,z(t))}\right) = g(t,z(t)) + h(t)z(t), \quad 1 \leq t \leq T,\ 0 < \alpha \leq 1, \\ {}^{H}I^{1-\alpha}z(t)|_{t=1} = \eta, \end{cases} \tag{2.39}$$

We take note that $f \in C([1,T]\times\mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in C([1,T]\times\mathbb{R},\mathbb{R})$, $h \in C([1,T],\mathbb{R})$, ${}^{H}D^{\alpha}$ is the derivative of order α with Hadamard approach, ${}^{H}I^{1-\alpha}$ is the integral of order $1-\alpha$ with Hadamard approach, and $\eta \in \mathbb{R}$.

It is to note that in the case where h is identically zero, the associated problem has been discussed by B. Ahmed et al., see [2].

Thanks to [2], the integral equation that is equivalent to (2.39) is given by:

$$z(t) = f(t,z(t))\left(\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_1^t(log\frac{t}{s})^{\alpha-1}(g(s,z(s))+h(s)z(s))\frac{ds}{s}\right), \quad t \in [1,T] \tag{2.40}$$

Introducing the following two hypotheses,

$(\mathcal{H}.1)$ As $t \in [1,T]$, there exist a positive constant F, such that $|f(t,z(t))| \leq F$.

$(\mathcal{H}.2)$ For $b, p \in C([1,T],\mathbb{R}^+)$, $0<\gamma<1$, we assume that $|h(t)z(t)+g(t,z(t)| \leq b(t)|z(t)|+p(t)|z(t)|^{\gamma}$,

we prove the theorem:

Theorem 2.4 *[19] Let us consider that the hypothesis $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$ are valid. If $z(t)$ is a solution of (2.39), then the following estimations hold:*

(i) Suppose that $\alpha > 1/2$. Then

$$\begin{aligned} |z(t)| &\leq \Big[\tilde{A}_1^{1-\gamma}(t)+(1-\gamma)\frac{G_1F^2}{\Gamma^2(\alpha)} \\ &\times \int_1^t \exp\left((\gamma-1)\frac{G_1F^2}{\Gamma^2(\alpha)}\int_1^s b^2(\sigma)\sigma^{-1}d\sigma\right)p^2(s)s^{3\gamma-4)}ds\Big]^{1/2(1-\gamma)} \\ &\times t^{3/2}\exp\left(\frac{G_1F^2}{2\Gamma^2(\alpha)}\int_1^t b^2(s)s^{-1}ds\right), \qquad t\in I \end{aligned} \tag{2.41}$$

where $\tilde{A}_1(t)=\max\limits_{1\leq s\leq t} 3s^{-3}\frac{\eta^2F^2}{\Gamma^2(\alpha)}(logs)^{2(\alpha-1)}$, and $G_1=\Gamma(2\alpha-1)/9^{\alpha-1}$.

(ii) Suppose that $\alpha\in(0,1/2]$, $q=(1+\alpha)/\alpha$, and $p=1+\alpha$. Then

$$\begin{aligned} |z(t)| &\leq \Big[\tilde{A}_2^{1-\gamma}(t)+(1-\gamma)\frac{G_2F^q}{\Gamma^q(\alpha)} \\ &\times \int_1^t p^q(s)s^{q(\gamma(\frac{p+1}{p})-2)}\exp\left((\gamma-1)\frac{G_2F^q}{\Gamma^q(\alpha)}\int_1^s b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right)ds\Big]^{1/q(1-\gamma)} \\ &\times t^{\frac{p+1}{p}}\exp\left(\frac{G_2F^q}{q\Gamma^q(\alpha)}\int_1^t b^q(\sigma)\sigma^{-q(\frac{p-1}{p})}d\sigma\right) \end{aligned} \tag{2.42}$$

where $\tilde{A}_2(t)=\max\limits_{1\leq s\leq t} 3^{q-1}s^{-q(\frac{p+1}{p})}\frac{|\eta|^qF^q}{\Gamma^q(\alpha)}(logs)^{q(\alpha-1)}$, and $G_2=3^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p}$.

Proof:

Let $t\in[1,T]$. Then, we can have:

$$\begin{aligned} |z(t)| &\leq |f(t,z(t)|\Big(|\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}|+\frac{1}{\Gamma(\alpha)} \\ &\times \int_1^t (log\frac{t}{s})^{\alpha-1}|h(s)z(s)+g(s,z(s))|\frac{ds}{s}\Big) \end{aligned} \tag{2.43}$$

Thanks to hypothesis $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$, we get:

$$\begin{aligned} |z(t)| &\leq F|\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1}|+\frac{F}{\Gamma(\alpha)} \\ &\times \int_1^t (log\frac{t}{s})^{\alpha-1}b(s)|z(s)|+p(s)|z(s)|^{\gamma}\frac{ds}{s} \end{aligned} \tag{2.44}$$

If we rewrite (2.44) as

$$|z(t)| \leq \hat{A}(t) + \int_1^t (log\frac{t}{s})^{\alpha-1} B(s)|z(s)|\frac{ds}{s} + \int_1^t (log\frac{t}{s})^{\alpha-1} P(s)|z(s)|^{\gamma}\frac{ds}{s} \tag{2.45}$$

where $\hat{A}(t) = F\frac{|\eta|}{\Gamma(\alpha)}|(logt)^{\alpha-1}|$, $B(t) = \frac{F}{\Gamma(\alpha)}b(t)$, and $P(t) = \frac{F}{\Gamma(\alpha)}p(t)$, we notice that the inequality is similar to (2.10).

Using Theorem 2.2, we get the desired results. ■

Now, let's consider the following equation:

$$\begin{cases} {}^{H}D^{\alpha}\left(\dfrac{z(t)}{f(t,z(t))}\right) = \displaystyle\sum_{i=1}^{i=n} g_i(t,z(t)), \quad 1 \leq t \leq T, \ 0 < \alpha \leq 1, \\ \\ {}^{H}I^{1-\alpha} z(t)|_{t=1} = \eta, \end{cases} \tag{2.46}$$

We take note that $f \in C([1,T]\times\mathbb{R}, \mathbb{R}\backslash\{0\})$, $g_i \in C([1,T]\times\mathbb{R},\mathbb{R})$ $(i = 1,...,n)$, ${}^{H}D^{\alpha}$ is the derivative of order α with Hadamard approach, ${}^{H}I^{1-\alpha}$ is the integral of order $1-\alpha$ with Hadamard approach, and $\eta \in \mathbb{R}$.

The equivalent integral representation of (2.46) can be represented as follow:

$$z(t) = f(t,z(t))\Big(\frac{\eta}{\Gamma(\alpha)}(logt)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}\sum_{i=1}^{i=n} g_i(s,z(s))\frac{ds}{s}\Big), \ \ t\in[1,T]. \tag{2.47}$$

Introducing the following hypothesis,

$(\mathcal{H}.3)$ For $i = 1,...n$, $k_i \in C([1,T],\mathbb{R}^{+})$, $0 < \delta_i < 1$, we have $|g_i(t,z(t)| \leq k_i(t)|z(t)|^{\delta_i}$,

we present to the reader the following main result.

Theorem 2.5 *[19] Suppose that $(\mathcal{H}.1)$ and $(\mathcal{H}.3)$ are satisfied.*
Then, the following two cases are valid:
(i) *If $\alpha > 1/2$, we have*

$$\begin{aligned} |z(t)| \leq & \left\{ 2\hat{A}^2(t) + \frac{6t^3F^2\Gamma(2\alpha-1)}{9^{\alpha}\Gamma^2(\alpha)} \int_1^t \sum_{i=1}^{n} nk_i^2(s)s^{-4}\left(\delta_i 2\hat{A}^2(s) + 1 - \delta_i\right) \right. \\ & \left. \times exp\left(\int_s^t \frac{6\sigma^3F^2\Gamma(2\alpha-1)}{9^{\alpha}\Gamma^2(\alpha)} \sum_{i=1}^{n} n\delta_i k_i^2(\sigma)(\sigma)^{-4}d\sigma\right) ds \right\}^{1/2}, \end{aligned} \tag{2.48}$$

where, $\hat{A}(t) = \left(F\frac{|\eta|}{\Gamma(\alpha)}|(logt)|^{\alpha-1}\right)$.

(ii) *Suppose that* $\alpha \in (0, 1/2]$, $q = (1+\alpha)/\alpha$, *and* $p = 1+\alpha$. *Then, we have*

$$\begin{aligned} |z(t)| \leq & \left\{ 2^{q-1}\hat{A}^q(t) + 2^{q-1}\left(\frac{F}{\Gamma(\alpha)}\right)^q \left(\frac{t^{p+1}\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p} \right. \\ & \times \int_1^t \sum_{i=1}^{n} n^{q-1} k_i^q(s) s^{-2q} \left(\delta_i 2^{q-1}\hat{A}^q(s) + (1-\delta_i)\right) \\ & \left. \times exp\left(\int_s^t 2^{q-1}\left(\frac{F}{\Gamma(\alpha)}\right)^q \left(\frac{\sigma^{p+1}\Gamma(p(\alpha-1)+1)}{(p+1)^{p(\alpha-1)+1}}\right)^{q/p} \sum_{i=1}^{n} n^{q-1}\delta_i k_i^q(\sigma)\sigma^{-2q}\right) ds \right\}^{1/q}. \end{aligned} \tag{2.49}$$

Proof: Let $t \in [1, T]$. Accordingly, it is obvious that

$$|z(t)| \leq |f(t, z(t))| \Big(\frac{|\eta|}{\Gamma(\alpha)} |(logt)|^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t (log\frac{t}{s})^{\alpha-1} \sum_{i=1}^{i=n} |g_i(s, z(s))| \frac{ds}{s} \Big) \tag{2.50}$$

Thanks to $(\mathcal{H}.1)$ and $(\mathcal{H}.3)$, we can write:

$$|z(t)| \leq \Big(F\frac{|\eta|}{\Gamma(\alpha)} |(logt)|^{\alpha-1} + \frac{F}{\Gamma(\alpha)} \int_1^t (log\frac{t}{s})^{\alpha-1} \sum_{i=1}^{i=n} |k_i(s)||z^{\delta_i}(s)| \frac{ds}{s} \Big) \tag{2.51}$$

If we rewrite (2.51) as

$$|z(t)| \leq \hat{A}(t) + \int_1^t (log\frac{t}{s})^{\alpha-1} \sum_{i=1}^{i=n} K_i(t) |z^{\delta_i}(s)| \frac{ds}{s} \tag{2.52}$$

where $\hat{A}(t) = F\frac{|\eta|}{\Gamma(\alpha)}|(logt)|^{\alpha-1}$, and $K_i(t) = \frac{F}{\Gamma(\alpha)}|k_i(s)|$ $(i = 1, ..., n)$, we notice that the inequality is similar to (2.25).

Thanks to Theorem 2.3, we achieve the proof of this theorem. ■

Chapter 3

Investigation of Existence of Solutions of a Boundary Value Problem

In recent years, researchers focused on developing as many types of fractional differential equations as possible. They have shown special interest to this domain. One of these types of equations that has got special attention is hybrid differential equations. We see a lot of good works that addressed both types quadratic and linear as we see in [2], [3], [4], [16], [23], [24], and also in [25], [28], [37], [38], [40], [41], [63].

In this chapter, we mainly present another way of how we can apply integral inequalities to highlight the existence/ existence and uniqueness of the solution of a certain class of hybrid differential equations of fractional order.

1 Boundary Value Problem

$$\begin{cases} {}^c\mathbf{D}^{\alpha_1}\left(\dfrac{\chi_1(t)}{f_1(t,\chi_1(t),\chi_2(t),...,\chi_n(t))}\right) = h_1(t,\chi_1(t),\chi_2(t),...,\chi_n(t)) \\ \qquad +I^{\delta_1}k_1(t,\chi_1(t),\chi_2(t),...,\chi_n(t)), \quad t\in J \\ {}^c\mathbf{D}^{\alpha_2}\left(\dfrac{\chi_2(t)}{f_2(t,\chi_1(t),\chi_2(t),...,\chi_n(t))}\right) = h_2(t,\chi_1(t),\chi_2(t),...,\chi_n(t)) \\ \qquad +I^{\delta_2}k_2(t,\chi_1(t),\chi_2(t),...,\chi_n(t)), \quad t\in J \\ \qquad \cdots \\ {}^c\mathbf{D}^{\alpha_n}\left(\dfrac{\chi_n(t)}{f_n(t,\chi_1(t),\chi_2(t),...,\chi_n(t))}\right) = h_n(t,\chi_1(t),\chi_2(t),...,\chi_n(t)) \\ \qquad +I^{\delta_n}k_n(t,\chi_1(t),\chi_2(t),...,\chi_n(t)), \quad t\in J \\ \chi_i(0) = \theta_i\displaystyle\int_0^{\beta_i}\varphi_i(s)\chi_i(s)ds, \\ \qquad 0<\beta_i<1, i=1,2,...,n. \end{cases} \tag{3.1}$$

For our problem, we consider ${}^cD^{\alpha_i}$ the Caputo derivatives with $0<\alpha_i<1$, the symbols I^{δ_i} denote the RL-(Riemann-Liouville) fractional integrals of order δ_i, with $0 < \delta_i < 1, i = 1, ..., n$, $J = [0, 1]$ represents the time interval, θ_i are real numbers, φ_i are continuous functions on $[0, \beta_i]$, $f_i \in C(J \times \mathbb{R}^n, \mathbb{R} - \{0\})$ and $h_i, k_i \in C(J \times \mathbb{R}^n, \mathbb{R})$.

The Integral Representation

Since the system we have is not linear, we consider what is called "integral representation" or "integral solution" of the given problem.

We also note that in order to make it easy to concentrate, we note: $\chi = (\chi_1, \chi_2, ..., \chi_n)$ and $\chi(t) = (\chi_1(t), \chi_2(t), ..., \chi_n(t))$.

The following lemma is an auxiliary result that highlights the integral representation of the system (3.1) which is very important for the main results.

Lemma 3.1 *[17] Let $i = 1, 2, ..., n$ and $0 < \alpha_i, \delta_i < 1$. For $f_i \in C(J \times \mathbb{R}^n, \mathbb{R} - \{0\})$ and $h_i, k_i \in C(J \times \mathbb{R}^n, \mathbb{R})$, we can consider as a solution for the equation:*

$$ {}^cD^{\alpha_i}\left(\frac{\chi_i(t)}{f_i(t,\chi(t))}\right) = h_i(t,\chi(t)) + I^{\delta_i}k_i(t,\chi(t)) \tag{3.2}$$

with the associated condition:

$$\chi_i(0) = \theta_i \int_0^{\beta_i} \varphi_i(s)\chi_i(s)ds, \quad 0 < \beta_i < 1, i = 1, 2, ..., n. \tag{3.3}$$

the integral equation given by:

$$\begin{aligned}\chi_i(t) \quad = \quad & f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau \\ & + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds}\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s) \\ & \times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\end{aligned} \tag{3.4}$$

with: $f_i(0,\chi(0)) \neq \theta_i \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds$.

Proof.

For $i = 1, ..., n$, we consider:

$$^{c}D^{\alpha_i}\left(\frac{\chi_i(t)}{f_i(t,\chi(t))}\right) = h_i(t,\chi(t)) + I^{\delta_i}k_i(t,\chi(t)), t \in J \tag{3.5}$$

To be able to obtain the general solution for the equation (3.5), we put lemmas 1.3 and 1.4 into perspective and we get:

$$\frac{\chi_i(t)}{f_i(t,\chi(t))} = I^{\alpha_i}h_i(t,\chi(t)) + I^{\alpha_i+\delta_i}k_i(t,\chi(t)) - c_0 \tag{3.6}$$

where $c_0 \in \mathbb{R}$ is an arbitrary constant.

From (3.6), we get:

$$\chi_i(t) = f_i(t,\chi(t))[I^{\alpha_i}h_i(t,\chi(t)) + I^{\alpha_i+\delta_i}k_i(t,\chi(t)) - c_0] \tag{3.7}$$

On the other hand, we multiply both sides of (3.7) by $\theta_i\varphi_i(s)$, we get:

$$\begin{aligned}\theta_i\varphi_i(s)\chi_i(s) &= \theta_i\varphi_i(s)f_i(s,\chi(s)) \\ &\quad \times[I^{\alpha_i}h_i(s,\chi(s)) + I^{\alpha_i+\delta_i}k_i(s,\chi(s))] - c_0\theta_i f_i(s,\chi(s))\varphi_i(s)\end{aligned} \tag{3.8}$$

The equation (3.8) gives us the privilege to have:

$$\begin{aligned}\theta_i\int_0^{\beta_i}\varphi_i(s)\chi_i(s)ds &= \theta_i\int_0^{\beta_i}\varphi_i(s)f_i(s,\chi(s))[I^{\alpha_i}h_i(s,\chi(s)) + I^{\alpha_i+\delta_i}k_i(s,\chi(s))]ds \\ &\quad -c_0\int_0^{\beta_i}\theta_i f_i(s,\chi(s))\varphi_i(s)ds\end{aligned} \tag{3.9}$$

With the help of (3.7) and the condition given in (3.3), we get

$$\begin{aligned}c_0\left(f_i(0,\chi(0)) - \int_0^{\beta_i}\theta_i f_i(s,\chi(s))\varphi_i(s)ds\right) &= \theta_i\int_0^{\beta_i}\varphi_i(s)f_i(s,\chi(s)) \\ &\quad \times[I^{\alpha_i}h_i(s,\chi(s)) + I^{\alpha_i+\delta_i}k_i(s,\chi(s))]ds\end{aligned} \tag{3.10}$$

and therefore, we establish that

$$\begin{aligned}c_0 &= \frac{\theta_i}{\left(f_i(0,\chi(0)) - \int_0^{\beta_i}\theta_i f_i(s,\chi(s))\varphi_i(s)ds\right)} \\ &\quad \int_0^{\beta_i}\varphi_i(s)f_i(s,\chi(s))[I^{\alpha_i}h_i(s,\chi(s) + I^{\alpha_i+\delta_i}k_i(s,\chi(s))]ds\end{aligned} \tag{3.11}$$

Replacing c_0 by its value in (3.7), we obtain (3.4). ■

Now, we introduce the following Banach spaces:

$$X_i = \{\chi_i, i = 1, ..., n : \chi_i \in C(J, \mathbb{R})\} \tag{3.12}$$

with the norm:

$$\|\chi_i\|_{X_i} = sup\{|\chi_i(t)| : t \in J\} \tag{3.13}$$

where $i = 1, ..., n$.

We bring to the attention that for $i = 1, 2, ..., n$, $\left(X_i, \|.\|_{X_i}\right)$ is a Banach space [58]. The product space with its norm

$$\left(\prod_{i=1}^{n} X_i, \|.\|_{\prod_{i=1}^{n} X_i}\right) \quad \text{with} \quad \|\chi\|_{\prod_{i=1}^{n} X_i} = \sum_{i=1}^{n} \|\chi_i\|_{X_i} \tag{3.14}$$

is also a Banach space [58].

Let $\mathcal{Q}$ be an operator defined by:

$$\mathcal{Q} : \quad \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} X_i$$

$$\chi \longmapsto \mathcal{Q}\chi$$

such that for $t \in J$,

$$\mathcal{Q}\chi(t) = \left(\mathcal{Q}_1\chi(t), \mathcal{Q}_2\chi(t), ..., \mathcal{Q}_n\chi(t)\right) \tag{3.15}$$

where:

$$\begin{aligned}
\mathcal{Q}_i\chi(t) \quad = \quad & f_i(t, \chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} h_i(\tau, \chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^t (t-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, \chi(\tau))d\tau \\
& + \frac{\theta_i}{f_i(0, \chi(0)) - \theta_i \int_0^{\beta_i} f_i(s, \chi(s))\varphi_i(s)ds} \int_0^{\beta_i} f_i(s, \chi(s))\varphi_i(s) \\
& \times \Bigg[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i - 1} h_i(\tau, \chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^s (s-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, \chi(\tau))d\tau\Bigg] ds\Bigg)
\end{aligned} \tag{3.16}$$

for $i = 1, ..., n$.

2 Existence and Uniqueness

Theorem 3.1 *[17] We suppose that:*

H1. *There exist constants ξ_{ij} , ζ_{ij} for $i, j = 1, ..., n$ where we'd have:*

$$|h_i(t, \chi_1, ..., \chi_n) - h_i(t, y_1, ..., y_n)| \leq \sum_{j=1}^{n} \xi_{ij}|\chi_j - y_j| \tag{3.17}$$

and

$$|k_i(t, \chi_1, ..., \chi_n) - k_i(t, y_1, ..., y_n)| \leq \sum_{j=1}^{n} \zeta_{ij}|\chi_j - y_j| \tag{3.18}$$

when all values of t are in J, and $\chi, y \in \mathbb{R}^n$.

H2. *There exist nonnegative constants $F_i, i = 1, ..., n$ such that for all $t \in J$ and $\chi(t) \in \mathbb{R}^n$ $|f_i(t, \chi(t))| \leq F_i$.*

H3. $\sum_{i=1}^{n} \left(\Phi_i \sum_{j=1}^{n} \xi_{ij} + \Psi_i \sum_{j=1}^{n} \zeta_{ij} \right) < 1$, *where:*

$$\Phi_i \quad := \quad \frac{F_i}{\Gamma(\alpha_i + 1)} + \frac{F_i^2 |\theta_i| \sup\limits_{s\in J} |\varphi_i(s)| \beta_i^{\alpha_i + 1}}{\Gamma(\alpha_i + 2)|f_i(0, \chi(0)) - \theta_i \int_0^{\beta_i} f_i(s, \chi(s))\varphi_i(s)ds|}$$

$$\Psi_i \quad := \quad \frac{F_i}{\Gamma(\alpha_i + \delta_i + 1)} + \frac{F_i^2 |\theta_i| \sup\limits_{s\in J} |\varphi_i(s)| \beta_i^{\alpha_i + \delta_i + 1}}{\Gamma(\alpha_i + \delta_i + 2)|f_i(0, \chi(0)) - \theta_i \int_0^{\beta_i} f_i(s, \chi(s))\varphi_i(s)ds|}$$

are satisfied. Then, there exists a unique solution to (3.1) provided that θ_i and $f_i(0, \chi(0))$ satisfy the condition of Lemma 3.1.

Proof

To achieve the desired results, we chose to proceed on two steps:

Step 1: Let $\mathfrak{B}_r$ be given by $\mathfrak{B}_r = \{\chi \in \prod_{i=1}^{n} X_i : \|\chi\|_{\prod_{i=1}^{n} X_i} \leq r\}$ where r is defined by:

$$r \geq \frac{\sum_{i=1}^{n} \Phi_i H_i^0 + \Psi_i K_i^0}{1 - \sum_{i=1}^{n} (\Phi_i \sum_{j=1}^{n} \xi ij + \Psi_i \sum_{j=1}^{n} \zeta_{ij})} \tag{3.19}$$

Let us have H_i and K_i which are constants given by $H_i^0 := \sup\limits_{t\in J} |h_i(t, 0, ..., 0)| < \infty$ and $K_i^0 := \sup\limits_{t\in J} |k_i(t, 0, ..., 0)| < \infty$, for $i = 1, ..., n$.

We notice that using $(\mathbf{H}1)$, for $\chi \in \mathfrak{B}_r$, we can write:

$$\begin{aligned} |h_i(t,\chi_1,...,\chi_n)| &\leq |h_i(t,\chi_1,...,\chi_n) - h_i(t,0,...,0)| + |h_i(t,0,...,0)| \\ &\leq \textstyle\sum_{j=1}^n \xi_{ij}|\chi_j| + H_i^0 \\ &\leq \textstyle\sum_{j=1}^n \xi_{ij} r + H_i^0 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} |k_i(t,\chi_1,...,\chi_n)| &\leq |k_i(t,\chi_1,...,\chi_n) - k_i(t,0,...,0)| + |k_i(t,0,...,0)| \\ &\leq \textstyle\sum_{j=1}^n \zeta_{ij}|\chi_j| + K_i^0 \\ &\leq \textstyle\sum_{j=1}^n \zeta_{ij} r + K_i^0 \end{aligned} \tag{3.21}$$

On the other hand, we have:

$$\begin{aligned} |\mathcal{Q}_i\chi(t)| \leq\ & |f_i(t,\chi(t))|\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}|h_i(\tau,\chi(\tau))|d\tau \\ &+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,\chi(\tau))|d\tau \\ &+\frac{|\theta_i|}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\int_0^{\beta_i} |f_i(s,\chi(s))||\varphi_i(s)| \\ &\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}|h_i(\tau,\chi(\tau))|d\tau \\ &+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,\chi(\tau))|d\tau\Bigg]ds\Bigg) \end{aligned} \tag{3.22}$$

So, using (**H**1), (**H**2), (3.20), and (3.21), we get:

$$\begin{aligned}|\mathcal{Q}_i\chi(t)| \;\leq\; & F_i\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau\left(\sum_{j=1}^{n}\xi_{ij}r+H_i^0\right) \\ & +\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau\left(\sum_{j=1}^{n}\zeta_{ij}r+K_i^0\right) \\ & +\frac{F_i|\theta_i|\sup_{s\in J}|\varphi_i(s)|}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i}f_i(s,\chi(s))\varphi_i(s)ds|} \\ & \times\int_0^{\beta_i}\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}d\tau\left(\sum_{j=1}^{n}\xi_{ij}r+H_i^0\right)\right. \\ & \left.+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}d\tau\left(\sum_{j=1}^{n}\zeta_{ij}r+K_i^0\right)\right]ds\Bigg)\end{aligned} \tag{3.23}$$

From this, we can easily conclude that

$$\begin{aligned}\|\mathcal{Q}_i\chi\|_{X_i} \;\leq\; & \left(\frac{F_i}{\Gamma(\alpha_i+1)}+\frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i}f_i(s,\chi(s))\varphi_i(s)ds|}\right)\left(\sum_{j=1}^{n}\xi_{ij}r+H_i^0\right) \\ & +\left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)}+\frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i}f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ & \times\left(\sum_{j=1}^{n}\zeta_{ij}r+K_i^0\right) \\ = \; & \Phi_i\left(\sum_{j=1}^{n}\xi_{ij}r+H_i^0\right)+\Psi_i\left(\sum_{j=1}^{n}\zeta_{ij}r+K_i^0\right)\end{aligned} \tag{3.24}$$

for $i=1,...,n$.

So (4.17) implies that:

$$\|\mathcal{Q}_i\chi\|_{X_i} \;\leq\; \Phi_i\left(\sum_{j=1}^{n}\xi_{ij}r+H_i^0\right)+\Psi_i\left(\sum_{j=1}^{n}\zeta_{ij}r+K_i^0\right), i=1,...,n. \tag{3.25}$$

Hence,

$$\|\mathcal{Q}_i\chi\|_{\prod_{i=1}^{n}X_i} \;\leq\; r. \tag{3.26}$$

which leads to the conclusion that $\mathcal{Q}_i(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Step 2: Let $\chi, y \in X_i$. For t with all its values in J, we have:

$$\begin{aligned} |\mathcal{Q}_i\chi(t) - \mathcal{Q}_i y(t)| \leq & \ |f_i(t,\chi(t))| \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |h_i(\tau,\chi(\tau)) - h_i(\tau, y(\tau))| d\tau \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} |k_i(\tau,\chi(\tau)) - k_i(\tau,y(\tau))| d\tau \\ & + \frac{F_i|\theta_i|}{|f_i(0,\chi(0)) - \theta_i \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|} \int_0^{\beta_i} |\varphi_i(s)| \\ & \times \Bigg[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i-1} |h_i(\tau,\chi(\tau)) - h_i(\tau,\chi(\tau))| d\tau \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^s (s-\tau)^{\alpha_i+\delta_i-1} |k_i(\tau,\chi(\tau)) - k_i(\tau,y(\tau))| d\tau \Bigg] ds \Bigg) \end{aligned} \tag{3.27}$$

Thanks to ($\mathbf{H}1$) and ($\mathbf{H}2$), we get:

$$\begin{aligned} \|\mathcal{Q}_i\chi - \mathcal{Q}_i y\|_{X_i} \leq & \ F_i \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i-1} d\tau \sum_{j=1}^n \xi_{ij} \|\chi_j - y_j\|_{X_j} \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} d\tau \sum_{j=1}^n \zeta_{ij} \|\chi_j - y_j\|_{X_j} \\ & + \frac{F_i|\theta_i|}{|f_i(0,\chi(0)) - \theta_i \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|} \int_0^{\beta_i} \sup_{s\in J} |\varphi_i(s)| \\ & \times \Bigg[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i-1} d\tau \sum_{j=1}^n \xi_{ij} \|\chi_j - y_j\|_{X_j} \\ & + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^s (s-\tau)^{\alpha_i+\delta_i-1} d\tau \sum_{j=1}^n \zeta_{ij} \|\chi_j - y_j\|_{X_j} \Bigg] ds \Bigg) \end{aligned} \tag{3.28}$$

This leads to

$$\begin{aligned}
\|\mathcal{Q}_i\chi - \mathcal{Q}_i y\|_{X_i} &\leq \left(\frac{F_i}{\Gamma(\alpha_i+1)} + \frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\
&\quad \times \left(\sum_{j=1}^{n}\xi_{ij}\|\chi_j - y_j\|_{X_j}\right) \\
&\quad + \left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)} + \frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\
&\quad \times \left(\sum_{j=1}^{n}\zeta_{ij}\|\chi_j - y_j\|_{X_j}\right) \\
&= \Phi_i\left(\sum_{j=1}^{n}\xi_{ij}\|\chi_j - y_j\|_{X_j}\right) + \Psi_i\left(\sum_{j=1}^{n}\zeta_{ij}\|\chi_j - y_j\|_{X_j}\right)
\end{aligned} \tag{3.29}$$

From (3.29), we have:

$$\|\mathcal{Q}_i\chi - \mathcal{Q}_i y\|_{X_i} \leq \left(\Phi_i\sum_{j=1}^{n}\xi_{ij} + \Psi_i\sum_{j=1}^{n}\zeta_{ij}\right) \times \left(\sum_{j=1}^{n}\|\chi_j - y_j\|_{X_j}\right) \tag{3.30}$$

for $i = 1, ..., n$.

Therefore,

$$\|\mathcal{Q}\chi - \mathcal{Q}y\|_{\sum_{i=1}^{n}X_i} \leq \sum_{i=1}^{n}\left(\Phi_i\sum_{j=1}^{n}\xi_{ij} + \Psi_i\sum_{j=1}^{n}\zeta_{ij}\right) \times \left(\sum_{j=1}^{n}\|\chi_j - y_j\|_{X_j}\right) \tag{3.31}$$

Since (**H**3) assures that $\sum_{i=1}^{n}\left(\Phi_i\sum_{j=1}^{n}\xi_{ij} + \Psi_i\sum_{j=1}^{n}\zeta_{ij}\right) < 1$, then the operator $\mathcal{Q}$ is contractive. According to Banach contraction principle, the system (3.1) has a unique solution on $[0, 1]$.

3 Existence : First Approach

Let us introduce the following hypotheses:

H4. For $i = 1, ..., n$, the functions f_i, h_i, and k_i are continuous in J $\times \mathbb{R}^n$.

H5.There exist nonnegative constants H_i and K_i, for $i = 1, ..., n$, such that for all $t \in J$ and $\chi \in \mathbb{R}^n$, $|h_i(t, \chi)| \leq H_i$ and $|k_i(t, \chi)| \leq K_i$.

H6. There exist constants γ_{ij}, for $i, j = 1, ..., n$, when for all $t \in J$ and $\chi, y \in \mathbb{R}^n$, we have:

$$|f_i(t, \chi) - f_i(t, y)| \leq \sum_{j=1}^{n} \gamma_{ij} |\chi_j - y_j|.$$

Theorem 3.2 *[18] Under the hypotheses* (**H**2), (**H**4) *and* (**H**5), *the problem* (3.1) *has at least one solution.*

Proof.

We use Schaefer fixed point theorem to prove that (3.1) has at least a solution. To do that, we go through the following steps:

Step.1 : We show that $\mathcal{Q}$ is a continuous operator.

Due to the fact that f_i, h_i and k_i are continuous functions on $J \times \mathbb{R}^n$, then $\mathcal{Q}_i$ is a continuous operator for $i = 1..., n$. As a consequence, $\mathcal{Q}$ is continuous.

Step.2 : We make sure that the operator $\mathcal{Q}$ maps a bounded set into another bounded set.

Let $\mathfrak{B}_r$ be given by $\mathfrak{B}_r = \{\chi \in \prod_{i=1}^{n} X_i : \|\chi\|_{\prod_{i=1}^{n} X_i} \leq r\}$. Therefore, for $\chi \in \mathfrak{B}_r$, for $t \in J$, we have:

$$\begin{aligned}
|\mathcal{Q}_i \chi(t)| \leq\ & |f_i(t, \chi(t))| \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |h_i(\tau, \chi(\tau))| d\tau \\
& + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^t (t-\tau)^{\alpha_i + \delta_i - 1} |k_i(\tau, \chi(\tau))| d\tau \\
& + \frac{|\theta_i|}{|f_i(0, \chi(0)) - \theta_i \int_0^{\beta_i} f_i(s, \chi(s)) \varphi_i(s) ds|} \int_0^{\beta_i} |f_i(s, \chi(s))| |\varphi_i(s)| \\
& \times \left[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i - 1} |h_i(\tau, \chi(\tau))| d\tau \right. \\
& \left. + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^s (s-\tau)^{\alpha_i + \delta_i - 1} |k_i(\tau, \chi(\tau))| d\tau \right] ds \Bigg)
\end{aligned} \tag{3.32}$$

Thanks to (**H**2) and (**H**5), we can write:

$$\begin{aligned}|\mathcal{Q}_i\chi(t)| &\leq F_i\Bigg(\frac{H_i}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau + \frac{K_i}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau \\ &+\frac{F_i|\theta_i|}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\int_0^{\beta_i}|\varphi_i(s)| \\ &\times\left[\frac{H_i}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}d\tau + \frac{K_i}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}d\tau\right]ds\Bigg),\end{aligned} \tag{3.33}$$

The above step leads to having the following estimation

$$\begin{aligned}\|\mathcal{Q}_i\chi\|_{X_i} &\leq H_i\left(\frac{F_i}{\Gamma(\alpha_i+1)} + \frac{F_i^2|\theta_i|\beta_i^{\alpha_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ &+K_i\left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)} + \frac{F_i^2|\theta_i|\beta_i^{\alpha_i+\delta_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ &:= H_i\Phi_i + K_i\Psi_i < +\infty\end{aligned} \tag{3.34}$$

for $i = 1, ..., n$.
From (3.34), we can deduce that

$$\|\mathcal{Q}_i\chi\|_{\prod_{i=1}^n X_i} \leq \sum_{i=1}^n H_i\Phi_i + K_i\Psi_i < +\infty.$$

Consequently, $\mathcal{Q}$ is bounded.

Step.3: We prove that $\mathcal{Q}$ maps bounded sets into equicontinuous sets.
Let $t_1, t_2 \in J$, such that $t_1 \leq t_2$. We have:

$$
\begin{aligned}
|Q_i\chi(t_2) - Q_i\chi(t_1)| \quad = \quad & \Bigg| f_i(t_2, \chi(t_2))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}(t_2-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{t_2}(t_2-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau \\
& + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds}\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s) \\
& \times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^{s}(s-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{s}(s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg) \\
& - f_i(t_1, \chi(t_1))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}(t_1-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{t_1}(t_1-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau \\
& + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds}\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s) \\
& \times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^{s}(s-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{s}(s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\Bigg|
\end{aligned}
\tag{3.35}
$$

Therefore, we get

$$
\begin{aligned}
|\mathcal{Q}_i\chi(t_2) - \mathcal{Q}_i\chi(t_1)| \quad &\leq \quad \max(|f_i(t_2,\chi(t_2))|, |f_i(t_1,\chi(t_1))|)\Bigg|\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}(t_2-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{t_2}(t_2-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau \\
&-\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}(t_1-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^{t_1}(t_1-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau\Bigg)\Bigg|
\end{aligned}
$$

Using some easy integral calculations and manipulations, we obtain:

$$
\begin{aligned}
|\mathcal{Q}_i\chi(t_2) - \mathcal{Q}_i\chi(t_1)| \quad &\leq \quad \max(|f_i(t_2,\chi(t_2))|, |f_i(t_1,\chi(t_1))|)\Big(\frac{1}{\Gamma(\alpha_i+1)}\Big(t_2^{\alpha_i}-t_1^{\alpha_i}\Big)H_i \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i+1})\Big(t_2^{\alpha_i}-t_1^{\alpha_i}\Big)K_i\Big)
\end{aligned}
$$

We can see that when $t_1 \to t_2$, it follows as a natural consequence that $|\mathcal{Q}_i\chi(t_2) - \mathcal{Q}_i\chi(t_1)| \longrightarrow 0$, for all $i = 1, ..., n$. From this, we can say that $\mathcal{Q}$ is equicontinuous. Thanks to the steps $1, 2$ and 3, and according to Arzela-Ascoli theorem, the operator $\mathcal{Q}$ is completely continuous.

Step.4: Now, what we shall do is to prove that the set

$$\varDelta = \{\chi \in \prod_{i=1}^{n} X_i : \chi = \lambda\mathcal{Q}(\chi), \lambda \in]0;1[\} \tag{3.36}$$

is bounded.
For $i = 1, ..., n$ and $t \in J$, we have

$$|\chi_i(t)| \quad = \quad |\lambda||\mathcal{Q}_i\chi(t)| \tag{3.37}$$

Taking in consideration that λ has her values in $]0,1[$, then, we can find that

$$\begin{aligned}|\chi_i(t)| &\leq |\mathcal{Q}_i\chi(t)| \\ &\leq H_i\left(\frac{F_i}{\Gamma(\alpha_i+1)}+\frac{F_i^2|\theta_i|\beta_i^{\alpha_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i}f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ &+K_i\left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)}+\frac{F_i^2|\theta_i|\beta_i^{\alpha_i+\delta_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i}f_i(s,\chi(s))\varphi_i(s)ds|}\right)\end{aligned}$$

and so, for all $i=1,...,n$, we have:

$$\|\chi_i\|_{X_i} \leq H_i\Phi_i+K_i\Psi_i<+\infty \tag{3.38}$$

which implies that

$$\|\chi\|_{\prod_{i=1}^n X_i} \leq \sum_{i=1}^n H_i\Phi_i+K_i\Psi_i<+\infty \tag{3.39}$$

We conclude that Δ is a bounded set.
Thanks to steps (1) to (4), and according to Schaefer fixed point theorem, (3.1) has at least one solution χ, for $t\in J$.

■

4 Existence: Second Approach

Theorem 3.3 *[18] Let's suppose that the hypotheses* (**H**2), (**H**4), (**H**5) *and* (**H**6) *are satisfied.*

If $\gamma M<1$; *where:* $\gamma=\sum_{i=1}^n\sum_{j=1}^n\gamma_{ij}$, *and* $M=\sum_{i=1}^n\frac{1}{F_i}(H_i\Phi_i+K_i\Psi_i)$, *then,* (3.1) *has at least one solution* χ *over* J.

Proof.

let $S=\{\chi\in\prod_{i=1}^n X_i:\|\chi\|_{\prod_{i=1}^n X_i}\leq R\}$ be a subset of $\prod_{i=1}^n X_i$, where R is given by:

$$R=\frac{F^0M}{1-\gamma M}, \tag{3.40}$$

such that, $F^0=\sum_{i=1}^n\sup\limits_{s\in J}|f_i(t,0,...,0)|$.

We define the operators $A:\prod_{i=1}^n X_i\longrightarrow\prod_{i=1}^n X_i$ and $B:S\longrightarrow\prod_{i=1}^n X_i$ by:

$$A\chi(t)=(A_1\chi(t),A_2\chi(t),...,A_n\chi(t))$$

and,

$$B\chi(t) = (B_1\chi(t), B_2\chi(t), ..., B_n\chi(t)),$$

where, for i having the values from 1 to n, we have

$$\begin{aligned}
A_i\chi(t) &= f_i(t, \chi(t)) \\
B_i\chi(t) &= \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} h_i(\tau, \chi(\tau)) d\tau + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^t (t-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, \chi(\tau)) d\tau \\
&+ \frac{\theta_i}{f_i(0, \chi(0)) - \theta_i \int_0^{\beta_i} f_i(s, \chi(s)) \varphi_i(s) ds} \int_0^{\beta_i} f_i(s, \chi(s)) \varphi_i(s) \\
&\left[\frac{1}{\Gamma(\alpha_i)} \int_0^s (s-\tau)^{\alpha_i - 1} h_i(\tau, \chi(\tau)) d\tau + \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^s (s-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, \chi(\tau)) d\tau \right] ds
\end{aligned}$$

We proceed the proof through the following steps:

Step.1: We show that the operator A is lipschitzian operator, with the lipschitz constant γ.

Let $\chi, y \in \prod_{i=1}^n X_i$. For $t \in J$, using (**H**6.), we have :

$$\begin{aligned}
|A_i\chi(t) - A_iy(t)| &= |f_i(t, \chi(t)) - f_i(t, y(t))| \\
&\leq \sum_{j=1}^n \gamma_{ij} |\chi_j(t) - y_j(t)|
\end{aligned}$$

which leads to

$$\|A_i\chi - A_iy\|_{X_i} \leq \sum_{j=1}^n \gamma_{ij} \|\chi_j - y_j\|_{X_j} \tag{3.41}$$

for $i = 1, ..., n$.

This implies that

$$\|A\chi - Ay\|_{\prod_{i=1}^n X_i} \leq \gamma \|\chi - y\|_{\prod_{i=1}^n X_i}, \tag{3.42}$$

where, $\gamma = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij}$.

Hence, A is a lipschizian operator.

Step.2: We show that B is a completely continuous operator.

To satisfy that, we just need to prove that B is: a) continuous, b) bounded, and c) equicontinuous operator.

a)- According to the second hypothesis, it is clear that the operator B is a continuous operator.

b)- For $\chi \in S$, for all $t \in J$, we have

$$\begin{aligned}|B_i\chi(t)| \quad \leq \quad & \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}|h_i(\tau,\chi(\tau))|d\tau \\ & +\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,\chi(\tau))|d\tau \\ & +\frac{|\theta_i|}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\int_0^{\beta_i}|f_i(s,\chi(s))||\varphi_i(s)| \\ & \left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}|h_i(\tau,\chi(\tau))|d\tau\right. \\ & \left.+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,\chi(\tau))|d\tau\right] ds\end{aligned} \tag{3.43}$$

Then, thanks to (**H**2.) and (**H**5.), (3.43) can be written as:

$$\begin{aligned}|B_i\chi(t)| \quad \leq \quad & \frac{H_i}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau+\frac{K_i}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau \\ & +\frac{|\theta_i|F_i}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\int_0^{\beta_i}\sup_{s\in J}|\varphi_i(s)| \\ & \left[\frac{H_i}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}d\tau+\frac{K_i}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}d\tau\right] ds\end{aligned} \tag{3.44}$$

which leads to

$$\begin{aligned}\|B_i\chi\|_{X_i} \quad \leq \quad & \frac{H_i}{\Gamma(\alpha_i+1)}+\frac{H_i|\theta_i|F_i\beta_i^{\alpha_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|} \\ & +\frac{K_i}{\Gamma(\alpha_i+\delta_i+1)}+\frac{K_i|\theta_i|F_i\beta_i^{\alpha_i+\delta_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|} \\ \leq \quad & \frac{1}{F_i}\left(H_i\Phi_i+K_i\Psi_i\right) \\ := \quad & M_i<+\infty\end{aligned} \tag{3.45}$$

and that for $i=1,...,n$.

Therefore,

$$\|B_i\chi\|_{\prod_{i=1}^{n} X_i} = M < +\infty, \tag{3.46}$$

where, $M = \sum_{i=1}^{n} Mi$.

Then B is bounded.

c)- Let $t_1, t_2 \in J$ with $t_1 \leq t_2$. Then:

$$\begin{aligned}
|B_i\chi(t_2) - B_i\chi(t_1)| &= \Bigg| \frac{1}{\Gamma(\alpha_i)} \int_0^{t_2} (t_2-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&- \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} (t_1-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&+ \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^{t_2} (t_2-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau \\
&- \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^{t_1} (t_1-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau \Bigg|.
\end{aligned}$$

As a consequence, we can obtain

$$\begin{aligned}
|B_i\chi(t_2) - B_i\chi(t_1)| &\leq \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} (t_2-\tau)^{\alpha_i-1} d\tau - \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} (t_1-\tau)^{\alpha_i-1} d\tau \\
&+ \frac{1}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha_i-1} d\tau \Bigg) H_i \\
&\Bigg(\frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^{t_1} (t_2-\tau)^{\alpha_i+\delta_i-1} d\tau - \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^{t_1} (t_1-\tau)^{\alpha_i+\delta_i-1} d\tau \\
&\frac{1}{\Gamma(\alpha_i+\delta_i)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha_i+\delta_i-1} d\tau \Bigg) K_i.
\end{aligned} \tag{3.47}$$

Logically speaking, through some calculations, we easily get

$$\begin{aligned}
|B_i\chi(t_2) - B_i\chi(t_1)| &\leq \frac{1}{\Gamma(\alpha_i+1)} \left(t_2^{\alpha_i} - t_1^{\alpha_i} \right) H_i \\
&\frac{1}{\Gamma(\alpha_i+\delta_i+1)} \left(t_2^{\alpha_i+\delta_i} - t_1^{\alpha_i+\delta_i} \right) K_i
\end{aligned} \tag{3.48}$$

Accordingly, when $t_1 \longrightarrow t_2$, we get that $|B_i\chi(t_2) - B_i\chi(t_1)| \longrightarrow 0$, for $i = 1, ..., n$, for $\chi \in S$. Therefore, B is equicontinuous.

With a), b), and c) and thanks to Arzela-Ascoli theorem, we can state that B is completely continuous.

Step.3: We show that if $\chi = A\chi By$ with $y \in S$, then we have χ is in S.
Let $\chi \in \prod_{i=1}^{n} X_i$ and $y \in S$. So, for $i = 1, ..., n$, $t \in J$, the following formula takes place :

$$\begin{aligned}
|\chi_i(t)| &= |A_i(\chi(t))||B_i(y(t))| \\
&= |f_i(t,\chi(t))|\Bigg|\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,y(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,y(\tau))d\tau \\
&+\frac{\theta_i}{f_i(0,y(0))-\theta_i\int_0^{\beta_i} f_i(s,y(s))\varphi_i(s)ds}\int_0^{\beta_i} f_i(s,y(s))\varphi_i(s) \\
&\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}h_i(\tau,y(\tau))d\tau + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,y(\tau))d\tau\right] ds\Bigg|
\end{aligned} \tag{3.49}$$

We get

$$\begin{aligned}
|\chi(t)| \leq &\Big(|f_i(t,\chi(t)) - f_i(t,0,...,0)| + |f_i(t,0,...,0)|\Big) \\
&\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}|h_i(\tau,y(\tau))|d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,y(\tau))|d\tau \\
&+\frac{|\theta_i|}{|f_i(0,y(0))-\theta_i\int_0^{\beta_i} f_i(s,y(s))\varphi_i(s)ds|}\int_0^{\beta_i} |f_i(s,y(s))||\varphi_i(s)| \\
&\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}|h_i(\tau,y(\tau))|d\tau \right.\\
&\left.+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,y(\tau))|d\tau\right] ds\Bigg]
\end{aligned}$$

Noting $F_i^0 = |f_i(t, 0, ..., 0)|$, and thanks to ($\mathbf{H}6.$), it follows that

$$\begin{aligned}
\|\chi(t)\|_{X_i} \leq & \left(\sum_{j=1}^{n} \gamma_{ij}\|\chi_j\|_{X_j} + F_i^0\right)\Bigg(\frac{H_i}{\Gamma(\alpha_i+1)} \\
& + \frac{H_i|\theta_i|F_i\beta_i^{\alpha_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+2)|f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|} \\
& + \frac{K_i}{\Gamma(\alpha_i+\delta_i+1)} + \frac{K_i|\theta_i|F_i\beta_i^{\alpha_i+\delta_i+1}\sup\limits_{s\in J}|\varphi_i(s)|}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\Bigg) \\
\leq & \left(\sum_{j=1}^{n} \gamma_{ij}\|\chi_j\|_{X_j} + F_i^0\right)M_i
\end{aligned} \tag{3.50}$$

Therefore, we observe that

$$\|\chi\|_{\prod_{i=1}^n X_i} \leq (\gamma\|\chi\|_{\prod_{i=1}^n X_i} + F^0)M. \tag{3.51}$$

That is to say that

$$\|\chi\|_{\prod_{i=1}^n X_i} \leq \frac{F^0 M}{1-\gamma M} = R. \tag{3.52}$$

We deduce that χ is in S.

Since $M = \|B(S)\|_{\prod_{i=1}^n X_i}$ and $\gamma M < 1$, then the 4th hypothesis of Lemma 4 is also fulfilled.

Since all conditions are satisfied, so according to Lemma 4, the problem (3.1) has at least one solution. ■

5 Illustrative Examples

First approach:

For $t \in [0,1]$, let us have the following system:

$$\begin{cases} \mathbf{D}^{1/2}\left(\chi_1(t)\left(\frac{1+|\chi_1(t)|+|\chi_2(t)|}{t|\chi_1(t)|+|\chi_2(t)|+(1+|\chi_1(t)|+|\chi_2(t)|)sin(2|\chi_3(t)|)}\right)\right) = \dfrac{2t-1}{24-sin(|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|)} \\ \qquad +I^{1/12}\dfrac{cos(|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|)}{e^{3t}}, \\ \mathbf{D}^{3/4}\left(\chi_2(t)\left(\frac{3(1+sin(|\chi_2(t)|+|\chi_3(t)|))}{(t^2-3cos|\chi_2(t)|)(1+sin(|\chi_2(t)|+|\chi_3(t)|))+3sin|\chi_2(t)|}\right)\right) = \dfrac{t+10+cos(|\chi_1(t)|+|\chi_2(t)|)}{2t^2+1} \\ \qquad -\dfrac{sin|\chi_3(t)|}{2t^2+1}+I^{3/4}\dfrac{1}{25}(24t^2-1)sin(|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|), \\ \mathbf{D}^{2/3}\left(\chi_3(t)\left(\frac{t^2+4t+1}{(t^2+4t+1)sin|\chi_1(t)|-sin(|\chi_2(t)|+|\chi_3(t)|)}\right)\right) = \dfrac{|\chi_1(t)|+|\chi_2(t)|-|\chi_3(t)|}{3(1+|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|)} \\ \qquad +I^{4/7}(\dfrac{t}{12}+1)cos(|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|), \\ \chi_1(0) = \sqrt{2}\displaystyle\int_0^{\pi/4} sin(s))\chi_1(s)ds, \\ \chi_2(0) = 3\displaystyle\int_0^{1/2} e^s)\chi_2(s)ds \\ \chi_3(0) = 12\displaystyle\int_0^{3/4} (s+1)\chi_3(s)ds \end{cases} \tag{3.53}$$

We have that f_i are given by

$$f_1(t,\chi_1(t),\chi_2(t),\chi_3(t)) = \frac{t|\chi_1(t)|+|\chi_2(t)|}{1+|\chi_1(t)|+|\chi_2(t)|}+sin(2|\chi_3(t)|), \tag{3.54}$$

$$f_2(t,\chi_1(t),\chi_2(t),\chi_3(t)) = \frac{t^2}{3}-cos|\chi_1(t)|+\frac{sin|\chi_2(t)|}{1+sin(|\chi_2(t)|+|\chi_3(t)|)}, \tag{3.55}$$

$$f_3(t,\chi_1(t),\chi_2(t),\chi_3(t)) = sin|\chi_1(t)|-\frac{sin(|\chi_2(t)|+|\chi_3(t)|)}{t^2+4t+1}, \tag{3.56}$$

and for the h_i,

$$h_1(t,\chi_1(t),\chi_2(t),\chi_3(t)) = \frac{2t-1}{24-sin(|\chi_1(t)|+|\chi_2(t)|+|\chi_3(t)|)}, \tag{3.57}$$

$$h_2(t,\chi_1(t),\chi_2(t),\chi_3(t)) = \frac{t+10+cos(|\chi_1(t)|+|\chi_2(t)|)-sin|\chi_3(t)|}{2t^2+1}, \tag{3.58}$$

$$h_3(t, \chi_1(t), \chi_2(t), \chi_3(t)) = \frac{|\chi_1(t)| + |\chi_2(t)| - |\chi_3(t)|}{3(1 + |\chi_1(t)| + |\chi_2(t)| + |\chi_3(t)|)}, \tag{3.59}$$

and for k_i we have:

$$k_1(t, \chi_1(t), \chi_2(t), \chi_3(t)) = \frac{cos(|\chi_1(t)| + |\chi_2(t)| + |\chi_3(t)|)}{e^{3t}}, \tag{3.60}$$

$$k_2(t, \chi_1(t), \chi_2(t), \chi_3(t)) = \frac{1}{25}(24t^2 - 1)sin(|\chi_1(t)| + |\chi_2(t)| + |\chi_3(t)|), \tag{3.61}$$

$$k_3(t, \chi_1(t), \chi_2(t), \chi_3(t)) = (\frac{t}{12} + 1)cos(|\chi_1(t)| + |\chi_2(t)| + |\chi_3(t)|), \tag{3.62}$$

It is clear that f_i, h_i, and k_i are continuous functions on $J = [0, 1]$. For the hypothesis (H.4), it is also fulfilled since we have

$$|f_1(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq 2 := F_1 \tag{3.63}$$

$$|f_2(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq \frac{4}{3} := F_2 \tag{3.64}$$

$$|f_3(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq 2 := F_3 \tag{3.65}$$

and

$$|h_1(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq \frac{3}{23} := H_1 \tag{3.66}$$

$$|h_2(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq 12 := H_2 \tag{3.67}$$

$$|h_3(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq \frac{1}{3} := H_3 \tag{3.68}$$

and

$$|k_1(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq 1 := K_1 \tag{3.69}$$

$$|k_2(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq \frac{23}{25} := K_2 \tag{3.70}$$

$$|k_3(t, \chi_1(t), \chi_2(t), \chi_3(t))| \leq \frac{13}{12} := K_3 \tag{3.71}$$

Since, the hypothesis (**H**2), (**H**4), and (**H**5) are satisfied, then, as a direct consequence, we can assume that by the means of Theorem 3.2 the system has at least one solution.

Chapter 4

Stability of a System of Fractional Hybrid Differential Equations

1 About Ulam-Hyers Stability

Ulam-Hyers stability is a concept that was started around 1940s in a Mathematical Colloquium at the University of Wisconsin. It was the result of answering a question that was put out by S. M. Ulam about the stability of homomorphisms. [29]

A year later, around 1941, D. H. Hyers presented an answer to Ulam's question by his results about the stability of functional equations in the case where G_1 and G_2 are assumed to be Banach spaces. [29]

Theorem 4.1 (Hyers) [26]

Let $f : E_1 \to E_2$ be a function between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{4.1}$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{4.2}$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta \tag{4.3}$$

for every $x \in E_1$.

"From this result, the additive Cauchy equation $f(x+y) = f(x)+f(y)$ is said to have the *Ulam-Hyers stability* on (E_1, E_2) if for every function $f : E_1 \to E_2$ satisfying the inequality (4.1) for some $\delta \geq 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1." [29]

Based on this method, many works have been done and results have been found that address the stability of functional equations. This extension has also covered the differential equations with arbitrary order. It is the most used method to show that a fractional differential equation is stable according to Ulam-Hyers theorem. We can see many interesting works in [27], [29],[30],[31],[32][61], and so many others.

2 Stability of a System of Fractional Hybrid Differential Equations

The classical definition of Ulam-Hyers stability is always adjusted to the given problem. To be more clear, we give the following example:

Let us assume that we have the following problem:

$$\begin{cases} {}^cD^\alpha v(t) &= f(t, v(t)), \qquad 0<\alpha<1, t\in J_1=[0,T] \\ v(t_0) &= v_0 \end{cases} \tag{4.4}$$

where ${}^cD^\alpha$ is the Caputo derivative of order α, $0<\alpha<1$, $t\in J_1=[0,T]$, $f\in C(J_1\times\mathbb{R},\mathbb{R})$, and $v_0\in\mathbb{R}$.

We consider that the solution of the problem (4.4) is in $C(J_1,\mathbb{R})$.

Definition 4.1 *We say that the equation in (4.4) is Ulam-Hyers stable if there exists a positive constant C $(C>0)$ such that for all $\varepsilon>0$, and for every solution $v\in C(J_1,\mathbb{R})$ that satisfies:*

$$|{}^cD^\alpha v(t) - f(t,v(t))| \leq \varepsilon, \quad 0<\alpha<1, t\in J_1=[0,T] \tag{4.5}$$

there exists a solution $v* \in C(J_1,\mathbb{R})$ that satisfies:

$$|{}^cD^\alpha v^*(t) - f(t,v^*(t))| = 0, \quad 0<\alpha<1, t\in J_1=[0,T] v^*(0)=v_0, \tag{4.6}$$

such that

$$\|v-v^*\| \leq C\varepsilon, \quad 0<\alpha<1, t\in J_1=[0,T]. \tag{4.7}$$

where $\|.\|$ represents the infinity norm.

Since hybrid differential equations are not conventional equations in their form, an adjustment needed to take place in the definition. We see this as well in the work of [32]. We give the following customized definition:

Definition 4.2 *[18] For $i = 1, ..., n$, the problem (3.1) is Hyers-Ulam stable if there exists a positive constant $\mathcal{C}$, such that for every $\varepsilon_i > 0$, if:*

$$\Bigg|\chi_i(t) - f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds} \times \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}h_i(\tau,\chi(\tau))d\tau + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi(\tau))d\tau\right]ds\Bigg)\Bigg| \le \varepsilon_i, \tag{4.8}$$

then, there exists $\chi^* \in \prod_{i=1}^n X_i$, for $t \in J$, satisfying:

$$\chi_i^*(t) = f_i(t,\chi^*(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,\chi^*(\tau))d\tau + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi^*(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi^*(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi^*(s))\varphi_i(s)ds} \times \int_0^{\beta_i} f_i(s,\chi^*(s))\varphi_i(s)\left[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}h_i(\tau,\chi^*(\tau))d\tau + \frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,\chi^*(\tau))d\tau\right]ds\Bigg) \tag{4.9}$$

for $i = 1, ..., n$, such that

$$\|\chi - \chi^*\|_{\sum_{i=0}^n X_i} \le \mathcal{C}\varepsilon \tag{4.10}$$

where, $\varepsilon = \sum_{i=1}^n \varepsilon_i$.

Theorem 4.2 *[18] Under the hypothesis of Theorem 3.1, the problem (3.1) is Ulam-Hyers stable.*

Proof. Let $\varepsilon_i > 0$ and χ_i that satisfy (4.8), for all $i = 1, ..., n$ and $t \in J$. Let also $\chi^* \in \prod_{i=1}^n X_i$ solutions of (4.9), for all $i = 1, ..., n$ and $t \in J$. We mention that Theorem 3.1 establishes conditions for the existence and uniqueness of the solution χ^* for(3.1).

So, let's consider the following mathematical statement:

$$\Bigg|\chi_i(t) - \chi_i^*(t) + \chi_i^*(t) - f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau$$

$$+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds}$$

$$\times\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau$$

$$+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\Bigg| \le \varepsilon_i$$

which implies:

$$|\chi_i(t) - \chi_i^*(t)| - \Bigg| - \chi_i^*(t) + f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau$$

$$+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds}$$

$$\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau$$

$$+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\Bigg| \le \varepsilon_i \tag{4.11}$$

From this, we can write the above inequality in the following form:

$$|\chi_i(t) - \chi_i^*(t)| \le \varepsilon_i + Y, \tag{4.12}$$

where:

$$
\begin{aligned}
Y = \Bigg| &- \chi_i^*(t) + f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds} \\
&\times \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\Bigg|.
\end{aligned}
\tag{4.13}
$$

In order to keep the process of reasoning as clear as possible, we work on Y separately. Then, we finish the steps of the proof. For this reason, we have

$$
\begin{aligned}
Y = \Bigg| &- \chi_i^*(t) + f_i(t,\chi(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau + \frac{\theta_i}{f_i(0,\chi(0)) - \theta_i \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds} \\
&\times \int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1} h_i(\tau,\chi(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1} k_i(\tau,\chi(\tau))d\tau\Bigg]ds\Bigg)\Bigg|
\end{aligned}
\tag{4.14}
$$

for $i = 1, ..., n$ and $t \in J$. We replace $\chi_i^*(t)$ by its value in (4.9) and by using (**H**1.)

and (**H**2.) and some calculations, we get:

$$\begin{aligned} Y &\leq F_i\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau \sum_{j=1}^n \xi_{ij}\|\chi_j-\chi_j^*\|_{X_j} \\ &+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau \sum_{j=1}^n \zeta_{ij}\|\chi_j-\chi_j^*\|_{X_j} \\ &+\frac{F_i|\theta_i|}{|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\int_0^{\beta_i}\sup_{s\in J}|\varphi_i(s)| \\ &\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\tau)^{\alpha_i-1}d\tau \sum_{j=1}^n \xi_{ij}\|\chi_j-\chi_j^*\|_{X_j} \\ &+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^s (s-\tau)^{\alpha_i+\delta_i-1}d\tau \sum_{j=1}^n \zeta_{ij}\|\chi_j-\chi_j^*\|_{X_j}\Bigg]ds\Bigg) \end{aligned} \tag{4.15}$$

As a direct consequence, we have

$$\begin{aligned} Y &\leq \left(\frac{F_i}{\Gamma(\alpha_i+1)}+\frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ &\times\left(\sum_{j=1}^n \xi_{ij}\|\chi_j-\chi_j^*\|_{X_j}\right) \\ &+\left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)}+\frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,\chi(0))-\theta_i\int_0^{\beta_i} f_i(s,\chi(s))\varphi_i(s)ds|}\right) \\ &\times\left(\sum_{j=1}^n \zeta_{ij}\|\chi_j-\chi_j^*\|_{X_j}\right) \\ &= \Phi_i\left(\sum_{j=1}^n \xi_{ij}\|\chi_j-\chi_j^*\|_{X_j}\right)+\Psi_i\left(\sum_{j=1}^n \zeta_{ij}\|\chi_j-\chi_j^*\|_{X_i}\right) \end{aligned} \tag{4.16}$$

Thanks to (4.16), we observe that:

$$Y \leq \left(\Phi_i\sum_{j=1}^n \xi_{ij}+\Psi_i\sum_{j=1}^n \zeta_{ij}\right)\times\left(\sum_{j=1}^n \|\chi_j-\chi_j^*\|_{X_j}\right), \tag{4.17}$$

for $i=1,...,n$.

Replacing Y by its value in the inequality (4.12), we get

$$\|\chi_i - \chi_i^*\|_{X_i} \leq \varepsilon_i + \left(\Phi_i \sum_{j=1}^{n} \xi_{ij} + \Psi_i \sum_{j=1}^{n} \zeta_{ij} \right) \times \left(\sum_{j=1}^{n} \|\chi_j - \chi_j^*\|_{X_j} \right). \tag{4.18}$$

By mathematical means, we get

$$\sum_{i=1}^{n} \|\chi_i - \chi_i^*\|_{X_i} \leq \sum_{i=1}^{n} \left(\varepsilon_i + \left(\Phi_i \sum_{j=1}^{n} \xi_{ij} + \Psi_i \sum_{j=1}^{n} \zeta_{ij} \right) \times \left(\sum_{j=1}^{n} \|\chi_j - \chi_j^*\|_{X_j} \right) \right). \tag{4.19}$$

Hence, we have

$$\begin{aligned} \|\chi - \chi^*\|_{\prod_{i=1}^{n} X_i} &\leq \frac{\varepsilon}{1 - \sum_{i=1}^{n} \left(\Phi_i \sum_{j=1}^{n} \xi_{ij} + \Psi_i \sum_{j=1}^{n} \zeta_{ij} \right)} \\ &:= \mathcal{C}\varepsilon. \end{aligned} \tag{4.20}$$

Considering $\mathcal{C} = \frac{1}{1 - \sum_{i=1}^{n} \left(\Phi_i \sum_{j=1}^{n} \xi_{ij} + \Psi_i \sum_{j=1}^{n} \zeta_{ij} \right)}$, such that (**H**3.) is verified, the problem (3.1) is Ulam-Hyers stable. ■

Conclusion

In this thesis, we have seen the concept of applications of integral inequalities to a certain class of hybrid differential equations of fractional order.

First, we have explored some of the basics of fractional calculus by addressing different approaches of integration and differentiation. We have also highlighted some necessary tools for the logical progress of each chapter.

Then, we have seen some generalizations of certain theorems based on Gronwall-Ballman type inequalities for the purpose of adjusting them to hybrid differential equations. Moreover, we have applied those generalizations to a certain class of hybrid differential equations with Hdamard derivative.

After that, we have presented a boundary value problem that consists of n-fractional hybrid differential equations with nonlocal conditions. The existence and uniqueness of solution of the system has been addressed. In addition to that, we have presented two different approaches that lead to the existence of one solution at least.

Finally, the stability of the studied system has been addressed. According to Ulam-Hyers stability theorem, the boundary value problem with nonlocal conditions is stable.

As for future perspectives, this work opens some new possibilities for us to explore. For instance, the achieved approximations through applying Gronwall-Ballman type inequalities, is there a possibility for them to be useful in some numerical methods? Does these approximations have a meaning in a more concrete way? There are many aspects that still need to be investigated, hopefully for the next coming works.

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A note from the author:

Dear reader,

Since this is an academic work, it didn't seem appropriate to share feelings and personal thoughts on any part of this work so I decided a side note would be a better idea.

First, congratulations to you dear reader. You must be in a certain level of Master's degree or PhD degree yourself to give this modest work a look. Or you might be a researcher. Whatever it is, you did a great job so far so allow yourself a moment to appreciate the efforts you made to reach where you are right now.

The path of academics is a bit rough to walk through. You need to have courage, discipline and tenacity to keep going.

You were chosen, not by an exam or by people or by your own effort. No, my dear reader. You were chosen by Allah Almighty for a mission only he knows its horizons. Out of couple of hundreds of people, you were selected for this task. So when days get darker, and the nights so full of shadows and fear and worry, when you question if you are on the right path, or if you chose well, remember you chose but you also been chosen. You made the decision and the decision was made for you as well.

The point of all of this is: you have something unique to offer that no other person has. I wish you'd keep that in mind as you go through the sea of articles, original results, and the waves of new works that didn't even cross your mind. When you are overwhelmed, remember: you are looking at the sea but your are not seeing the drops of water that made the sea. And you yourself is part of the sea and you matter to the sea and you are important to the sea regardless of how you see yourself and your work. So do your best, be yourself, and let what's meant to happen flow through you and let your drops of water find their way to enlarge the sea that they came from and which they always belonged to.

Take care of your health, your prayers, and your loved ones. That's what truly matter. And make everything else you do a part of your prayer because prayer comes from the heart regardless what form in takes as an action.

Fi aman Allah wa hifdhih.