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Stability and Stabilization of Positive Infinite-Dimensional Systems

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T H E S I S

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Contents

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Notations

$\times$	:	Inner product.
$\oplus$	:	Direct sum.
$\in$	:	Belongs to.
$\subset$	:	Subset.
$\cap$	:	Intersection.
$\lvert . \rvert$	:	Absolute value.
$< .,. >$	:	Scalar product.
$p \Longleftrightarrow q$	:	Statements p and q are equivalent.
$\mathbb{N}^*$	:	Set of natural numbers starting from 1, i.e., $\{1, 2, 3, \cdots\}$.
$\mathbb{R}$	:	Set of real numbers.
$\mathbb{C}$	:	Set of complex numbers.
F	:	Field of real or complex spaces.
$\mathbb{R}^n$	:	Space of n-dimensional real vectors.
$\mathbb{R}^{n\times m}$	:	Space of $n \times m$ real matrices.
$\mathrm{Re}(z)$	:	Real part of the complex number z.
I_d	:	Identity matrix.
$\mathrm{rank}(\mathrm{A})$	:	Rank of the matrix A.
$\det(\mathrm{A})$	:	Determinant of the matrix A.
deg	:	Degree of a polynomial.
C_0	:	Strongly continuous semi-group.
$\mathrm{S}(t)$	:	Semi-group.
X	:	Hilbert space.
X^*	:	Dual of a Hilbert space X.
$\mathrm{X}^{\perp}$	:	Orthogonal of a Hilbert space X.
$\mathbf{D}(\mathrm{T})$	:	Domain of the operator T.
I_{X}	:	Identity operator a Hilbert space X.
T^*	:	Adjoint of the operator T.
$\lVert \mathrm{T} \rVert$	:	Norm of the operator T.
$\mathscr{L}(\mathrm{X},\mathrm{Y})$	:	Set of linear operators from X to Y.
$\mathbf{B}(\mathrm{X})$	:	Set of bounded operators on X.

$\mathrm{Im}(T)$	:	Image of the operator T.
$\ker(T)$	:	Kernel of the operator T.
$G(T)$	:	Graph of the operator T.
$\rho(T)$	:	Resolvent set of the operator T.
$\sigma(T)$	:	Spectrum of the operator T.
$\sigma_p(T)$	:	Punctual spectrum of the operator T.
$\sigma_c(T)$	:	Continuous spectrum of the operator T.
$\sigma_{res}(T)$	:	Residual spectrum of the operator T.

Introduction

In the literature, numerous mathematical models of invariant time degenerate systems in infinite-dimension, known also as singular dynamical systems with operators coefficients, can be described by differential equations, wherein abstract forms are written as differential operator equations in Hilbert space. These models can be extracted from electrodynamics, social sciences fields, micro and macro economy, biology, communication, and information science, industrial processors involving chemical reactors to name a few [7, 8, 17, 20, 32, 42] and [44].

However, in the last few years, considerable attention has been paid to infinite-dimensional degenerate systems, also called the degenerate Cauchy problem, by the development of a mathematical framework for generalizing finite dimensional results to infinite dimension since it is one of the main research problems in control theory and its effectiveness in modelling many practical problems of physics, geometry, and applied mathematics [4, 10, 15, 16, 20, 28, 30] and [40].

It must be emphasized that the first results in the infinite-dimensional case were extended by *Lions* in [29], and, later, in other monographs [4, 15] and [35].

Many efforts have been done to develop degenerate dynamical systems in finite dimension, originated in 1976 with the fundamental paper of *Campbell* [9], and later on the paper of *Luenberger* [31], since, unlike the non-singular case, models of this form have some important advantages in comparison with models in the standard case, more details can be found in many books and papers as [15, 18, 20] and [22].

It is important to note that several research areas on degenerate dynamical systems in finite dimension remain uncompleted, among them the problem of minimum energy control which belongs to the field of control theory appears. This problem is the first contribution of the present thesis.

Moreover, the problem of minimum energy control for infinite-dimensional degenerate Cauchy problem with skew-hermitian pencil and bounded input will be addressed and resolved. The key idea is the use of the concept exact controllability, since it plays an essential role in the development of modern mathematical control theory, and some notions of operator theory as the orthogonal decomposition. Furthermore, a procedure for computing the minimum energy control will be proposed.

In addition, the problem of stability, which has been founded by the Russian scientist *Aleksandr Mikhailovich Lyapunov* in his famous work [34], is essential and crucial problem in control theory whether in finite or infinite dimension. Recently, the stability and robustness of such a class of systems have been extensively studied from both an algebraic and analytical point of view. Despite intensive research, many difficult and unresolved problems with the stability and stabilization for infinite-dimensional systems remain open issues. Another contribution of this thesis is to introduce some new results on the analysis of the condition of stability and stabilization for systems with different types of operators.

The remainder of this thesis is organized as follows;

- The first chapter is devoted to a remainder of the general definitions and important results in operator and semi-group theories. Then, we will present few interesting concepts of a skew-hermitian pencil which will be used for the decomposition of a degenerate systems.

- In the second chapter, we will begin by recalling the definitions of some particular matrices. Next, we will present some examples of modelling real problems in finite and infinite-dimensional cases to motivate the readers for the interest of their studies. Then, some definitions and properties of positive systems followed by the different concepts of controllability and characterizations for the both cases will be given.

- The problem of minimum energy control for a finite dimensional singular dynamical system with rectangular inputs will be studied in the third chapter. We will start by presenting the main problem, then, we will transform the control that is used for the formulation of the minimum energy control problem in order to determine the minimal energy of the system and the minimum control. A procedure for calculating the optimal control and the minimal energy for this problem is proposed in the last part of this chapter.

- In the fourth chapter, a formulation of the minimum energy control for a degenerate Cauchy problem with variable operator coefficients, skew-hermitian pencil taking into account the bounded input condition will be processed. Then, the solution to our problem is obtained thanks to some techniques. Finally, we will propose an algorithm to calculate the minimum energy of the system and the minimum energy control.

- The last chapter will be dedicated to the problem of stability and stabilization of infinite-dimensional dynamical systems. The general definitions and properties of stability will be recalled, followed by few important results of the weak, asymptotic, and exponential stabilities. Finally, we will focus on the stabilization problem in infinite-dimensional dynamical systems with bounded operators.

The last part of this manuscript presents a general conclusion of our work, followed by some perspectives of our future research. The different papers and books that have been used in the development of our research are described in the bibliography which closes the manuscript.

Chapter 1

Mathematical background

1 Introduction

In this chapter, we recall the general definitions, concepts, and characterizations related to linear operators, and semi-group theory.

2 Basic mathematical tools for linear operators

Let us start by recalling some definitions, and characterizations of operators, we will base ourselves on [2, 6, 14, 15, 21, 45] and [47].

2.1 Norm and normed space

Definition 2.1 [41] *Let* X *be a vector space over* F*. A norm on* X *is a function* $\|.\| : X \to \mathbb{R}$ *such that for all* $x, y \in X$ *and for all* $\alpha \in F$*, we have*

- **i)** $\|x\|_X \geq 0$*;*
- **ii)** $\|x\|_X = 0$*, if and only if,* $x = 0_X$*;*
- **iii)** $\|\alpha x\| = |\alpha| \|x\|_X$*;*
- **iv)** $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

If such function exists, then, X *is called a normed vector space or just a normed space and noted by* $(X, \|.\|)$ *or simply by* X.

Definition 2.2 [41] *Let* X *be a normed space over* F*. The space* $\mathbf{B}(X, F)$ *is called the dual space of* X *and is denoted by* X^*.

Definition 2.3 *A sequence* $(x_n)_n$ *of elements of normed space* X *is said to be convergent, if there exists* $x \in X$ *such that*

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : n > N_\epsilon \Rightarrow \|x_n - x\| < \epsilon.$$

In this case, we write $x = \lim_{n \to +\infty} x_n$.

Definition 2.4 *A sequence* $(x_n)_n$ *of elements of normed space* X *is said to be of Cauchy, if there exists* $x \in X$ *such that*

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} : n, m > N_\epsilon \Rightarrow \|x_n - x_m\| < \epsilon. \tag{1.1}$$

It is well known that every convergent sequence is a Cauchy one. The converse is generally not true.

Definition 2.5 *A normed space* X *in which every Cauchy sequence is convergent is called Banach space.*

2.2 Inner product spaces and Hilbert spaces

Definition 2.6 [2] *Let* X *be a complex vector space. An inner product on* X*, denoted by* $\langle .,. \rangle$*, is a function*

$$\begin{array}{rcl} \langle .,. \rangle : X \times X & \longrightarrow & \mathbb{C} \\ (x, y) & \longmapsto & \langle .,. \rangle, \end{array}$$

such that for all $x, y, z \in X$ *and for all* $\alpha, \beta \in \mathbb{C}$*, we have*

i) $\langle x, x \rangle \geq 0$;

ii) $\langle x, x \rangle = 0$ *, if and only if,* $x = 0$;

iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

iv) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Remark 2.7 *If* $\langle .,. \rangle$ *is an inner product on* X*, then, we obtain a norm on* X *by setting*

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{1.2}$$

Thus, we say that the norm (1.2) *derives from the inner product of* X.

Definition 2.8 *A Banach space* X *is called a Hilbert space if, its norm derives from an inner product.*

Remark 2.9 [41] *In a Hilbert space* X, *the Cauchy-Schwarz inequality holds*

$$\forall x, y \in X : |\langle x, y\rangle| \leq \|x\| \|y\|.$$

In the case of Hilbert space, we have the following result, stating that X and X^* are isometrically identifiable to each other.

Theorem 2.10 [41] *If* X *is a Hilbert space and* $f \in X^*$, *then, there is a unique* $y \in X$ *such that*

$$f(x) = \langle x, y\rangle,$$

for all $x \in X$. *Moreover,*

$$\|f\| = \|y\|.$$

2.3 Linear operators

Definition 2.11 [41] *Let* X *and* Y *be normed spaces and let* $T : X \to Y$ *be a linear transformation.* T *is said to be bounded if there exists a positive real number* k, *such that*

$$\|T(x)\| \leq k\|x\|, \quad \textit{for all } x \in X.$$

If T is bounded, then, the set of all positive constant k satisfying the precedent inequality is minored and we have

$$\mathcal{N}(T) = \inf\left\{k \geq 0 : \|Tx\| \leq k\|x\|\right\}, \quad \text{for all } x \in X.$$

It is clear that the mapping $T \longmapsto \mathcal{N}$ satisfies all conditions of a norm. For this reason, it is called the norm of operator T and noted $\|T\|$. It is well known that equipped with this norm, $\mathbf{B}(X, Y)$ stands a normed space [41]. Moreover, if Y is a Banach space, then, $T \in \mathbf{B}(X, Y)$. Thus,

$$\begin{aligned} \|T\| &= \sup_{\|x\| \leq 1} \|Tx\|, \\ &= \sup_{\|x\| = 1} \|Tx\|, \\ &= \sup_{\|x\| \neq 0} \frac{\|Tx\|}{\|x\|}. \end{aligned}$$

Lemma 2.12 [41] *Let* X *and* Y *be normed linear spaces and let* $T : X \to Y$ *be a linear transformation. The following assertions are equivalent :*

i) T *is uniformly continuous;*

ii) T *is continuous;*

iii) T *is continuous at* 0*;*

iv) T *is bounded.*

Remark 2.13 *By definition, elements of* $\mathbf{B}(\mathrm{X},\mathrm{Y})$ *are supposed defined on the whole space* X*. In other words,*

$$\mathrm{T} \in \mathbf{B}(\mathrm{X},\mathrm{Y}) \Rightarrow \mathbf{D}(\mathrm{T}) = \mathrm{X}.$$

An example of bounded operators is given in the following.

Example 2.14 *Let us consider* $\mathrm{X} = \mathrm{Y} = l^p(\mathbb{N},\mathbb{C})$ *(for* $1 \leq p \leq +\infty$*), where*

$$l^p(\mathbb{N},\mathbb{C}) = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \ : \ \sum_{i=1}^{n} |x_i|^p < +\infty \right\}, \quad \text{for} \quad 1 \leq p < +\infty,$$

and

$$l^\infty(\mathbb{N},\mathbb{C}) = \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \ : \ \sup_{i=\overline{1,n}} |x_i| < +\infty \right\}.$$

Now, we define the operator of shift to the right T_d *and the operator of shift to left* T_g *of domain* $l^p(\mathbb{N},\mathbb{C})$ *respectively by*

$$\mathrm{T}_d(x_1, x_2, \cdots, x_n) = (0, x_1, x_2, \cdots, x_n),$$

and

$$\mathrm{T}_g(x_1, x_2, \cdots, x_n) = (x_2, x_3, \cdots, x_n).$$

T_d *and* T_g *are two bounded operators, therefore,*

$$\|\mathrm{T}_d x\| = \|x\|, \quad \text{for all} \quad x \in l^p(\mathbb{N},\mathbb{C}),$$

and if $x = (0, 1, 0, \cdots, 0)$*, we have*

$$\|\mathrm{T}_d\| = 1.$$

Moreover, for the operator of the shift to the left T_g*, we note*

$$\|\mathrm{T}_g x\| \leq \|x\|,$$

and if $x = (0, 1, 0, \cdots, 0)$*, we have*

$$\|\mathrm{T}_g\| = 1.$$

Definition 2.15 [41] *Let* T *be an operator defined on* $\mathbf{D}(\mathrm{T})$*. Its range is defined by*

$$\mathrm{rank}(\mathrm{T}) = \mathrm{Im}(\mathrm{T}) = \left\{ y \in \mathrm{Y} \,\middle|\, \exists x \in \mathrm{D}(\mathrm{A}), \ y = \mathrm{T}x \right\},$$

and its kernel is

$$\ker(\mathrm{T}) = \left\{ x \in \mathrm{D}(\mathrm{T}) \,\middle|\, \mathrm{T}x = 0 \right\}.$$

The operator T *is*

- ***Injective :*** *if and only if,*
$$\ker(\mathrm{T}) = \{0\};$$

- ***Surjective :*** *if and only if,*
$$\mathrm{rank}(\mathrm{T}) = \mathrm{Y};$$

- ***Bijective :*** *if and only if,* T *is injective and surjective in the same time.*

In the following, we recall the definition of the inverse operator.

Definition 2.16 [41] *Let* X *and* Y *be a normed linear spaces. An operator* $\mathrm{T} \in \mathbf{B}(\mathrm{X},\mathrm{Y})$ *is said to be invertible if there exists an operator* $\mathrm{S} \in \mathbf{B}(\mathrm{Y},\mathrm{X})$ *such that* $\mathrm{ST} = \mathrm{I_X}$ *and* $\mathrm{TS} = \mathrm{I_Y}$.

In this case, the operator S *is called the inverse of* T *and it is noted by* T^{-1}.

Remark 2.17 *It is well-known by the Banach theorem for the inverse operator* [41] *that if* $\mathrm{T} \in \mathbf{B}(\mathrm{X},\mathrm{Y})$ *is a bijection, then,* T *is invertible.*

2.4 Adjoint operator

Definition 2.18 [41] *Let* X *and* Y *be two Hilbert spaces and let* $\mathrm{T} \in \mathbf{B}(\mathrm{X},\mathrm{Y})$. *The adjoint* T^* *of* T *is the unique linear operator that satisfies the conditions* $\mathrm{T}^* \in \mathbf{B}(\mathrm{X},\mathrm{Y})$ *and*

$$\forall x \in \mathrm{X},\ \forall y \in \mathrm{Y}\ :\ \langle \mathrm{T}x, y\rangle_{\mathrm{Y}} = \langle x, \mathrm{T}^* y\rangle_{\mathrm{X}}.$$

Some useful properties of the adjoint are listed by in the following theorem.

Theorem 2.19 [41] *Let* X, Y, *and* Z *be complex Hilbert spaces and let* $\mathrm{T} \in \mathbf{B}(\mathrm{X},\mathrm{Y})$, *and* $\mathrm{S} \in \mathbf{B}(\mathrm{Y},\mathrm{Z})$. *Then,*

i) $\|\mathrm{T}^*\| = \|\mathrm{T}\|$;

ii) $\ker(\mathrm{T}) = \left(\mathrm{Im}\left(\mathrm{T}^*\right)\right)^{\perp}$;

iii) $\overline{\mathrm{Im}(\mathrm{T})} = (\ker(\mathrm{T}^*))^{\perp}$;

iv) $\ker(\mathrm{TT})^* = \ker(\mathrm{T})^*$;

v) *If* T *is invertible, then,* $\left(\mathrm{T}^{-1}\right)^* = \left(\mathrm{T}^*\right)^{-1}$;

vi) $(\mathrm{ST})^* = (\mathrm{T})^*(\mathrm{S})^*$ *and* $\|\mathrm{ST}\| \le \|\mathrm{S}\|\|\mathrm{T}\|$;

vii) *The function* $f : \mathbf{B}(\mathrm{X},\mathrm{Y}) \to \mathbf{B}(\mathrm{Y},\mathrm{X})$ *defined by* $f(\mathrm{T}) = \mathrm{T}^*$ *is continuous.*

2.5 Spectrum of operators

Definition 2.20 [41] *Let* X *be a complex Hilbert space and let* $I_X \in \mathbf{B}(X)$ *be the identity operator. The spectrum of* $T \in \mathbf{B}(X)$*, denoted by* $\sigma(T)$*, is defined as*

$$\sigma(T) = \left\{\lambda \in \mathbb{C} : T - \lambda I_X \textit{ is not invertible}\right\}. \tag{1.3}$$

Remark 2.21 *If* $T \in \mathbf{B}(X)$*, then,*

i) $\sigma(T)$ *is a nonempty and compact subset of* $\mathbb{C}$ *contented in the closed ball with radius* $\|T\|$ *and centered at the origin;*

ii) *The complement in* $\mathbb{C}$ *of the spectrum* $\sigma(T)$ *of* T *is called the resolvent set and is noted by* $\rho(T)$*. So, according to the Banach theorem for the inverse operator, a complex number* λ *belongs to the resolvent set* $\rho(T)$*, if and only if, the operator* $T - \lambda I_X$ *is bijective.*

iii) *Form the relation* $\left(T^* - \lambda I_X\right)^{-1} = \left(\left(T - \overline{\lambda} I_X\right)^{-1}\right)^*$*, it is follows that*

$$\sigma\left(T^*\right) = \overline{\sigma(T)} = \left\{\overline{\lambda} : \lambda \in \sigma(T)\right\}. \tag{1.4}$$

If $\lambda \in \sigma(T)$, then, one of the following three situations holds:

Situation 1 : If the operator $(T - \lambda I_X)$ is not injective, then, there exists at least one vector $x_\lambda \neq 0$ such that $Tx_\lambda = \lambda x_\lambda$. x_λ is called the eigenvector of the operator T associated with the eigenvalue λ. The set of all eigenvalues $\sigma_p(T)$ is called the punctual spectrum of the operator T.

Situation 2 : If the operator $(T - \lambda I_X)$ is injective and $\overline{\text{rank}(T - \lambda I_X)} \neq X$, then, λ belongs to the residual spectrum $\sigma_{res}(T)$.

Situation 3 : If the operator $(T - \lambda I_X)$ is injective and $\overline{\text{rank}(T - \lambda I)} = X$, then, λ belongs to the continuous spectrum $\sigma_c(T)$.

Some characterizations on the residual and continuous spectrums are presented in the following.

Theorem 2.22 [41] *Let* X *be a Hilbert space and let* $T \in \mathbf{B}(X)$*. Then,*

i)

$$\lambda \in \sigma_{res}(T) \Longleftrightarrow \left[\lambda \notin \sigma_p(T) \quad \textit{and} \quad \lambda \in \sigma_p(T^*)\right];$$

ii)

$$\lambda \in \sigma_c(T) \Longleftrightarrow \left[\lambda \notin \sigma_p(T), \lambda \notin \sigma_p(T^*) \quad \textit{and} \quad \lambda \in \sigma(T)\right].$$

Definition 2.23 [41] *Let* X *be a Hilbert space and let* $\mathrm{T} \in \mathbf{B}(\mathrm{X})$. T *is called self-adjoint operator if,*

$$\mathrm{T} = \mathrm{T}^*.$$

Remark 2.24 *For a bounded self-adjoint operator* T, *we have*

$$\|\mathrm{T}\| = \sup_{\|x\|=1} |\langle \mathrm{T}x, x\rangle|. \tag{1.5}$$

Example 2.25 I_{X} *is self-adjoint operator.*

For a bounded self-adjoint operator, we have the following characterizations of the resolvent set and spectrum.

Theorem 2.26 [41] *Let* T *be a bounded self-adjoint operator in the Hilbert space* X. *Then,*

i) $\lambda \in \rho(\mathrm{T}) \Longleftrightarrow \mathrm{rank}(\mathrm{T} - \lambda \mathrm{I}_{\mathrm{X}}) = \mathrm{X}$;

ii) $\sigma_c(\mathrm{T}) \subset \mathbb{R}$ *and* $\sigma_{res}(\mathrm{T}) = \emptyset$;

iii) $\lambda \in \sigma_c(\mathrm{T}) \Longleftrightarrow \mathrm{rank}(\mathrm{T} - \lambda \mathrm{I}_{\mathrm{X}}) \neq \overline{\mathrm{rank}(\mathrm{T} - \lambda \mathrm{I}_{\mathrm{X}})} = \mathrm{X}$;

iv) $\lambda \in \sigma_c(\mathrm{T}) \Longleftrightarrow \overline{\mathrm{rank}(\mathrm{T} - \lambda \mathrm{I}_{\mathrm{X}})} \neq \mathrm{X}$;

v) *Two eigenvectors corresponding to two distinct eigenvalues are orthogonal;*

vi) *At least one of real numbers* $\|\mathrm{T}\|$ *and* $-\|\mathrm{T}\|$ *belongs to* $\sigma_p(\mathrm{T})$.

2.6 Positive operators

Definition 2.27 [41] *Let* X *be a Hilbert space and let* $\mathrm{T} \in \mathbf{B}(\mathrm{X})$. T *is positive operator if* T *is self-adjoint operator and*

$$\langle \mathrm{T}x, x\rangle \geq 0, \quad \forall x \in \mathrm{X}. \tag{1.6}$$

Proposition 2.28 [20, 41] *For positive operators, we have*

i) *The spectrum of a bounded positive operator is contained in the real half-line* $[0, +\infty[$;

ii) *A finite sum of positive operators* (*acting in the same Hilbert space*) *is positive;*

iii) *If* T *is a positive operator on* X, *then,*

$$\langle \mathrm{T}x, y\rangle \leq \langle \mathrm{T}x, x\rangle \to \langle \mathrm{T}y, y\rangle, \quad \forall x, y \in \mathrm{X};$$

iv) *The product of positive operators is positive;*

v) *The inverse of a positive and invertible operator is also positive.*

It must be emphasized that for all the previous results the domain $\mathbf{D}(\mathrm{T})$ of the operator T coincides with the whole space X. However, in the following, we will introduce an important class of operators which do not necessarily satisfy this domain condition.

2.7 Graph and closed operators

Definition 2.29 [41] *Let* X *and* Y *be two normed spaces and let the sequence* $(x_n)_n$ *of elements of normed space* X*. A linear operator* $\mathrm{T}\,:\,\mathrm{X}\longrightarrow\mathrm{Y}$ *with domain* $\mathbf{D}(\mathrm{T})$ *is called closed if, it satisfies the following implication*

$$\begin{cases} (x_n)_n \subset \mathbf{D}(\mathrm{T}) \\ \lim\limits_{n\to+\infty} x_n = x \\ \lim\limits_{n\to+\infty} \mathrm{T}x_n = y \end{cases} \Rightarrow \begin{cases} x \subset \mathbf{D}(\mathrm{T}). \\ y = \mathrm{T}x. \end{cases}$$

It is well known that every bounded operator is closed. The converse is, in general, not true. However, we have the following result called the closed graph theorem.

Theorem 2.30 [41] *If* X *and* Y *are Banach spaces, then, every closed operator* $\mathrm{T}\,:\,\mathrm{X}\longrightarrow\mathrm{Y}$ *satisfying the condition* $\mathbf{D}(\mathrm{T})=\mathrm{X}$ *is bounded.*

We have also the following useful result.

Definition 2.31 [41] *Let* X *and* Y *be a normed spaces and let* $\mathrm{T}\,:\,\mathrm{X}\to\mathrm{Y}$ *be a linear operator. Then, the operator* T *is closed, if and only if, its graph*

$$\mathbf{G}(\mathrm{T}) = \Big\{(x,\mathrm{T}x)\;:\;x\in\mathrm{X}\Big\},$$

is a closed subspace of the cartesian product $\mathrm{X}\times\mathrm{Y}$.

3 Semi-groups of linear operators

In this section, we will present some results on the semi-group theory for linear operators which we will use them latter.

Definition 3.1 [36] *A family* $(\mathrm{S}(t))_{t\geq 0}$ *of bounded linear operators acting in the Banach space* X *is called a semi-group if the following conditions are satisfied*

i) $\mathrm{S}(t+\tau)=\mathrm{S}(t)\mathrm{S}(\tau)$, *for all* $t,\tau\geq 0$;

ii) $\mathrm{S}(0)=\mathrm{I}_{\mathrm{X}}$, *where* I_{X} *is the identity operator in* X.

Definition 3.2 [36] *A semi-group of bounded linear operators* $(\mathrm{S}(t))_{t\geq 0}$ *is uniformly continuous if*

$$\lim_{t\to 0^+} \|\mathrm{S}(t)x - \mathrm{I_X}\| = 0.$$

$(\mathrm{S}(t))_{t\geq 0}$ *is called strongly continuous, or* C_0 *semi-group if*

$$\lim_{t\to 0^+} \|\mathrm{S}(t)x - x\| = 0, \quad \forall x \in \mathrm{X}.$$

Proposition 3.3 [36]

- **i)** *Every uniformly continuous semi-group is a* C_0 *semi-group;*

- **ii)** *If* $(\mathrm{S}(t))_{t\geq 0}$ *on* X *is a uniformly continuous semi-group, then, there exists a unique linear operator* $\mathrm{A} \in \mathbf{B}(\mathrm{X})$ *such that*

$$\mathrm{S}(t) \;=\; e^{t\mathrm{A}} \;:=\sum_{n=0}^{+\infty} \frac{t^n \mathrm{A}^n}{n!}.$$

The following theorem is one of the most important results in the theory of C_0 semi-groups.

Theorem 3.4 [36] *If* $(\mathrm{S}(t))_{t\geq 0}$ *is a* C_0 *semi-group, then, there exists two real numbers* $\omega \geq 0$ *and* $\mathrm{M} \geq 1$ *such that*

$$\|\mathrm{S}(t)\| \leq \mathrm{M}e^{\omega t}, \quad \forall t \geq 0.$$

If $(\mathrm{M}, \omega) = (1, 0)$, *then, the semi-group* $(\mathrm{S}(t))_{t\geq 0}$ *is called contractive.*

Definition 3.5 *Let* $(\mathrm{S}(t))_{t\geq 0}$ *be a semi-group on* X. *The generate of* $(\mathrm{S}(t))_{t\geq 0}$ *is the linear operator* A *defined in* X *by*

$$\mathrm{A}x = \lim_{t\to 0^+} \frac{\mathrm{S}(t)x - x}{t}, \quad \textit{for } x \in \mathbf{D}(\mathrm{A}),$$

where,

$$\mathbf{D}(\mathrm{A}) = \left\{ x \in \mathrm{X} \,:\, \lim_{t\to 0^+} \frac{\mathrm{S}(t)x - x}{t} \;\; \textit{exists} \right\}.$$

Example 3.6 *If* $\mathrm{X} = \mathbf{L^2}\,(]-\infty, +\infty[)$,*then, the family* $\mathrm{S}(t)f(x) = f(x+t)$, $t \geq 0$ *is a* C_0 *semi-group.*

Its generator is the operator A *with domain* $\mathbf{D}(\mathrm{A}) \;=\; \mathbf{H^1}\,(]-\infty, +\infty[)$ *and acting the rule*

$$\mathrm{A}f = f'.$$

The following result, known as Hille-Yosida theorem, gives a characterization of the generator operator of contractive semi-group.

Theorem 3.7 [36] *Let* $(S(t))_{t\geq 0}$ *be a contractive* C_0 *semi-group such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0, \quad \omega \geq 0, \quad M \geq 1.$$

A linear (not necessarily bounded) operator A *in* X *is the generator of* $(S(t))_{t\geq 0}$, *if and only if, the following conditions are satisfied :*

i) A *is closed operator and* $\overline{\mathbf{D}(A)} = X$;

ii) *For every real number* $\lambda > 0$, *the operator* $(A - \lambda I_X)^{-1}$ *exists and it is bounded from* X *into* $\mathbf{D}(A) \subset X$ *and*

$$\|(A - \lambda I_X)^{-1}\| \leq \frac{1}{\lambda}.$$

For the general case, we have the Feller-Miyadera-Phillip theorem.

Theorem 3.8 [36] *Let* $(S(t))_{t\geq 0}$ *be a contractive* C_0 *semi-group such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0, \quad \omega \geq 0, \quad M \geq 1. \tag{1.7}$$

A linear (not necessarily bounded) operator A *in* X *is the generator of* $(S(t))_{t\geq 0}$, *if and only if, the following conditions are satisfied :*

i) *The operator* A *is closed and* $\overline{\mathbf{D}(A)} = X$;

ii) *For every real number* $\lambda > 0$, *the operator* $(A - \lambda I_X)^{-1}$ *exists, it is bounded from* X *into* $\mathbf{D}(A) \subset X$, *and*

$$\|(A - \lambda I_X)^{-n}\| \leq \frac{M}{(\omega - \lambda)^n}, \quad \forall n \in \mathbb{N}^*. \tag{1.8}$$

Some useful properties of C_0 semi-groups are given below.

Theorem 3.9 [10] *Let* $A : \mathbf{D}(A) \subset X \longrightarrow X$ *be the generator operator of a* C_0 *semi-group* $(S(t))_{t\geq 0}$. *Then,*

i) *For each* $t \geq 0$ *and each* $x \in X$, *we have*

$$\lim_{t\to 0^+} \frac{1}{h} \int_t^{t+h} S(\tau)x\,d\tau = S(t)x; \tag{1.9}$$

ii) *For each* $t \geq 0$ *and each* $x \in X$, *we have*

$$\left(\int_0^t S(\tau)x\,d\tau\right) \in D(A), \quad \textit{and} \quad A\int_0^t S(\tau)x\,d\tau = S(t)x - x;$$

iii) *For each* $t \geq 0$ *and each* $x \in \mathbf{D}(A)$, *we have* $S(t)x \in \mathbf{D}(A)$. *Moreover, the mapping* $t \longmapsto S(t)x$ *is of class* $\mathscr{C}^1$ *on* $[0, +\infty[$, *and satisfies*

$$\frac{d}{dt}\big(S(t)x\big) = AS(t)x = S(t)Ax; \tag{1.10}$$

iv) *For each* $x \in \mathbf{D}(A)$ *and each* $0 \leq s \leq t < +\infty$, *we have*

$$\int_s^t AS(\tau)x d\tau = \int_s^t S(\tau)Ax d\tau = S(t)x - S(s)x. \tag{1.11}$$

There is no doubt that the most important application of semi-group theory is in resolution of linear abstract differential equations.

Theorem 3.10 [10] *Let* $A : \mathbf{D}(A) \subset X \longrightarrow X$ *be the generator of a* C_0 *semi-group of linear operators. Let* $f : [0, T_0] \longrightarrow X$ *be a function of class* $\mathscr{C}^1$. *Then, for every* $x_0 \in \mathbf{D}(A)$, *the Cauchy problem*

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + f, \quad \textit{for} \quad 0 \leq t \leq T_0, \\ x(0) &= x_0, \end{cases}$$

where $T_0 \in \mathbb{R}_+^*$, *has the unique solution of the form*

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)f(\tau)d\tau, \quad \textit{for} \quad 0 \leq t \leq T_0.$$

4 Skew-hermitian pencil

The main goal of this section is present some concepts of skew-hermitian pencil.

For this purpose, let us consider the two Hilbert spaces X and Y and let E and A be two linear closed operators from X into Y with domains $\mathbf{D}(E)$ and $\mathbf{D}(A)$ respectively, such that

$$\overline{\mathbf{D}} = X \quad \text{and} \quad \overline{\mathbf{D}_*} = Y,$$

where

$$\mathbf{D} = \mathbf{D}(E) \cap \mathbf{D}(A) \quad \text{and} \quad \mathbf{D}_* = \mathbf{D}(E^*) \cap \mathbf{D}(A^*).$$

Note that, the operators E^* and A^* are well-defined since their domains are dense in X and Y respectively. Moreover, we have

$$E^* : \mathbf{D}(E^*) \subseteq Y \longrightarrow X \quad \text{and} \quad A^* : \mathbf{D}(A^*) \subseteq Y \longrightarrow X.$$

Definition 4.1 *The set of all operators of the form*

$$(\lambda E + A), \quad \lambda \in \mathbb{C},$$

is called the pencil generated by E *and* A*. The conjugate pencil is the set of all operators of the form*

$$(\mu \mathrm{E}^* + \mathrm{A}^*), \quad \mu \in \mathbb{C},$$

it is generated by the operators E^* *and* A^*.

Definition 4.2 [40] *We say that the complex number* α *is a regular point for the pencil* $(\lambda\mathrm{E}+\mathrm{A})$ *if the operator* $(\alpha\mathrm{E}+\mathrm{A})^{-1}$ *exists, is bounded, and its domain is the whole space* Y.

The set of all regular points for the pencil $(\lambda\mathrm{E}+\mathrm{A})$ *will be noted by* $\rho(\mathrm{E},\mathrm{A})$. *Its complementary in* $\mathbb{C}$, *which is called the spectrum of the pencil* $(\lambda\mathrm{E}+\mathrm{A})$, *is noted by* $\sigma(\mathrm{E},\mathrm{A})$.

Remark 4.3 *For* $\mathrm{X}=\mathrm{Y}$ *and* $\mathrm{E}=\mathrm{I}_{\mathrm{X}}$, *one retrieves the classical concepts of resolvent set and spectrum of the operator* A.

Now, we consider the bilinear form

$$\psi(x,y) \;=\; \langle \mathrm{E}x, \mathrm{A}y\rangle \;+\; \langle \mathrm{A}x, \mathrm{E}y\rangle, \quad (x,y)\in \mathbf{D}\times\mathbf{D}. \tag{1.12}$$

Definition 4.4 [40] *The pencil* $(\lambda\mathrm{E}+\mathrm{A})$ *is said to be skew-symmertric if the bilinear form* (1.12) *is identically null.*

Remark 4.5 *Since the operators* E *and* A *are densely defined, then, it is not difficult to see that the pencil* $(\lambda\mathrm{E}+\mathrm{A})$ *is skew-hermitian, if and only if,*

$$\mathrm{E}^*\mathrm{A}+\mathrm{A}^*\mathrm{E}=0.$$

Analogically, the pencil $(\mu\mathrm{E}^*+\mathrm{A}^*)$ *is skew-hermitian, if and only if,*

$$\mathrm{E}\mathrm{A}^*+\mathrm{A}\mathrm{E}^*=0.$$

Theorem 4.6 [40] *Suppose that there exists at least one complex number* α, *such that*

$$\mathrm{Re}(\alpha)>0, \quad \alpha\in\rho(\mathrm{E},\mathrm{A}), \quad \textit{and} \quad \overline{\alpha}\in\rho\left(\mathrm{E}^*,\mathrm{A}^*\right). \tag{1.13}$$

Then, all point of the complex half plan $\mathrm{Re}(\lambda)>0$ *are regular for* $(\lambda\mathrm{E}+\mathrm{A})$ *and* $(\mu\mathrm{E}^*+\mathrm{A}^*)$.

In addition, if the complex half plan $\mathrm{Re}(\mu)<0$ *contains at least one regular value of the pencil* $(\mu\mathrm{E}^*+\mathrm{A}^*)$, *then, all point of this half plan* $\left(\mathrm{Re}(\mu)<0\right)$ *are, also, regular for the pencils* $(\lambda\mathrm{E}+\mathrm{A})$ *and* $(\mu\mathrm{E}^*+\mathrm{A}^*)$.

Definition 4.7 *The pencil* $(\lambda\mathrm{E}+\mathrm{A})$ *is called skew-hermitian if,*

herm1. *The pencil* $(\lambda\mathrm{E}+\mathrm{A})$ *is skew-symmetric;*

herm2. *The complex half-plan* $\mathrm{Re}(\mu) < 0$ *meets* $\rho(E^*, A^*)$*;*

herm3. *The complex half-plan* $\mathrm{Re}(\mu) < 0$ *contains at least one value* α*, satisfying* (1.13).

Remark 4.8 *It follows from theorem* 4.6 *that for a skew-hermitian pencil* $(\lambda E + A)$ *all complex numbers* δ *satisfying the condition* $\mathrm{Re}(\delta) \neq 0$ *are regular for* $(\lambda E + A)$ *and* $(\lambda E^* + A^*)$*, i.e.,*

$$\mathrm{Re}(\delta) \neq 0 \;\Rightarrow\; \delta \in \rho(E, A) \cap \rho(E^*, A^*).$$

Moreover, if E *or* A *is bounded, then,*

$$\rho(E, A) = \rho(E^*, A^*).$$

Proposition 4.9 *If the pencil* $(\mu E^* + A^*)$ *is skew-hermitian, then,*

$$[\mathrm{Re}(\alpha) \neq 0 \,\wedge\, x \in \mathbf{D}(E)] \Rightarrow \left\| (\alpha E + A)^{-1} A(x) \right\| \leq \frac{\|x\|}{|\mathrm{Re}(\alpha)|}.$$

Proposition 4.10 *Suppose that the conjugate pencil* $(\mu E^* + A^*)$ *is skew-hermitian. If* $C : X \longrightarrow \mathbf{D}(E)$ *is continuous operator, then, the perturbation* $\tilde{A} = A + EC$ *of* A *defines a pencil* $(\lambda E + \tilde{A})$ *whose spectrum* $\rho(E, \tilde{A})$ *is contained in the strip* $|\mathrm{Re}(\lambda)| < \dfrac{\|C\|}{q}$*, for all* $0 < q < 1$.

Moreover,

$$|\mathrm{Re}(\lambda)| \geq \frac{\|C\|}{q} \;\Rightarrow\; \left\| (\alpha E + \tilde{A})^{-1} E(x) \right\| \leq \frac{(1-q)^{-1}}{|\mathrm{Re}(\alpha)|} \|x\|, \quad \forall x \in \mathbf{D}(E). \tag{1.14}$$

Let us suppose that the pencil $(\mu E^* + A^*)$ is skew-hermitian, by the orthogonal decomposition, we get

$$X = X_1 \oplus X_2, \quad X_1 = \ker(E), \quad Y = Y_1 \oplus Y_2, \quad \text{and} \quad Y_1 = \ker(E^*). \tag{1.15}$$

Then, the operators A and E can be represented as

$$E = \begin{bmatrix} 0 & 0 \\ 0 & E_{22} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

As the pencil $(\mu E^* + A^*)$ is skew-hermitian, then, the operator $A_{21} : X_2 \longrightarrow Y_1$ is a null operator. Therefore,

$$\lambda E + A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & \lambda E_{22} + A_{22} \end{bmatrix} \quad \text{and} \quad E_{22}A_{22}^* + E_{22}^* A_{22} = 0.$$

Consequently,

i) The operator A_{11} is invertible;

ii) $\rho(E, A) = \rho(E_{22}, A_{22})$, and the pencil $\mu E_{22}^* + A_{22}^*$ is, also, skew-hermitian.

The use of the method defect indices for the symmetric operators [40] shows that the operator $\hat{E}$, defined from X_2 into X_2 by $\hat{E} = -iE_{22}^{-1}A_{22}^{-1}$, is self-adjoint. This result will be used for the resolution of the singular systems.

5 Conclusion

The basic definitions, theorems, and properties in operators and semi-group theories have been recalled in this chapter, followed by some concepts on the skew-hermitian pencil. We will use these tools in the following chapters.

Chapter 2

Positivity and controllability of finite dimensional dynamical systems and infinite-dimensional dynamical systems

1 Introduction

Positive linear systems are of great partial importance for control theory and their applications, they are linear dynamical systems whose state trajectories are positive for each positive initial state and for all positive inputs functions. A variety of models with positive linear behavior can be found in economics, social sciences, biology, engineering, management science, and medicine [7, 8, 17, 42] and [44].

Nevertheless, the importance of the positivity property for infinite-dimensional systems has been revealed by the storage and industrial systems which involve chemical reactions and heat exchangers, for instance, distributed parameter models of tubular reactors [26] and [27].

In this chapter, firstly, we introduce definitions of some particular positives matrices, then, we present examples of modelling real problems in finite and infinite dimensions. Next, we define the positivity property for finite dimensional dynamical systems and infinite-dimensional dynamical systems. Finally, important properties of exact controllability, exactly null controllable, approximately controllable and their characterizations are given for both cases.

2 Particular positive matrices

Definitions and properties of some particular positive matrices will be introduced in this section.

2.1 Non-negative matrices

Definition 2.1 [20] *A matrix* $A \in \mathbb{R}^{n \times m}$ *is called non-negative if its entries* a_{ij} *are non-negative. The non-negative matrix* A *will be denoted by* $A \geq 0$.

Example 2.2 *The matrix* A *given by*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

is non-negative matrix.

Remark 2.3 *The null matrix is considered non-negative matrix.*

2.2 Positive matrices

Definition 2.4 [20] *A non-negative matrix* $A \in \mathbb{R}^{n \times m}$ *is called positive if at least one of its entries* a_{ij} *is strictly positive . The positive matrix* A *will be denoted by* $A > 0$.

Example 2.5 *The matrix* A

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.2}$$

is a positive matrix.

2.3 Strictly positive matrices

Definition 2.6 [20] *A matrix* $A \in \mathbb{R}^{n \times m}$ *is called strictly positive if all its entries* a_{ij} *are strictly positive. The strictly positive matrix* A *will be denoted by* $A >> 0$.

Example 2.7 *The matrix* A

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix},$$

is a strictly positive matrix.

2.4 Metzler matrices

Definition 2.8 [20] *A matrix* $A \in \mathbb{R}^{n \times m}$ *is called Metzler if its off diagonal entries* a_{ij} *are non-negative.*

Example 2.9 *The matrix* A *defined by*

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & 4 \\ 3 & 0 & -6 \end{bmatrix},$$

is a Metzler matrix.

2.5 Monomial matrices

Definition 2.10 [20] *A matrix* $A \in \mathbb{R}^{n\times m}$ *is called monomial or generalized permutation if its every row and its every column contains only one strictly positive entry and the remaining entries are zero.*

Example 2.11 *The matrix*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix},$$

is a monomial matrix.

3 Examples of finite dimensional dynamical systems and infinite-dimensional dynamical systems

In the section, we will give some examples of finite dimensional dynamical systems and infinite-dimensional dynamical systems.

3.1 Finite dimensional dynamical systems

Example 3.1 [5] *Let us consider the electrical circuit represented by the following figure*

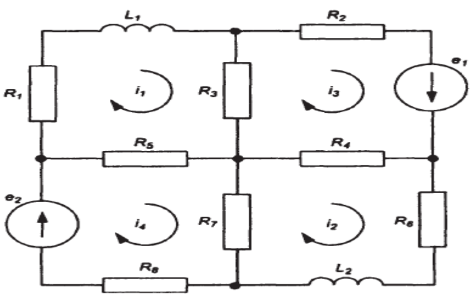

Figure 2.1: RCL circuit.

where

- R_j, $j = 1, 2, \cdots, 8$ *are the voltage on the resistance given;*
- L_j, $j = 1, 2$ *are the inductances;*
- e_j, $j = 1, 2$ *are the voltages on the sources.*

Denote the current intensities in the four meshs by i_1, i_2, i_3, *and* i_4. *The application of the Kirchoff laws to the circuit give*

$$\begin{cases} L_1 \dfrac{di_1(t)}{dt} &=& -\big(R_1 + R_3 + R_5\big) i_1(t) + R_3 i_3(t) + R_5 i_4(t), \\ L_2 \dfrac{di_2(t)}{dt} &=& -\big(R_4 + R_6 + R_7\big) i_2(t) + R_4 i_3(t) + R_7 i_4(t), \\ 0 &=& R_3 i_1(t) + R_4 i_2(t) - \big(R_2 + R_3 + R_4\big) i_3(t) + e_1, \\ 0 &=& R_5 i_1(t) + R_7 i_2(t) - \big(R_5 + R_7 + R_8\big) i_4(t) + e_2. \end{cases} \tag{2.3}$$

The system (2.3) *can be written in the following form*

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2.4}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \dfrac{-R_{11}}{L_1} & 0 & \dfrac{R_3}{L_1} & \dfrac{R_5}{L_1} \\ 0 & \dfrac{-R_{22}}{L_2} & \dfrac{R_4}{L_2} & \dfrac{R_7}{L_2} \\ R_3 & R_4 & -R_{33} & 0 \\ R_5 & R_7 & 0 & -R_{44} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \\ i_4(t) \end{bmatrix}, \quad \textit{and} \quad u(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

with,

$$R_{11} = R_1 + R_3 + R_5, \quad R_{22} = R_4 + R_6 + R_7, \quad R_{33} = R_2 + R_3 + R_4, \quad \textit{and} \quad R_{44} = R_5 + R_7 + R_8.$$

We chosse

$$y_1(t) = L_1 \frac{di_1(t)}{dt} + R_{11} i_1(t) \quad \textit{and} \quad y_2(t) = R_6 i_2(t),$$

as the output. Then, the output equation has the form

$$y = \mathrm{C}x(t) + \mathrm{D}u(t),$$

where

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 0 & 0 & \mathrm{R}_3 & \mathrm{R}_5 \\ 0 & \mathrm{R}_6 & 0 & 0 \end{bmatrix}, \textit{ and } \mathrm{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The system (2.4) *is known as a finite dimensional singular dynamical system.*

Example 3.2 [20] *The following figure represents an electrical circuit*

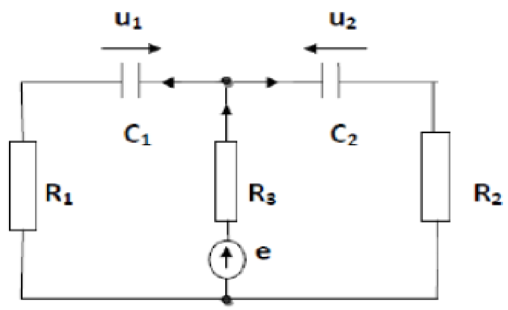

Figure 2.2: The electrical circuit

with a given resistances R_1, R_2, *and* R_3, *the two capacitances* C_1 *and* C_2, *and the source voltage* $e(t)$.

Thanks to the Kirchhoff's laws, this electrical circuit can be represented by the following equations

$$\begin{array}{rcl} \mathrm{R}_1\mathrm{C}_1\dot{u}_1(t) + u_1(t) + \mathrm{R}_3\big(\mathrm{C}_1\dot{u}_1(t) + \mathrm{C}_2\dot{u}_2(t)\big) & = & e(t), \\ \mathrm{R}_3\big(\mathrm{C}_1\dot{u}_1(t) + \mathrm{C}_2\dot{u}_2(t)\big) + u_2(t) + \mathrm{R}_2\mathrm{C}_2\dot{u}_2(t) & = & e(t), \end{array} \tag{2.5}$$

and

$$y(t) = u_1(t) + u_2(t). \tag{2.6}$$

Note that $u_1(t)$ *and* $u_2(t)$ *are the state variables,* $e(t)$ *is the input, and* $y(t)$ *presents the output.*

From the equations (2.5) *and* (2.6), *yields*

$$\begin{bmatrix} (\mathrm{R}_1 + \mathrm{R}_3)\mathrm{C}_1 & \mathrm{R}_3\mathrm{C}_2 \\ \mathrm{R}_3\mathrm{C}_1 & (\mathrm{R}_2 + \mathrm{R}_3)\mathrm{C}_2 \end{bmatrix} \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e(t),$$

and

$$y = [1 \;\; 1] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

or even by

$$\begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \mathrm{A} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \mathrm{B}e(t), \quad \textit{and} \quad y = \mathrm{C} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

where,

$$\mathrm{A} = \begin{bmatrix} -\dfrac{\mathrm{R}_2+\mathrm{R}_3}{\mathrm{C}_1\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} & \dfrac{\mathrm{R}_3}{\mathrm{C}_1\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} \\ \dfrac{\mathrm{R}_3}{\mathrm{C}_2\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} & -\dfrac{\mathrm{R}_1+\mathrm{R}_3}{\mathrm{C}_2\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} \end{bmatrix},$$

$$\mathrm{B} = \begin{bmatrix} -\dfrac{\mathrm{R}_2}{\mathrm{C}_1\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} \\ \dfrac{\mathrm{R}_1}{\mathrm{C}_2\left[\mathrm{R}_1(\mathrm{R}_2+\mathrm{R}_3)+\mathrm{R}_2\mathrm{R}_3\right]} \end{bmatrix}, \quad \textit{and} \quad \mathrm{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Note that $\mathrm{A} \in \mathbb{R}^{2\times 2}$ *is a Metzler matrix,* $\mathrm{B} \in \mathbb{R}_+^{2\times 1}$*, and* $\mathrm{C} \in \mathbb{R}_+^{1\times 2}$*. Therefore, the* RCL *circuit presented by the figure 2.2 is the best example of the positive continuous-time dynamical system for all* $u_1(0) \geq 0$, $u_2(0) \geq 0$, *and* $e(t) \geq 0$. *Note that for* $t > 0$, *we have* $u_1(t) \geq 0$, $u_2(t) \geq 0$, *and* $y(t) \geq 0$.

The next part is dedicated for the presentation of the most important examples of control theory problems in infinite dimension [11, 12, 13] and [14]. Among these problems, we can find those who arise to the delay and the distributed parameter and are modelled by partial differential equations. Many open problems and questions remain to be discussed, among them, the problems of positivity, controllability, and the minimum energy control appear and will be covered later.

3.2 Infinite-dimensional dynamical systems

Example 3.3 [15] *Let us consider a stretched nonuniform string whose motion is described by*

$$\begin{cases} \rho(x)\dfrac{\partial^2 z(x,t)}{\partial t^2} - \alpha(x)\dfrac{\partial^2 z(x,t)}{\partial x^2} &= v(x,t), \\ z(0,t) &= 0, \\ z(1,t) &= u(t), \end{cases} \tag{2.7}$$

where

- $z(x,t)$: *is the displacement of the string at position* x*;*
- $\rho(x)$: *represents the density of the string;*
- $\alpha(x)$: *is the scaled tensile parameter;*
- $v(x,t)$: *represents the distributed control;*

- $u(t)$: *is the control that we can apply along the length of the string.*

The problem presented by the system (2.7) *is an infinite-dimensional dynamical system and the question to be asked is can we find the control* $u(t)$ *that brings the string to rest.*

Example 3.4 [15] *The evolution of the population of a country can be described by the following linear hyperbolic partial differential equation (PDE).*

$$\begin{cases} \dfrac{\partial P(x,t)}{\partial t} + \dfrac{\partial P(x,t)}{\partial x} &= -\mu(x,t)P(x,t), & \\ P(x,0) &= P_0(x), & x \geq 0, \\ P(0,t) &= u(t), & t \geq 0, \end{cases}$$

where

- $P(x,t)$: *is the number of individuals of age* x *at time* t*;*
- $\mu(x,t)$: *represents the mortality function;*
- $P_0(x)$: *is the initial age distribution;*
- $u(t)$: *represents the number of individuals born at time instant* t.

In this example, the problem to be asked is can we find the control $u(t)$ *to achieve a desired age profile* $q(x)$ *at the final time* t_1. *Mathematically, it can be interpreted as minimizing the expression*

$$J(u) = \int_0^1 \left|P(x,t_1) - q(x)\right|^2 dx + \int_0^{t_1} \lambda |u(s)|^2 ds,$$

where the second term measures the social cost of controlling birth-rate. This is again a linear quadratic control problem, but with a boundary control input, for more detail see [15].

Example 3.5 [15] *In steel making plants, it is necessary to estimate the temperature distribution of metal slabs based on measurements at certain points on the surface. A possible model for the temperature distribution is*

$$\begin{cases} \rho C_1 \dfrac{\partial z(x,t)}{\partial t} &= K\dfrac{\partial^2 z(x,t)}{\partial x^2} - \alpha\left[z(x,t) - z_0(x,t)\right], & 0 < x < 1, \\ \dfrac{\partial z(0,t)}{\partial x} &= 0, & \\ \dfrac{\partial z(l,t)}{\partial x} &= 0, & \end{cases}$$

where

- ρ : *is the density;*

- C_1 : *represents the heat capacity;*
- K : *is the effective thermal conductivity of the metal slab;*
- α : *is the heat transfer parameter;*
- z_0 : *represents the average coolant temperature.*

The main problem is to estimate the temperature profile $z(x,t)$ *with* $0 \le x \le l$ *and* $t \ge 0$ *based on the noisy measurements*

$$Y_i(t) = z(x_i, t) + n_i(t), \quad i = 1, 2, \cdots, k,$$

where x_i, $i = 1, 2, \cdots, k$ *are points on the surface of the slab and* $n_i(t)$ *represents the measurement error.*

4 Various concepts of positivity

The main objective of this section is to present, on the one hand, the fundamental definitions, properties, and results of the positivity of finite dimensional dynamical systems. In another hand, the most important outcomes of the positivity of infinite-dimensional dynamical systems will be described.

4.1 Positivity of finite dimensional dynamical systems

4.1.1 Solvability of dynamical systems

Consider the linear continuous dynamical system described by following equations

$$\begin{cases} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{cases} \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^P$ is the output. E, A, B, C, and D are real matrices of appropriate dimensions. x_0 is the initial condition at $t = 0$.

To ensure the solution of the system (2.8), the condition of regularity must be satisfied.

Definition 4.1 *The system described by* (2.8) *is called regular, if and only if, for some* $s \in \mathbb{C}$, *we have*

$$\det\left[sE - A\right] \neq 0. \tag{2.9}$$

In order to deal with to solvability of the system (2.8), two cases may arise.

- **First case :** If $\det \mathrm{E} \neq 0$, then, the system (2.8) turns into

$$\begin{cases} \dot{x}(t) &= \mathrm{E}^{-1}\mathrm{A}x(t) + \mathrm{E}^{-1}\mathrm{B}u(t), \\ y(t) &= \mathrm{C}x(t) + \mathrm{D}u(t), \\ x(0) &= x_0. \end{cases} \tag{2.10}$$

The system (2.10) is called standard linear continuous-time dynamical system. Its solution has the form

$$x(t) = e^{\mathrm{E}^{-1}\mathrm{A}t}x_0 + \int_0^t e^{\mathrm{E}^{-1}\mathrm{A}(t-\tau)}\mathrm{E}^{-1}\mathrm{B}u(\tau)d\tau,$$

and its response is

$$y(t) = \mathrm{C}e^{\mathrm{E}^{-1}\mathrm{A}t}x_0 + \int_0^t \mathrm{C}e^{\mathrm{E}^{-1}\mathrm{A}(t-\tau)}\mathrm{E}^{-1}\mathrm{B}u(\tau)d\tau + \mathrm{D}u(t).$$

- **Second case :** If $\det \mathrm{E} = 0$, then, the system

$$\begin{cases} \mathrm{E}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) &= \mathrm{C}x(t) + \mathrm{D}u(t), \\ x(0) &= x_0, \end{cases} \tag{2.11}$$

is called singular linear continuous-time dynamical system. Its state has the form

$$x(t) = e^{\phi_0\mathrm{A}t}\phi_0\mathrm{E}x_0 + \int_0^t e^{\phi_0\mathrm{A}(t-\tau)}\phi_0\mathrm{B}u(\tau)d(\tau) + \sum_{i=1}^{\nu}\phi_{-i}\Big(\mathrm{B}u^{(i-1)}(t) + \mathrm{E}x_0\delta^{(i-1)}(t)\Big), \tag{2.12}$$

where ϕ_i, obtained using the Laurent series in the neighborhood of ∞

$$\left[s\mathrm{E} - \mathrm{A}\right]^{-1} = \sum_{i=-\nu}^{\infty}\phi_i s^{-i-1}, \tag{2.13}$$

are known as fundamental matrices, $\nu = \mathrm{rank}(\mathrm{E}) - \deg\left[\det[s\mathrm{E}-\mathrm{A}]\right] + 1$ is the nilpotent index of pencil (E,A), $u^{(i)}(t) = \dfrac{d^i u(t)}{dt^i}$, $i = 0, \cdots, \nu - 1$, and δ represents the Dirac delta function.

The output of the singular system (2.11) is given by

$$\begin{aligned} y(t) =& \mathrm{C}e^{\phi_0\mathrm{A}t}\phi_0\mathrm{E}x_0 + \int_0^t \mathrm{C}e^{\phi_0\mathrm{A}(t-\tau)}\phi_0\mathrm{B}u(\tau)d\tau \\ &+ \sum_{i=1}^{\nu}\mathrm{C}\phi_{-i}\big(\mathrm{B}u^{(i-1)}(t) + \mathrm{E}x_0\delta^{(i-1)}(t)\big) + \mathrm{CD}u(t). \end{aligned} \tag{2.14}$$

Substituting $x_0 = 0$ and $u(t) = \delta(t)$ into (2.14), we obtain the impulse response $g(t)$

of the system (2.11)

$$g(t) = \begin{cases} \mathrm{C}e^{\phi_0 \mathrm{A}t}\phi_0\mathrm{B} & \text{for} \quad t > 0, \\ \mathrm{C}e^{\phi_0 \mathrm{A}t}\phi_0\mathrm{B} + \sum_{i=1}^{\nu} \mathrm{C}\phi_{-i}\mathrm{B}\delta^{(i-1)}(t) + \mathrm{CD}\delta(t) & \text{for} \quad t = 0. \end{cases} \tag{2.15}$$

4.1.2 Externally and internally positive dynamical systems

Definition 4.2 [17] *A linear system described by the representation* (2.10) *is said to be positive, if and only if, for every non-negative initial state and for every non-negative input, its state and output are non-negative.*

Definition 4.3 [20] *The standard system* (2.10) *is called externally positive if for every* $x_0 = x(0) = 0$ *and every* $u(t) \in \mathbb{R}_+^m$, $t \geq 0$, *we have,* $y(t) \in \mathbb{R}_+^p$, *for* $t \geq 0$.

Theorem 4.4 [19] *The standard system* (2.10) *is externally positive, if and only if, its matrix of impulse response*

$$g(t) = \begin{cases} \mathrm{C}e^{\mathrm{E}^{-1}\mathrm{A}t}\mathrm{E}^{-1}\mathrm{B} & \textit{for} \quad t > 0, \\ \mathrm{D}\delta(t) & \textit{for} \quad t = 0, \end{cases} \tag{2.16}$$

is non-negative, i.e., $g(t) \in \mathbb{R}_+^{p\times m}$ *for* $t \geq 0$, *where* $\delta(t)$ *is the Dirac delta function.*

Definition 4.5 [19] *The standard system* (2.10) *is called internally positive if for every* $x_0 \in \mathbb{R}_+^n$ *and all inputs* $u(t) \in \mathbb{R}_+^m$, $t \geq 0$, *we have* $x(t) \in \mathbb{R}_+^n$ *and* $y(t) \in \mathbb{R}_+^p$ *for* $t \geq 0$.

Theorem 4.6 [19] *The standard system* (2.10) *is internally positive, if and only if,* A *is a Metzler matrix,* $\mathrm{B} \in \mathbb{R}_+^{n\times m}$, $\mathrm{C} \in \mathbb{R}_+^{p\times n}$, *and* $\mathrm{D} \in \mathbb{R}_+^{p\times m}$.

Remark 4.7 [19] *The standard internally positive system* (2.10) *is always externally positive.*

Definition 4.8 [19] *The singular system* (2.11) *is called externally positive if for* $x_0 = 0$, *and any non-negative input* $u(t) \geq 0$ *with* $u^{(i)}(t) \geq 0, i = 1, \cdots, \nu - 1$ *for* $t \in \mathbb{R}_+$, *the output* $y(t)$ *is also non-negative, i.e.,* $y(t) \geq 0$ *for* $t > 0$.

Theorem 4.9 [5] *The singular system* (2.11) *with* $\mathrm{D} = 0$ *is externally positive, if and only if, its impulse response* $g(t)$, *given by the expression* (2.15), *is non-negative, i.e.,* $g(t) \in \mathbb{R}_+$ *for* $t \in \mathbb{R}_+$.

Definition 4.10 [19] *The singular system* (2.11) *is called internally positive if for every admissible* $x_0 \in \mathbb{R}_+^n$ *and any non-negative input* $u(t) \geq 0$ *with* $u^{(i)}(t) \geq 0, i = 1, \cdots, \nu - 1$ *for* $t \in \mathbb{R}_+$, *the state vector* $x(t) \in \mathbb{R}_+^n$ *and the output* $y(t) \in \mathbb{R}_+^p$ *for* $t > 0$.

Definition 4.11 [19] *The singular system described by* (2.11) *is weakly positive if* A *is Metzler matrix,* $\mathrm{E} \in \mathbb{R}_+^{n\times n}$, $\mathrm{B} \in \mathbb{R}_+^{n\times m}$, $\mathrm{C} \in \mathbb{R}_+^{p\times n}$, *and* $\mathrm{D} \in \mathbb{R}_+^{p\times m}$.

Remark 4.12 [5] *The singular internally positive system* (2.11) *is always externally positive.*

4.2 Positivity of infinite-dimensional dynamical systems

4.2.1 Solvability of dynamical systems

Consider the continuous-time dynamical system described by the following abstract differential equation

$$\begin{cases} E\dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \end{cases} \tag{2.17}$$

where X is a Hilbert space provided with the inner product $\langle .,. \rangle$ and associated with the norm $\|.\|_X$. E, A are two operators defined from X to X, and $B \in \mathbf{B}(U,X)$, where U is a Hilbert space called control space.

In order to solve of the above problem, we need to estimate the resolvent $R_\lambda = (\lambda E + A)^{-1}E$ on half-plans $\mathrm{Re}\,\lambda > \alpha$, for $\alpha < \infty$, since, it plays a significant role during the analysis of the system (2.17). It can be solved by different methods, as the Laplace transform or the decomposition of spaces and operators [25, 39] and [40].

It should be noted that the system (2.17) admits several solutions depending on the operators E, A, and B and their natures. Among them, we can find the two following cases.

- **First case :** If E, A, and B are bounded operators with E invertible operator, then, the system (2.17) becomes

$$\begin{cases} \dot{x}(t) &= E^{-1}Ax(t) + E^{-1}Bu(t), \\ x(0) &= x_0. \end{cases} \tag{2.18}$$

  The trajectory of the system (2.18) is [4, 14]

$$x(t) = e^{E^{-1}At}x_0 + \int_0^t e^{E^{-1}A(t-\tau)}Bu(\tau)d\tau.$$

- **Second case :** If A is an infinitesimal generator of a strongly continuous semi-group $S(t)$ on X, E is the identity operator on X, and $B \in \mathbf{B}(U,X)$. Then, the system (2.17) turns into

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0. \end{cases} \tag{2.19}$$

  The solution of the system (2.19) is written as following [14]

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau.$$

4.2.2 Positivity of dynamical systems

The study of the positivity of infinite-dimensional dynamical systems requires various specific spaces like ordered vector space and ordered Banach space, with positive cone.

For more details, we refer the reader to [3].

From now, we assume that X, Y, and U are ordered Banach spaces with positive cones X^+, Y^+, and U^+ respectively, such that

$$X^+ = \left\{x \in X \mid x \geq 0\right\}, \quad Y^+ = \left\{y \in Y \mid y \geq 0\right\}, \quad \text{and} \quad U^+ = \left\{u \in U \mid u \geq 0\right\},$$

and $int(X^+) = \emptyset$, where $int(X^+)$ is the interior of X^+ (for the strong topology).

Let $\{e_n\}_{n\geq 1}$ be a positive Schauder basis of X, i.e., each element x of X has a unique representation of the from $x = \sum_{n=1}^{\infty} \alpha_n e_n$, such that the linear functional

$$\begin{aligned} \alpha_n : X &\longrightarrow F \\ x &\longmapsto \alpha_n =: \langle x, e_n \rangle, \end{aligned}$$

is bounded, where α_n denotes the n^{th} coordinate of x with respect to the basis $\{e_n\}_{n\geq 1}$ [46] and

$$X^+ = \left\{x = \sum_{n=1}^{\infty} \alpha_n e_n \mid \alpha \geq 0, \ \forall n\right\}.$$

Consider a closed linear operator A defined by

$$A : D(A) \subset X \longrightarrow X,$$

where $\{e_n\}_{n\geq 1} \subset D(A)$ and A is the infinitesimal generator of a C_0 semi-group $S_A(t)_{t\geq 0}$

Definition 4.13 [1]

i) *The operator* A *is said to be Metzler if*

$$a_{nk} = \langle Ae_k, e_n \rangle \geq 0, \quad \forall n \neq k;$$

ii) *The system* $\dot{x}(t) = Ax(t)$ *is said to be positive if* X^+ *is* $S_A(t)$*-invariant, i.e.,*

$$S_A(t)X^+ \subset X^+, \ \forall t \geq 0.$$

Definition 4.14 [1] *The system* (2.19)*, i.e., the pair* (A, B)*, is said to be positive if for every* $\forall x_0 \in X^+$ *and all inputs* $u \in \mathcal{U}^+$*, i.e.,* $\forall u \in \mathcal{U}$ *such that* $u(t) \in U^+$, $\forall t \geq 0$*, the state trajectories* $x(t)$ *remain in* X^+ *for all* $t \geq 0$.

Remark 4.15 *Note that* $\mathcal{U} = \{u : \mathbb{R}^+ \longrightarrow U, \textit{continuous}\}$.

Definition 4.16 [1] *A* C_0 *semi-group* $(S(t))_{t\geq 0}$ *is said to be positive if all the operators* $S(t)$*,* $t \geq 0$*, are positive, i.e.,* $S(t)X^+ \subset X^+$ *for all* $t \geq 0$.

Proposition 4.17 [1] *A* C_0 *semi-group* $(S(t))_{t\geq 0}$ *is positive, if and only if, its resolvent* $R(\lambda, A) := (\lambda I - A)^{-1}$ *is positive for all* $\lambda > \omega_0$*, where*

$$\begin{aligned} \omega_0 &:= \inf_{t>0} \frac{\log \|S(t)\|}{t}, \\ &= \lim_{t\to\infty} \frac{\log \|S(t)\|}{t}, \end{aligned}$$

is the growth constant of $(S(t))_{t\geq 0}$.

Theorem 4.18 [1] *The system* (2.19) *is positive, if and only if,* A *is the infinitesimal generator of a* C_0 *positive semi-group and* B *is a positive operator.*

Inspired by [20], we get the following lemma.

Lemma 4.19 A *is a Metzler operator, if and only if,* $\forall t \geq 0,\ e^{At}x_0 \in X^+$.

5 Various concepts of controllability

In this section, various concepts of controllability for finite dimensinal dynamical systems and infinite-dimensional dynamical systems will be presented. It is question of defining the controllabilibilty matrix and controllability notions for finite dimensinal dynamical systems and the controllability operator, exact contronllability, null contronllability, and approximate contronllability for infinite-dimensional dynamical systems case.

5.1 Controllability of finite dimensional dynamical systems

5.1.1 Controllability matrix

Consider the linear finite dimensional dynamical system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \quad \text{in } [0, T], \\ x(0) &= x_0, \end{cases} \tag{2.20}$$

where the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, and the initial condition $x_0 = 0$.

The solution of the linear differential equation (2.20) can be written as

$$L_t u := x(t, u(t)) = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau, \tag{2.21}$$

where L_t is a linear bounded transformation from $L^2(0, t; \mathbb{R}^m)$ into $\mathbb{R}^n$.

The controllability matrix $W(0,T)$, also called Gramian matrix, can be obtained using the expression (2.21) and its adjoint. Indeed, from (2.21) and

$$\begin{aligned} L_t^* : \mathbb{R}^n &\longrightarrow L^2(0,t;\mathbb{R}^m) \\ x &\longmapsto L_t^*(x) = B^* e^{A^*(t-\tau)} x, \end{aligned}$$

we get

$$\begin{aligned} W(0,T) &:= L_T L_T^*, \\ &= \int_0^T e^{A(T-\tau)} BB^* e^{A^*(T-\tau)} d\tau, \end{aligned} \tag{2.22}$$

which is symmetric and positive definite matrix.

5.1.2 Controllability notions

Definition 5.1 [4] *The system* (2.20) *is controllable (from $x_0 \in \mathbb{R}^n$) in* $[0,T]$ *if given any* $\overline{x} \in \mathbb{R}^n$*, there exists a control function* $u(.) \in L^2(0,T;\mathbb{R}^m)$ *such that*

$$x(T, u(.)) = \overline{x}.$$

Theorem 5.2 [4] *The system* (2.20) *is controllable in* $[0,T]$*, if and only if,* $W(0,T) > 0$*, i.e.,* $W(0,T)$ *is positive definite matrix.*

Theorem 5.3 [4] *The following conditions are equivalent*

i) *The system* (2.20) *is controllable in* $[0,T]$*;*

ii) $W(0,T) > 0$*, i.e.,* $W(0,T)$ *is positive definite matrix;*

iii) $\operatorname{rank}[B : AB : \cdots : A^{n-1}B] = n$.

Remark 5.4 *When the system is controllable, we will refer to* (A,B) *as a controllable pair.*

Proposition 5.5 [48] *Assume that for some* $T > 0$ *the matrix* $W(0,T)$ *is nonsingular. Then,*

i) *For arbitrary* $x_0, x_f \in \mathbb{R}^n$ *the control*

$$\hat{u}(\tau) = -B^* e^{A^*(T-\tau)} W^{-1}(0,T)(e^{AT} x_0 - x_f), \quad \tau \in [0,T],$$

transfers x_0 *to* x_f *at time* t*;*

ii) *Among all controls* $u(.)$ *steering* x_0 *to* x_f *at time* T *the control* $\hat{u}$ *minimizes the integral* $\int_0^T |u(\tau)|^2 d\tau$*. Moreover,*

$$\int_0^T |\hat{u}(\tau)|^2 d\tau = \langle W^{-1}(0,T)(e^{AT} x_0 - x_f), e^{AT} x_0 - x_f \rangle.$$

Theorem 5.6 [48] *The following conditions are equivalent*

i) *An arbitrary state* $x_f \in \mathbb{R}^n$ *is attainable from* 0*;*

ii) *The system* (2.20) *is controllable;*

iii) *The system* (2.20) *is controllable at a given time* $\mathrm{T} > 0$*;*

iv) *The matrix* $\mathrm{W}(0,\mathrm{T})$ *is nonsingular for some* $\mathrm{T} > 0$*;*

v) *The matrix* $\mathrm{W}(0,\mathrm{T})$ *is nonsingular for an arbitrary* $\mathrm{T} > 0$*;*

vi) $\mathrm{rank}[\mathrm{B} : \mathrm{AB} : \cdots : \mathrm{A}^{n-1}\mathrm{B}] = n$.

Remark 5.7 *The condition* (**vi**) *is called the Kalman rank condition.*

In this paragraph, we are interested to present some notions of the controllability for a singular dynamical system described by

$$\begin{cases} \mathrm{E}\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) & \text{in } [0,\mathrm{T}], \\ x(0) = x_0, \end{cases} \tag{2.23}$$

where the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^m$, $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{m\times n}$ with $\det \mathrm{E} = 0$, and the initial condition $x_0 = 0$.

In order to simplify the study of controllability, the system (2.23) must be transformed. For this purpose, we will use the Weierstrass decomposition [20]. Hence, the differential equation associated with the system (2.23) turns into

$$\begin{cases} \dot{x}_1(t) = \mathrm{A}_1 x_1(t) + \mathrm{B}_1 u(t), \\ \mathrm{N}\dot{x}_2(t) = x_2(t) + \mathrm{B}_2 u(t) \end{cases} \tag{2.24}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, $u \in \mathbb{R}^m$, $\mathrm{A}_1 \in \mathbb{R}^{n_1\times n_1}$, $\mathrm{B}_1 \in \mathbb{R}^{n_1\times m}$, $\mathrm{B}_2 \in \mathbb{R}^{n_2\times m}$, $\mathrm{N} \in \mathbb{R}^{n_2\times n_2}$ is nilpotent matrix of the index ν, and

$$\mathrm{QEP} = \mathrm{diag}(\mathrm{I}_{n_1}, \mathrm{N}),\ \ \mathrm{QAP} = \mathrm{diag}(\mathrm{A}_1, \mathrm{I}_{n_2}),\ \ \mathrm{QB} = \begin{pmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{pmatrix}, \text{ and } \mathrm{P}^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then, the system (2.23) can be written as two subsystems by

$$\begin{cases} \dot{x}_1(t) = \mathrm{A}_1 x_1(t) + \mathrm{B}_1 u(t), \\ x_1(0) = x_{01}, \end{cases} \tag{2.25}$$

and

$$\begin{cases} \mathrm{N}\dot{x}_2(t) = x_2(t) + \mathrm{B}_2 u(t), \\ x_2(0) = x_{02}. \end{cases} \tag{2.26}$$

Definition 5.8 [16] *The system* (2.24) *is called controllable if, for any* $t > 0$, $x_0 \in \mathbb{R}^n$ *and* $\overline{x} \in \mathbb{R}^n$, *there exists a control input* $u(t) \in \mathbb{R}^{\nu-1}$ *such that* $x(t) = \overline{x}$.

Then, we have the following theorem

Theorem 5.9 [16]

i) *The subsystems* (2.24) *is controllable, if and only if,*

$$\operatorname{rank}[s\mathrm{E}-\mathrm{A},\mathrm{B}] = n, \quad \forall s \in \mathbb{C},\ s \text{ finite};$$

ii) *The following statements are equivalent*

- *The subsystem* (2.26) *is controllable.*
- $\operatorname{rank}[\mathrm{B}_2 : \mathrm{NB}_2 : \cdots : \mathrm{N}^{\nu-1}\mathrm{B}_2] = n_2$;
- $\operatorname{rank}[\mathrm{N}\ \ \mathrm{B}_2] = n_2$;
- $\operatorname{rank}[\mathrm{E}\ \ \mathrm{B}] = n$;
- *for any nonsingular matrices* Q_1 *and* P_1 *satisfying* $\mathrm{E} = \mathrm{Q}_1 \operatorname{diag}(\mathrm{I}_d, 0)\mathrm{P}_1$, *and* $\mathrm{Q}_1\mathrm{B} = [\tilde{\mathrm{B}}_1/\tilde{\mathrm{B}}_2]$. *Then,* $\tilde{\mathrm{B}}_2$ *is of full row rank,*

$$\operatorname{rank}\tilde{\mathrm{B}}_2 = n - \operatorname{rank}\mathrm{E}.$$

iii) *The following statements are equivalent.*

- *System* (2.24) *is controllable;*
- *Both its two subsystems* (2.25) *and* (2.26) *are controllable;*
- $\operatorname{rank}[\mathrm{B}_1 : \mathrm{A}_1\mathrm{B}_1 : \cdots : \mathrm{A}_1^{n_1-1}\mathrm{B}_1] = n_1$ *and* $\operatorname{rank}[\mathrm{B}_2 : \mathrm{NB}_2 : \cdots : \mathrm{N}^{\nu-1}\mathrm{B}_2] = n_2$;
- $\operatorname{rank}[s\mathrm{E}-\mathrm{A}\ \ \mathrm{B}] = n$, $\forall s \in \mathbb{C}$, s *finite, and* $\operatorname{rank}[\mathrm{E}\ \ \mathrm{B}] = n$.

5.2 Controllability of infinite-dimensional dynamical systems

5.2.1 Controllability operator

Consider the linear system described by the following abstract Cauchy problem on a Hilbert space X

$$\begin{cases} \dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t), \quad \mathrm{T} > 0, \\ x(0) &= x_0 \in \mathrm{X}, \end{cases} \tag{2.27}$$

where $\mathrm{A} : \mathrm{D(A)} \subset \mathrm{X} \longrightarrow \mathrm{X}$ is linear operator generator a C_0 semi-group $\mathrm{S}(t)$ on X, and B is a bounded linear operator, such that

$$\mathrm{D(A)} = \left\{x \in \mathrm{X} \,\middle|\, \mathrm{A}x \in \mathrm{X}\right\}.$$

The trajectory of the dynamical system (2.27) is

$$x(\mathrm{T}) = \mathrm{S}(\mathrm{T})x_0 + \int_0^{\mathrm{T}} \mathrm{S}(\mathrm{T}-\tau)\mathrm{B}u(\tau)d\tau. \tag{2.28}$$

Let us consider the operator $\mathrm{L_T}$ as defined in [15] by

$$\begin{aligned} \mathrm{L_T} : \mathrm{L}^2(0,\mathrm{T};\mathrm{U}) &\longrightarrow \mathrm{X} \\ u &\longmapsto \mathrm{L_T}(u) = \int_0^{\mathrm{T}} \mathrm{S}(\mathrm{T}-\tau)\mathrm{B}u(\tau)d\tau. \end{aligned}$$

Note that $\mathrm{L_T}$ is a bounded linear operator, then, its adjoint operator is written as

$$\begin{aligned} \mathrm{L_T^*} : \mathrm{X} &\longrightarrow \mathrm{L}^2(0,\mathrm{T};\mathrm{U}) \\ x &\longmapsto \mathrm{L_T^*}(x) = \mathrm{B^*S^*}(\mathrm{T}-\tau)x. \end{aligned}$$

Consequently, the controllability operator is defined by the following equation [48]

$$\begin{aligned} \mathrm{Q_T}x &= \mathrm{L_T L_T^*}(x), \\ &= \int_0^{\mathrm{T}} \mathrm{S}(\tau)\mathrm{BB^*S^*}(\tau)x d\tau, \quad \forall x \in \mathrm{X}, \end{aligned} \tag{2.29}$$

where the function $\mathrm{S}(\tau)\mathrm{BB^*S}(\tau)x, \tau \in [0,\mathrm{T}]$ is continuous and the integral is well defined. Moreover, for a constant $c > 0$

$$\int_0^{\mathrm{T}} |\mathrm{S}(\tau)\mathrm{BB^*S^*}(\tau)x| d\tau \le c|x|, \quad x \in \mathrm{X}. \tag{2.30}$$

Hence, the operator $\mathrm{Q_T}$ is linear and continuous. It is also self-adjoint and nonnegative define, such that

$$\langle \mathrm{Q_T}x, x\rangle = \int_0^{\mathrm{T}} |\mathrm{B^*S^*}(\tau)x|^2 d\tau \ge 0; \quad x \in \mathrm{X}. \tag{2.31}$$

In what follows, we will denoted by $(\mathrm{Q_T})^{\frac{1}{2}}$ the unique self-adjoint and non-negative operator whose square is equal to $\mathrm{Q_T}$. There exists exactly one such operator. Well defined are the pseudoinverse operators $(\mathrm{Q_T})^{-1}$ and $\left(\mathrm{Q_T^{\frac{1}{2}}}\right)^{-1} := (\mathrm{Q_T})^{-\frac{1}{2}}$.

Theorem 5.10 [48]

i) *There exists a strategy $u(.) \in \mathrm{U}$ transferring $x_0 \in \mathrm{X}$ to $x_f \in \mathrm{X}$ in time T, if and only if,*

$$\left[\mathrm{S}(\mathrm{T})x_0 - x_f\right] \in \mathrm{Im}\,(\mathrm{Q_T})^{\frac{1}{2}};$$

ii) *Among the strategies transferring x_0 to x_f in time T, there exists exactly one strategy*

$\hat{u}$ which minimizes the functional

$$J_T(u) = \int_0^T |u(\tau)|^2 d\tau.$$

Moreover,

$$J_T(\hat{u}) = |\big(Q_T\big)^{-\frac{1}{2}}(S(T)x_0 - x_f)|^2;$$

iii) *If* $(S(T)x_0 - x_f) \in \operatorname{Im} Q_T$*, then, the stategy $\hat{u}$ is given by*

$$\hat{u}(t) = -B^* S^*(T-t)Q_T^{-1}(S(T)x_0 - x_f), \quad t \in [0,T].$$

5.2.2 Exact controllability

Definition 5.11 [14] *Given any two points $x_0, x_f \in X$, we say the system (2.27) is exactly controllable on $[0,T]$ if there exists a control $u \in L^p(0,T;U)$ such that $x(T) = x_f$, i.e.,*

$$\forall x_0 \in X,\ \forall x_f \in X,\ \exists u \in L^p(0,T;U) \quad \textit{such that} \quad x(T) = x_f.$$

Definition 5.12 [15] *The system (2.27) is exactly controllable $[0,T]$ (for some finite $T > 0$) if all points in X can be reached from origin at time T, i.e., if*

$$\operatorname{Im} L_T = X.$$

Theorem 5.13 [15] *The system (2.27) is exactly controllable from 0 in time T, if and only if,*

$$\operatorname{Im} L_T = X,$$

which means that the operator L_T is surjective.

Theorem 5.14 [15] *The system (2.27) is exactly controllable from 0 to T, if and only if,*

$$\exists \gamma > 0,\ \textit{such that}:\ \|L_T^* x\|_{L^2(0,T;U)} \geq \gamma \|x\|_X, \quad \forall x \in X.$$

where L_T^* is the adjoint operator of L_T.

Theorem 5.15 [48] *The following conditions are equivalent*

i) *The system (2.27) is exactly controllable from an arbitrary state in time $T > 0$;*

ii) *There exists $c > 0$ such that for arbitrary $x \in X$*

$$\int_0^T |B^* S^*(\tau)x|^2 d\tau \geq c|x|^2; \tag{2.32}$$

iii) $\mathrm{Im}\left(Q_T\right)^{\frac{1}{2}} = X.$

Theorem 5.16 [15] *The system* (2.27) *is exactly controllable from* 0 *to* T*, if and only if,*

$$\mathrm{Ker}\, L_T^* = \{0\} \quad \textit{and} \quad \mathrm{Im}\, L_T^* \textit{ is closed.}$$

Lemma 5.17 [15]

- *The system* (2.27) *is exactly controllable on* $[0,T]$*, if and only if, the system*

$$\begin{cases} \dot{x}(t) &= (sI_X - A)x(t) + Bu(t), \\ x(0) &= x_0, \end{cases}$$

  is for any $s \in \mathbb{C}$*;*

- *The system* (2.27) *is exactly controllable on* $[0,T]$*, if and only if, the system*

$$\begin{cases} \dot{x}(t) &= (A + BF)x(t) + Bu(t), \\ x(0) &= x_0, \end{cases}$$

  is for any $s \in \mathbb{C}$ *and for any feedback* $F \in \mathscr{L}(X,U)$.

5.2.3 Exact null controllability

Definition 5.18 [48] *The system* (2.27) *is exactly null controllable on* $[0,T]$*, if for*

$$\forall x_0 \in X,\ \exists u \in L^2(0,T;U) \quad \textit{such that} \quad x(T) = 0.$$

Definition 5.19 [48] *The system* (2.27) *is null controllable on* $[0,T]$ *if an arbitrary state can be transferred to* 0 *in time* T *or, equivalent, if and only if,*

$$\mathrm{Im}\, S(T) \subset \mathrm{Im}\, L_T.$$

Theorem 5.20 [48] *The following conditions are equivalent*

i) *The system* (2.27) *is null controllable in time* $T > 0$;

ii) *There exists* $c > 0$ *such that for all* $x \in X$

$$\int_0^T |B^* S^*(\tau)x|^2 d\tau \geq c|S^*(T)x|^2; \tag{2.33}$$

iii) $\mathrm{Im}\left(Q_T\right)^{\frac{1}{2}} \supset \mathrm{Im}\, S(T).$

5.2.4 Approximate controllability

Definition 5.21 [14] *The system (2.27) is approximately controllable for any $x_f \in \mathrm{X}$, and any $\epsilon > 0$, if there exists a control $u \in \mathrm{L}^2(0,\mathrm{T};\mathrm{U})$ such that $\|x(t) - x_f\|_{\mathrm{X}} \leq \epsilon$, i.e.,*

$$\forall \epsilon > 0, \quad \forall x_f \in \mathrm{X}, \exists u \in \mathrm{L}^2(0,\mathrm{T};\mathrm{U}) \quad \textit{such that} \quad \|x(t) - x_f\|_{\mathrm{X}} \leq \epsilon.$$

Definition 5.22 [15] *The system (2.27) is approximately controllable on $[0,\mathrm{T}]$ (for some finite $\mathrm{T} > 0$) if given an arbitrary $\epsilon > 0$, it is possible to steer from the origin to within a distance ϵ from all points in the state space at time T , i.e., if*

$$\overline{\mathrm{Im}\,\mathrm{L}_\mathrm{T}} = \mathrm{X}.$$

Theorem 5.23 [48] *If $\overline{\mathrm{Im}\,\mathrm{L}_\mathrm{T}} = \mathrm{X}$. Then, the system (2.27) is approximately controllable on $[0,\mathrm{T}]$.*

Theorem 5.24 [15] *The system (2.27) is approximately controllable on $[0,\mathrm{T}]$, if and only if,*

$$\ker \mathrm{L}_\mathrm{T}^* = \{0\}.$$

Theorem 5.25 [48] *The following conditions are equivalent*

i) *The system (2.27) is approximately controllable in time $\mathrm{T} > 0$ from an arbitrary state;*

ii) *If $\mathrm{B}^*\mathrm{S}^*(\tau)x = 0$ for almost all $\tau \in [0,\mathrm{T}]$, then, $x = 0$;*

iii) $\mathrm{Im}\left(\mathrm{Q}_\mathrm{T}\right)^{\frac{1}{2}}$ *is dense in* X.

Lemma 5.26 [15]

- *The system (2.27) is approximately controllable on $[0,\mathrm{T}]$, if and only if, the system*

$$\begin{cases} \dot{x}(t) &= (s\mathrm{I}_\mathrm{X} - \mathrm{A})x(t) + \mathrm{B}u(t), \\ x(0) &= x_0, \end{cases}$$

  is for any $s \in \mathbb{C}$;

- *The system (2.27) is approximately controllable on $[0,\mathrm{T}]$, if and only if, the system*

$$\begin{cases} \dot{x}(t) &= (\mathrm{A} + \mathrm{BF})x(t) + \mathrm{B}u(t), \\ x(0) &= x_0, \end{cases}$$

  is for any $s \in \mathbb{C}$ and for any feedback $\mathrm{F} \in \mathscr{L}(\mathrm{X},\mathrm{U})$.

6 Conclusion

The first part of this chapter was dedicated to the presentation of some particular positives matrices followed by some real examples for different problems in the finite dimensional dynamical systems and infinite-dimensional dynamical systems. Then, we have recalled some notions of positivity for both dynamical systems. Finally, various results on controllability for finite dimensional dynamical systems and for infinite-dimensional dynamical systems has been given.

Chapter 3

Minimum energy control of finite dimensional singular dynamical systems with rectangular inputs

1 Introduction

The problem of controllability, reachability and minimum energy control of finite dimensional singular dynamical systems with rectangular type inputs vector will be studied in this chapter. Necessary and sufficient conditions for the existence of such type of rectangular inputs that steers the system from zero initial conditions to the desired state will be established and proved by the use of the solution of the singular dynamical systems with rectangular type inputs and the application of the pseudo-inverse of the command matrix where the expression of the inputs vector can be found. Then, a new formulation of the minimum energy control problem for the singular dynamical systems with rectangular inputs using the Weierstrass-Kronecker decomposition is discussed followed by a procedure to solve the problem. Finally, in the last section, the computation of the optimal input and the minimal value of the performance index that represents the minimum energy of dynamical systems are presented and illustrated using a numerical example.

2 Position of the problem

Let us consider the singular continuous-time linear dynamical systems described by the following state equation

$$\begin{cases} \mathrm{E}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t), \qquad t \in [0, +\infty[, \\ x(0) &= 0, \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{n\times n}$, and $\mathrm{B} \in \mathbb{R}^{n\times m}$ with $\det \mathrm{E} = 0$. We assume that the system (3.1) is regular, i.e., for some $\lambda \in \mathbb{C}$

$$\det\left[\lambda \mathrm{E} - \mathrm{A}\right] \neq 0. \tag{3.2}$$

The rectangular input $u(t)$ has the form

$$u(t) = \begin{cases} \mathrm{U} & \text{for} \quad (i-1)\mathrm{T} \geq t \geq (i-1)\mathrm{T} + \Delta\mathrm{T}, \\ 0 & \text{for} \quad (i-1)\mathrm{T} + \Delta\mathrm{T} > t > i\mathrm{T}, \end{cases} \tag{3.3}$$

for $i = 1, 2, \cdots$. $\mathrm{U} \in \mathbb{R}^m$ is a vector of m constant, T is the period of the periodic signal and $\Delta\mathrm{T}$ is the pulse width.

As the expression (3.2) is well defined, then, the Weierstrass-Kronecker decomposition theorem [20] can be applied to the system (3.1). Hence, there exist a non singular matrices $\mathrm{P} \in \mathbb{R}^{n\times n}$ and $\mathrm{R} \in \mathbb{R}^{n\times n}$ that can be determined by the use the procedure give in [20]. Thus, thanks to the Weierstrass-Kronecker decomposition and some manipulations on the equation associated with the system (3.1), we get

$$\begin{cases} \dot{\tilde{x}}_1(t) &= \tilde{\mathrm{A}}_1 \tilde{x}_1(t) + \tilde{\mathrm{B}}_1 u(t), \\ \tilde{x}_1(0) &= 0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \mathrm{N}\dot{\tilde{x}}_2(t) &= \tilde{x}_2(t) + \tilde{\mathrm{B}}_2 u(t), \\ \tilde{x}_2(0) &= 0, \end{cases} \tag{3.5}$$

where, the new state vector is

$$\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \mathrm{R}^{-1} x(t), \quad \tilde{x}_1(t) \in \mathbb{R}^{n_1}, \quad \tilde{x}_2(t) \in \mathbb{R}^{n_2}, \tag{3.6}$$

and

$$\tilde{\mathrm{E}} = \mathrm{PER} = \begin{bmatrix} \mathrm{I}_{n_1} & 0 \\ 0 & \mathrm{N} \end{bmatrix}, \quad \tilde{\mathrm{A}} = \mathrm{PAR} = \begin{bmatrix} \tilde{\mathrm{A}}_1 & 0 \\ 0 & \mathrm{I}_{n_2} \end{bmatrix}, \quad \tilde{\mathrm{B}} = \mathrm{PB} = \begin{bmatrix} \tilde{\mathrm{B}}_1 \\ \tilde{\mathrm{B}}_2 \end{bmatrix}, \tag{3.7}$$

with $\tilde{\mathrm{A}}_1 \in \mathbb{R}^{n_1\times n_1}$, $\tilde{\mathrm{B}}_1 \in \mathbb{R}^{n_1\times m}$, $\tilde{\mathrm{B}}_2 \in \mathbb{R}^{n_2\times m}$, $n_1 + n_2 = n$, and $\mathrm{N} \in \mathbb{R}^{n_2\times n_2}$ is nilpotent matrix with the index μ such that, for some $\lambda \in \mathbb{C}$

$$\mu = \mathrm{rank}(\mathrm{E}) - \deg\left(\det\left[\lambda \mathrm{E} - \mathrm{A}\right]\right) + 1. \tag{3.8}$$

The solution of the standard subsystem (3.4) has the from

$$\tilde{x}_1(t) = e^{\tilde{\mathrm{A}}_1 t} \tilde{x}_1(0) + \int_0^t e^{\tilde{\mathrm{A}}_1 \tau} \tilde{\mathrm{B}}_1 u(t-\tau) d\tau, \tag{3.9}$$

and the one of the singular subsystem (3.5) is

$$\tilde{x}_2(t) = \sum_{i=0}^{\mu-1} \mathrm{N}^{(i)}\tilde{\mathrm{B}}_2 u^{(i)}(t). \tag{3.10}$$

Finally, the general solution of the singular dynamical system (3.1) can be written as

$$x(t) = \begin{bmatrix} \mathrm{R}e^{\tilde{\mathrm{A}}_1 t}\tilde{x}_1(0) + \int_0^t \mathrm{R}e^{\tilde{\mathrm{A}}_1 \tau}\tilde{\mathrm{B}}_1 u(t-\tau)d\tau \\ \sum_{i=0}^{\mu-1} \mathrm{R}\mathrm{N}^{(i)}\tilde{\mathrm{B}}_2 u^{(i)}(t) \end{bmatrix}. \tag{3.11}$$

Let us recall that the main goal of this chapter is to minimize the integral performance index

$$\mathrm{I}(u(t_f)) = \int_0^{t_f} u^{\mathrm{T}}(\tau)\mathrm{Q}u(\tau)d\tau, \tag{3.12}$$

of the system (3.1) where $\mathrm{Q} \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, i.e., $\mathrm{Q} = \mathrm{Q}^{\mathrm{T}}$ and $v^{\mathrm{T}}\mathrm{Q}v > 0$ for nonzero vector $v \in \mathbb{R}^m$. To deal with this problem, we will determine the input vector for each subsystem (3.4) and (3.5), that minimizes their performance index $\mathrm{I}_1(u(t_f))$ and $\mathrm{I}_2(u(t_f))$ respectively

$$\mathrm{I}_1(u(t_f)) = \int_0^{t_f} u^{\mathrm{T}}(\tau)\mathrm{Q}_1 u(\tau)d\tau, \tag{3.13}$$

and

$$\mathrm{I}_2(u(t_f)) = \sum_{i=0}^{\mu-1} \left(u^{(i)}\right)^{\mathrm{T}}(t_f)\mathrm{Q}_2 u^{(i)}(t_f), \tag{3.14}$$

where $\mathrm{Q}_1, \mathrm{Q}_2 \in \mathbb{R}^{m\times m}$ are a symmetric positive definite matrices.

However, before to deal with this problem, we need to ensure the existence of the inputs. This is the first goal of the next section.

3 Minimum energy control problem

In this section, we will determine the minimum energy control and the optimum values of the performance index for each subsystems (3.4) and (3.5), and the singular dynamical system (3.1).

3.1 Control problem

To find the minimum energy control of the subsystems (3.4) and (3.5), we must check whether the inputs that minimize the performance index (3.13) and (3.14) respectively, exist.

First, let us emphasized that our subsystems (3.4) and (3.5) must be controllable. In order to establish the input that steer the first system

$$\begin{cases} \dot{\tilde{x}}_1(t) &= \tilde{A}_1\tilde{x}_1(t)+\tilde{B}_1 u(t), \\ \tilde{x}_1(0) &= 0, \end{cases} \tag{3.15}$$

with the rectangular inputs

$$u(t) = \begin{cases} U_1 & \text{for} \quad (i-1)T \geq t \geq (i-1)T+\Delta T, \\ 0 & \text{for} \quad (i-1)T+\Delta T > t > iT, \end{cases} \tag{3.16}$$

where $U_1 \in \mathbb{R}^m$ is a vector of m constant, T is the period of the periodic signal and ΔT is the pulse width with for $i = 1,2,\cdots$, from zero initial condition to desired final state in time $t_f = q\Delta T$, we will use the expression of its solution.

Indeed,

$$\tilde{x}_{1f} = \tilde{x}_1(t_f) = e^{\tilde{A}_1 t_f}\tilde{x}_{01} + \int_0^{t_f} e^{\tilde{A}_1\tau}\tilde{B}_1 u(t_f-\tau)d\tau, \tag{3.17}$$

then,

$$\begin{aligned} \tilde{x}_1(t_f) &= \int_0^{t_f} e^{\tilde{A}_1\tau}\tilde{B}_1 u(t_f-\tau)d\tau, \\ &= \int_{T-\Delta T}^{T} e^{\tilde{A}_1\tau}\tilde{B}_1 U_1 d\tau + \int_{2T-\Delta T}^{2T} e^{\tilde{A}_1\tau}\tilde{B}_1 U_1 d\tau + \cdots + \int_{qT-\Delta T}^{qT} e^{\tilde{A}_1\tau}\tilde{B}_1 U_1 d\tau. \end{aligned} \tag{3.18}$$

Hence,

$$\begin{aligned} \tilde{A}_1\tilde{x}_{1f} &= \tilde{A}_1\left[\int_{T-\Delta T}^{T} e^{\tilde{A}_1\tau}d\tau + \int_{2T-\Delta T}^{2T} e^{\tilde{A}_1\tau}d\tau + \cdots + \int_{qT-\Delta T}^{qT} e^{\tilde{A}_1\tau}d\tau\right]\tilde{B}_1 U_1, \\ &= \left[e^{\tilde{A}_1 T} - e^{\tilde{A}_1\left(T-\Delta T\right)} + e^{2\tilde{A}_1 T} + \cdots + e^{q\tilde{A}_1 T} - e^{\tilde{A}_1\left(qT-\Delta T\right)}\right]\tilde{B}_1 U_1, \end{aligned} \tag{3.19}$$

multiplying both sides of the expression (3.19) by $\left[I_{n_1} - e^{-\tilde{A}_1\Delta T}\right]^{-1}$, we get

$$\begin{aligned} \left[I_{n_1} - e^{-\tilde{A}_1\Delta T}\right]^{-1}\tilde{A}_1\tilde{x}_{1f} &= \left[e^{\tilde{A}_1 T} + e^{2\tilde{A}_1 T} + \cdots + e^{q\tilde{A}_1 T}\right]\tilde{B}_1 U_1, \\ &= \left[\sum_{k=1}^{q}\left(e^{\tilde{A}_1 T}\right)^k\right]\tilde{B}_1 U_1. \end{aligned} \tag{3.20}$$

Thus,

$$\left[\sum_{k=1}^{q}\left(e^{\tilde{A}_1 T}\right)^k\right]^{-1}\left[I_{n_1} - e^{-\tilde{A}_1\Delta T}\right]^{-1}\tilde{A}_1\tilde{x}_{1f} = \tilde{B}_1 U_1. \tag{3.21}$$

Because of the invertibility of the exponential function for any matrix and since the subsystem (3.15) is controllable, then, the input U_1 can be calculated by the following formula

$$\mathrm{U}_1 = \tilde{\mathrm{B}}_1^{+}\left[\sum_{k=1}^{q}\left(e^{\tilde{\mathrm{A}}_1\mathrm{T}}\right)^{k}\right]^{-1}\left[\mathrm{I}_{n_1} - e^{-\tilde{\mathrm{A}}_1\Delta\mathrm{T}}\right]^{-1}\tilde{\mathrm{A}}_1\tilde{x}_{1f}, \tag{3.22}$$

where $\tilde{\mathrm{B}}_1^{+} \in \mathbb{R}^{m\times n_1}$ is the right pseudo-inverse of the rectangular matrix $\tilde{\mathrm{B}}_1 \in \mathbb{R}^{n_1\times m}$, which can be calculated by

$$\tilde{\mathrm{B}}_1^{+} = \tilde{\mathrm{B}}_1^{\mathrm{T}}\left[\tilde{\mathrm{B}}_1\tilde{\mathrm{B}}_1^{\mathrm{T}}\right]^{-1} + \left[\mathrm{I}_m - \tilde{\mathrm{B}}_1^{\mathrm{T}}\left[\tilde{\mathrm{B}}_1\tilde{\mathrm{B}}_1^{\mathrm{T}}\right]^{-1}\tilde{\mathrm{B}}_1\right]\mathrm{K}_1,\ \text{for an arbitrary}\ \mathrm{K}_1 \in \mathbb{R}^{m\times n_1}, \tag{3.23}$$

or even by

$$\tilde{\mathrm{B}}_1^{+} = \mathrm{K}_2\left[\tilde{\mathrm{B}}_1\mathrm{K}_2\right]^{-1},\ \text{for an arbitrary}\ \mathrm{K}_2 \in \mathbb{R}^{m\times n_1},\ \text{with}\ \det\left[\tilde{\mathrm{B}}_1\mathrm{K}_2\right] \neq 0.$$

Now, let us consider the subsystem

$$\begin{cases} \mathrm{N}\dot{\tilde{x}}_2(t) &= \tilde{x}_2(t) + \tilde{\mathrm{B}}_2 u(t), \\ \tilde{x}_2(0) &= 0, \end{cases} \tag{3.24}$$

with the rectangular inputs

$$u(t) = \begin{cases} \mathrm{U}_2 & \text{for} \quad (i-1)\mathrm{T} \geq t \geq (i-1)\mathrm{T} + \Delta\mathrm{T}, \\ 0 & \text{for} \quad (i-1)\mathrm{T} + \Delta\mathrm{T} > t > i\mathrm{T}, \end{cases} \tag{3.25}$$

where $\mathrm{U}_2 \in \mathbb{R}^m$ is a vector of m constant, T is the period of the periodic signal and $\Delta\mathrm{T}$ is the pulse width with for $i = 1, 2, \cdots$. To find the input U_2 that steer the subsystem (3.24) from the initial condition to final state in time $t_f = q\Delta\mathrm{T}$, we will use the solution of the subsystem (3.24).

Hence, the use of the solution of the subsystem (3.24) and taking in account the fact that the input $u(t)$ is a constant function, it yields

$$\begin{aligned} \tilde{x}_2(t_f) &= -\sum_{i=0}^{\mu-1}\mathrm{N}^{(i)}\tilde{\mathrm{B}}_2 u^{(i)}(t), \\ &= -\tilde{\mathrm{B}}_2 u^{(0)}(t_f), \\ &= -\tilde{\mathrm{B}}_2\mathrm{U}_2. \end{aligned} \tag{3.26}$$

As the subsystem (3.24) is controllable in time $t_f = q\Delta\mathrm{T}$, then, the input U_2 can be computed as following

$$\mathrm{U}_2 = -\tilde{\mathrm{B}}_2^{+}\tilde{x}_2(t_f), \tag{3.27}$$

where the right pseudo-inverse $\tilde{\mathrm{B}}_2^{+}$ of the rectangular matrix $\tilde{\mathrm{B}}_2 \in \mathbb{R}^{n_2\times m}$, can be obtained using

$$\tilde{B}_2^+ = \tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1} + \left[I_m - \tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{B}_2\right]K_1, \text{ for an arbitrary } K_1 \in \mathbb{R}^{m\times n_2}, \tag{3.28}$$

or

$$\tilde{B}_2^+ = K_2\left[\tilde{B}_2K_2\right]^{-1}, \text{ for an arbitrary } K_2 \in \mathbb{R}^{m\times n_2}, \text{ with } \det\left[\tilde{B}_2K_2\right] \neq 0. \tag{3.29}$$

Note that, the input U_2 expressed by the formula (3.27) steers the subsystem (3.24) from $\tilde{x}_{02} = 0$ to $\tilde{x}_{2f} \in \mathbb{R}^{n_2}$.

The problem of minimum energy control for each subsystems (3.15) and (3.24) require to find an optimum input vector from the whole set, i.e., we need to find the values of matrices K_1 and K_2 such that the performance indexes given by (3.13) and (3.14) respectively take the minimal value. This will be dealt in the following part.

3.2 Index of the performance problem

In order to get the optimum value of inputs vector for the subsystems (3.15) and (3.24) we need to find their performance indexes (3.13) and (3.14) respectively.

It is well know that, the performance index (3.13) for the first subsystem (3.15) in the case of rectangular type input signal (3.16) takes the form [38]

$$I_1(u(t_f)) = q\Delta T U_1^T Q_1 U_1. \tag{3.30}$$

Let us denote

$$M = \left[\sum_{k=1}^{q}\left(e^{\tilde{A}_1 T}\right)^k\right]^{-1}\left[I_{n_1} - e^{-\tilde{A}_1\Delta T}\right]^{-1}\tilde{A}_1\tilde{x}_{1f}. \tag{3.31}$$

Substituting (3.22) and (3.30) with the use of the right pseudo-inverse (3.23) for arbitrary matrix K_1, yields

$$\begin{aligned}
I_1(u(t_f)) &= q\Delta T U_1^T Q_1 U_1,\\
&= q\Delta T M^T\left[\tilde{B}_1^+\right] Q_1 \tilde{B}_1^+ M,\\
&= q\Delta T M^T\left[\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\tilde{B}_1 + K_1^T\left[I_m - \tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\tilde{B}_1\right]\right]Q_1\\
&\quad\times\left[\tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1} + \left[I_m - \tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\tilde{B}_1\right]K_1\right]M.
\end{aligned}$$

Consequently, the performance index takes the minimum value for any positive definite matrix $Q_1 \in \mathbb{R}^{m\times m}$ and $K_1 = 0_{m\times n_1}$. Hence,

$$\begin{aligned}
I_{1opt} &= I_1(u(t_f)),\\
&= q\varDelta T M^T\left[\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\tilde{B}_1\right]Q_1\left[\tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\right]M. \qquad (3.32)
\end{aligned}$$

And by the formula (3.22), the optimal value of the input U_1 is

$$\begin{aligned}
U_{1opt} &= \tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}\left[\sum_{k=1}^{q}(e^{\tilde{A}_1 T})^k\right]^{-1}\left[I_{n_1} - e^{-\tilde{A}_1\varDelta T}\right]^{-1}\tilde{A}_1\tilde{x}_{1f},\\
&= \tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}M.
\end{aligned} \qquad (3.33)$$

Furthermore, let us recall that for the subsystem (3.24), the performance index takes the form

$$\begin{aligned}
I_2(u(t_f)) &= \sum_{i=0}^{\mu-1}\left(u^{(i)}\right)^T(t_f)Q_2u^{(i)}(t_f),\\
&= U_2^T Q_2 U_2.
\end{aligned} \qquad (3.34)$$

Using (3.27) and (3.34), we get

$$\begin{aligned}
I_2(u(t_f)) &= \left[-\tilde{B}_2^+\tilde{x}_2(t_f)\right]^T Q_2\left[-\tilde{B}_2^+\tilde{x}_2(t_f)\right],\\
&= \tilde{x}_2^T(t_f)\left[\tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1} + \left[I_m - \tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{B}_2\right]K_1\right]^T\\
&\quad\times Q_2\left[\tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1} + \left[I_m - \tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{B}_2\right]K_1\right]\tilde{x}_2(t_f).
\end{aligned}$$

Finally, the minimal value of the performance index (3.34) takes the form

$$I_{2opt} = \tilde{x}_2^T(t_f)\left[\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{B}_2\right]Q_2\left[\tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\right]\tilde{x}_2(t_f), \qquad (3.35)$$

and the optimal value of the input can be calculated by the use of the formula (3.27) with the right pseudo-inverse $\tilde{B}_2^+$ and $K_1 = 0_{m\times n_2}$ as

$$U_{2opt} = -\tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{x}_{2f}. \qquad (3.36)$$

Combining the above results, it follows

$$\begin{aligned}
U_{opt} &= U_{1opt} + U_{2opt},\\
&= \tilde{B}_1^T\left[\tilde{B}_1\tilde{B}_1^T\right]^{-1}F\tilde{x}_{1f} + \tilde{B}_2^T\left[\tilde{B}_2\tilde{B}_2^T\right]^{-1}\tilde{x}_{2f},\\
&= B^+L\tilde{x}_f, \qquad (3.37)
\end{aligned}$$

that represents the optimal value of the input for the general singular system (3.1) with

$$\mathrm{L} = \begin{bmatrix} \mathrm{F} & 0 \\ 0 & \mathrm{I}_{n_2} \end{bmatrix}, \quad \mathrm{F} = \left[\sum_{k=1}^{q} \left(e^{\tilde{\mathrm{A}}_1 \mathrm{T}}\right)^k\right]^{-1} \left[\mathrm{I}_{n_1} - e^{-\tilde{\mathrm{A}}_1 \Delta \mathrm{T}}\right]^{-1} \tilde{\mathrm{A}}_1,$$

$$\mathrm{B}^+ = \left[\tilde{\mathrm{B}}_1^{\mathrm{T}} \left[\tilde{\mathrm{B}}_1 \tilde{\mathrm{B}}_1^{\mathrm{T}}\right]^{-1} \quad \tilde{\mathrm{B}}_2^{\mathrm{T}} \left[\tilde{\mathrm{B}}_2 \tilde{\mathrm{B}}_2^{\mathrm{T}}\right]^{-1}\right], \quad \text{and} \quad \tilde{x}_f = \begin{bmatrix} \tilde{x}_{1f} \\ \tilde{x}_{2f} \end{bmatrix}.$$

Furthermore, the minimal value of the performance index for the general singular system (3.1) is given by

$$\begin{aligned} \mathrm{I}_{opt} &= \mathrm{I}_{1opt} + \mathrm{I}_{2opt}, \\ &= q \Delta \mathrm{T} \mathrm{U}_{1opt}^{\mathrm{T}} \mathrm{Q}_1 \mathrm{U}_{1opt} + \mathrm{U}_{2opt}^{\mathrm{T}} \mathrm{Q}_2 \mathrm{U}_{2opt}, \\ &= \begin{bmatrix} \mathrm{U}_{1opt} \\ \mathrm{U}_{2opt} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} q \Delta \mathrm{T} \mathrm{Q}_1 & 0 \\ 0 & \mathrm{Q}_2 \end{bmatrix} \begin{bmatrix} \mathrm{U}_{1opt} \\ \mathrm{U}_{2opt} \end{bmatrix}. \end{aligned} \tag{3.38}$$

4 Procedure for computing the minimum energy control

A procedure for computing the minimum energy control of the finite-dimensional singular dynamical systems with rectangular inputs will be presented in this section.

The following steps are used to find the optimal value of the rectangular inputs and the minimal value of the performance index for the singular dynamical system (3.1).

- **Step 1 :** Knowing E, $\mathrm{A} \in \mathbb{R}^{n \times n}$, and $\mathrm{B} \in \mathbb{R}^{n \times m}$, find the matrices P, $\mathrm{R} \in \mathbb{R}^{n \times n}$, then, the matrices $\tilde{\mathrm{A}}_1 \in \mathbb{R}^{n_1 \times n_1}$, $\mathrm{N} \in \mathbb{R}^{n_2 \times n_2}$, $\tilde{\mathrm{B}}_1 \in \mathbb{R}^{n_1 \times m}$, and $\tilde{\mathrm{B}}_2 \in \mathbb{R}^{n_2 \times m}$ will be computed by the use of the decomposition presented by (3.7).

- **Step 2 :** Using the right pseudo-inverse (3.23), the input U_{1opt} will be calculated.

- **Step 3 :** Using the input (3.33), the minimal value of the performance index of the first subsystem (3.15) can be computed.

- **Step 4 :** Thanks to the right pseudo-inverse (3.28), the input U_{2opt} can be calculated.

- **Step 5 :** Using the input (3.36), the minimal value of the performance index of the second subsystem (3.24) can be computed.

- **Step 6 :** Knowing U_{1opt} and U_{2opt} compute the optimal input U_{opt} for the singular dynamical system (3.1) through the formula (3.37).

- **Step 7 :** Using (3.38), the minimal value of the performance index of the singular dynamical system (3.1) can be computed.

5 Example

Consider the singular dynamical system (3.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$x_0 = [0\,0\,0]^T$, $\quad x_f = [1\,1\,1]^T$, $\quad t_f = 1.25\,s$, $\quad T = 0.25\,s$, and $\quad \Delta T = 0.15\,s$.

In this case, we have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{E} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PER = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \end{bmatrix}, \tag{3.39}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PAR = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.40}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, PB = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \text{and } \tilde{B}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}. \tag{3.41}$$

It is clear that the system (3.1) is controllable for the rectangular inputs since the conditions presented on theorem 5.9 are satisfied, then, we have

$$\operatorname{rank}(\tilde{A}_1) = \operatorname{rank}(\tilde{B}_1) = 2 \quad \text{and} \quad \operatorname{rank}(\lambda N - I_{n_2}) = \operatorname{rank}(\tilde{B}_2) = 1. \tag{3.42}$$

Thus, the subsystems (3.15) and (3.24) are controllable.

The use of the procedure presented in section 4, gives

- **Step 1 :** The matrices $N \in \mathbb{R}^{n_2 \times n_2}$, $\tilde{A}_1 \in \mathbb{R}^{n_1 \times n_1}$, $\tilde{B}_1 \in \mathbb{R}^{n_1 \times m}$, and $\tilde{B}_2 \in \mathbb{R}^{n_2 \times m}$ are given, respectively, by (3.39), (3.40), and (3.41).

- **Step 2 :** Using the formula of U_{1opt} together with the right pseudo-inverse of rectangular matrix $\tilde{B}_1$ and the performance index (3.30), we obtain

$$U_{1opt} = \begin{bmatrix} 27.130 \\ 13.565 \\ 13.565 \\ 0 \end{bmatrix}. \tag{3.43}$$

- **Step 3 :** The minimal value of the performance index for the first subsystem (3.15) is

$$I_{1opt} = 828.04. \tag{3.44}$$

- **Step 4 :** Using (3.36) with (3.28), we obtain

$$U_{2opt} = \begin{bmatrix} -0.33 \\ 0 \\ -0.33 \\ -0.33 \end{bmatrix}. \tag{3.45}$$

- **Step 5 :** The minimal value of the performance index for the second subsystem (3.24) is

$$I_{2opt} = \frac{1}{3}. \tag{3.46}$$

- **Step 6 :** Using (3.37), we get

$$\begin{aligned} U_{opt} &= U_{1opt} + U_{2opt}, \\ &= \begin{bmatrix} 26.797 \\ 13.565 \\ 13.232 \\ -0.333 \end{bmatrix}. \end{aligned} \tag{3.47}$$

- **Step 7 :** Using (4.11), we get the minimal value of the performance index for the singular system

$$\begin{aligned} I_{opt} &= I_{1opt} + I_{2opt}, \\ &= 828.04 + 0.33, \\ &= 828.37. \end{aligned} \tag{3.48}$$

6 Conclusion

In this chapter, the problem of minimum energy control of the finite dimensional singular continuous-time linear dynamical systems with rectangular inputs has been studied. First, thanks to Weierstrass-Kronecker decomposition, the singular system has been decomposed on two subsystems. Then, the obtained subsystems help us to find the optimal control and the minimal value of the performance index of the singular dynamical system. Finally, a procedure for computing the minimum energy control of the finite dimensional singular system with rectangular inputs has been presented, where its effectiveness has been showed through an academic example.

Chapter 4

Minimum energy control of infinite-dimensional degenerate Cauchy problem with skew-hermitian pencil

1 Introduction

The minimum energy control problem is strongly connected with the controllability concept [22] and [23]. It is well know, in the state-of-the-art, that there exists, generally, many different admissible controls $u(t)$, defined for t belongs to $[0, t_f]$, which transfer the given initial state x_0 to the desired final state x_{t_f} at time t_f. However, we must know which of these possible admissible controls $u(t)$ are optimal according to the given a priori criterion. This problem will be dealt in this chapter, where we will find the expression of the minimum energy control for the infinite-dimensional degenerate Cauchy problem with variable operator coefficients, skew-hermitian pencil, and bounded-input conditions.

2 Transformation of degenerate Cauchy problem by the use of an orthogonal decomposition

This section is devoted to recall some mathematical background that are used along with this chapter and to present a decomposition for an the infinite-dimensional degenerate dynamical system represented by Cauchy problem with operators coefficients, skew-hermitian pencil, a given initial state, and bounded input in order to simplify its study.

Let us consider X, Y, U, and $\mathrm{L}^2\big(0, t; \mathrm{U}\big)$ as Hilbert spaces with $t \in]0, +\infty[$, and the following infinite-dimensional degenerate Cauchy problem

$$\begin{cases} E\dot{x}(t) + Ax(t) &= Bu(t), \\ x(0) &= x_0, \end{cases} \tag{4.1}$$

where $x(t)$ and $u(t)$ are, respectively, the state and the input vectors. $E \in \mathscr{L}(X, Y)$, $A \in \mathscr{L}(X, Y)$, and $B \in \mathscr{L}(X, Y)$ are bounded operators, with E a singular operator.

Based on [40], if the pencil $(\mu E^* + A^*)$, for some $\mu \in \mathbb{C}$, is a skew-hermitian operator, then, the condition

$$EA^* + AE^* = 0,$$

holds where A^* and E^* are respectively the adjoint operators of the operators A and E.

The following paragraph prescribes the decomposition of the infinite-dimensional degenerate Cauchy problem given by the system (4.1) to two dynamical subsystems. It must be mentioned that the decomposition used has various advantageous in the field of infinite-dimensional systems and help us to investigate them [40].

Indeed, by the use of an orthogonal decomposition for the spaces X and Y and a corresponding decomposition into blocks for the operators E, A, and B, it follows

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2,$$

and

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ and } \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

with

$$X_1 = \ker(E),\ X_2 = (\ker(E))^\perp,\ Y_1 = \ker(E^*),\ \text{and}\ Y_2 = (\ker(E^*))^\perp,$$

E^* is the adjoint operator of the operator E, $(\ker(E))^\perp$ and $(\ker(E^*))^\perp$ represent, respectively, the orthogonal of the kernel of E and the orthogonal kernel of E^*. Hence, the operator E becomes

$$E = \begin{bmatrix} 0 & 0 \\ 0 & E_{22} \end{bmatrix}.$$

However, the condition of the skew-hermitian of the pencil $(\mu E^* + A^*)$ and issue on *Rutkas* [40] involve that $A_{12} = 0$. Then, the decomposition of the operator A becomes

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}.$$

Finally, the system (4.1) is equivalent to the following system,

$$\begin{cases} A_{11}x_1(t) &= B_1u(t), \\ E_{22}\dot{x}_2(t) &= -A_{21}x_1(t) - A_{22}x_2(t) + B_2u(t), \\ x_1(0) &= x_{01}, \\ x_2(0) &= x_{02}. \end{cases} \tag{4.2}$$

The relations

$$\lambda E + A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & \lambda E_{22} + A_{22} \end{bmatrix} \quad \text{and} \quad E_{22}A_{22}^* + A_{22}E_{22}^* = 0,$$

where λ is the regular point of the pencil $(\lambda E_{22} + A_{22})$ and ensures that the operator A_{11} is invertible allow as to turn the system (4.2) into

$$\begin{cases} x_1(t) &= A_{11}^{-1}B_1u(t), \\ E_{22}\dot{x}_2(t) &= -A_{21}x_1(t) - A_{22}x_2(t) + B_2u(t), \\ x_1(0) &= x_{01}, \\ x_2(0) &= x_{02}, \end{cases}$$

or even into

$$\begin{cases} x_1(t) &= A_{11}^{-1}B_1u(t), \\ E_{22}\dot{x}_2(t) &= -A_{22}x_2(t) + \left(B_2 - A_{21}A_{11}^{-1}B_1\right)u(t), \\ x_1(0) &= x_{01}, \\ x_2(0) &= x_{02}. \end{cases}$$

Nevertheless, the operator's invertibility of E_{22} is ensured by the density of the space X_2. Therefore, it follows

$$\begin{cases} x_1(t) &= A_{11}^{-1}B_1u(t), \\ x_1(0) &= x_{01}, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dot{x}_2(t) &= \tilde{A}x_2(t) + \tilde{B}u(t), \\ x_2(0) &= x_{02}, \end{cases} \tag{4.4}$$

where $\tilde{A}$ and $\tilde{B}$ are two bounded operators, such that

$$\tilde{A} = -E_{22}^{-1}A_{22}, \quad \text{and} \quad \tilde{B} = E_{22}^{-1}\left(B_2 - A_{21}A_{11}^{-1}B_1\right).$$

The solutions of the systems (4.3) and (4.4) have the forms

$$\begin{aligned} x_1(t) &= A_{11}^{-1}B_1u(t), \\ x_2(t) &= e^{\tilde{A}t}x_{02} + \int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)d\tau, \end{aligned} \tag{4.5}$$

where $t > 0$.

3 Some results on exact controllability

In this section, some results on exact controllability of the infinite-dimensional degenerate Cauchy problem are established in order to find its minimum energy control and, then, to get the optimal control. However, we start by recalling the definition and criteria for the exact controllability.

Let us consider the operators $\widehat{\mathrm{L}}_{t_f}$ and L_{t_f} as defined in [15] by

$$\begin{array}{rcl} \widehat{\mathrm{L}}_{t_f} : \mathrm{L}^2(0,t_f;\mathrm{U}) & \longrightarrow & \mathrm{X}_2 \\ u & \longmapsto & \widehat{\mathrm{L}}_{t_f}(u) = \displaystyle\int_0^{t_f} e^{\tilde{\mathrm{A}}(t_f-\tau)}\tilde{\mathrm{B}}u(\tau)d\tau, \end{array}$$

and

$$\begin{array}{rcl} \mathrm{L}_{t_f} : \mathrm{L}^2(0,t_f;\mathrm{U}) & \longrightarrow & \mathrm{X} \\ u & \longmapsto & \mathrm{L}_{t_f}(u) = \displaystyle\int_0^{t_f} e^{\mathrm{A}(t_f-\tau)}\mathrm{B}u(\tau)d\tau. \end{array}$$

$\widehat{\mathrm{L}}_{t_f}$ and L_{t_f} are two bounded linear operators, then, their adjoint operators are written, respectively, as

$$\begin{array}{rcl} \widehat{\mathrm{L}}^*_{t_f} : \mathrm{X}_2 & \longrightarrow & \mathrm{L}^2(0,t_f;\mathrm{U}) \\ y & \longmapsto & \widehat{\mathrm{L}}^*_{t_f}(y) = \tilde{\mathrm{B}}^* e^{\tilde{\mathrm{A}}^*(t_f-\tau)}y, \end{array}$$

and

$$\begin{array}{rcl} \mathrm{L}^*_{t_f} : \mathrm{X} & \longrightarrow & \mathrm{L}^2(0,t_f;\mathrm{U}) \\ z & \longmapsto & \mathrm{L}^*_{t_f}(z) = \mathrm{B}^* e^{\mathrm{A}^*(t_f-\tau)}z. \end{array}$$

Based on [4, 6, 15, 24, 35] and [48], we present, in the following, some fundamental results that guarantee the exact controllability of the infinite-dimensional singular dynamical system (4.1). The obtained result, which helps us to solve the different problems as minimum energy control, is established thanks to the decomposition of the degenerate Cauchy problem (4.1) into standard Cauchy problem (4.4) together with the controllability Gramian operator in time $[0,t_f]$.

Definition 3.1 [15] *The system* (4.1) *is exactly controllable at time t_f if for all $x_0, x_f \in \mathrm{X}$, it exists $u(t) \in \mathrm{L}^2(0,t_f;\mathrm{U})$ such that*

$$x\left(t_f, x_0, u(.)\right) = x_f.$$

Theorem 3.2 [48] *The system* (4.4) *is exactly controllable in the interval $[0,t_f]$, if and only if,*

$$\mathrm{Im}(\widehat{\mathrm{L}}_{t_f}) = \mathrm{X}_2.$$

Theorem 3.3 [15] *The system* (4.4) *is exactly controllable in the interval* $[0, t_f]$*, if and only if, any one of the following conditions hold for some* $\gamma > 0$ *and for all* $z \in X_2$

- $\langle \widehat{L}_{t_f} \widehat{L}^*_{t_f} z, z \rangle \geq \gamma \|z\|^2_{X_2}$;
- $\displaystyle\int_0^{t_f} \|\tilde{B}^* e^{\tilde{A}^*(t_f-\tau)} z\|^2_U d\tau \geq \gamma \|z\|^2_{X_2}$;
- $\ker(\widehat{L}^*_{t_f}) = \{0\}$ *and* $\mathrm{Im}(\widehat{L}^*_{t_f})$ *is closed.*

Corollary 3.4 [41] *Let* X *be a Hilbert space and let* $T \in \mathscr{L}(X)$*. Then,* T *is invertible, if and only if,*

$$\exists \alpha > 0 \text{ such that } \|Tx\| \geq \alpha \|x\|, \quad \forall x \in X,$$

and

$$\ker(T^*) = \{0\},$$

where $\mathscr{L}(X)$ *represents the space of all bounded linear operators in* X.

Theorem 3.5 [43] *Let* X *and* Y *be Banach spaces, and let* $P : X \to Y$ *be a continuous linear operator. Then,* P *is surjective operator, if and only if, it exists* $\gamma > 0$*, for all* $y^* \in Y^*$*, we have*

$$\|P^* y^*\| \geq \gamma \|y^*\|.$$

Theorem 3.6 [14] *Let* $F \in \mathscr{L}(X, Z)$ *and* $G \in \mathscr{L}(Y, Z)$*, where* X*,* Y*, and* Z *are Banach spaces. If*

$$\mathrm{rank}(F) \subset \mathrm{rank}(G),$$

then, there exists $\gamma > 0$*, such that*

$$\|F^* z^*\|_{X^*} \leq \gamma \|G^* z^*\|_{Y^*}.$$

The following proposition which investigates the exact controllability of the system (4.1) arises from the above results.

Proposition 3.7 *The system* (4.1) *is exactly controllable, if and only if, the subsystem* (4.4) *is exactly controllable.*

Proof. Let us suppose that the system (4.1) is exactly controllable and the system (4.4) is not exactly controllable, that means that the operator $\widehat{L}_{t_f}$ is not surjective. Then, by theorem 3.2, we get $\mathrm{Im}\widehat{L}_{t_f} \neq X_2$; this is equivalent to

$$\forall \gamma > 0, \exists x \in X_2, \quad \text{such that} \quad \|\widehat{L}^*_{t_f} x\|_{L^2(0,t_f;U)} < \gamma \|x\|_{X_2}.$$

And, then, $\forall \gamma > 0, \exists x \in X$

$$\|L_{t_f}^* x\|_{L^2(0,t_f;U)} < \gamma \|x\|_X.$$

Thus, the system (4.1) is not exactly controllable, that contradicts our supposition.

Inversely and following the same partten, we suppose that the subsystem (4.4) is exactly controllable and the system (4.1) is not exactly controllable on $[0, t_f]$. Hence,

$$\forall \alpha > 0, \exists x \in X, \quad \text{such that} \quad \|L_{t_f}^* x\|_{L^2(0,t_f;U)} < \alpha \|x\|_X.$$

Since $\mathrm{Im}(\widehat{L}_{t_f}) \subset \mathrm{Im}(L_{t_f})$, then, the application of theorem 3.6 implies that $\exists\, \delta > 0$ such that

$$\|\widehat{L}_{t_f}^* x\|_{L^2(0,t_f;U)} < \delta \|L_{t_f}^* x\|_{L^2(0,t_f;U)}.$$

Thus,

$$\|\widehat{L}_{t_f}^* x\|_{L^2(0,t_f;U)} < \delta\alpha \|x\|_X,$$

i.e., $\exists\, \beta = \delta\alpha$, such that

$$\|\widehat{L}_{t_f}^* x\|_{L^2([0,t_f],U)} < \beta \|x\|_{X_2}.$$

Consequently, by the application of theorem 3.5 $\widehat{L}_{t_f}^*$ is not surjective, that means the subsystem (4.4) is not exactly controllable, which contradicts our supposition.

Finally, the system (4.1) is exactly controllable, if and only if, the subsystem (4.4) is exactly controllable. ■

Inspired by the results established in [4] and [15], the concept of the controllability Gramian operator will be introduced followed by an additional test for the exact controllability that we will develop.

Definition 3.8 [4] *The controllability Gramian operator of the infinite-dimensional dynamical subsystem* (4.4) *for the initial time $t_0 = 0$ and the final time t_f is the operator*

$$W(0, t_f) := \int_0^{t_f} e^{\tilde{A}(t_f - \tau)} \tilde{B}\tilde{B}^* e^{\tilde{A}^*(t_f - \tau)} d\tau,$$

where

$$\tilde{A} = -E_{22}^{-1} A_{22} \quad \textit{and} \quad \tilde{B} = E_{22}^{-1}\left(B_2 - A_{21}A_{11}^{-1}B_1\right).$$

Theorem 3.9 *The subsystem* (4.4) *is exactly controllable in the interval $[0, t_f]$, if and only if, the controllability Gramian operator*

$$\begin{aligned} W(0, t_f) &= \int_0^{t_f} e^{\tilde{A}(t_f - \tau)} \tilde{B}\tilde{B}^* e^{\tilde{A}^*(t_f - \tau)} d\tau, \\ &= \widehat{L}_{t_f} \widehat{L}_{t_f}^*, \end{aligned}$$

is invertible.

Proof. Let us start our demonstration by proving the sufficiency of the condition and, then, its necessity.

- **Sufficiency**

  Let us assume that $W(0,t_f)$ is invertible and show that the system (4.4) is exactly controllable. In this case, we can define the input $u(t)$ by

$$u(t) = \tilde{B}^* e^{\tilde{A}^*(t_f - t)} W^{-1}(0,t_f)(x_{2f} - e^{\tilde{A}t_f} x_{02}),$$

  for all $x_{02},\ x_{2f} \in X_2$ that ensure $u(t) \in L^2\left(0, t_f; U\right)$.

  The input $u(t)$ steers the state $x_2(t)$ of system (4.4) from x_{02} to x_{2f}. Hence,

$$\begin{aligned}
x_2(t_f) &= e^{\tilde{A}t_f} x_{02} + \int_0^{t_f} e^{\tilde{A}(t_f-\tau)} \tilde{B} u(\tau) d\tau, \\
&= e^{\tilde{A}t_f} x_{02} + \int_0^{t_f} e^{\tilde{A}(t_f-\tau)} \tilde{B}\tilde{B}^* e^{\tilde{A}^*(t_f-\tau)} W^{-1}(0,t_f)(x_{2f} - e^{\tilde{A}t_f} x_{02}) d\tau, \\
&= e^{\tilde{A}t_f} x_{02} + W(0,t_f) W^{-1}(0,t_f)(x_{2f} - e^{\tilde{A}t_f} x_{02}), \\
&= x_{2f}.
\end{aligned}$$

  Thus, the system (4.4) is exactly controllable.

- **Necessity**

  Now, we show that if the system (4.4) is exactly controllable, then, the Gramian operator $W(0,t_f)$ is invertible.

  As the system (4.4) is exactly controllable on $[0,t_f]$, then, by theorem 3.3 there exists $\gamma > 0$ and for all $z \in X_2$ we have

$$\int_0^{t_f} \| \tilde{B}^* e^{\tilde{A}^*(t_f-\tau)} z \|^2 d\tau \geq \gamma \|z\|^2,$$

  and

$$\ker\left(\widehat{L}^*_{t_f}\right) = \{0\}, \quad \text{and} \quad \mathrm{Im}\left(\widehat{L}^*_{t_f}\right) \text{ is closed.}$$

  Duo to lemma 3.4, we can, easily, prove that the operator $\widehat{L}^*_{t_f}$ is invertible, which ensures that $\widehat{L}_{t_f}$ is, also, invertible. Consequently,

$$[\widehat{L}^*_{t_f}]^{-1} [\widehat{L}_{t_f}]^{-1} = [\widehat{L}_{t_f} \widehat{L}^*_{t_f}]^{-1},$$

  that means, the Gramian operator $W(0,t_f)$ is invertible.

Finally, the system (4.4) is exactly controllable $\Longleftrightarrow$ the Gramian operator $W(0,t_f)$ is invertible. ∎

4 Minimum energy control problem with bounded input

This section is designed to present one of the main results of this thesis that concerns the minimum energy control problem for the infinite-dimensional degenerate Cauchy problem with variable operator coefficients, shew-hermitian pencil, and bounded input (4.1). To achieve the desired result, we start by formulating the main problem, afterwards, the optimal control of the system (4.1) is obtained by solving the problem (4.4) under some assumptions on the exact controllability in time $[0, t_f]$, followed by the determination of the value of the performance index that ensures the value of the minimum energy control of the system (4.4), and therefore the one of the system (4.1).

4.1 Problem formulation

Let us consider the system (4.4) which is the reduced form of the system (4.1). As it has been shown in section 3, if the system is exactly controllable, then, there exist many inputs that steer the state $x_2(t)$ of the system (4.4) from $x_{02} = 0$ to the given final state $x_{2f} \in \mathrm{X}_2$. Among these inputs, we are looking for an input $u(t) \in \mathrm{L}^2(0, t_f; \mathrm{U})$ satisfying the condition

$$u(t) < u_1 \in \mathrm{L}^2(0, t_f; \mathrm{U}), \tag{4.6}$$

with $u_1 \in \mathrm{U}$ is a given control, and minimizing the performance index $\mathrm{I}(u)$,

$$\mathrm{I}(u) = \int_0^{t_f} u(\tau)^{\mathrm{T}} \mathrm{Q}_2 u(\tau) d\tau, \tag{4.7}$$

where Q_2 is hermitian, positive, and invertible operator such that

$$\mathrm{Q}_2^{-1} \in \mathscr{L}(\mathrm{X}_2).$$

It is important to emphasize that the performance index $\mathrm{I}(u)$ defines the energy control in $[0, t_f]$ and the control $u(t)$ which minimizes the performance index $\mathrm{I}(u)$ is called the minimum energy control.

Hence, the minimum energy control problem of the system (4.4) can be stated as follows : for a given operators E, A, and B associated with the degenerate Cauchy problem (4.1), $u_1 \in \mathrm{L}^2(0, t_f; \mathrm{U})$, $\mathrm{Q}_2 \in \mathscr{L}(\mathrm{X}_2)$, and a final state $x_f \in \mathrm{X}$ with $t_f > 0$, find an input $u(t) \in \mathrm{L}^2(0, t_f; \mathrm{U})$ satisfying the condition (4.6) and steers the state $x(t)$ of the system (4.1) from $x_0 = 0$ to $x_f \in \mathrm{X}$ that minimizes the performance index (4.7).

4.2 Main results

To solve the problem of minimum energy control, we define the operator

$$\mathrm{W}_2(t_f,\mathrm{Q}_2) = \int_0^{t_f} e^{\tilde{\mathrm{A}}(t_f-\tau)}\tilde{\mathrm{B}}\mathrm{Q}_2^{-1}\tilde{\mathrm{B}}^* e^{\tilde{\mathrm{A}}^*(t_f-\tau)}\,d\tau, \tag{4.8}$$

where

$$\tilde{\mathrm{A}} = \mathrm{E}_{22}^{-1}\mathrm{A}_{22} \quad \text{and} \quad \tilde{\mathrm{B}} = \mathrm{E}_{22}^{-1}\left(\mathrm{B}_2 - \mathrm{A}_{21}\mathrm{A}_{11}^{-1}\mathrm{B}_1\right).$$

By theorem 3.9, the operator $\mathrm{W}_2(t_f,\mathrm{Q}_2)$ is invertible, if and only if, the system (4.1) is exactly controllable in time $[0, t_f]$. In this case, the input can be defined as

$$u(t) = \mathrm{Q}_2^{-1}\tilde{\mathrm{B}}^* e^{\tilde{\mathrm{A}}^*(t_f-t)}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}), \tag{4.9}$$

for all $t_f > 0$ and $t \in [0, t_f]$, where

$$\mathrm{Q}_2^{-1} \in \mathscr{L}(\mathrm{X}_2) \qquad \text{and} \qquad \mathrm{W}_2^{-1}(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}) \in \mathscr{L}(\mathrm{X}_2). \tag{4.10}$$

The minimum value of the index of the performance which guarantees the minimum energy control of the system (4.1) with skew-hermitian pencil and bounded input is presented by the following theorem.

Theorem 4.1 *Let the degenerate Cauchy problem (4.1) be exactly controllable in time $[0, t_f]$, and let the conditions (4.10) be satisfied. Moreover, let $\overline{u}(t) \in \mathrm{L}^2(0, t_f; \mathrm{U})$ be an input that steers the state $x(t)$ of the system from $x_0 = 0$ to $x_f \in \mathrm{X}$, and satisfying the condition (4.6). Then, the input $u(t)$ defined by (4.9), also, steers the initial state of the system from $x_0 = 0$ to the final state $x_f \in \mathrm{X}$ and minimizes the performance index (4.7), i.e.,*

$$\mathrm{I}(u) \leq \mathrm{I}(\overline{u}).$$

Thereafter, the minimal value of the performance index (4.7) of the system (4.1) is given by

$$\begin{aligned} \forall t_f > 0 : \mathrm{I}(u) &= x_f^{\mathrm{T}}\mathrm{W}^{-1}(t_f,\mathrm{Q}_2)x_f, \\ &= x_{2f}^{\mathrm{T}}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)x_{2f}, \end{aligned} \tag{4.11}$$

Proof. If the conditions (4.10) are satisfied and the system (4.4) is exactly controllable, then, the input (4.9) is well-defined. Indeed, by the use of (4.5) and (4.9), we have

$$\begin{aligned}
x_2(t_f) &= e^{\tilde{A}t_f}x_{02} + \int_0^{t_f} e^{\tilde{A}(t_f-\tau)}\tilde{B}u(\tau)d\tau,\\
&= e^{\tilde{A}t_f}x_{02} + \int_0^{t_f} e^{\tilde{A}(t_f-\tau)}\tilde{B}Q_2^{-1}\tilde{B}^* e^{\tilde{A}^*(t_f-\tau)}W_2^{-1}(t_f,Q_2)(x_{2f}-e^{\tilde{A}t_f}x_{02})d\tau,\\
&= e^{\tilde{A}t_f}x_{02} + W_2(t_f,Q_2)W_2^{-1}(t_f,Q_2)(x_{2f}-e^{\tilde{A}t_f}x_{02}),\\
&= x_{2f}.
\end{aligned}$$

As the expression (4.8) holds, we assume that the inputs $u(t)$ and $\overline{u}(t)$, $t \in [0,t_f]$ steer the system state from $x_{02}=0$ to $x_{2f} \in X_2$, therefore,

$$\begin{aligned}
x_2(t_f) &= \int_0^{t_f} e^{\tilde{A}(t_f-\tau)}\tilde{B}\overline{u}(\tau)d\tau,\\
&= \int_0^{t_f} e^{\tilde{A}(t_f-\tau)}\tilde{B}u(\tau)d\tau.
\end{aligned}$$

The subtraction of the two terms gives

$$\int_0^{t_f} e^{\tilde{A}(t_f-\tau)}\tilde{B}\left[\overline{u}(\tau)-u(\tau)\right]d\tau = 0,$$

where its transposition is

$$\int_0^{t_f} \left[\overline{u}(\tau)-u(\tau)\right]^{T}\tilde{B}^* e^{\tilde{A}^*(t_f-\tau)}d\tau = 0. \tag{4.12}$$

Post-multiplying the equality (4.12) by $W_2^{-1}(t_f,Q_2)\left[x_{2f}-e^{\tilde{A}t_f}x_{02}\right]$, gives

$$\int_0^{t_f} \left[\overline{u}(\tau)-u(\tau)\right]^{T}\tilde{B}^* e^{\tilde{A}^*(t_f-\tau)}W_2^{-1}(t_f,Q_2)\left[x_{2f}-e^{\tilde{A}t_f}x_{02}\right]d\tau = 0,$$

hence,

$$\int_0^{t_f} \left[\overline{u}(\tau)-u(\tau)\right]^{T}Q_2u(\tau)d\tau = 0. \tag{4.13}$$

As

$$\begin{aligned}
\int_0^{t_f} < \overline{u}(\tau), Q_2\overline{u}(\tau) >_U d\tau &= \int_0^{t_f} < \left[\overline{u}(\tau)-u(\tau)\right], Q_2\left[\overline{u}(\tau)-u(\tau)\right] >_U d\tau\\
&\quad + \int_0^{t_f} < u(\tau), Q_2u(\tau) >_U d\tau,
\end{aligned}$$

then, by the equation (4.13), it follows

$$\int_0^{t_f} \overline{u}(\tau)^{T}Q_2u(\tau)d\tau = \int_0^{t_f} u(\tau)^{T}Q_2u(\tau)d\tau,$$

Hence,

$$\mathrm{I}(u) \leq \mathrm{I}(\overline{u}),$$

since the second term on the right-hand side of the inequality is positive.

The final step of the proof is about determining the minimum value of the performance index. By the substitution (4.9) into (4.7), we obtain

$$\begin{aligned}
\mathrm{I}(u(t_f)) &= \int_0^{t_f} u^{\mathrm{T}}(\tau)\mathrm{Q}_2 u(\tau)d\tau, \\
&= \int_0^{t_f} \left[\mathrm{Q}_2^{-1}\tilde{\mathrm{B}}^* e^{\tilde{\mathrm{A}}^*(t_f-\tau)}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)\left(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}\right)\right]^{\mathrm{T}}\mathrm{Q}_2 \\
&\quad \times \left[\mathrm{Q}_2^{-1}\tilde{\mathrm{B}}^* e^{\tilde{\mathrm{A}}^*(t_f-\tau)}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)\left(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}\right)\right]d\tau, \\
&= \left(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}\right)^{\mathrm{T}}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)\left(x_{2f} - e^{\tilde{\mathrm{A}}t_f}x_{02}\right), \\
&= x_{2f}^{\mathrm{T}}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)x_{2f}, \quad \text{since} \quad x_{02} = 0.
\end{aligned}$$

Finally, we define the controllability Gramian operator $\mathrm{W}^{-1}(t_f,\mathrm{Q}_2)$ and the state vector $x(t_f)$ by

$$\mathrm{W}^{-1}(t_f,\mathrm{Q}_2) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2) \end{bmatrix}, \text{ and } x(t_f) = \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix}.$$

Then, the minimal value of the performance index of the infinite-dimensional degenerate Cauchy problem (4.1) is

$$\begin{aligned}
\mathrm{I}_{opt}(u(t_f)) &= \mathrm{I}(u(t_f)), \\
&= x_{2f}^{\mathrm{T}}\mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2)x_{2f}, \\
&= \begin{bmatrix} x_{1f} & x_{2f} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{W}_2^{-1}(t_f,\mathrm{Q}_2) \end{bmatrix}\begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix}, \\
&= x_f^{\mathrm{T}}\mathrm{W}^{-1}(t_f,\mathrm{Q}_2)x_f.
\end{aligned}$$

■

5 Procedure for computing the minimum energy control

In order to determine the optimal input $u(t)$ of infinite-dimensional degenerate Cauchy problem with operators coefficients, skew-hermitian pencil, and bounded input (4.1), we will present in this section a procedure. The optimal input $u(t)$, satisfies the condition (4.6), steers the system (4.1) from the initial state $x_0 = 0$ to the final state $x_f \in \mathrm{X}$ and minimizes the performance index (4.7).

The steps are as follows

- **Step 1 :** Knowing the operators E, A, and B, compute the operators $\tilde{A}$, $\tilde{B}$, and $e^{\tilde{A}t_f}$.
- **Step 2 :** For a given operator Q_2, compute the controllability Gramian operator $W_2(t_f, Q_2)$ using formula (4.8).
- **Step 3 :** For a given $x_f \in X$ and $u_1 \in L^2(0, t_f; U)$, compute the input u satisfying the condition (4.6). The obtained input u represents the minimum energy control of system (4.1).
- **Step 4 :** Using (4.11), compute the minimal value of the performance index $I(u)$ of the system (4.1).

6 Conclusion

In this chapter, we investigated the interesting problem of minimum energy control for an infinite-dimensional degenerate Cauchy problem with variable operator coefficients, skew-hermitian pencil, a given initial state, and bounded input. First, we focused on the orthogonal decomposition of degenerate Cauchy problem with a skew-hermitian pencil in order to establish necessary and sufficient conditions for the exact controllability of a degenerate Cauchy problem. Then, the minimum energy control problem for the degenerate Cauchy system with bounded input is formulated and solved where sufficient conditions for the existence of the solution of the problem has been given. Finally, a procedure for the computation of the optimal input satisfying the condition given above and the minimum value of the performance index is proposed.

Chapter 5

Stability and stabilization of infinite-dimensional dynamical systems

1 Introduction

The general problems of stability and stabilization of linear dynamical systems in infinite dimension with operator coefficients and initial condition are considered in this chapter. It consists of designing a controller that uses informations on a measurable input to influence the behavior of the state considered as a deviation from the desired equilibrium. One of the greatest important remarkable facts in modern control theory is the connection between stabilization and property of control systems named exact controllability.

In the next section, we will recall basic notions and some results of stability, the weak, the exponential, the asymptotic stabilities, and Liaponov's equation condition for infinite-dimensional dynamical systems. Then, in the third section, we will present basic concepts and properties of stabilization, and we deal with the stabilization problem of the infinite-dimensional dynamical systems with bounded operator coefficients.

2 Stability problem

The most known and important definitions and properties of stability, weak stability, asymptotic stability, exponential stability, and Liapunov's equation condition for infinite-dimensional dynamical systems are recalled in this section.

2.1 Problem formulation

Consider the dynamical system described by the following equation

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \in \mathbb{R}_+, \\ x(0) &= x_0, \end{cases} \tag{5.1}$$

where A is an infinitesimal generator of a C_0 semi-group $(S(t))_{t\geq 0}$ in a Hilbert space X, and $B \in \mathscr{L}(U, X)$ with U is the control space assumed to be a Hilbert space. $x(0) = x_0$ represents the initial condition.

The solution of the system (5.1) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau. \tag{5.2}$$

Now, we consider the dynamical system (5.1) without control, i.e., $u(t) = 0$, we obtain

$$\begin{cases} \dot{x}(t) &= Ax(t), \\ x(0) &= x_0, \end{cases} \tag{5.3}$$

Then, the trajectory of the system (5.3) is

$$x(t) = S(t)x_0. \tag{5.4}$$

2.2 Weak, asymptotic, and exponential stabilities

2.2.1 Basic definitions

Definition 2.1 [33] *The system* (5.3) *is said to be*

- ***Weakly stable*** *if for every* $x \in X$ *and* $y \in X^*$, *we have*

$$\langle S(t)x, y\rangle \longrightarrow 0, \qquad as \qquad t \longrightarrow \infty;$$

- ***Asymptotically stable*** *if for every* $x \in X$, *we*

$$\|S(t)x\| \longrightarrow 0, \qquad as \qquad t \longrightarrow \infty;$$

- ***Exponentially stable*** *if there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that,*

$$\|S(t)\| \leq Me^{-\omega t}. \tag{5.5}$$

Remark 2.2 [49]

- *In the case of finite dimension, the three types of stability coincide;*
- *In the case of infinite-dimension, we have*

$$\text{Exponential stability} \quad \Rightarrow \quad \text{Asymptotic stability} \quad \Rightarrow \quad \text{Weak stability.}$$

However, the converse is not true.

The following example illustrated the remark

Example 2.3 [33] *Let* $X = l^2$ *the Hilbert space of all square-summable sequences, and define*

$$S(t)x = \left(e^{-t}x_1, e^{\frac{-t}{2}}x_2, \cdots, e^{\frac{-t}{n}}x_n, \cdots\right), \qquad t \geq 0, \tag{5.6}$$

for all $(x_1, x_2, \cdots, x_n, \cdots) \in X$. *Then,* $S(t)$ *is a* C_0 *semi-group on* X *and for every* $x \in X$

$$\|S(t)x\|^2 = \sum_{n=1}^{\infty} e^{\frac{-2t}{n}} x_n^2 \longrightarrow 0, \qquad as \qquad t \longrightarrow \infty. \tag{5.7}$$

Thus, $S(t)$ *is asymptotically stable. However, for any* $t \in [0, +\infty[$

$$\begin{aligned} \|S(t)\| &= \sup_{||x||=1} \|S(t)x\|, \\ &= \sup_{||x||=1} \left(\sum_{n=1}^{\infty} e^{\frac{-2t}{n}} x_n^2\right)^{\frac{1}{2}}, \\ &= \lim_{n\to\infty} e^{\frac{-t}{n}}, \\ &= 1. \end{aligned} \tag{5.8}$$

This indicates that $S(t)$ *is not exponentially stable. The infinitesimal generator of* $S(t)$ *is found to be*

$$Ax = \left.\frac{d^+S(t)}{dt}\right|_{t=0} = -\left(x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots\right), \tag{5.9}$$

and

$$\sigma(A) = \left\{ \frac{-1}{n} \;\middle|\; n \geq 1 \right\}, \tag{5.10}$$

where $\sigma(A)$ *is the spectrum of the operator* A.

2.2.2 Characterizations of stabilities

Let us recall that, in infinite dimension, the point spectrum of A is noted by $\sigma(A)$ and

$$\sigma_0 = \lim_{t\to\infty} \frac{1}{t} \ln |||S(t)|||. \tag{5.11}$$

More than that, the upper stability index of the operator A is

$$\sigma_{-} = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathrm{A})\}, \tag{5.12}$$

and the lower stability index of the operator A is

$$\sigma_{+} = \inf\left\{\mu \mid \exists \mathrm{M} > 0;\ |||\mathrm{S}(t)||| \leq \mathrm{M}e^{\mu t},\ \forall t \geq 0\right\}. \tag{5.13}$$

For the study of the asymptotic stability of C_0 semi-groups, we need to recall the following properties of C_0 semi-group of isometrics on Banach spaces.

Note that, a C_0 semi-group on a Banach space X is called a C_0 semi-group of isometrics if

$$\|\mathrm{S}(t)x\| = \|x\|, \quad \text{for all} \quad x \in \mathrm{X} \quad \text{and} \quad t \geq 0. \tag{5.14}$$

Lemma 2.4 [33] *Let* $\mathrm{S}(t)$ *be a* C_0 *semi-group of isometrics on a Banach space* X *with generator* A. *Then,*

i) *If* $\mathrm{Re}(\lambda) < 0$, *so,*

$$\|(\lambda \mathrm{I} - \mathrm{A})x\| \geq |\mathrm{Re}(\lambda)|\|x\| \quad \textit{for all} \quad x \in \mathbf{D}(\mathrm{A});$$

ii) *If* $\mathrm{S}(t)$ *does not extend to a* C_0*-group of isometrics on* X, *then,* $\lambda \in \sigma(\mathrm{A})$ *for all* λ *with* $\mathrm{Re}(\lambda) \leq 0$ *and* $\lambda \in \sigma_r(\mathrm{A})$ *if* $\mathrm{Re}(\lambda) < 0$*;*

iii) *If* $\mathrm{X} \neq \{0\}$ *and* $\mathrm{S}(t)$ *is a* C_0*-group of isometrics on* X, *then,*

$$\sigma(\mathrm{A}) \cap i\mathbb{R} \neq \emptyset.$$

Theorem 2.5 [33] *Let* $\mathrm{S}(t)$ *be a uniformly bounded* C_0 *semi-group on a Banach space* X *and let* A *be its generator. Then,*

i) *If* $\mathrm{S}(t)$ *is asymptotically stable, then,* $\sigma(\mathrm{A}) \cap i\mathbb{R} \subset \sigma_c(\mathrm{A})$ *which represents the continuous spectrum of* A*;*

ii) *If* $\sigma(\mathrm{A}) \cap i\mathbb{R} \subset \sigma_c(\mathrm{A})$ *and* $\sigma_c(\mathrm{A})$ *is countable. Then,* $\mathrm{S}(t)$ *is asymptotically stable;*

iii) *If* $\rho(\mathrm{A})$ *is compact. Then,* $\mathrm{S}(t)$ *is asymptotically stable, if and only if,* $\mathrm{Re}(\lambda) < 0$ *for all* $\lambda \in \sigma(\mathrm{A})$.

The following proposition establishes the link between the weak stability and the asymptotic stability.

Proposition 2.6 [33] *Let* X *be a Hilbert space. Suppose that* $S(t)$ *is a weakly stable* C_0 *semi-group on* X, *i.e.,*

$$\langle S(t)x, y\rangle \longrightarrow 0 \quad as \quad t \longrightarrow \infty \quad for\ all\ \ x, y \in X.$$

If its infinitesimal generator A *has compact resolvent. Then,* $S(t)$ *is asymptotically stable, i.e.,*

$$\|S(t)x\| \longrightarrow 0 \quad as \quad t \longrightarrow \infty \quad for\ all\ \ x \in X.$$

Theorem 2.7 [49] *The system* (5.3) *is exponentially stable, if and only if,*

$$\int_0^{\infty} \|S(t)x\|^2 dt < \infty \quad for\ all \quad x \in X. \tag{5.15}$$

Corollary 2.8 [49] *If there exists* $t_0 > 0$ *such that* $|||S(t_0)||| \leq 1$, *then, the system* (5.3) *is exponentially stable.*

Remark 2.9 [49]

i) *In the case of finite dimension, the exponential stability is examined from the spectrum of the system dynamics. So, the system* (5.3) *is exponentially stable, if and only if,*

$$\sup\{\mathrm{Re}(\lambda),\ \ \lambda \in \sigma(A)\} \leq 0; \tag{5.16}$$

ii) *In infinite-dimension, we always have the inequality*

$$\sup\{\mathrm{Re}(\lambda),\ \ \lambda \in \sigma(A)\} \leq \sigma_0. \tag{5.17}$$

However, we do not need equality to obtain exponential stability as shown in the above theorem.

Theorem 2.10 [33] *Let* A *be the infinitesimal generator of a* C_0 *semi-group* $S(t)$ *on a Banach space* X. *If for some* $p \geq 1$

$$\int_0^{\infty} \|S(t)x\|^p dt < \infty, \quad for\ every \quad x \in X,$$

then, $S(t)$ *is exponentially stable.*

Theorem 2.11 [33] *Let* $S(t)$ *be a* C_0 *semi-group with infinitesimal generator* A. *The following statements are equivalent*

i) $S(t)$ *is exponentially stable, i.e.,*

$$\|S(t)\| \leq Me^{-\omega t}, \quad for \quad M \geq 1, \quad \omega > 0; \tag{5.18}$$

ii) $\lim_{t\to\infty} \|S(t)\| = 0$;

iii) *There exists a* $t_0 > 0$ *such that*

$$\|S(t_0)\| < 1. \tag{5.19}$$

Remark 2.12 [33] *We say that* $S(t)$ *is exponentially asymptotically stable if for every* $x \in X$, *there exist* M_x, $\omega_x > 0$ *depending on* x *such that*

$$\|S(t)x\| \le M_x e^{-\omega_x t}. \tag{5.20}$$

2.3 Liapunov's equation condition

Let us recall that a matrix A is stable, if and only if, the Liapunov equation

$$A^* Q + QA = -I, \tag{5.21}$$

has a positive solution Q [48]. However, in infinite-dimensional case, the generalization of this result is addressed by the following theorem.

Theorem 2.13 [48] *Assume that* X *is a real Hilbert space. The infinitesimal generator* A *of the semi-group* $S(t)$ *is exponentially stable, if and only if, there exists a non-negative, linear and continuous operator* Q *such that*

$$2\langle QAx, x\rangle = -|x|^2 \quad \textit{for all} \quad x \in \mathbf{D}(A). \tag{5.22}$$

If the generator A *is exponentially stable, then, the equation* (5.22) *has exactly one non-negative, linear and continuous solution* Q.

3 Stabilization problem

In the first part of this section, we proceed to the stabilization problem of the system (5.1), where, it is solution given by

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau. \tag{5.23}$$

Afterwards, the second the part is devoted to stabilization problem of infinite-dimensional regular dynamical system with bounded operator coefficients.

3.1 Definition and characterizations

Definition 3.1 [49] *The system* (5.1) *is said to be weakly* (*respectively strongly and exponentially*) *stabilizable, if there exists a bounded operator* $\mathrm{K} \in \mathscr{L}(\mathrm{X},\mathrm{U})$ *such that the system*

$$\begin{cases} \dot{x}(t) &=& (\mathrm{A}+\mathrm{BK})x(t), \\ x(0) &=& x_0 \in \mathrm{X}, \end{cases} \tag{5.24}$$

be weakly (*respectively strongly and exponentially*).

Theorem 3.2 [49] *The system* (5.1) *is exponentially stabilizable, if and only if,*

$$\int_0^{\infty} \|\mathrm{S}(t)x\|^2 dt < \infty \quad \textit{for all} \quad x \in \mathrm{X}, \tag{5.25}$$

where $\mathrm{S}(t)$ *is a semi-group generated by the operator* $(\mathrm{A}+\mathrm{BK})$.

Corollary 3.3 [49] *If there exists* $t_0 > 0$ *such that* $|||\mathrm{S}(t_0)||| \leq 1$, *then, the system* (5.1) *is exponentially stabilizable.*

An important characterization of the stabilization of the system (5.1) is given by the following theorem.

Theorem 3.4 [37] *The following conditions are equivalent*

i) *The system* (5.1) *is exponentially stabilizable;*

ii) *For every initial condition* $x_0 \in \mathrm{X}$ *there exists a control* $u(.)$ *such that for the corresponding mild solution of* (5.23)

$$\int_0^{\infty} \left(\|x(t)\|^2 + \|u(t)\|^2\right) dt < +\infty; \tag{5.26}$$

iii) *There exists a non-negative operator* P *satisfying the following Riccati's equation*

$$2\langle \mathrm{PA}x, x\rangle + \langle x, x\rangle - \langle \mathrm{P}^2 x, x\rangle = 0, \quad x \in \mathbf{D}(\mathrm{A}); \tag{5.27}$$

iv) *For every initial condition* $x_0 \in \mathrm{X}$ *there exists a control* $u(.)$ *such that the control* $u(t)$ *and the corresponding mild solution* $x(t)$ *tend to zero exponentially as* $t \longrightarrow +\infty$.

3.2 Stabilization problem for infinite-dimensional systems with bounded operator coefficients

In this part, we consider the following dynamical system

$$\begin{cases} \dot{x}(t) &=& \mathrm{A}x(t)+\mathrm{B}u(t), \\ x(0) &=& x_0, \end{cases} \tag{5.28}$$

with A and B are two bounded operators defined in $\mathscr{L}(X)$ and $\mathscr{L}(U,X)$ receptively.

Let us emphasize that the definition 3.1 remains valid for the system (5.28). Therefore, The system (5.28) becomes

$$\begin{cases} \dot{x}(t) &= \mathrm{A_K}x(t), \\ x(0) &= x_0, \end{cases} \tag{5.29}$$

where, $\mathrm{A_K} = \mathrm{A} + \mathrm{BK}$ with the domain $\mathbf{D}(\mathrm{A_K}) = \mathbf{D}(\mathrm{A})$ and $u(t) = \mathrm{K}x(t), \quad t \geq 0$ for the existence of a linear continuous operator $\mathrm{K} : \mathrm{X} \longrightarrow \mathrm{U}$.

Hence, the operator $\mathrm{A_K}$ generates a semi-group $\mathrm{S_K}(t)$, such that

$$x(t) = \mathrm{S_K}(t)x_0 = e^{(\mathrm{A}+\mathrm{BK})t}x_0. \tag{5.30}$$

The application of different results and properties of stability and stabilization, which have been presented in the previous part, to analyze the system (5.29) is straightforward.

4 Conclusion

In this chapter, we have been interested in the study of stability and stabilization problem for the infinite-dimensional dynamical systems. Firstly, we have presented several results and properties of the stability within sens weak, asymptotic, exponential, and Liapunov's equation condition. Secondly, we have recalled definitions and some characterizations of stabilization problems for controlled infinite-dimensional dynamical systems. Finally, we have applied the notions of stability and stabilization on dynamical systems with bounded operator coefficients in infinite dimension.

Conclusion

This thesis is devoted to extend and present new results on the minimum energy control, stability, and stabilization problems by the use of many concepts of operators theory, semi-group theory, control theory, and matrix theory. The whole document is structured around three themes:

- Minimum energy control problem for finite dimensional singular dynamical systems with rectangular inputs. To carry out our study, we have established sufficient conditions for the formulations of the minimum energy control problem by involving the Weierstrass theorem. Then, we have presented rigorous proof of the solvency of the minimum energy control problem using a new technique that we have developed taking into account the rectangular inputs. A procedure for computing the optimal control and the minimum values of the performance index has been proposed. Finally, the effectiveness of our results has been illustrated through an example.

- Minimum energy control problem for the infinite-dimensional degenerate Cauchy problem with variables operator coefficients, skew-hermitian pencil, and bounded input. To acquire the desired results, we firstly have used the orthogonal decomposition combined with a new technique to handle the problem of solvability and cover the problem of exact controllability. Then, the problem of minimum energy control was solved thanks to the obtained solution, the Gramian operator, and suitable formula of the control. Finally, a procedure for calculating the optimal input which satisfies a condition, and the minimum value of the performance index has been proposed.

- Stability and stabilization problem for infinite-dimensional dynamical systems. We have started by considering the case where the state coefficient is an infinitesimal generator of a C_0 semi-group and the input coefficient is a linear operator. Then, the case where the coefficients are bounded operators has been processed.

Based on the results given by the present thesis, several perspectives should be considered, among them,

- The study of the minimum energy control problem for other types of infinite-dimensional dynamical systems by involving other conditions and types of inputs.
- The extension of existing results and a deep study of the stability and stabilization of dynamical systems in the infinite dimension.

References

[1] **Abouzaid, B., Winkin, J. J., and Wertz, V.** (2010), *Positive stabilization of infinite-dimensional linear systems.* 49th IEEE conference on Decision and Control (CDC), 845–850. https://doi.org/10.1109/CDC.2010.5717896 29, 30

[2] **Akhieser, N. I. and Glazman, I. M.** (1993), *Theory of linear operators in Hilbert spaces.* Dover Publications, INC. New York. (Translated from the Russian by Nestell, M.). 4, 5

[3] **Becker, R.** (2008), *Ordered Banach spaces.* Collection travaux en cours : mathématiques, Hermann, Paris. 29

[4] **Bensoussan, A., Da Prato, G., Delfour, M. C., and Mitter, S.** (2007), *Representation and control of infinite-dimensional systems.* Systems & Control : Foundations & Applications, Birkhäuser, Boston. https://doi.org/10.1007/978-0-8176-4581-6 1, 28, 31, 53, 55

[5] **Bouagada, D.** (2007), *Systèmes diffèrentiel singuliers positifs et LMIS.* Doctoral thesis, Université dÉs-Sénia, Algeria. 20, 27

[6] **Brezis, H.** (2005), *Analyse fonctionnelle : Théorie et applications.* Sciences sup, Dunod, Paris. 4, 53

[7] **Brown, R. F.** (1985), *Biomedical systems analysis via compartemental concept.* Abacus Press, USA. 1, 18

[8] **Caccetta, L., and Rumchev, V.** (2000), *A survey of reachability and controllability for positive linear systems.* Annals of Operation Research, **98**, 101–122. https://doi.org/10.1023/A:1019244121533 1, 18

[9] **Campbell, S. L., Meyer, Jr. C. D., and Rose, N. J.** (1976), *Application of the Drazin inverse to linear systems of differential equation with singular constant coefficients.* SIAM Journal on Applied Mathematics, **31(3)**, 411–425. https://doi.org/10.1137/0131035 1

[10] **Carroll, R. W. and Showalteri, R. E.** (1976), *Singular and degenerate Cauchy problems.* **127**, Mathematics in Science and Engineering, Academic Press, Inc, USA. https://doi.org/10.1016/s0076-5392(08)x6195-8 1, 13, 14

[11] **Curtain, R. F. and Falb, P. L.** (1971), *Stochastic differencial equations in Hilbert space.* Jornal of Differential Equations, **10(3)**, 412–430. https://doi.org/10.1016/0022-0396(71)90004-0 23

[12] **Curtain, R. F. and Pritchard, A. J.** (1975), *The infinite-dimensional Riccati equation for systems defined by evolution operators.* SIAM Journal of Control and Optimization, **14(5)**, 951–983. https://doi.org/10.1137/0314061 23

[13] **Curtain, R. F.** (1976), *Estimation theory for abstract evolution equations exicited by general white noise processes.* SIAM Journal of Control and Optimization, **14(6)**, 1124–1150. https://doi.org/10.1137/0314070 23

[14] **Curtain, R. F. and Pritchard, A. J.** (1978), *Infinite-dimensional linear systems theory.* Lecture Notes in Control and Information Sciences, Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0006761 4, 23, 28, 35, 37, 54

[15] **Curtain, R. F. and Zwart, H. J.** (1995), *An introcution to infinite-dimensional linear systems theory.* Texts in Applied Mathematics, Springer-Verlag, New York. https://doi.org/10.1007/BFb0006761 1, 4, 23, 24, 34, 35, 36, 37, 53, 54, 55

[16] **Dai, L.** (1989), *Singular control systems.* Lecture Notes in Control and Information Sciences, Springer-Verlag. https://doi.org/10.1007/BFb0002475 1, 33

[17] **Farina, L. and Rinaldi, S.** (2000), *Positive linear systems : theory and applications.* **50**, John Wiley & Sons. https://doi.org/10.1002/9781118033029 1, 18, 27

[18] **Kaczorek, T.** (1992), *Linear control systems : analysis of multivariable systems.* Industrial Control, Computers and Communications Series, **6**, Research Studies Press. 1

[19] **Kaczorek, T.** (2001), *Standard and singular 1D and 2D positive linear systems.* Electronic version prepared by CJSC AVTEKS Saint Petersburg. 27

[20] **Kaczorek, T.** (2002), *Positive 1D and 2D systems.* Communications and Control Engineering, Springer-Verlag, London. https://doi.org/10.1007/978-1-4471-0221-2 1, 10, 19, 20, 22, 27, 30, 32, 40

[21] **Kato, T.** (1995), *Perturbution theory for linear operators.* Classics in Mathematics, **132**, Springer-Verlag, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-66282-9 4

[22] **Klamka, J.** (1991), *Controllability of dynamical systems.* Mathematics and its Applications, **48**, Springer, Netherlands. 1, 50

[23] **Klamka, J.** (1993), *Controllability of dynamical systems. A survey.* Bulletin of the Polish Academy of Sciences, Technical Sciences, **61(2)**, 335–342. 50

[24] **Komornik, V.** (1994), *Exact controllability and stabilization : The multiplier method.* Research in Applied Mathematics, Masson-John, Wiley, Paris. 53

[25] **Krein, S. G. and Khazan, M.** (1985), *Differential equations in a Banach space.* Journal of Soviet Mathematics, **30(3)**, 2154–2239. https://doi.org/10.1007/bf02105398 28

[26] **Laabissi, M., Achhab, M. E., Winkin, J. J., and Dochain, D.** (2001), *Trajectory analysis of nonisothermal tubular reactor nonlinear models.* Systems and Control Letters, **42(3)**, 169–184. https://doi.org/10.1016/S0167-6911(00)00088-8 18

[27] **Laabissi, M., Achhab, M. E., Winkin, J., and Dochain, D.** (2003), *Positivity and invariance properties of nonisothermal tubular reactor nonlinear models.* In : Benvenuti, L., De Santis, A., and Farina, L. (eds.), *Positive Systems*, Springer-Verlag, Berlin, Heidelberg, 159–166. https://doi.org/10.1007/978-3-540-44928-7_22 18

[28] **Lagnese, J.** (1973), *Singular differential equations in Hilbert space.* SIAM Journal on Mathematical Analysis, **4(4)**, 623–637. https://doi.org/10.1137/0504054 1

[29] **Lions, J. L.** (1961), *Equations differentielles operationnelles : Et problèmes aux Limites.* Grundlehren der mathematischen Wissenschaften, **111**, Springer-Verlag, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-25839-2 1

[30] **Lions, J. L.** (1969), *Quelques méthodes de résolution des problèmes aux limites nonlinéaires.* Collection études mathématiques, Dunod, Paris. 1

[31] **Luenberger, D.** (1977), *Dynamic equations in descriptor form.* IEEE Transactions on Automatic Control, **22(3)**, 312–321. https://doi.org/10.1109/TAC.1977.1101502 1

[32] **Luenberger, D. G.** (1979), *Introduction to dynamic systems : Theory, models, and applications.* Wiley. 1

[33] **Luo, Z. H., Guo, B.Z., and Mergül. Ö.** (1999), *Stability and stabilization of infinite dimensional systems with applications.* Communications and Control Engineering, Springer-Verlag, London. https://doi.org/10.1007/978-1-4471-0419-3 63, 64, 65, 66, 67

[34] **Lyapunov, A. M.** (1992), *The general problem of the stability of motion.* International Journal of Control, **55(3)**, 531–534. https://doi.org/10.1080/00207179208934253 2

[35] **Naso, M. G.** (2000), *Extension of the controllability concept from the finite dimension to the infinite dimension: Exact, approximate, null Controllability.* D.E.A Mathématiques et Applications, UFR des Sciences et des Thechniques, Université de Franche-Comté, Besançon, France. 1, 53

[36] **Pazy, A.** (1983), *Semigroup of linear operators and applications to partial differential equations.* Applied Mathematical Sciences, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-5561-1 11, 12, 13

[37] **Pritchard, A. J. and Zabczyk, J.** (1981), *Stability and stabilizability of infinite dimensional systems.* SIAM Review, **23(1)**, 25–52. https://doi.org/10.1137/1023003 68

[38] **Rogowski, K.** (2020), *Minimum energy control and reachability of continuous-time linear systems with rectangular inputs* In : Szewczyk, R., Zieliński, C., and Kaliczyńska, M. (eds.) *Automation 2019 : Progress in Automation, Robotics and Measurement Techniques*, Advances in Intelligent Systems and Computing, **920**, Springer International Publishing, 208–217. 44

[39] **Rutkas, A. G.** (1975), *The Cauchy problem for the equation* $\mathrm{A}\dot{x}(t)+\mathrm{B}x(t)=f(t)$. Differentsial'nye Uravneniya, **11(11)**, 1996–2010. 28

[40] **Rutkas, A. G.** (1989), *Perturbations of skew-Hermitian bundles and the degenerate Cauchy problem.* Ukrainian Mathematical Journal, **41(8)**, 931–935. https://doi.org/10.1007/bf01058310 1, 15, 17, 28, 51

[41] **Rynne, B. and Youngson, M. A.** (2008), *Linear functional analysis.* Springer Undergraduate Mathematics Series, Springer-Verlag, London. https://doi.org/10.1007/978-1-84800-005-6 4, 6, 7, 8, 9, 10, 11, 54

[42] **Sethi, S.** (2019), *Optimal control theory : Applications to management science and economics.* Springer International Publishing, Switzerland. https://doi.org/10.1007/978-3-319-98237-3 1, 18

[43] **Vasudeva, H. L.** (2017) *Elements of Hilbert spaces and operator theory.* Springer, Singapore. https://doi.org/10.1007/978-981-10-3020-8 54

[44] **Vollmann, T. E., Berry, W. L., and Whybark, D. C.** (1992), *Manufacturing planning and control systems.* Business One Irwin. https://doi.org/10.1007/1-4020-0612-8 1, 18

[45] **Weidmann, J.** (1980), *Linear operators in Hilbert spaces.* Graduate Texts in Mathematics, Springer-Verlag, New York Inc. https://doi.org/10.1007/978-1-4612-6027-1 4

[46] **Young, R. M.** (2001), *An introduction to non-harmonic Fourier series.* Academic Press. 29

[47] **Yosida, K.**(1995), *Functional analysis.* Classics in Mathematics, Springer-Verlag, Berlin, Heidelberg. 4

[48] **Zabczyk, J.** (2008), *Mathematical control theory : An introduction.* Modern Birkhäuser Classics, Birkhäuser, Basel, Boston. https://doi.org/10.1007/978-0-8176-4733-9 31, 32, 34, 35, 36, 37, 53, 67

[49] **Zerrik, E. H. and Castillo, O.** (2021), *Stabilization of infinite dimensional systems.* Studies in Systems, Decision and Control, **355**, Springer International Publishing, Switzerland. https://doi.org/10.1007/978-3-030-68600-0 63, 66, 68

استقرار وتثبيت الأنظمة الإيجابية في أبعاد لا نهائية

الملخص : الغرض من هذه الأطروحة هو حل مسألة التحكم في الطاقة الدنيا لمشكلة كوشي المشوهة ذات الأبعاد اللانهائية و المعاملات من نوع المؤثرات المتغيرة، والحزمة الضد الهيرمايتية، والإدخالات المحدودة و أيضا لأنظمة ديناميكية مفردة ذات أبعاد محدودة والإدخالات مستطيلة من ناحية، ومن ناحية أخرى، دراسة مشكلة الاستقرار والتثبيت للأنظمة الديناميكية اللانهائية الأبعاد. بالنسبة للمشكلة الأولى، تتبع الدراسة مجموعة من الأساليب والتقنيات، من بينها نظرية وايرستراس وبعض المفاهيم لقابلية التحكم في حالة البعد المحدود، والتحليل المتعامد، والمؤثر الجرامي، وتعبير مناسب للمدخلات في حالة البعد اللانهائي. أكثر من ذلك، تم اقتراح منهاج لحساب الإدخال الأمثل وتقليل مؤشر الأداء لكلتا الحالتين. بالنسبة للمشكلة الثانية، تم تمديد بعض النتائج الحالية حول الاستقرار والتثبيت للأنظمة الديناميكية غير المحدودة الأبعاد ذات معاملات مؤاثرات محدودة. شجعتنا النتائج الواعدة المتحصل عليها على دراسة مشكلة الحد الأدنى من التحكم في الطاقة للأنظمة الديناميكية ذات الأبعاد اللانهائية اخرى وتحليل تثبيتها واستقرارها.

الكلمات المفتاحية : الحد الأدنى من التحكم ، الاستقرار ، التثبيت، أنظمة ديناميكية مفردة ذات أبعاد محدودة، الإدخالات مستطيلة، مشكلة كوشي المشوهة ذات الأبعاد اللانهائية، الحزمة الضد الهيرمايتية، الإدخالات المحدودة.

Stabilité et Stabilisation des Systèmes Positifs en Dimension-Infinie

Résumé : Cette thèse explore, d'une part, le problème du contrôle de l'énergie minimale pour un problème de Cauchy dégénéré de dimension infinie à coefficients opérateurs variables, avec un faisceau anti−hermitien et une entrée bornée, et pour un système dynamique singulier de dimension finie avec des entrées rectangulaires, et d'autre part, le problème de stabilité et de stabilisation pour les systèmes dynamiques de dimension infinie. Pour le premier problème, l'étude suit un ensemble de méthodes et de techniques, parmi lesquelles, le théorème de Weierstrass et quelques concepts de contrôlabilité dans le cas de dimension finie, et la décomposition orthogonale, l'opérateur de Gramien, et une expression appropriée de l'entrée pour le cas de la dimension infinie. Plus que cela, une procédure pour calculer l'entrée optimale et minimiser l'indice de performance est proposée pour les deux cas. Ensuite, pour le deuxième problème, nous étendons certains résultats existant de la stabilité et de la stabilisation pour les systèmes dynamiques de dimension infinie à coefficients opérateurs bornés. Les résultats prometteurs que nous avons obtenus nous ont encouragés à étudier le problème du contrôle de l'énergie minimale pour d'autres systèmes dynamiques de dimension infinie et à analyser leurs stabilités et stabilisations.

Mots−Clés : Contrôle de l'énergie minimale, Stabilité, Stabilisation, Systèmes dynamiques singuliers de dimension finie, Entrées rectangulaires, Problème de Cauchy dégénéré de dimension infinie, Faisceau anti−hermitien, Entrée bornée.

Stability and Stabilization of Positive Infinite-Dimensional Systems

Abstract : This thesis explores, on one hand, the minimum energy control problem for an infinite−dimensional degenerate Cauchy problem with variable operator coefficients, skew−hermitian pencil, and bounded input and for a finite dimensional singular dynamical systems with rectangular inputs, and on the other hand, the problem of stability and stabilization for the infinite−dimensional dynamical systems. For the first problem, the investigation follows a set of methods and techniques, among them, the Weierstrass theorem and some concepts of controllability in the case of finite dimension, and the orthogonal decomposition, the Gramian operator, and a suitable expression of the input in the case of infinite dimension. More than that, a procedure for computing the optimal input and minimizing the performance index is proposed for both cases. Then, for the second problem, some existing results on stability and stabilization have been extended for infinite−dimensional dynamical systems with bounded operator coefficients. The promising results that we have obtained encouraged us to study the problem of minimum energy control for infinite−dimensional dynamical systems and to analyse their stabilities and stabilizations.

Key Words : Minimum energy control, Stability, Stabilization, Finite dimensional singular dynamical systems, Rectangular inputs, Infinite−dimensional degenerate Cauchy problem, Skew−hermitian pencil, Bounded input.