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Title

ON CERTAIN SINGULAR BOUNDARY VALUE PROBLEMS AND INTEGRAL INEQUALITIES

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Résumé

Dans cette thèse, nous nous intéressons à certaines classes d'équations différentielles qui sont singulières sur l'axe du temps. Nous utilisons quelques moyens sophistiqués de l'analyse fonctionnelle et le calcul fractionnaire, tels que les inégalités intégrales, qui sont très présentes dans cette thèse. Ces moyens sont aussi les dérivées fractionnaires, la théorie des opérateurs ainsi que la théorie des points fixes et la méthode de Runge Kutta. Nous étudions les questions d'existence de solutions, d'existence et d'unicité, d'analyse des stabilités au sens d'Ulam-Hyers. Nous présentons également quelques simulations numériques sur les dérivées de Caputo pour étudier le deuxième problème de cette thèse. En particulier, nous nous intéressons, d'abord, à une problème singulier plus général combiné avec des notions séquentielles à n dérivées de Caputo. Certaines des questions ci-dessus sont étudiées et plusieurs exemples sont présentés. Aussi, dans cette thèse, nous étudions une classe d'EDFs singulières impliquant le calcul fractionnaire et les séries. En particulier, nous étudions la question de l'existence et l'unicité des solutions en utilisant à la fois la théorie des points fixes et les inégalités intégrales. Puis, nous passons à l'étude de la question de la stabilité des solutions au sens d'Ulam-Hyers. Quelques exemples sont présentés dans cette partie. A la fin de notre thèse, nous présentons une étude sur la question des approximations de solutions en utilisant des résultats récents sur les approximations de Caputo à l'aide de la méthode numérique de Rung Kutta.

Mots clés: Riemann-Liouville, Caputo derivative, séquentiel, inégalités intégrales, point fixe, existence, unicité, stabilité Ulam-Hyers, EDF singulière.

Abstract

In this thesis, we are concerned with some classes of differential equations that are singular on the time axis. With the help of some sophisticated means of functional analysis and fractional calculus, like for instance, the integral inequalities theory which are very present, the fractional derivatives, the operator theory as well as the fixed point theory and the well known Runge Kutta method, we study the questions of existence of solutions, the existence and uniqueness, the analysis of stabilities in the sense of Ulam-Hyers. We also present some numerical simulations on Caputo derivatives to study the second problem that is presented in this thesis. In particular, we are concerned, first, with a more general singular problem which is combined with some sequential notions with n Caputo derivatives. Some of the above questions are studied and several examples are illustrated. Also, we study a class of singular differential equations involving fractional calculus and convergent series. Especially, we study the question of existence and uniqueness of solutions by using both fixed point theory and integral inequalities. Then, we pass to study the question of stability of solutions in the sense of Ulam-Hyers. Some examples are presented in this part. At the end, we investigate the question of approximations of solutions by using some recent results on Caputo approximations and Rung Kutta numerical Method.

Keywords: Riemann-Liouville, Caputo derivative, sequential, fixed point,integral inequalities, fixed point, existence, uniqueness, Ulam-Hyers stability, singular FDE.

ملخص

في هذه الأطروحة، نهتم بدراسة بعض الأصناف من المعادلات التفاضلية ذات مشتقات برتبة كيفية بحيث تملك هذه المعادلات، في أحد حدودها، نقاطا شاذة على محور الزمن.

باستعمال بعض التقنيات والأفكار الخاصة بالمتراجحات التكاملية، التحليل الدالي وكذا الحساب الكسري ومقاربة النقطة الثابتة وكذا الطريقة العددية لرونج–كوطارتية 4، نقوم بدراسة مشاكل الحلول الوحيدة، الوجود للحلول. نقوم كذلك بدراسة استقرار الحلول بمعنى أولام–هيارز. كذلك نقارب مشتقات كابوتو من أجل إعطاء بعض الدراسة العددية لأحد المشاكل المطروحة. بصفة خاصة، نقوم أولا بدراسة صنفا من المعادلات الكسرية الشاذة بمشتقات كابوتو متتالية: بعض المسائل المطروحة أعلاه نجيب عنها وكذلك نعطي بعض من الأمثلة على النتائج المحصل عليها.

كذلك في هذه الأطروحة، ندرس صنفا آخر من المعادلات الشاذة ذات السلاسل المتقاربة، حيث نهتم بوحدانية الحلول وكذا استقرارها وإعطاء أمثلة ثم ندرس تقريبات لكابوتو وكذا المشكل المطروح في هذه الحالة.

الكلمات المفتاحية: مشتق كابوتو، تكامل ريمان وليوفيل، نقطة صامدة، متراجحات تكاملية، وجود، وحدانية، معادلة تفاضلية شاذة.

General Introduction

Fractional calculus is an important topic in mathematics with its models of real-world problems in various fields of science, technology, and engineering [35, 55, 60]. Its roots extend back to more than three centuries, perhaps one of the beginnings of its appearance was since the regular calculus, with the first reference probably being associated with Leibniz and L'Hôspital in 1695 where half-order derivative was discussed. Then, many works were made: Lagrange developed the law of exponents for differential operators and Laplace defined the fractional derivative by using of integral. In the early 19th century, Abel used fractional operations to the solution of tautochrome problem and Liouville touched on fractional calculus[54]. Since the beginning of the nineties of XXth century, the fractional calculus attracted the attention of many mathematicians, and engineers that have been supporting its development and originating many formulations and mainly using it to explain some natural and engineering phenomena[53]. At present, the number of applications of fractional calculus rapidly grows, we refer the reader to the following papers of applications in effects of economy crises, hydro-magnetic in plasma, hydro magnetic waves and vibration with large membranes [14, 15, 29, 40, 41, 42].

To investigate fractional differential problems, in our opinion, there are two important approaches. The first approach is the Riemann Liouville definition in which fractional derivative of a constant is not zero. The second one is the Caputo approach, which is characterized by fractional derivative of a constant to be equal to zero. It is used in cases of initial value problems of fractional differential equations [31]. We recall that the Caputo fractional derivative is very useful in many applied problems, because it saisfies its initial data which contains $y(0)$, $y^{'}(0)$, etc., as well as the same data for boundary conditions [50].
The main objective in this project is to complete the content of other works in fractional calculus, the focus is on studying certain classes of nonlinear singular differential equations of arbitrary order. We study the questions of existence, existence and uniqueness, stability

of solutions, approximation of solution. All these notions are investigated using inequalities.
This thesis is organized as follows:
The first chapter includs basic concepts in fractional calculus, which are important tools for the other main chapters.
The second chapter gives some properties of functional analysis, the focus is on integral inequalities and their applications in existence and uniqueness (and the existence of at least) of fixed points. This theory is very present in the last two chapters.
In the third chapter, a nonlinear singular differential problem is dealt with. It involves n fractional Caputo derivatives under the conditions that neither commutativity nor semi group property is satisfied for the derivatives. We demonstrate an existence and uniqueness result by application of Banach contraction principle. Then, another result that deals with the existence of at least one solution is delivered and some sufficient conditions related to this result are established by means of the fixed point theorem of Schaefer. We conclude the chapter by providing some illustrative examples in order to show the validity of the results.
The fourth chapter is concerned with a new type of nonlinear fractional integro-differential equations with nonlocal integral conditions, having one nonlinearity with time variable singularity. It involves also some convergent series combined to Riemann-Liouville integrals. The uniqueness of the solutions to the proposed problem is demonstrated, and some examples are provided to illustrate this result. Also, we review the Ulam-Hyers stability for the problem. Some numerical simulations, using Rung Kutta method, are discussed too.
Finally, a conclusion follows. It explains what we have done in our project and what we will be able to do in the future.

Contents

Chapter 1

Preliminaries

1.1 Some important functions

In this part, we are interested in introducing both the gamma and beta functions, as they are important tools on which the theory of fractional differential equations is based. So we will present, in the following, some important properties of these two functions and the relationship between them.

1.1.1 Gamma function

Définition 1.1.1 *Let $\zeta \in \mathbb{R}_+^*$. The gamma function (Γ) is defined as:*

$$\Gamma(\zeta) = \int_0^{+\infty} e^{-t} t^{\zeta-1} dt.$$

Proposition 1.1.1 *The Gamma function is well defined on $\mathbb{R}_+^*$.*

Proof.

The function $\Gamma(\zeta)$ is written as

$$\Gamma(\zeta) := \int_0^{+\infty} e^{-t} t^{\zeta-1} dt = \int_0^1 e^{-t} t^{\zeta-1} dt + \int_1^{+\infty} e^{-t} t^{\zeta-1} dt.$$

We put $S_1 = \int_0^1 e^{-t} t^{\zeta-1} dt$ and $S_2 = \int_1^{+\infty} e^{-t} t^{\zeta-1} dt$, so $\Gamma(\zeta) = S_1 + S_2$.
We have

$$S_1 = \int_0^1 e^{-t} t^{\zeta-1} dt < \int_0^1 t^{\zeta-1} dt = \frac{1}{\zeta},$$

from where S_1 is convergent for $0 < \zeta \leq 1$.
Let us study the convergence of S_2. We have

$$\frac{t^{\zeta-1}}{e^{-\frac{t}{2}}} \leq 1, \text{ because } \lim_{t \longrightarrow +\infty} \frac{t^{\zeta-1}}{e^{-\frac{t}{2}}} = 0.$$

Then, we can write

$$S_2 = \int_1^{+\infty} e^{-t} t^{\zeta-1} dt < \int_1^{+\infty} e^{-\frac{t}{2}} dt = 2e^{-\frac{1}{2}}.$$

Hence the Gamma function is defined for every $\zeta > 0$.

Proposition 1.1.2 *Let $\zeta \in \mathbb{R}$ such as $\zeta > 0$, then the Gamma function satisfies the following properties:*

(P_1): $\Gamma(\zeta+1) = \zeta\Gamma(\zeta)$.
(P_2): $\Gamma(n+1) = n!, n \in \mathbb{Z}_+$.
Proof.
(P_1): Let $\zeta > 0$, we have

$$\Gamma(\zeta+1) = \int_0^{+\infty} t^{\zeta} e^{-t}.$$

We put

$$\begin{cases} u = t^{\zeta} \\ dv = e^{-t}. \end{cases}$$

So, it yields that

$$\begin{cases} du = \zeta t^{\zeta-1} \\ v = -e^{-t}. \end{cases}$$

By integration, we find

$$\begin{aligned} \Gamma(\zeta+1) &= \int_0^{+\infty} t^{\zeta} e^{-t} dt \\ &= \left[-t^{\zeta} e^{-t}\right]_0^{+\infty} + \zeta \int_0^{+\infty} t^{\zeta-1} e^{-t} dt \\ &= \zeta \int_0^{+\infty} t^{\zeta-1} e^{-t} dt \\ &= \zeta\Gamma(\zeta). \end{aligned}$$

(P_2): Using the property (P_1), we will be able to write

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)\Gamma(n-2)\\ &\vdots\\ &= n(n-1)(n-2)\times\cdots\times 2\times 1\times \underbrace{\Gamma(1)}_{=1}\\ &= n(n-1)(n-2)\times\cdots\times 2\times 1\\ &= n!.\end{aligned}$$

Proposition 1.1.3 *We have the following two properties*

(P_1^*): $\Gamma(\zeta) = \lim\limits_{n\longrightarrow+\infty} \dfrac{n!\; n^{\zeta}}{\zeta(\zeta+1)\ldots(\zeta+n)}$, $\zeta \neq 0,-1,-2,\ldots$

(P_2^*): $\dfrac{1}{\Gamma(\zeta)} = \zeta e^{\psi\zeta}\prod\limits_{n=1}^{\infty}\left(1+\dfrac{\zeta}{n}\right)e^{-\dfrac{\zeta}{n}}$, ($\psi$ is the Euler-Mascheroni constant.)

Proof.

(P_1^*): We consider the following function

$$\phi_n(\zeta) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{\zeta-1}dt.$$

We put $s=\frac{t}{n}$, so we have

$$\phi_n(\zeta) = n^{\zeta}\int_0^1 (1-s)^n s^{\zeta-1}ds.$$

By integration by parts, we put

$$\begin{cases} u=(1-s)^n\\ dv = s^{\zeta-1}.\end{cases}$$

So, it yields that

$$\begin{cases} du = -n(1-s)^{n-1}\\ v=\frac{1}{\zeta}s^{\zeta}.\end{cases}$$

We have

$$\begin{aligned}\phi_n(\zeta) &= n^\zeta\left(\left[\frac{(1-s)^n)}{\zeta}s^\zeta\right]_0^1 + \frac{n}{\zeta}\int_0^1 (1-s)^{n-1}s^\zeta\right)ds\\ &= \frac{n^\zeta}{\zeta}n\int_0^1 (1-s)^{n-1}s^\zeta ds.\end{aligned}$$

By integrating n times, we find that

$$\begin{aligned}\phi_n(\zeta) &= \frac{n^\zeta n!}{\zeta(\zeta+1)\cdots(\zeta+n)}\int_0^1 (1-s)^{n-n}s^{\zeta+n-1}ds\\ &= \frac{n^\zeta n!}{\zeta(\zeta+1)\cdots(\zeta+n)}\int_0^1 s^{\zeta+n-1}\\ &= \frac{n^\zeta n!}{\zeta(\zeta+1)\cdots(\zeta+n-1)}\left[\frac{s^{\zeta+n}}{\zeta+n}\right]_0^1\\ &= \frac{n^\zeta n!}{\zeta(\zeta+1)\cdots(\zeta+n)} \quad (*).\end{aligned}$$

By definition, we can write

$$\begin{aligned}\lim_{n\to+\infty}\phi_n(\zeta) &= \lim_{n\to+\infty}\int_0^n\left(1-\frac{t^n}{n}\right)t^{\zeta-1}dt\\ &= \int_0^{+\infty} e^{-t}t^{\zeta-1}\\ &= \Gamma(\zeta) \quad (**).\end{aligned}$$

From $(*)$ and $(**)$, we find

$$\Gamma(\zeta) = \lim_{n\to+\infty}\frac{n^\zeta n!}{\zeta(\zeta+1)(\zeta+2)\cdots(\zeta+n)}.$$

(P_2^*): Please see [56] for more details.

1.1.2 Beta function

Définition 1.1.2 *Let $\zeta,\zeta^*\in\mathbb{R}_+^*$. The Beta function is defined as:*

$$B(\zeta,\zeta^*) := \int_0^1 t^{\zeta-1}(1-t)^{\zeta^*-1}dt.$$

Proposition 1.1.4

$$B(\zeta,\zeta^*) = 2\int_0^{\frac{\pi}{2}} \sin^{2\zeta-1}\theta\cos^{2\zeta^*-1}\theta d\theta.$$

Proof.

We put $t=\sin^2\theta$. We have

$$\begin{cases} 1-t = 1-\sin^2\theta = \cos^2\theta \\ dt = 2\cos\theta\sin\theta. \end{cases}$$

So, its yields

$$\begin{aligned} B(\zeta,\zeta^*) &= \int_0^1 t^{\zeta-1}(1-t)^{\zeta^*-1}dt \\ &= 2\int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{\zeta-1}\left(\cos^2\theta\right)^{\zeta^*-1}\cos\theta\sin\theta d\theta \\ &= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2\zeta-2}(\cos\theta)^{2\zeta^*-2}\cos\theta\sin\theta d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2\zeta-1}\theta\cos^{2\zeta^*-1}\theta d\theta. \end{aligned}$$

This ends the proof.

1.1.3 Gamma and Beta relation

Proposition 1.1.5 *Let $\zeta,\zeta^*\in\mathbb{R}$ such as $\zeta,\zeta^*>0$. Then we have*

$$B(\zeta,\zeta^*) = \frac{\Gamma(\zeta)\Gamma(\zeta^*)}{\Gamma(\zeta+\zeta^*)}. \tag{1.1}$$

Proof.

Let $\zeta,\zeta^*>0$. Then we obtain

$$\begin{aligned} \Gamma(\zeta)\Gamma(\zeta^*) &= \int_0^{+\infty} t^{\zeta-1}e^{-t}dt\int_0^{+\infty} s^{\zeta^*-1}e^{-s}ds \\ &= \int_0^{+\infty}\int_0^{+\infty} t^{\zeta-1}s^{\zeta^*-1}e^{-(t+s)}dtds. \end{aligned}$$

We put $r = t + s$, hence we can write

$$\begin{aligned}
\Gamma(\zeta)\Gamma(\zeta^*) &= \int_0^{+\infty}\int_0^{+\infty} (r-s)^{\zeta-1}s^{\zeta^*-1}e^{-r}drds \\
&= \int_0^{+\infty}\int_0^{+\infty} r^{\zeta-1}\left(1-\frac{s}{r}\right)^{\zeta-1} s^{\zeta^*-1}e^{-r}drds \\
&= \int_0^{+\infty} r^{\zeta+\zeta^*-2}e^{-r}dr\int_0^{r}\left(1-\frac{s}{r}\right)^{\zeta-1}\left(\frac{s}{r}\right)^{\zeta^*-1}ds.
\end{aligned}$$

After we use the change of variable $z = \frac{s}{r}$, we get

$$\begin{aligned}
\Gamma(\zeta)\Gamma(\zeta^*) &= \int_0^{+\infty} r^{\zeta+\zeta^*-2}e^{-r}dr\int_0^{1}(1-z)^2 \\
&= \int_0^{+\infty} r^{\zeta+\zeta^*-1}e^{-r}dr\int_0^{1}(1-z)^2 \\
&= \left(\int_0^{+\infty} r^{\zeta+\zeta^*-1}e^{-r}dr\right)\left(\int_0^{1}(1-z)^2.\right.
\end{aligned}$$

Remark 1.1.1

1) The Beta function verifies the property of symmetry, i.e.

$$B\left(\zeta,\zeta^*\right) = B\left(\zeta^*,\zeta\right).$$

Also, we have

2) $B(\zeta+1,\zeta^*) = \frac{\zeta}{\zeta+\zeta^*}B(\zeta,\zeta^*)$, $\qquad B(\zeta,\zeta^*+1) = \frac{\zeta^*}{\zeta+\zeta^*}B(\zeta,\zeta^*)$.

3) $B(\zeta,\zeta^*) = B(\zeta,\zeta^*+1) + B(\zeta+1,\zeta^*)$.

1.2 Fractionalisation of integrations and derivatives

In what follow, we shall present the two important approches of fractional calculs; we present, first, the approch of Riemann-Liouville. Then, we introduce the approch of Caputo.

1.2.1 Fractionalisation of Riemann-Liouville

Définition 1.2.1 *[48] Let $f \in L^1([a,b])$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of f is given by*

$$\left({}^{RL}I_a^{\alpha} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds,\ t \in [a,b]. \tag{1.2}$$

Remark 1.2.1

1) In the case $\alpha = 0$, the fractional integral I^0 is interpreted as an identity operator.

2) If $\alpha = n \in \mathbb{N}$, then definition 1.2.1 coincids with the integral:

$$\begin{aligned} \left(I_a^n f\right)(t) &= \int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_n} f(t_n)\, dt_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt. \end{aligned}$$

Example 1.2.1

Let $f(\xi) = (\xi - a)^{\lambda}$, , $\lambda > -1$, then for $\alpha > 0$, we have

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^{\xi} (\xi-\tau)^{\alpha-1} (\xi-a)^{\lambda}\, d\tau, \xi \in [a,b]. \tag{1.3}$$

We put $\xi = a + \rho(\xi - \tau)$, $0 \leq \rho \leq 1$. Then the formula (1.3) is written in the form

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{(\xi-a)^{\alpha+\lambda}}{\Gamma(\alpha)} \int_0^1 \rho^{\lambda} (1-\rho)^{\alpha-1}\, d\rho,$$

Thanks to (1.1), we get

$$\left({}^{RL}I_a^{\alpha} f\right)(\xi) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} (\xi-a)^{\alpha+\lambda}.$$

If $\lambda = 0$, then $\left({}^{RL}I_a^{\alpha} 1\right)(\xi) = \dfrac{1}{\Gamma(\alpha)} (\xi - a)^{\alpha}$.

Proposition 1.2.1 *Let $\alpha \in \mathbb{R}$ such as $\alpha > 0$, then the operator ${}^{RL}I_a^{\alpha}$ is well defined.*

Proof.

Let $f \in L^1([a,b])$ and $\alpha \in \mathbb{R}$ $(\alpha > 0)$. According to Fubini theorem, we have

$$\begin{aligned}
\int_a^t |I_a^\alpha f(t)|\, dt &\leq \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^t (t-s)^{\alpha-1} |f(s)|\, ds dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^b |f(s)| \left(\int_s^b (t-s)^{\alpha-1}\, dt \right) ds \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} \int_a^b |f(s)|\, (b-s)^\alpha\, ds \\
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \int_a^b |f(s)|\, ds < \infty.
\end{aligned}$$

Proposition 1.2.2 *Let $f \in L^1([a,b])$. Then, we have*

$$^{RL}I_a^\alpha \left(I_a^\beta f(x) \right) =^{RL} I_a^{\alpha+\beta} f(t),\ \alpha > 0, \beta > 0.$$

Proof.

By definition, we have

$$\begin{aligned}
^{RL}I_a^\alpha I_a^\beta f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1}\, du \frac{1}{\Gamma(\beta)} \int_a^u (u-t)^{\beta-1} f(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(t)\, dt \int_t^x (x-u)^{\alpha-1} (u-t)^{\beta-1}\, du.
\end{aligned}$$

We put $y = \dfrac{u-t}{x-t}$, so we can write

$$\begin{aligned}
^{RL}I_a^\alpha I_a^\beta f(x) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^x f(t)\, dt\, (x-t)^{\alpha+\beta-1} \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy \\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha+\beta-1} f(t)\, dt =^{RL} I_a^{\alpha+\beta} f(t).
\end{aligned}$$

The relation is thus proved.

Définition 1.2.2 *Let $f \in L^1([a,b])$. The fractional derivative of Riemann-Liouville of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is given by*

$$\begin{aligned}
\left(^{RL}D_a^\alpha f \right)(t) &:= \left(\frac{d}{dt} \right)^n \left(I_a^{n-\alpha} f \right)(t) \qquad (1.4) \\
&= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds,\ n \in \mathbb{N}^*, n-1 < \alpha \leq n, t \in [a,b].
\end{aligned}$$

Remark 1.2.2

If $\alpha = n \in \mathbb{Z}_{+}$, then

$$\left({}^{RL}D_{a}^{\alpha} f\right)(t) \quad = \quad f^{(n)}(t),$$

where $f^{(n)}$ is the standard derivative of order n of the function f.

Example 1.2.2

Let f be the function defined by $f(t) = t^{\lambda}, t \in [0, b], b > 0, \lambda > -1$ and $n-1 < \alpha \leq n, n \in \mathbb{N}^{*}$.
So, we have

$$\begin{aligned}
\left({}^{RL}D_{0}^{\alpha} f\right)(x) \quad : &= \left(\frac{d}{dx}\right)^{n} \left(\frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-s)^{n-\alpha-1} s^{\lambda} ds\right) \\
&= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^{n} \left(x^{n-\alpha+\lambda} \int_{0}^{1} (1-u)^{n-\alpha-1} u^{\lambda} du\right) \\
&= \frac{1}{\Gamma(m-\alpha)} B(\lambda+1, n-\alpha) \left(\frac{d}{dx}\right)^{n} x^{n-\alpha+\lambda}.
\end{aligned}$$

Since

$$\left(\frac{d}{dx}\right)^{n} x^{p} = p(p-1)\ldots(p-n+1)\, x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n},$$

for anny $p \in \mathbb{R} \backslash \{-1, -2, -3, \ldots\}$.
Therefore, we obtain

$$\begin{aligned}
\left({}^{RL}D_{0}^{\alpha} f\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \times \frac{\Gamma(\lambda+1)\Gamma(n-\alpha)}{\Gamma(\lambda+1+n-\alpha)} \times \frac{\Gamma(n-\alpha+\lambda+1)}{\Gamma(n-\alpha+\lambda-m+1)} x^{\lambda-\alpha} \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} x^{\lambda-\alpha}.
\end{aligned}$$

Remark 1.2.3

As a special case, if $\lambda = 0$, then we get

$$\begin{aligned}
{}^{RL}D_{0}^{\alpha} 1 &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \ \forall \alpha \in \mathbb{R}^{+} \backslash \{0, 1, 2, 3, \ldots\}. \\
{}^{RL}D_{0}^{\alpha} 1 &= 0, \forall \alpha \in \mathbb{Z}_{+}.
\end{aligned}$$

Remark 1.2.4 *The fractional derivative in the sense of Riemann-Liouville of a constant function is not zero.*

Proposition 1.2.3 *Let $\alpha, \beta > 0$ such as $n-1 < \alpha \leq n$ and $m-1 < \beta \leq m, n, m \in \mathbb{N}^*$. If $\alpha > \beta > 0$, then for $f \in L^1([a,b])$, we have*

$$\left({}^{RL}D^{\beta} I_a^{\alpha} f\right)(t) = I_a^{\alpha-\beta} f(t).$$

Proposition 1.2.4 *Let $\alpha > 0$ such as $n-1 < \alpha \leq n, n \in \mathbb{N}^*$. For $f \in L^1([a,b])$, we have*

$$\left({}^{RL}D^{\alpha} I_a^{\alpha} f\right)(t) = f(t).$$

Proposition 1.2.5 *Let $n-1 < \alpha \leq n, n \in \mathbb{N}^*, m \in \mathbb{N}^*$ and $f \in L^1([a,b])$. If the fractional derivatives $\left({}^{RL}D^{\alpha} f\right)(t)$ and $(D^{\alpha+m} f)(t)$ exist, then we have*

$$(D^m D^{\alpha} f)(t) = \left(D^{\alpha+m} f\right)(t).$$

1.2.2 Fractionalisation of Caputo

Définition 1.2.3 *The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ of a function $f \in C^n([a,b])$ is defined by*

$${}^{c}D_a^{\alpha} f(t) :={}^{RL} I_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,\ n \in \mathbb{N}^*, n-1 < \alpha < n, t > a.$$

Remark 1.2.5

1) In particular, when $0 < \alpha < 1$ and $f \in C([a,b])$, then

$${}^{c}D_a^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s)\, ds = I_a^{1-\alpha} f'(t).$$

2) If $\alpha \in \mathbb{N}$, then we have

$${}^{c}D_a^{\alpha} f(t) = f^{(n)}(t).$$

Example 1.2.3

Let $f(t) = C, t \in [a,b]$, the constant function, then we have

$${}^{c}D^{\alpha} f(t) = 0 \text{ but } {}^{RL}D^{\alpha} f(t) \neq 0.$$

Proposition 1.2.6 *Let f and g be two functions such that ${}^{c}D^{\alpha} f(t),{}^{c} D^{\alpha} g(t)$ exist. Then the Caputo fractional derivation is a linear operator:*

$$^{c}D_{a}^{\alpha}\left(\lambda f+\gamma g\right)\left(t\right)=\lambda^{c}D_{a}^{\alpha}f\left(t\right)+\gamma^{c}D_{a}^{\alpha}g(t),\ \forall\lambda,\gamma\in\mathbb{R}.$$

Proposition 1.2.7 *Let $n-1<\alpha<n, n\in\mathbb{N}^{*}, m\in\mathbb{N}$ and let the function f such that $^{c}D^{\alpha}f(t)$ exists. Then:*

$$^{c}D^{\alpha}D^{m}f(t)=^{c}D^{\alpha+m}f(t)\neq D^{mc}D^{\alpha}f(t).$$

The following theorem establishes the relation between the fractional derivative in the sense of Caputo and that in the sense of Riemann-Liouville.

Théorème 1.1 *Let $\alpha>0$ with $n-1<\alpha<n, n\in\mathbb{N}^{*}$, and let f be a function such that $^{c}D_{a}^{\alpha}f\left(t\right)$ et $^{RL}D_{a}^{\alpha}f\left(t\right)$ exist. Then, we have:*

$$^{c}D_{a}^{\alpha}f\left(t\right)=^{RL}D_{a}^{\alpha}f\left(t\right)-\sum_{k=0}^{n-1}\frac{f^{(k)}\left(a\right)}{\Gamma\left(k-\alpha+1\right)}\left(t-a\right)^{k-\alpha}.$$

1.2.3 Important notes

In this section, we provide some lemmas of fractional derivatives, witch will play major roles in our analysis, see [8, 10, 25].

Lemma 1.1 *Let $\alpha>0$. Then the general solution of the equation $^{c}D_{a}^{\alpha}x\left(t\right)=0, t\in[a,b]$ can be given by:*

$$x\left(t\right)=\sum_{i=0}^{n-1}c_{i}\left(t-a\right)^{i}, t\in[a,b],$$

such that $c_{i}\in\mathbb{R}, i=0,1,2,..,n-1,\ n=[\alpha]+1$.

Lemma 1.2 *We consider an $\alpha>0$. Then, it yields that*

$$^{RL}I^{\alpha}D^{\alpha}x\left(t\right)=x\left(t\right)+\sum_{i=0}^{n-1}c_{i}\left(t-a\right)^{i}, t\in[a,b],$$

for $c_{i}\in\mathbb{R}, i=0,1,2,...,n-1,\ n=[\alpha]+1$.

Chapter 2

Integral Inequalities for Fixed Points

2.1 Introduction

In this chapter, we are concerned with some important notions on fixed point theory. Some integral inequalities are shown to the reader in order to be used in the two last chapters [28, 43, 61].

2.2 Some needed concepts

2.2.1 Banach space

Définition 2.2.1 *Let B be a vector normed space and σ a metric on B. A metric space (B, σ) is complete if every Cauchy sequence in B has a limit.*

Définition 2.2.2 *We call a Banach space every normed vector space where the induced metric is complete.*

2.2.2 Completely continuous operators

Définition 2.2.3 *A function $f : X \to Y$ (between metric spaces) is continuous when it preserves convergence, this means:*

$$\chi_n \to \chi \in X \Rightarrow f(\chi_n) \to f(\chi) \in Y, \tag{2.1}$$

where $\{\chi_n\}_{n\in\mathbb{N}}$.
In this case, $f\left(\lim_{n\to+\infty}\chi_n\right)=\lim_{n\to+\infty}f\left(\chi_n\right)$.

Définition 2.2.4 *Any set* B *is bounded when the distance between any two points in* B *has an upper bound,*

$$\exists r>0, \quad \forall \chi, y\in B, \quad d(\chi,y)\leq r. \tag{2.2}$$

Définition 2.2.5 *Let us have the spaces* X *and* Y *that are of Banach and let* $T:D\subset X\to Y$

1) *We say that the operator* T *is bounded if it any bounded application subset of* D *into a bounded subset of* Y.

2) *We say that the operator* T *is completely continuous if it is continuous and any bounded application subset of* D *into a relatively compact subset of* Y.

2.3 Around fixed points

2.3.1 Banach Contraction Principle (BCP)

Définition 2.3.1 *Let* (X,d) *a complete metric space and* T *an application of* X *in* X. *We say that* T *is an Lipschitizienne application if it exists a positive constant* k *as we have:*

$$\forall x,y\in X: d(T(x),T(y))\leq kd(x,y).$$

If $k<1$, T *is then called a contraction.*

Théorème 2.1 *Let* T *be a continuous application on a Banach space* X. *Then the following assertions are true:*
1)If there exist $x,y\in X$ *with*

$$\lim_{n\to+\infty}T^n(x)=y,$$

then, $T(y)=y$.
2) If $T(X)$ *is a compact set on* X *and for all* $\epsilon>0$ *there is a* $x_\epsilon\in X$ *with*

$$\|T\left(x_\epsilon\right)-x_\epsilon\|<\epsilon$$

hence, T *admits a fixed point.*

Proof.

1) Let $y_n = T^n(x), n = 1, 2, \ldots$. If T is a continuous application, so

$$T(y) = T\left(\lim_{n\to+\infty} y_n\right) = \lim_{n\to+\infty} T(y_n) = \lim_{n\to+\infty} y_{n+1} = y,$$

which ends the proof of the first assertion.

2) Suppose that the assumptions of 2) are fulfilled. Hence, for $n = 1, 2, \ldots,$ we have $x_n \in X$ and:

$$\|T(x_n) - x_n\| < \tfrac{1}{n} \tag{2.3}$$

$T(X)$ is a compact set implies that there exists a convergent subsequence $(T(x_{n_k}))_{n=1}^{+\infty}$ of $(T(x_n))_{n=1}^{+\infty}$ of limit x. So thanks to (2.3) and the fact that T is continuous, we deduce that x is a fixed point of T.

Théorème 2.2 *(Banach BCP) Let be X a Banach space and $T : X \longrightarrow X$ be a contracting application. Then T has a unique fixed point.*

Proof.

<u>Existence:</u>

We consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} x_n = T(x_n), & n \geq 1 \\ x_0 \in X. \end{cases}$$

We demonstrate that (x_n) is a Cauchy sequence in X. For $m < n$, we have :

$$\|x_n - x_m\| \leq \|x_{m+1} - x_m\| + \|x_{m+2} - x_{m+1}\| + \ldots\ldots + \|x_n - x_{n-1}\|$$

Since T is a contraction, so:

$$\|x_{p+1} - x_p\| = \|Tx_p - Tx_{p-1}\| \leq k\,\|x_p - x_{p-1}\|,\ p \geq 1.$$

Repeating this inequality, we get:

$$\begin{aligned}\|x_n - x_m\| &\leq \left(k^m + k^{m+1} + \ldots\ldots k^{n-1}\right)\|x_1 - x_0\| \\ &\leq k^m \left(1 + k + \ldots\ldots + k^{n-m-1}\right)\|x_1 - x_0\| \\ &\leq \frac{k^m}{1-k}\|x_1 - x_0\|.\end{aligned}$$

We deduce that $(x_n)_n$ is Cauchy in X which is complete, hence, $(x_n)_n$ converges to x in X. Since T is continuous, so:

$$x = \lim_{x\to+\infty} x_n = \lim_{x\to+\infty} T(x_{n-1}) = T\left(\lim_{x\to+\infty} x_{n-1}\right) = Tx.$$

Therefore, x is a fixed point of T.
Uniqueness:
We suppose that $Tx = x$ and $Ty = y$. Thus, it yields that

$$\|x - y\| = \|Tx - Ty\| \leq k\|x - y\|.$$

Since $k < 1$, we deduce that $\|x - y\| = 0$, it means $x = y$, therefore the uniqueness of the fixed point of T is guaranted.

We propose to the reader also the following theorem:

Théorème 2.3 *Let T be an application on a Banach space X, such as T^N is contraction on X for a positive integer N. So T admits a unique fixed point.*

Proof.

The Banach BCP implies that there exists a fixed point for T^N. let us call it x_0. Now, we just note:

$$\|T(x_0) - x_0\| = \left\|T^N(T(x_0)) - T^N(x_0)\right\| \leq k\|T(x_0) - x_0\|.$$

This implies that $T(x_0) = x_0$, this is because $0 < k < 1$. The uniqueness is evidently since a fixed point of T is also a fixed point for T^N.

2.3.2 Schaefer Fixed Point Theorem

We recall the theorem.

Théorème 2.4 *Let B be a Banach space and $T : B \to B$ be a completely continuous operator. If the set:*

$$\Omega := \{u \in B : u = \mu T u, \mu \in]0, 1[\}$$

is bounded, hence T has at least one fixed point.

2.3.3 Arzela-Ascoli and relative compactness

We also present the result.

Théorème 2.5 *Let $A \subset C(K, \mathbb{R}^n), (K = [a, b] \subset \mathbb{R})$. A is relatively compact if and only if:*

1. *A is uniformly bounded.*
2. *A is equicontinuous.*

we remember that a function f is uniformly bounded in A if there is a constant $M > 0$ with:

$$\|f\| = \sup_{x \in K} |f(x)| \leq M, \quad \forall f \in A.$$

2.3.4 Finite dimension theorem and inequalities

The following Brower theorem is well used with its estimates for proving existence of fixed points in finite dimension.

Définition 2.3.2 *We say that a topological space X has the property of the fixed point if any application continues $T : X \to X$ has a fixed point.*

Théorème 2.6 *(Brouwer theorem) Let B_n be the closed unit ball of $\mathbb{R}^N$. It has the property of the fixed point for all $n \in \mathbb{N}^*$.*

2.3.5 Infinite dimension theorem and inequalities

This theorem uses inequality theory to prove and to extend the result of Brouwer for the proof of the existence of a fixed point of a continuous application on a compact convex in a Banach space. It is important to be recalled in what follows.

Théorème 2.7 *(Schauder theorem) Let K be a compact and convex subset of a Banach space X and $T : K \to K$ be a continuous application. So, T admits a fixed point.*

Proof.

Let $T : K \to K$ be a continuous application. Since K is compact, so T is uniformly continuous. Thus for ε fixed, there exists $\delta > 0$; for all $x, y \in K$, we have the inequality:

$$\|x - y\| \leq \delta \Longrightarrow \|T(x) - T(y)\| \leq \varepsilon.$$

Moreover, there exists a finite set of points $\{x_1, x_2, \ldots, x_p\} \subset K$ such as open radius ball δ centered at point x_i recover K; that means: $K \subset \bigcup_{1 \leq j \leq p} B(x_j, \delta)$. If we put $L = \operatorname{vect}(T(x_j))_{1 \leq j \leq p}$, so L est is of finite dimension, and $K^* = K \cap L$ is compact convex of finite dimension. For $1 \leq j \leq p$, we define the continuous function $\psi_j : X \to \mathbb{R}$ by :

$$\begin{cases} 0 & \text{if } \|x - x_j\| \geq \delta \\ 1 - \frac{\|x - x_j\|}{\delta} & \text{if not} \end{cases}$$

It's clear that ψ_j is strictly positive on $B(x_j, \delta)$ and nul outsite. So we have, for all $x \in K, \sum_{j=1}^{p} \psi_j(x) > 0$, we can define on K the positive continuous functions φ_j by :

$$\varphi_j(x) = \frac{\psi_j(x)}{\sum_{k=1}^{p} \psi_k(x)},$$

for which we have $\sum_{j=1}^{p} \varphi_j(x) = 1$, for all $x \in K$.
Let us now pose, for $x \in K$,

$$g(x) = \sum_{j=1}^{p} \varphi_j(x) T(x_j).$$

The function g is continuous (sum of continuous functions) and takes its values in K^* (because g is a barycenter of $T(x_j)$). If we take the restriction $g/K^* : K^* \to K^*$, (according to Brouwer theorem) g has a fixed point $y \in K^*$. Further:

$$\begin{aligned} T(y) - y &= T(y) - g(y) \\ &= \sum_{j=1}^{p} \varphi_j(y) T(y) - \sum_{j=1}^{p} \varphi_j(y) T(x_j) \\ &= \sum_{j=1}^{p} \varphi_j(y) (T(y) - T(x_j)). \end{aligned}$$

But if $\varphi_j(y) \neq 0$ so $\|y - x_j\| < \delta$, and consequently $\|T(y) - T(x_j)\| < \varepsilon$.

so, we have for all j,

$$\begin{aligned}\|T(y)-y\| &\leq \sum_{j=1}^{p} \varphi_j(y)\,\|T(y)-T(x_j)\| \\ &\leq \sum_{j=1}^{p} \varepsilon\varphi_j(y)=\varepsilon.\end{aligned}$$

So, for all whole number m , we can find a point $y_m \in K$ in which $\|T(y_m)-y_m\| \leq 2^{-m}$. and since K is compact, then from the sequence $(y_m)_{m\in\mathbb{Z}}$ we can extract a sub-sequence (y_{m_k}) which converges to a point $y^* \in K$. So T being continuous, The sequence $(T(y_{m_k}))$ converge to $T(y^*)$, and we conclude that $T(y^*)=y^*$, that's means y^* is a fixed point of T on K.

Théorème 2.8 *We suppose that T is a continuous application between two Banach space X et Y. If K is a compact set in X so, $T(K)$ is a compact set in Y.*

Let $T: X \to Y$ an application between twho Banach space. The different notions of continuity used in this chapter are:
We say that T is
- Continuous: if for all $x \in X$ and for $\epsilon > 0$, it exists $\delta = \delta(x,\epsilon)$ in which whatever $y \in X$:

$$\|y-x\|_X < \delta \Rightarrow \|T(y)-T(x)\|_Y < \epsilon.$$

- Uniformly continuous on $A : (A \in X)$, if for all $\epsilon > 0$, it exists $\delta = \delta(\epsilon)$ in which whatever are $x, y \in A$ we have:

$$\|y-x\|_X < \delta \Rightarrow \|T(y)-T(x)\|_Y < \epsilon.$$

If $T_i : X \to Y$ is a set of applications between two Banach spaces. T_i is equicontinue on X, if for all $\epsilon > 0$, it exists $\delta = \delta(\epsilon)$ in which for any $x, y \in X$ and any $i \in I$, we have :

$$\|y-x\|_X < \delta \Rightarrow \|T_i(y)-T_i(x)\|_Y < \epsilon.$$

2.3.6 Inequalities using both Banach BCP and Schauder theorem

We have already presented the two main theorems of the fixed point theory, Schauder theorem and Banach BCP. The result of Krasnoselskii combines these two theorems. So, we have:

Théorème 2.9 *Let F be a closed and convex set of a Banach space X, and let T_1 and T_2 be two applications of F in X, with:*

1. $T_1(x) + T_2(y) \in F, \forall x, y \in F$,

2. *T_1 is a contraction.*

3. *T_2 is compact and continuous.*

So, $T_1 + T_2$ admits a fixed point in F.

Remark 2.3.1

In the proof, we use the inequality:

$$\|T_1(x) - T_1(y)\| \leq k\|x - y\|, x, y \in F, k \in (0, 1).$$

Also, we are invited to use:

$$\|\left(I - T_1\right)(x) - \left(I - T_1\right)y)\| \geq \|x - y\| - \|T_1(x) - T_1(y)\| \geq (1 - k)\|x - y\|$$

and

$$\|\left(I - T_1\right)(x) - \left(I - T_1\right)y)\| \leq \|x - y\| + \|T_1(x) - T_1(y)\| \leq (1 + k)\|x - y\|.$$

These two inequalities are very used in our main results.

Chapter 3

A Class of Time Singular Fractional BVP of Sequential Caputo Derivatives

3.1 Introduction

Research on the existence of unique solutions for fractional differential equations is of big importance since it help physician to better know on the behaviour of real phenomena. For more details, see the papers [2, 17, 21, 23, 24, 26, 36]. Moreover, the singular differential equations are also very important in applied sciences, see [5, 12, 13, 49]. Among these equations, we cite the standard Lane-Emden equation which is part of the present work but in a general case. This equation has a considerable importance in astrophysics, for more details, [38, 52, 57] and the reference therein. Before we begin recalling some other equations and problems that have motivated the present work, we invite the reader to know on the standard form of Lane Emden equation, it is written as follows:

$$y^{''}(t) + \frac{a}{t} y^{'}(t) + f(t, y(t)) = g(t), \ t \in]0,1], \tag{3.1}$$

by taking

$$y(0) = a_1, \ y^{'}(0) = a_2,$$

where f and g are continuous functions (see [62]).
In [47], the authors have worked on the following interesting problem:

$$\begin{cases} D^{\alpha}y(t)+\dfrac{k}{t^{\alpha-\beta}}D^{\beta}y(t)+f(t,y(t))=g(t),\ t\in(0,1], \\ \\ k\geq 0,\ 1<\alpha\leq 2,\ 0<\beta\leq 1, \end{cases} \tag{3.2}$$

with

$$y(0)=y_0,\ y^{'}(0)=y_1,$$

where y_0 and y_1 are real constants, f and g are some continuous functions and the derivatives are in the sense of Riemann-Liouville. They have used a numerical method to establish some solutions for the problem.

In [37, 38], Rabha W. Ibrahim has studied two equations. The equations are given by:

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\frac{a}{t}\right)u(t)+f(t,u(t))=g(t), \\ 0<\alpha,\beta\leq 1,\ 0<t\leq 1,\ a\geq 0. \end{cases} \tag{3.3}$$

For the first equation, Rabha W. Ibrahim has taken the conditions $u(0)=u(1)=u(r)=0, 0<r<1$; the existence of solutions by Krasnoselskii theorem has been studied in [37]. The second problem has the conditions $u(0)=\mu,\ u(1)=\nu$; for this second problem, the Ulam stability of solutions has been discussed in [21].
Also in [16], A. Bekkouche et al. have studied the existence of solutions and the $\Delta-$Ulam stabilities for the following two dimension system:

$$\begin{cases} D^{\beta_1}\left(D^{\alpha_1}+b_1g_1(t)\right)x_1(t)+f_1(t,x_1(t),x_2(t)) \\ \qquad =\omega_1S_1(t,x_1(t),x_2(t)),\ 0<t<1, \\ D^{\beta_2}\left(D^{\alpha_2}+b_2g_2(t)\right)x_2(t)+f_2(t,x_1(t),x_2(t)) \\ \qquad =\omega_2S_2(t,x_1(t),x_2(t)),\ 0<t<1, \\ \quad x_k(0)=0,\ D^{\alpha}x_k(1)+b_kg_k(1)x_k(1)=0, \end{cases} \tag{3.4}$$

under the conditions: $0<\beta_k<1, 0<\alpha_k<1, b_k\geq 0, 0<\omega_k<\infty, k=1,2$ and the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo. The functions $f_k:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ and $S_k:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous, $g_k:]0,1]\to[0,+\infty)$ is continuous and singular at $t=0$.
In the paper [17], A. Benzidane and Z. Dahmani have considered the following class of

nonlinear equations:

$$\begin{cases} D^{\beta_1}\Big(D^{\alpha_1}+g_1(t)\Big)x_1(t)+f_1(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)) \\ \quad =h_1(t,x_1(t),x_2(t)), \\ D^{\beta_2}\Big(D^{\alpha_2}+g_2(t)\Big)x_2(t)+f_2(t,x_1(t),x_2(t),D^{\delta_1}x_1(t),D^{\delta_2}x_2(t)) \\ \quad =h_2(t,x_1(t),x_2(t)), \\ x_k(0)=a_k,\ x_k(1)=b_k,\ t\in J, \end{cases} \tag{3.5}$$

where $J=[0,1]$, $0<\alpha_k,\ \beta_k<1$, $0<\delta_k<\alpha_k<1$, $k=1,2$; the functions $f_k:[0,1]\times\mathbb{R}^4$, $k=1,2$ are continuous, $g_k:(0,1]\longrightarrow[0,+\infty)$ are continuous functions, singular at $t=0$, and $\lim_{t\to 0^+}g_k(t)=\infty$; the operators D^{α_k}, D^{β_k} and D^{δ_k} $k=1,2$ are the derivatives in the sense of Caputo and the constants a_k, b_k are reals. The authors have studied the existence and uniqueness of solutions and the Ulam stability for the considered class.
Y. Bahous ans Z. Dahmani [11] have considered a problem involving both Caputo derivative and Riemann-Liouville integral. Thier problem is given by:

$$\begin{cases} D^{\beta}\Big(D^{\alpha}+\frac{k}{t^{\lambda}}\Big)y(t)+f(t,y(t),D^{\delta}y(t))+g(t,y(t),I^{\rho}y(t)) \\ =h(t),t\in]0,1[, \\ y(0)=\nu,\ y(1)=b\displaystyle\int_0^{\eta}q(s)y(s)ds,\ 0<\eta<1, \\ 0<\beta,\ \alpha<1,\ k>0,\ \lambda>0, \end{cases} \tag{3.6}$$

where D^{α} is of Caputo, I^{ρ} is of Riemann-Liouville integral of ρ, the functions $f,g:[0,1]\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous, and h and q are continuous on $[0,1]$. The authors have investigated the existence and uniqueness of solutions. Then, they have studied the Ulam-Hyers stability.
Also, Y. Gouari et al. [31] have presented the study of the following nonlinear singular

integro-differential equation:

$$\begin{cases} D^{\beta}\Big(D^{\alpha}+\frac{k}{t^{\lambda}}\Big)y(t)+\Delta_1 f(t,y(t),D^{\delta}y(t))+\Delta_2 g(t,y(t),I^{\rho}y(t)) \\ +h(t,y(t))=l(t) \\ y(0)=0 \\ y(1)=b\displaystyle\int_0^{\eta}y(s)ds,\ 0<\eta<1 \\ I^{q}y(u)=y(1),\ 0<u<1 \\ k>0,\ 0<\lambda\leq 1,\ 1\leq\beta\leq 2,\ 0\leq\alpha,\ \delta\leq 1, \end{cases} \tag{3.7}$$

where $\Delta_1>0,\Delta_2>0$ are positive real numbers, I^{ρ} is the Riemann-Liouville integral of order ρ, and f,g are two given functions defined on $[0,1]\times\mathbb{R}^2$, and h and l are two given functions defined over $[0,1]$. The authors have proved the existence and uniqueness of solutions by application of Banach contraction principle, then, by means of Schaefer fixed point theorem, they have studied the existence of at least one solution for the problem.
In this chapter, we are concerned with the following time-singular fractional problem[32]:

$$\begin{cases} D^{\alpha_1}(D^{\alpha_2}...(D^{\alpha_n}(D^{\beta}+\dfrac{k}{t^{\lambda}}))...)u(t)+f(t,u(t),D^{\delta}u(t))+g(t,u(t),I^{\rho}u(t)) \\ +h(t,u(t))=l(t),\quad t\in]0,1[, \\ u(0)=0, \\ u(1)=\theta,\theta\in\mathbb{R}, \\ D^{\alpha_n}(D^{\beta}u(0))=0, \\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0)))=0, \\ \vdots \\ D^{\alpha_3}(D^{\alpha_4}...(D^{\alpha_n}(D^{\beta}u(0)))...)=0, \\ D^{\alpha_2}(D^{\alpha_3}...(D^{\alpha_n}(D^{\beta}u(1)+\phi_{k,\lambda}(1)u(1)))...)=0, \\ k>0. \end{cases} \tag{3.8}$$

For (3.8), we need to consider $J:=[0,1],\ 0\leq\beta\leq 1, 0\leq\alpha_i<1;;i=1,2,...,n,\delta<min(\beta,\alpha_i),\phi_{k,\lambda}(t)=\dfrac{k}{t^{\lambda}}$, the derivatives are in the sense of Caputo, I^{ρ} denotes the Riemann-Liouville fractional integral of order ρ, and $f,g:J\times\mathbb{R}^2\to\mathbb{R}$ are two given functions, also $h:J\times\mathbb{R}\to\mathbb{R}$ is a given function and l is a function which is defined on J.

We need also to shed light on the following remarks:
1. We introduce the Caputo derivatives and the Riemann Liouville-integral in the problem.
3. The problem includes n parameters of Caputo derivations which allow us to introduce a problem with absence of commutativity and semi group properties between the introduced derivatives. So, we have to obtain some arguments to solve this problem.
4. Another important remark in this chapter is the time singularity at the origin for the above problem.
So based on the above conditions, we are concerned with a more general sequential problem of Lane Emden type; it is more general in the sense that it can be used to describe many problems that arise in mathematical physics, since it includes several particular types of equations with some applications. For example, our problem includes the standard LaneEmden equation as a special case. Also, it includes the EmdenFowler equation that was used to model several phenomena in mathematical physics and astrophysics, such as the theory of stellar structure and thermionic currents. Also, the fractional LaneEmden model proposed by Mechee and Senu [47] can be derived from the above problem under some special values on the parameters and the functions.
To the best of our knowledge this is the first time in the literature where such problem is investigated.

3.2 Solutions: existence, existence and uniqueness

3.2.1 Representation of the integral solution

Lemma 3.1 *Let $G \in C([0,1])$. Then, we can state that the problem*

$$\left\{\begin{array}{c} D^{\alpha_1}(D^{\alpha_2}...(D^{\alpha_n}(D^{\beta}+\frac{k}{t^{\lambda}}))...)u(t)=G(t),\ t\in]0,1[,\\ u(0)=0,\\ u(1)=\theta,\theta\in\mathbb{R},\\ D^{\alpha_n}(D^{\beta}u(0))=0,\\ D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0)))=0,\\ \vdots\\ D^{\alpha_3}(D^{\alpha_4}...(D^{\alpha_n}(D^{\beta}u(0)))...)=0,\\ D^{\alpha_2}(D^{\alpha_3}...(D^{\alpha_n}(D^{\beta}u(1)+\phi_{k,\lambda}(1)u(1)))...)=0,\\ k>0,0\leq\alpha_i<1,0\leq\beta\leq 1;i=1,2,...,n \end{array}\right. \tag{3.9}$$

admits the following representation as solution:

$$\begin{aligned} u(t)=&\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\int_0^s(s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}G(\tau)d\tau-\frac{k}{s^{\lambda}}u(s)\right)ds\\ &-\frac{\int_0^1(1-s)^{\alpha_1-1}G(s)ds}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}t^{\sum_{i=2}^{n}\alpha_i+\beta}+\left[\frac{\int_0^1(1-s)^{\alpha_1-1}G(s)ds}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}\right]t^{\beta}\\ &-\left[\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\int_0^s(s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}G(\tau)d\tau-\frac{k}{s^{\lambda}}u(s)\right)ds+\theta\right]t^{\beta}. \end{aligned} \tag{3.10}$$

Proof.

We use the property established in Lemma 1.2 to (3.8). So we have

$$u(t) = I^{\beta}\left(I^{\sum\limits_{i=1}^{n}\alpha_i} G(s) - \frac{k}{s^{\lambda}} \right)(t) + \frac{c_0}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)} t^{\sum\limits_{i=2}^{n}\alpha_i + \beta} + \frac{c_1}{\Gamma(\sum\limits_{i=3}^{n}\alpha_i + \beta + 1)} t^{\sum\limits_{i=3}^{n}\alpha_i + \beta}$$
$$+ \frac{c_2}{\Gamma(\sum\limits_{i=4}^{n}\alpha_i + \beta + 1)} t^{\sum\limits_{i=4}^{n}\alpha_i + \beta} + ... + \frac{c_{n-2}}{\Gamma(\alpha_n + \beta + 1)} t^{\alpha_n + \beta} + \frac{c_{n-1}}{\Gamma(\beta + 1)} t^{\beta} + c_n. \tag{3.11}$$

Some of our conditions allow us to get

$$\begin{gathered}
u(0) = 0 \Rightarrow c_n = 0 \\
D^{\alpha_2}(D^{\alpha_3}...(D^{\alpha_n}(D^{\beta}u(1) + \phi_{k,\lambda}(1)u(1)))...) = 0 \Rightarrow c_0 = -I^{\alpha_1}G(1) \\
D^{\alpha_n}(D^{\beta}u(0)) = 0 \Rightarrow c_{n-2} = 0 \\
D^{\alpha_{n-1}}(D^{\alpha_n}(D^{\beta}u(0))) = 0 \Rightarrow c_{n-3} = 0 \\
\vdots \\
D^{\alpha_4}(D^{\alpha_5}...(D^{\alpha_n}(D^{\beta}u(0)))...) = 0 \Rightarrow c_2 = 0 \\
D^{\alpha_3}(D^{\alpha_4}...(D^{\alpha_n}(D^{\beta}u(0)))...) = 0 \Rightarrow c_1 = 0 \\
u(1) = \theta \Rightarrow \\
\frac{c_{n-1}}{\Gamma(\beta+1)} = \frac{\int_0^1 (1-s)^{\alpha_1 - 1} G(s) ds}{\Gamma(\alpha_1)\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)} \\
-\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i)} \int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i - 1} G(\tau) d\tau - \frac{k}{s^{\lambda}} u(s) \right) ds + \theta.
\end{gathered} \tag{3.12}$$

Replacing $c_0, c_1, c_2, ..., c_n$ in (4.3), we end the proof of the result.

Let us now transforming the above problem to a fixed point one.
We begin by considering the Banach space:

$$X := \{x \in C(J,\mathbb{R}), D^{\delta}x \in C(J,\mathbb{R})\} \tag{3.13}$$

and its norm:

$$\|x\|_X = Max\{\|x\|_\infty\ , \|D^{\delta}x\|_\infty\}, \tag{3.14}$$

where by definition, we put:

$$\|x\|_\infty = \sup_{t\in J}|x(t)|\ ,\ \|D^{\delta}x\|_\infty = \sup_{t\in J}|D^{\delta}x(t)|. \tag{3.15}$$

Then, we pass to consider the nonlinear operator $H : X \to X$ defined by:

$$\begin{aligned}
Hu(t) \ &= \ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big[\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i - 1}\Big(l(\tau) - h(\tau,u(\tau)) - f(\tau,u(\tau),D^{\delta}u(\tau))\\
&-g(\tau,u(\tau),I^{\rho}u(\tau))\Big)d\tau - \frac{k}{s^{\lambda}}u(s)\Big]ds - \frac{t^{\sum_{i=2}^{n}\alpha_i+\beta} + t^{\beta}}{\Gamma(\alpha_1)\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}\int_0^1 (1-s)^{\alpha_1-1}\Big(l(s)\\
&-h(s,u(s)) - f(s,u(s),D^{\delta}u(s)) - g(s,u(s),I^{\rho}u(s))\Big)ds - \Big[\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\Big(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\\
&\times\int_0^s (s-\tau)^{\sum_{i=1}^{n}\alpha_i-1}\Big(l(\tau) - h(\tau,u(\tau)) - f(\tau,u(\tau),D^{\delta}u(\tau)) - g(\tau,u(\tau),I^{\rho}u(\tau))\Big)d\tau\\
&-\frac{k}{s^{\lambda}}u(s)\Big)ds + \theta\Big]t^{\beta},
\end{aligned} \tag{3.16}$$

such that $\lambda < 1$.

To prove the main results, we need to work with the following considerations:

$(A1)$: The functions f and g defined on $J\times\mathbb{R}^2$ are continuous and h defined on $J\times\mathbb{R}$ is also continuous.

$(A2)$: There exist nonnegative constants $L_{f1}, L_{f2}, L_{g1}, L_{g2}$ such that, for any t $\in J$, $x_i, x_i^* \in \mathbb{R}$,

$$|f(t, x_1, x_2) - f(t, {x_1}^*, {x_2}^*)| \leq \sum_{i=1}^{2} L_{fi}|x_i - {x_i}^*|, \tag{3.17}$$

$$|g(t, x_1, x_2) - g(t, {x_1}^*, {x_2}^*)| \leq \sum_{i=1}^{2} L_{gi}|x_i - {x_i}^*|, \tag{3.18}$$

and there is a positive number r_0 such that, for any $t \in J$, $x, y \in \mathbb{R}$,

$$|h(t, x) - h(t, y)| \leq r_0|x - y|. \tag{3.19}$$

We take, also, the quantities:

$$L'_f := Max(L_{f1}, L_{f2}), \;\; L'_g := Max(L_{g1}, L_{g2}). \tag{3.20}$$

$(A3)$: There exist non negative constants M_f, M_g, M_h, such that, for any $t \in J, \; x \in \mathbb{R}^2, y \in \mathbb{R}$, we have

$$|f(t, x)| \leq M_f, \;\; |g(t, x)| \leq M_g, |h(t, y)| \leq M_h. \tag{3.21}$$

$(A4)$: We take: $\|l\|_\infty = M_l$.

Also we consider the quantities:

$$\begin{aligned} D1 \;\; = \;\; & 2\Bigg[\left(r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho+1)}\right)\left(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)}\right) \\ & + \frac{k\Gamma(1-\lambda)}{\Gamma(\beta - \lambda + 1)}\Bigg], \end{aligned} \tag{3.22}$$

$$D2 = \Big(r_0 + 2L_f^{'} + L_g^{'} + \frac{L_g'}{\Gamma(\rho+1)}\Big)\Big(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta - \delta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\beta-\delta+1)\Gamma(\alpha_1+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)\Gamma(\beta-\delta+1)}\Big) + k\Gamma(1-\lambda)\Big(\frac{1}{\Gamma(\beta-\delta-\lambda+1)} + \frac{1}{\Gamma(\beta-\lambda+1)}\Big). \tag{3.23}$$

3.2.2 One solution

The first main result deals with the existence of a unique solution for (4.1). It is based on the application of BCP theorem. We prove:

Théorème 3.1 *If the conditions* $(A\ i)_{i=2,3,4}$ *are satisfied and* $D < 1, D := max\{D1, D2\}$, *then, the problem (3.8) has a unique solution on J.*

Proof.

It is sufficient for us to prove that H is a contraction mapping.
Let $(x, y) \in X^2$. Then, we can write

$$\|Hy - Hx\|_\infty \leq 2\Big[\Big(r_0 + 2L_f^{'} + L_g^{'} + \frac{L_g'}{\Gamma(\rho+1)}\Big)\Big(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1+1)}\Big) + \frac{k\Gamma(1-\lambda)}{\Gamma(\beta-\lambda+1)}\Big]\|y - x\|_X. \tag{3.24}$$

On the other hand, since

$$
\begin{aligned}
D^{\delta}Hu(t) \;=\; & \frac{1}{\Gamma(\beta-\delta)}\int_0^t (t-s)^{\beta-\delta-1}\Big[\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i)}\int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i-1}\Big(l(\tau)-h(\tau,u(\tau))\\
& -f(\tau,u(\tau),D^{\delta}u(\tau))-g(\tau,u(\tau),I^{\rho}u(\tau))\Big)d\tau-\frac{k}{s^{\lambda}}u(s)\Big]ds-\frac{t^{\sum\limits_{i=2}^{n}\alpha_i+\beta-\delta}}{\Gamma(\alpha_1)\Gamma(\sum\limits_{i=2}^{n}\alpha_i+\beta-\delta+1)}\\
& \times\int_0^1 (1-s)^{\alpha_1-1}\Big(l(s)-h(s,u(s))-f(s,u(s),D^{\delta}u(s))-g(s,u(s),I^{\rho}u(s))\Big)ds\\
& +\frac{\Gamma(\beta+1)t^{\beta-\delta}}{\Gamma(\alpha_1)\Gamma(\sum\limits_{i=2}^{n}\alpha_i+\beta+1)\Gamma(\beta-\delta+1)}\int_0^1 (1-s)^{\alpha_1-1}\Big(l(s)-h(s,u(s))\\
& -f(s,u(s),D^{\delta}u(s))-g(s,u(s),I^{\rho}u(s))\Big)ds-\Big[\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\Big(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i)}\\
& \times\int_0^s (s-\tau)^{\sum\limits_{i=1}^{n}\alpha_i-1}\Big(l(\tau)-h(\tau,u(\tau))-f(\tau,u(\tau),D^{\delta}u(\tau))-g(\tau,u(\tau),I^{\rho}u(\tau))\Big)d\tau\\
& -\frac{k}{s^{\lambda}}u(s)\Big)ds+\theta\Big]\frac{\Gamma(\beta+1)}{\Gamma(\beta-\delta+1)}t^{\beta-\delta},
\end{aligned}
\tag{3.25}
$$

then, with the same arguments as before, we have

$$
\begin{aligned}
\|D^{\delta}Hy - D^{\delta}Hx\|_{\infty} \leq & \Big(r_0 + 2L_f^{'} + L_g^{'} + \frac{L_g'}{\Gamma(\rho+1)}\Big)\Big(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta - \delta + 1)} \\
& + \frac{1}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
& + \frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)}\Big)\|y - x\|_X \\
& + k\Gamma(1-\lambda)\Big(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)}\Big)\|y - x\|_X.
\end{aligned}
\tag{3.26}
$$

Thanks to (3.24) and (3.26), we obtain

$$
\|Hy - Hx\|_X \leq D\ \|x - y\|_X.
$$

The proof is thus achieved.

3.2.3 At least one solution

The following main result deals with the existence of at least one solution of the studied problem.

Théorème 3.2 *Under the hypotheses* $(A1)$, $(A3)$ *and* $(A4)$, *the problem (3.8) has at least one solution* $u(t), t \in J$.

Proof.
Let us prove the result by considering the following main steps:

Continuous of H
The proof is evident then it is omitted.

Boundedness of H

Let us take $r > 0$ and consider the (bounded) ball $B_r := \{x \in X; \|x\|_X \leq r\}$. For $y \in B_r$, in virtue of $(A3)$ and $(A4)$, we can write

$$\begin{aligned}
\|Hy\|_\infty \;\leq\; & 2\Bigg[\Big(M_l + M_h + M_f + M_g\Big)\Big(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)}\Big) \\
& + \frac{k\Gamma(1-\lambda)}{\Gamma(\beta - \lambda + 1)}\Bigg] + |\theta| < +\infty
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
\|D^\delta Hy\|_\infty \;\leq\; & \Big(M_l + M_h + M_f + M_g\Big)\Big(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta - \delta + 1)} \\
& + \frac{1}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta + 1)}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
& + \frac{\Gamma(\beta + 1)}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)}\Big) + k\Gamma(1-\lambda)\Big(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)}\Big) \\
& + |\theta| < +\infty.
\end{aligned} \tag{3.28}$$

The above two inequalities show that $\|Hy\|_X < +\infty$.

Consequently H is uniformly bounded.

Ascolli Arzella for H

We prove that for any bounded set B_r for instance, we obtain that $H(B_r)$ is an equicontinuous set of X.

Taking $t_1, t_2 \in [0, 1], t_1 < t_2$ and consider the above (bounded) ball B_r of X. So by considering $y \in B_r$, we can state that

$$
\begin{aligned}
|Hy(t_1) - Hy(t_2)| \;\leq\;& \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum\limits_{i=1}^{n} \alpha_i + \beta + 1)} \, |{t_1}^{\sum\limits_{i=1}^{n} \alpha_i + \beta} - {t_2}^{\sum\limits_{i=1}^{n} \alpha_i + \beta}| \\
& + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta)} |{t_1}^{\beta-\lambda} - {t_2}^{\beta-\lambda}| \\
& + \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum\limits_{i=2}^{n} \alpha_i + \beta + 1)\Gamma(1+\alpha_1)} \, \Big(|{t_1}^{\sum\limits_{i=2}^{n} \alpha_i + \beta} - {t_2}^{\sum\limits_{i=2}^{n} \alpha_i + \beta}| + |{t_1}^{\beta} - {t_2}^{\beta}| \Big) \\
& + \Big(\frac{M_l + M_h + M_f + M_g}{\Gamma(\sum\limits_{i=1}^{n} \alpha_i + \beta + 1)} + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta)} + \theta \Big) |{t_1}^{\beta} - {t_2}^{\beta}|
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
|D^\delta Hy(t_1) - D^\delta Hy(t_2)| \;\leq\; & \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=1}^{n} \alpha_i + \beta - \delta + 1)} |t_1^{\sum_{i=1}^{n} \alpha_i - \delta + \beta} - t_2^{\sum_{i=1}^{n} \alpha_i - \delta + \beta}| \\
& + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta-\delta)} |t_1{}^{\beta-\delta-\lambda} - t_2{}^{\beta-\delta-\lambda}| \\
& + \frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=2}^{n} \alpha_i + \beta + 1)\Gamma(1+\alpha_1)} |t_1^{\sum_{i=2}^{n} \alpha_i + \beta - \delta} - t_2^{\sum_{i=2}^{n} \alpha_i + \beta - \delta}| \\
& + \frac{\Big(M_l + M_h + M_f + M_g\Big)\Gamma(\beta+1)}{\Gamma(\sum_{i=2}^{n} \alpha_i + \beta + 1)\Gamma(\beta-\delta+1)\Gamma(\alpha_1+1)} |t_1{}^{\beta-\delta} - t_2{}^{\beta-\delta}| \\
& + \left(\frac{M_l + M_h + M_f + M_g}{\Gamma(\sum_{i=1}^{n} \alpha_i + \beta + 1)} + kr \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\beta)} + \theta \right) \frac{\Gamma(\beta+1)}{\Gamma(\beta-\delta+1)} \\
& \times |t_1{}^{\beta-\delta} - t_2{}^{\beta-\delta}|.
\end{aligned}
\tag{3.30}
$$

For (3.29) and (3.30), their right hand sides tend to zero for $t_1 \to t_2$.
As a consequence the Ascoli-Arzela theorem, we conclude that H is completely continuous.

Boundedness of A_γ

The set $A_\gamma := \{x \in X : x = \gamma\ Hx, \gamma \in]0,1[\}$ is bounded.
Let $y \in A_\gamma$. Then we have $y = \gamma Hy$ for some $0 < \gamma < 1$. Hence we can write

$$
\begin{aligned}
\|y\|_{\infty} \;\; \leq \;\; & \gamma\Bigg(2\Bigg[\Bigg(M_l + M_h + M_f + M_g\Bigg)\Bigg(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + 1)} + \frac{1}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\alpha_1 + 1)}\Bigg) \\
& + \frac{k\Gamma(1-\lambda)}{\Gamma(\beta - \lambda + 1)}\Bigg] + |\theta|\Bigg).
\end{aligned} \tag{3.31}
$$

We have also

$$
\begin{aligned}
\|D^{\delta}y\|_{\infty} \;\; \leq \;\; & \gamma\Bigg(\Bigg(M_l + M_h + M_f + M_g\Bigg)\Bigg(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta - \delta + 1)} \\
& + \frac{1}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1 + 1)} + \frac{\Gamma(\beta+1)}{\Gamma(\sum\limits_{i=2}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)\Gamma(\alpha_1 + 1)} \\
& + \frac{\Gamma(\beta+1)}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + 1)\Gamma(\beta - \delta + 1)}\Bigg) + k\Gamma(1-\lambda)\Bigg(\frac{1}{\Gamma(\beta - \delta - \lambda + 1)} + \frac{1}{\Gamma(\beta - \lambda + 1)}\Bigg) \\
& + |\theta|\Bigg).
\end{aligned} \tag{3.32}
$$

Using (3.27) and (3.28), we state that $\|y\|_X < \infty$. The set is thus bounded. Consequently, thanks to Schaefer fixed point theorem, we deduce that H has at least one fixed point. Thus, the problem (4.1) has a solution.

3.3 Illustrative examples

let us give the following examples.

Example 3.3.1

We concider the following example with 11 sequential derivatives:

$$\begin{cases} D^{0.5}(D^{0.2}(D^{0.6}(D^{0.5}(D^{0.9}(D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}+\dfrac{2}{10t^{\frac{1}{2}}})))))))))))y(t) \\ +\dfrac{1}{80e^{t+2}}\left(\dfrac{|y(t)|}{2(1+|y(t)|)}+\dfrac{\cos D^{\frac{1}{100}}y(t)}{e^{t^2+1}}\right)+\dfrac{\cos y(t)+\cos I^{\frac{1}{2}}y(t)}{95(\pi^2+t)}+\dfrac{\sin y(t)}{300e^t}=e^t+t,\ t\in]0,1[, \\ y(0)=0, \\ y(1)=1, \\ D^{0.5}(D^{0.1}y(0))=0, \\ D^{0.8}(D^{0.5}(D^{0.1}y(0)))=0, \\ D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0))))=0, \\ D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0)))))=0, \\ D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0))))))=0, \\ D^{0.9}(D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0)))))))=0, \\ D^{0.5}(D^{0.9}(D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0))))))))=0, \\ D^{0.6}(D^{0.5}(D^{0.9}(D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(0)))))))))=0, \\ D^{0.2}(D^{0.6}(D^{0.5}(D^{0.9}(D^{0.8}(D^{0.8}(D^{0.4}(D^{0.8}(D^{0.5}(D^{0.1}y(1)+\phi_{\frac{2}{10},\frac{1}{2}}(1)y(1))))))))))=0, \\ \phi_{\frac{3}{10},0.1}(t)=\dfrac{2}{10t^{\frac{1}{2}}}. \end{cases} \tag{3.33}$$

We have:

$$\begin{aligned} &f(t,x_1,x_2)=\frac{1}{80e^{t+2}}\left(\frac{|x_1(t)|}{2(1+|x_1(t))|}+\frac{\cos x_2(t)}{e^{t^2+1}}\right), \\ &g(t,x_1,x_2)=\frac{\cos x_1(t)+\cos x_2(t)}{95(\pi^2+t)}, \\ &h(t,x)=\frac{\sin x(t)}{300e^t}, \\ &l(t)=e^t+t. \end{aligned}$$

We remark also that

$$\begin{aligned}
&L'_f = \frac{1}{80e^2}, L'_g = \frac{1}{95\pi^2}, r_0 = \frac{1}{300},\\
&r_0 + 2L'_f + L'_g + \frac{L'_g}{\Gamma(\rho+1)} = 0.009\\
&\frac{1}{\Gamma(\sum_{i=1}^{10}\alpha_i + \beta + 1)} = 0.007,\\
&\frac{1}{\Gamma(\sum_{i=2}^{10}\alpha_i + \beta + 1)\Gamma(\alpha_1+1)} = 0.0144,\\
&\frac{k\Gamma(1-\lambda)}{\Gamma(\beta-\lambda+1)} = 0.238.
\end{aligned}$$

Based on the above data, we have

$$\begin{aligned}
&\frac{1}{\Gamma(\sum_{i=1}^{10}\alpha_i + \beta - \delta + 1)} = 0.0071,\\
&\frac{1}{\Gamma(\sum_{i=2}^{10}\alpha_i + \beta - \delta + 1)\Gamma(\alpha_1+1)} = 0.0146,\\
&\frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=2}^{10}\alpha_i + \beta + 1)\Gamma(\beta-\delta+1)\Gamma(\alpha_1+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\sum_{i=1}^{10}\alpha_i + \beta + 1)\Gamma(\beta-\delta+1)} = 0.0213,\\
&k\Gamma(1-\lambda)\left(\frac{1}{\Gamma(\beta-\delta-\lambda+1)} + \frac{1}{\Gamma(\beta-\lambda+1)}\right) = 0.2344.
\end{aligned}$$

Hence, it yields that

$$D1 = 0.4765, \quad D2 = 0.4728,$$

$$D = max\,\{D1, D2\} = 0.4765.$$

The conditions of Theorem 3.1 hold. Therefore, the above example has a unique solution $y(t)$ on $[0,1]$.

Example 3.3.2

Now, we consider another example involving five sequential derivatives:

$$\begin{cases} D^{0.1}(D^{0.4}(D^{0.3}(D^{0.2}(D^{0.9} + \dfrac{3}{10t^{0.1}}))))y(t) + \dfrac{|y(t)|}{(44\pi + t)(1 + |y(t)|)} + \dfrac{|D^{\delta}y(t)|}{(200 + t)(1 + |D^{\delta}y(t)|)} \\ + \dfrac{cos(y(t) + I^{\rho}y(t))}{400(t^2 + 1)} + \dfrac{|y(t)|}{(72e^{t+2})(1 + |y(t)|)} = 3t, \ t \in]0, 1[, \\ y(0) = 0, \\ y(1) = 1, \\ D^{0.2}(D^{0.9}y(0)) = 0, \\ D^{0.3}(D^{0.2}(D^{0.9}y(0))) = 0, \\ D^{0.4}(D^{0.3}(D^{0.2}(D^{0.9}y(1) + \phi_{\frac{3}{10},0.1}(1)y(1)))) = 0, \\ \phi_{\frac{3}{10},0.1}(t) = \dfrac{3}{10t^{0.1}}. \end{cases} \tag{3.34}$$

we remrk that

$$\begin{aligned} &f(t, x_1, x_2) = \frac{|x_1|}{(44\pi + t)(1 + |x_1|)} + \frac{|x_2|}{(200 + t)(1 + |x_2|)}, \\ &g(t, x_1, x_2) = \frac{cos(x_1 + x_2)}{400(t^2 + 1)}, \\ &h(t, x) = \frac{|x|}{(72e^{t+2})(1 + |x|)}, \\ &l(t) = 3t. \end{aligned}$$

$$\begin{aligned} &Also, \\ &\delta = 0.1, \ \rho = 0.5, \\ &D1 = 0.9613, \quad D2 = 0.9924, \\ &D = max\left\{D1, D2\right\} = 0.9924. \end{aligned}$$

The conditions of Theorem 3.1 hold. Therefore, the above example has a unique solution $y(t)$ on $[0, 1]$.

Example 3.3.3

The flowing example is given to show the validity of the second main result. So we consider:

$$\begin{cases} D^{0.2}(D^{0.4}(D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}}+\frac{2}{5t^{0.2}})))))y(t)+\dfrac{\sin y(t)+\sin D^{\delta}y(t)}{100e^{t^2+1}}+ \\ +\dfrac{1}{48e^t}\left(\cos y(t)+\dfrac{|I^{\rho}y(t)|}{1+|I^{\rho}y(t)|}\right)+\dfrac{|y(t)|}{(144e^{t^2})(1+|y(t)|)}=4t,\ t\in]0,1[, \\ y(0)=0, \\ y(1)=1, \\ D^{0.2}(D^{\frac{1}{2}}y(0))=0, \\ D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(0)))=0, \\ D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(0))))=0, \\ D^{0.4}(D^{0.4}(D^{0.2}(D^{0.2}(D^{\frac{1}{2}}y(1)+\phi_{\frac{2}{5},0.2}(1)y(1)))))=0, \\ \phi_{\frac{2}{5},0.2}(t)=\dfrac{4}{10t^{0.2}}. \end{cases} \tag{3.35}$$

It is clear that

$$f(t,x_1,x_2)=\frac{\sin x_1+\sin x_2}{100e^{t^2+1}},\qquad g(t,x_1,x_2)=\frac{1}{48e^t}\left(\cos x_1+\frac{|x_2|}{1+|x_2|}\right),$$
$$h(t,x)=\frac{|x|}{(144e^{t^2})(1+|x|)},\qquad l(t)=4t.$$

Also, we have

$$\delta=\tfrac{1}{2},\quad \rho=\tfrac{1}{7}.$$

Hence, we remark that

$$|f(t,x_1,x_2)|\leq\frac{1}{50},\qquad |g(t,x_1,x_2)|\leq\frac{1}{24},\qquad |h(t,x)|\leq\frac{1}{144},\qquad \|l\|_{\infty}=4.$$

Since the functions f,g,h are continuous, then by Theorem 3.2, the problem presented in this example has at least one solution on $u(t), t\in[0,1]$.

Chapter 4

An Analytic and a Numerical Study for a Class of Singular BVPs With Series

4.1 Introduction

The differential equations, with time or space singularities, are of great interest since several physical situations are modelled by problems of this kind, (for example, problems in gas and fluid dynamics), see [17, 18, 46]. For this singular field theory, many authors have paid a great attention to the questions of the existence and uniqueness of solutions to this type of equations. For more details, we refer the reader to [30, 33, 38]. The reader can also point out that stability of solutions of such equations is useful in solving many problems in economics, mechanics, and also in control theory, see [38, 51, 52] and the reference therein.

4.2 The posed problem

Before introducing our problem, we need to cite some other results that have motivated our aim. We begin by [1], where the authors have studied, for the first time, the existence and uniqueness of solutions for the following non singular system involving series:

$$\begin{cases} D^{\alpha}u\,(t) = f_1(t,u(t),v(t)) + \displaystyle\sum_{i=1}^{\infty}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\varphi_i(s)g_i(s,u(s),v(s))ds, t\in[0,1], \\ D^{\beta}v\,(t) = f_2(t,u(t),v(t)) + \displaystyle\sum_{i=1}^{\infty}\int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\phi_i(s)h_i(s,u(s),v(s))ds, t\in[0,1], \\ \\ \displaystyle\sum_{k=0}^{n-2}(|u^{(k)}(0)|+|v^{(k)}(0)|)=0, \\ u^{(n-1)}(0)=\gamma I^p u(\eta), \eta\in[0,1], \\ v^{(n-1)}(0)=\delta I^p v(\zeta), \zeta\in[0,1]. \end{cases}$$

Then, based on the above paper, the authors in [58] have studied the following second non singular fractional differential problem:

$$\begin{cases} D^{\alpha_1}u\,(t) = \displaystyle\sum_{i=1}^{l} f_i(t,u(t),v(t),D^{\gamma_1}u(t),D^{\gamma_1}v(t)) \\ +\displaystyle\sum_{j=1}^{\infty}\int_0^t \frac{(t-s)^{\delta_j-1}}{\Gamma(\delta_j)}\varphi_i(s)g_i(s,u(s),v(s),D^{\gamma_2}u(s),D^{\gamma_2}v(s))ds, t\in[0,1], \\ D^{\alpha_2}v\,(t) = \displaystyle\sum_{i=1}^{l} k_i(t,u(t),v(t),D^{\gamma_2}u(t),D^{\gamma_2}v(t)) \\ +\displaystyle\sum_{j=1}^{\infty}\int_0^t \frac{(t-s)^{\theta_j-1}}{\Gamma(\theta_j)}\phi_i(s)h_i(s,u(s),v(s),D^{\gamma_2}u(s),D^{\gamma_2}v(s))ds, t\in[0,1], \\ \\ u(0)=a_0, v(0)=b_0, \\ u^{(j)}(0)=v^{(j)}(0)=0, j=1,2,...,n-2, \\ u^{(n-1)}(0)=J^p u(\tau), p>0, \tau\in]0,1[, \\ v^{(n-1)}(0)=J^q v(\rho), q>0, \rho\in]0,1[. \end{cases}$$

For the singular case without series, we can also cite the papers [31, 33], where the authors have studied the questions of existence of solutions as well as the stability for the problem:

$$\begin{cases} D^{\beta}(D^{\alpha}+\frac{k}{t^{\lambda}})y(t)+\Delta_1 f(t,y(t),D^{\delta}y(t))+\Delta_2 g(t,y(t),I^{\rho}y(t))+h(t,y(t)) \\ =l(t),\ t\in]0,1[, \\ \\ y(0)=0, \\ y(1)=b\int_0^{\eta}y(s)ds,\ 0<\eta<1, \\ I^{q}y(u)=y(1),\ 0<u<1, \\ k>0, 0<\lambda\leq 1,\ 1\leq\beta\leq 2,\ 0\leq\alpha,\ \delta\leq 1, \end{cases}$$

where, $\Delta_1>0$, $\Delta_2>0$, $J:=[0,1]$, the two fractional derivative of the problem are in the sense of Caputo, I^{ρ} is the Riemann-Liouville integral and f,g,h,l are some given functions. Motivated by both the above two series-works and by the applications of singular differential equations in fluid dynamics, in this paper, we study the following problem:

$$\begin{cases} D^{\alpha}u(t)+\lambda f(u(t),u^{''}(t))=\delta g(t,u(t),D^{\gamma}u(t))+\sum_{i=1}^{\infty}\nu_i\Phi_i(t)I^{\alpha}h_i(t,u(t)),t\in(0,1], \\ u^{''}(0)+u^{''}(1)=\kappa_1\int_0^{\xi}u(s)ds,\quad 0<\xi<1, \\ u^{'}(0)+u^{'}(1)=\kappa_2\int_0^{\theta}u(s)ds,\quad 0<\theta<1, \\ u(0)+u(1)=\kappa_3\int_0^{\eta}u(s)ds,\quad 0<\eta<1, \\ 2<\alpha\leq 3,\quad 0<\gamma<1,\quad \kappa_1,\kappa_2,\kappa_3,\lambda,\delta,\nu_i\in\mathbb{R}, \end{cases} \tag{4.1}$$

where we note that $J:=[0,1]$, the functions f, h_i and Φ_i will be specified later, g is singular at $t=0$, the operators D^{α} and D^{γ} are the derivatives in the sense of Caputo.

To the best of our knowledge, this is the first time in the literature where singular differential equations, involving fractional calculus and convergent series on Riemann-Liouville integrals and other terms, are investigated. So, in general, our aim is to present a first contribution in this direction and try to fill this gap. Especially, we study the question of existence and uniqueness of solutions by using both fixed point theory and integral inequalities, then we pass to the investigate the question of stability of solutions in the sense of Ulam-Hyers where the integral inequalities and estimates will allow us to prove the results. Our results will be concretized by some illustrated examples. Then, thanks to some numerical techniques that allow us to approximate the Caputo derivatives, (see the two papers [22, 44]), and by using Rung Kutta method, we present a numerical study with some simulations in order to present to the reader more comprehension on the proposed examples.

4.3 Uniqueness

4.3.1 Integral equation

We present to the reader the proof of the integral solution of the introduced problem.

Lemma 4.1 *Let G in $C(]0,1])$, $(H_i)_{i=1,\dots,r}$ and $(\Phi_i)_{i=1,\dots,r}$ in $C(J)$, $r \in \mathbb{N}^*$, such that $\sum_{i=1}^{\infty} ||\nu_i \Phi_i I^\alpha H_i||_\infty$ is finite, then, one has*

$$\begin{cases} D^\alpha u(t) = G(t) + \displaystyle\sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^\alpha H_i(t), t \in (0,1], \\ u''(0) + u''(1) = \kappa_1 \displaystyle\int_0^\xi u(s)ds, \quad 0 < \xi < 1, \\ u'(0) + u'(1) = \kappa_2 \displaystyle\int_0^\theta u(s)ds, \quad 0 < \theta < 1, \\ u(0) + u(1) = \kappa_3 \displaystyle\int_0^\eta u(s)ds, \quad 0 < \eta < 1, \\ 2 < \alpha \leq 3, \quad \kappa_1, \kappa_2, \kappa_3, \nu_i \in \mathbb{R} \end{cases}$$

if and only if

$$
\begin{aligned}
u(t) \qquad &= I^\alpha G(t) + \sum_{i=1}^{\infty} \nu_i I^\alpha \Big(\Phi_i(t) I^\alpha H_i(t)\Big) + \Big[\frac{\Lambda_1 t^2 + \psi_1 t + \Delta_1}{\varphi}\Big]\Big[\kappa_3 \int_0^\eta I^\alpha G(s) ds \\
&+ \sum_{i=1}^{\infty} \kappa_3 \nu_i \int_0^\eta I^\alpha \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds - \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} G(s) ds - \sum_{i=1}^{\infty} \frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times \int_0^1 (1-s)^{\alpha-3} \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds\Big] + \Big[\frac{\Lambda_2 t^2 + \psi_2 t + \Delta_2}{\varphi}\Big]\Big[\kappa_2 \int_0^\theta I^\alpha G(s) ds \\
&+ \sum_{i=1}^{\infty} \kappa_2 \nu_i \int_0^\theta I^\alpha \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} G(s) ds - \sum_{i=1}^{\infty} \frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times \int_0^1 (1-s)^{\alpha-2} \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds\Big] + \Big[\frac{\Lambda_3 t^2 + \psi_3 t + \Delta_3}{\varphi}\Big]\Big[\kappa_1 \int_0^\xi I^\alpha G(s) ds \\
&+ \sum_{i=1}^{\infty} \kappa_1 \nu_i \int_0^\xi I^\alpha \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} G(s) ds - \sum_{i=1}^{\infty} \frac{\nu_i}{\Gamma(\alpha)} \\
&\times \int_0^1 (1-s)^{\alpha-1} \Big(\Phi_i(s) I^\alpha H_i(s)\Big) ds\Big], \qquad (4.2)
\end{aligned}
$$

where, we need to take into consideration:

$$\varphi = F_1(E_2-2)(D_3-1) + E_1(F_3-2)(D_2-2) + (D_1-4)F_2(E_3-1) - F_1(E_3-1)(D_2-2)$$
$$-(F_3-2)(E_2-2)(D_1-4) - E_1F_2(D_3-1),$$

$$\Lambda_1 = (F_3-2)(E_2-2) - F_2(E_3-1),$$

$$\Lambda_2 = F_1(E_3-1) - E_1(F_3-2),$$

$$\Lambda_3 = E_1F_2 - F_1(E_2-2),$$

$$\psi_1 = F_2(D_3-1) - (F_3-2)(D_2-2),$$

$$\psi_2 = (F_3-2)(D_1-4) - F_1(D_3-1),$$

$$\psi_3 = F_1(D_2-2) - F_2(D_1-4),$$

$$\Delta_1 = (E_3-1)(D_2-2) - (E_2-2)(D_3-1),$$

$$\Delta_2 = E_1(D_3-1) - (E_3-1)(D_1-4),$$

$$\Delta_3 = (E_2-2)(D_1-4) - E_1(D_2-2),$$

$$\begin{aligned} D_1 &= \frac{\kappa_3\eta^3}{3}, E_1 = \frac{\kappa_3\eta^2}{2}, F_1 = \kappa_3\eta, \\ D_2 &= \frac{\kappa_2\theta^3}{3}, E_2 = \frac{\kappa_2\theta^2}{2}, F_2 = \kappa_2\theta, \\ D_3 &= \frac{\kappa_1\xi^3}{3}, E_3 = \frac{\kappa_1\xi^2}{2}, F_3 = \kappa_1\xi, \\ &\varphi \neq 0. \end{aligned}$$

Proof: We prove the first implication.

Thanks to Lemma 1.2, we observe that

$$\begin{aligned}
u(t) &= I^{\alpha}G(t) + \sum_{i=1}^{\infty} \nu_i I^{\alpha}\Big(\Phi_i(t) I^{\alpha} H_i(t)\Big) + c_2 t^2 + c_1 t + c_0, \\
u^{'}(t) &= I^{\alpha-1}G(t) + \sum_{i=1}^{\infty} \nu_i I^{\alpha-1}\Big(\Phi_i(t) I^{\alpha} H_i(t)\Big) + 2c_2 t + c_1, \\
u^{''}(t) &= I^{\alpha-2}G(t) + \sum_{i=1}^{\infty} \nu_i I^{\alpha-2}\Big(\Phi_i(t) I^{\alpha} H_i(t)\Big) + 2c_2,
\end{aligned} \tag{4.3}$$

By considering the conditions

$$u^{''}(0) + u^{''}(1) = \kappa_1 \int_0^{\xi} u(s)ds, \quad 0 < \xi < 1,$$

$$u^{'}(0) + u^{'}(1) = \kappa_2 \int_0^{\theta} u(s)ds, \quad 0 < \theta < 1,$$

$$u(0) + u(1) = \kappa_3 \int_0^{\eta} u(s)ds, \quad 0 < \eta < 1,$$

and thanks to Cramer rule, we achieve the proof.
The second implication is evident and hence it is omitted.
In what follows, we use fixed point theory to study the above problem. First, it is important to introduce the space:

$$X := \{x \in C(J, \mathbb{R}), x^{''} \in C(J, \mathbb{R}), D^{\gamma}x \in C(J, \mathbb{R})\}.$$

The norm:

$$\|x\|_X = \|x\|_{\infty} + \|x^{''}\|_{\infty} + \|D^{\gamma}x\|_{\infty}$$

is also to be introduced.

Then, we shall consider the nonlinear operator $H : X \to X$ defined by by:

$$
\begin{aligned}
Hu(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds \\
&+ \sum_{i=1}^{\infty} \nu_i \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} h_i(\tau,u(\tau))d\tau\Big)ds \\
&+\Big[\frac{\Lambda_1 t^2+\psi_1 t+\Delta_1}{\varphi}\Big]\Big[\kappa_3 \int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+ \sum_{i=1}^{\infty} \kappa_3\nu_i \int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&- \frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times \int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] + \Big[\frac{\Lambda_2 t^2+\psi_2 t+\Delta_2}{\varphi}\Big] \\
&\Big[\kappa_2 \int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+ \sum_{i=1}^{\infty} \kappa_2\nu_i \int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&- \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times \int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] + \Big[\frac{\Lambda_3 t^2+\psi_3 t+\Delta_3}{\varphi}\Big] \\
&\Big[\kappa_1 \int_0^{\xi}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+ \sum_{i=1}^{\infty} \kappa_1\nu_i \int_0^{\xi}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&- \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times \int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big].
\end{aligned}
$$

At the end of this section, it is important to note that, we will be concerned with singular differential equations, fixed point theory and integral inequalities to prove our main results.

4.3.2 Uniqueness of solutions

We consider the following sufficient hypotheses:

$(Q1)$: The functions f is defined on $\mathbb{R}^2$, g is defined on $(0,1]\times\mathbb{R}^2$ and $(h_i)_{i=1,\ldots,r}$, $r\in\mathbb{N}^*$ are defined on $J\times\mathbb{R}$; all these functions are supposed continuous.

$(Q2)$: There exist nonnegative constants ϖ_1,ϖ_2, such that for any $t\in J$, $u_1,v_1,u_2,v_2\in\mathbb{R}$,

$$|f(u_1,u_2)-f(v_1,v_2)| \leq \varpi_1\frac{|u_1-v_1|}{1+|u_1+u_2|}+\varpi_2\frac{|u_2-v_2|}{1+|v_1+v_2|},$$

There exist positive continuous functions $\iota_1(t),\iota_2(t)$, such that for any $t\in(0,1]$, $u_1,v_1,u_2,v_2\in\mathbb{R}$,

$$|g(t,u_1,u_2)-g(t,v_1,v_2)| \leq \iota_1(t)\sin(u_1-v_1)+\iota_2(t)\frac{|u_2-v_2|}{1+|u_2v_2|}.$$

And, There exist positive continuous functions $\varsigma_i(t)$, for any integer i and any $t\in J$, $u,v\in\mathbb{R}$,

$$|h_i(t,u)-h_i(t,v)|\leq\varsigma_i(t)\frac{|u-v|}{(1+t^2)(|u|+|v|)}.$$

We take the expressions:

$$\begin{aligned} N &= Max(\varpi_1,\varpi_2),\\ M &= Max(\sup_{t\in J}|\iota_1(t)|,\sup_{t\in J}|\iota_2(t)|),\\ O_i &= \sup_{t\in J}|\varsigma_i(t)|,\\ O &= \sup_{i\in\mathbb{N}^*}O_i \end{aligned}$$

$(Q3)$: Suppose that $(\Phi_i)_{i=1,\ldots,r}$, $r\in\mathbb{N}^*$ are defined on J, continuous and $\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty<+\infty$.

Also,, we consider the following three quantities:

$$\Upsilon_1 = \left[\frac{M|\delta| + N|\lambda|}{\Gamma(\alpha+1)} + \frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right] + \left[\frac{|\Lambda_1| + |\psi_1| + |\Delta_1|}{|\varphi|}\right] \times \left[\left(M|\delta| + N|\lambda|\right)\right.$$
$$\left.\times\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha-1)}\right)\right]$$

$$+\left[\frac{|\Lambda_2| + |\psi_2| + |\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta| + N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\right.$$
$$\left.\times\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\right] + \left[\frac{|\Lambda_3| + |\psi_3| + |\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta| + N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right)\right.$$
$$\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\right],$$

$$\Upsilon_2 = \left[\frac{M|\delta| + N|\lambda|}{\Gamma(\alpha-\gamma+1)} + \frac{O}{\Gamma(2\alpha-\gamma+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_1|}{\Gamma(3-\gamma)} + \frac{|\psi_1|}{\Gamma(2-\gamma)}\right]$$
$$\times\left[\left(M|\delta| + N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}\right.\right.$$
$$\left.\left.+\frac{1}{\Gamma(2\alpha-1)}\right)\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_2|}{\Gamma(3-\gamma)} + \frac{|\psi_2|}{\Gamma(2-\gamma)}\right]\left[\left(M|\delta| + N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right)\right.$$
$$\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\right] + \frac{1}{|\varphi|}\left[\frac{2|\Lambda_3|}{\Gamma(3-\gamma)} + \frac{|\psi_3|}{\Gamma(2-\gamma)}\right]$$
$$\times\left[\left(M|\delta| + N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\right.$$
$$\left.\times\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\right],$$

and

$$
\begin{aligned}
\Upsilon_3 =& \left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha-1)}+\frac{O}{\Gamma(2\alpha-1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right]+\frac{2|\Lambda_1|}{|\varphi|}\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}\right.\right.\\
&\left.+\frac{1}{\Gamma(\alpha-1)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left.\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]+\frac{2|\Lambda_2|}{|\varphi|}\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\\
&+\frac{2|\Lambda_3|}{|\varphi|}\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\right.\\
&\left.\times\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right].
\end{aligned}
$$

We pass to prove the following result for unique solution.

Théorème 4.1 *Assume that both the three hypotheses* $(Q_1),(Q_2),(Q_3)$ *and the condition* $\Upsilon<1;\Upsilon=\Upsilon_1+\Upsilon_2+\Upsilon_3$ *are satisfied. Then, the problem* (4.1) *has exactly one solution.*

Proof.

We begin this proof by showing that H satisfies the Banach BCP.

For $(u,v)\in X^2$, we can write

$$
\begin{aligned}
&\|Hu-Hv\|_\infty\\
&\le\left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha+1)}+\frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right]\|u-v\|_X+\left[\frac{|\Lambda_1|+|\psi_1|+|\Delta_1|}{|\varphi|}\right]\\
&\times\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-1)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\right.\\
&\left.\times\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]\|u-v\|_X+\left[\frac{|\Lambda_2|+|\psi_2|+|\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\right.\\
&\left.\times\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\|u-v\|_X\\
&+\left[\frac{|\Lambda_3|+|\psi_3|+|\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\right.\\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right]\|u-v\|_X.
\end{aligned}
\tag{4.4}
$$

On the other hand, we know that

$$
\begin{aligned}
D^{\gamma}Hu(t) \;=\;& \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}[\delta g(s,u(s),D^{\gamma}u(s))-\lambda f(u(s),u^{''}(s))]ds \\
&+\sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\Big)ds \\
&+\frac{1}{\varphi}\left[\frac{2\Lambda_1 t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_1 t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\Bigg[\kappa_3\int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^{\gamma}u(\tau))-\lambda f(u(\tau), \\
&u^{''}(\tau))]d\tau ds+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^{\eta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\delta g(s,u(s),D^{\gamma}u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times\int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Bigg]+\frac{1}{\varphi}\left[\frac{2\Lambda_2 t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_2 t^{1-\gamma}}{\Gamma(2-\gamma)}\right] \\
&\Bigg[\kappa_2\int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^{\gamma}u(\tau))-\lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^{\theta}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\delta g(s,u(s),D^{\gamma}u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times\int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Bigg]+\frac{1}{\varphi}\left[\frac{2\Lambda_3 t^{2-\gamma}}{\Gamma(3-\gamma)}+\frac{\psi_3 t^{1-\gamma}}{\Gamma(2-\gamma)}\right] \\
&\Bigg[\kappa_1\int_0^{\xi}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^{\gamma}u(\tau))-\lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^{\xi}\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\delta g(s,u(s),D^{\gamma}u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times\int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Bigg].
\end{aligned}
$$

Then, based on the bove quantities and using the same arguments as before, the following inequality

$$
\begin{aligned}
&\|D^{\gamma}Hu - D^{\gamma}Hv\|_{\infty} \\
&\le \left[\frac{M|\delta| + N|\lambda|}{\Gamma(\alpha-\gamma+1)} + \frac{O}{\Gamma(2\alpha-\gamma+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_{\infty}\right]\|u-v\|_X \\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_1|}{\Gamma(3-\gamma)} + \frac{|\psi_1|}{\Gamma(2-\gamma)}\right]\Bigg[\Bigg(M|\delta| + N|\lambda|\Bigg)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) \\
&+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_{\infty}\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha-1)}\right)\Bigg]\|u-v\|_X \\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_2|}{\Gamma(3-\gamma)} + \frac{|\psi_2|}{\Gamma(2-\gamma)}\right]\Bigg[\Bigg(M|\delta| + N|\lambda|\Bigg)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right) \\
&+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_{\infty}\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\Bigg]\|u-v\|_X \\
&+\frac{1}{|\varphi|}\left[\frac{2|\Lambda_3|}{\Gamma(3-\gamma)} + \frac{|\psi_3|}{\Gamma(2-\gamma)}\right]\Bigg[\Bigg(M|\delta| + N|\lambda|\Bigg)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right) \\
&+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_{\infty}\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\Bigg]\|u-v\|_X
\end{aligned} \tag{4.5}
$$

is valid.

Also, the second derivative, which is needed in this proof, is given the following quantity.

$$
\begin{aligned}
H^{''}u(t) \;=\;& \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}[\delta g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u^{''}(s))]ds \\
&+\sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\Big)ds \\
&+\Big[\frac{2\Lambda_1}{\varphi}\Big]\Big[\kappa_3\int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\delta g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times\int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]+\Big[\frac{2\Lambda_2}{\varphi}\Big] \\
&\Big[\kappa_2\int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\delta g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times\int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]+\Big[\frac{2\Lambda_3}{\varphi}\Big] \\
&\Big[\kappa_1\int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau))-\lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\delta g(s,u(s),D^\gamma u(s))-\lambda f(u(s),u^{''}(s))]ds-\sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times\int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big].
\end{aligned}
$$

Using the $H^{''}(t)$ quantity, we can write

$$
\begin{aligned}
\|H^{''}u - H^{''}v\|_\infty \ \le \ & \left[\frac{M|\delta| + N|\lambda|}{\Gamma(\alpha-1)} + \frac{O}{\Gamma(2\alpha-1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right]\|u-v\|_X + \frac{2|\Lambda_1|}{|\varphi|}\Bigg[\Bigg(M|\delta| \\
& +N|\lambda|\Bigg)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\Bigg(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
& +\frac{1}{\Gamma(2\alpha-1)}\Bigg)\Bigg]\|u-v\|_X + \frac{2|\Lambda_2|}{|\varphi|}\Bigg[\Bigg(M|\delta| + N|\lambda|\Bigg)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha)}\right) \\
& +\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha)}\right)\Bigg]\|u-v\|_X \\
& +\frac{2|\Lambda_3|}{|\varphi|}\Bigg[\Bigg(M|\delta| + N|\lambda|\Bigg)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right) + \left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right) \\
& \times\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{\Gamma(2\alpha+1)}\right)\Bigg]\|u-v\|_X.
\end{aligned}
\tag{4.6}
$$

From (4.4), (4.5) and (4.6) we conclude that

$$
\|Hu - Hv\|_X \le \Big(\Upsilon_1 + \Upsilon_2 + \Upsilon_3\Big)\|u-v\|_X.
$$

With Banach BCP and the condition on Υ, we have the contraction of H. So, H admits a unique fixed point x_0. The proof is thus complete.

4.3.3 Examples

In this section, we present two examples to illustrate the validity of the result dealing with the existence of exactly one solution.

Example 4.3.1

We consider the following $ln-problem$:

$$\begin{cases} D^{\frac{5}{2}}u(t)+\frac{1}{2}\frac{|u(t)+u^{''}(t)|}{10\pi(1+|u(t)+u^{''}(t)|)}=\frac{1}{20}\Big(\frac{\sin(u(t))}{e^{t^2+6}}+\frac{|D^{\frac{3}{2}}u(t)|}{200(1+|D^{\frac{3}{2}}u(t)|)}+|ln(t)|\Big) \\ \quad +\sum_{i=1}^{\infty}\frac{3e^{-it^2}}{125(i\pi)^2}I^{\frac{5}{2}}\Big(\frac{|u(t)|}{300i\big[(t^2+1)+|u(t)|\big]}\Big), t\in(0,1], \\ u(0)+u(1)=\int_0^{0.1}2u(s)ds, \\ u^{'}(0)+u^{'}(1)=\int_0^{0.3}3u(s)ds, \\ u^{''}(0)+u^{''}(1)=\int_0^{0.5}4u(s)ds, \end{cases}$$

Remrk that

$$\alpha=\tfrac{5}{2},\ \lambda=\tfrac{1}{2},\ \delta=\tfrac{1}{20},\ \gamma=\tfrac{3}{2},$$

$$\Upsilon_1=0.0416,\quad \Upsilon_2=0.1550,\quad \Upsilon_3=0.0397,$$

$$\Upsilon=\Upsilon_1+\Upsilon_2+\Upsilon_3=0.2363.$$

So, thanks to Theorem 4.1, we confirm that this example has a unique solution.

Example 4.3.2

As a second illustrative example, we consider the problem with the singular function $\dfrac{1}{t}$.

$$\begin{cases} D^{2.1}u(t)+\frac{3}{10}\frac{|2u(t)+2u^{''}(t)|}{\pi^4(t+2)(1+3|u(t)+u^{''}(t)|)}=\frac{3}{2}\Big(\frac{e^t+\sin(u(t))}{30(t^2+1)}+\frac{|D^{1.2}u(t)|}{20e^{t+1}(1+|D^{1.2}u(t)|)}+\frac{1}{t}\Big) \\ \quad +\sum_{i=1}^{\infty}\frac{e^{-it^2}}{50i^2}I^{2.1}\Big(\frac{|u(t)|}{200i\big[(t+1)+|u(t)|\big]}+e^t\Big), t\in(0,1], \\ u(0)+u(1)=\int_0^{0.3}u(s)ds, \\ u^{'}(0)+u^{'}(1)=\int_0^{0.5}2u(s)ds, \\ u^{''}(0)+u^{''}(1)=\int_0^{0.1}u(s)ds, \end{cases}$$

We see that

$$\alpha = 2.1,\ \lambda = \tfrac{3}{10},\ \delta = \tfrac{3}{2},\ \gamma = 1.2,$$

$$\Upsilon_1 = 0.1498, \quad \Upsilon_2 = 0.4170, \quad \Upsilon_3 = 0.0911,$$

$$\Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 = 0.6579.$$

Also, by Theorem 4.1, our example has a unique solution.

4.4 Stabilities of solutions

It is to mention that the Ulam-Hyers (UH) stabilities for fractional differential problems are useful for solving practical problems in biology, economics and mechanics. The examples of the application of this theory can be found also in [3, 20, 21, 39]. It is important to notice that there are many applications for UH stability in nonlinear analysis problems including differential equations and integral equations [7]. Insteade of finding explicit solutions for our BVPs, if there are UH stable, so all what is needed is to fing approximate solutions for some integral inequalities. These types of stability is very important with respect to Lyapounov/Lagrange one.

4.4.1 Basic concepts

We associate to our problem the following definitions with their integral inequalities.

Définition 4.4.1 *The equation* (4.1) *has the UH stability if there exists a real number* $\Theta > 0$*, such that for each* $\varepsilon > 0, t \in]0,1]$ *and for each* $u \in X$ *solution of the inequality*

$$\left|D^{\alpha}u(t) + \lambda f(u(t), u^{''}(t)) - \sigma g(t, u(t), D^{\gamma}u(t)) - \sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^{\alpha} h_i(t, u(t))\right| \leq \varepsilon, \tag{4.7}$$

there exists $v \in X$ *a solution of* (4.1)*, such that*

$$\|u - v\|_X \leq \Theta\varepsilon.$$

Définition 4.4.2 *The equation* (4.1) *has the UH stability in the generalized sense if there* *exists* $\Omega \in C(\mathbb{R}^+, \mathbb{R}^+)$; $\Omega(0) = 0$, *such that for each* $\varepsilon > 0$, *and for any* $u \in X$ *solution of* (4.7), *there exists a solution* $v \in X$ *of (4.1), such that*

$$\|u - v\|_X < \Omega(\varepsilon).$$

4.4.2 Ulam-Hyers

Now, we are able to prove the first main result.

Théorème 4.2 *Under the conditions of Theorem* 4.1, *we state that* (4.1) *is Ulam Hyers stable.*

Proof.

Let $u \in X$ be a solution of (4.7), and let, by Theorem 4.1, $v \in X$ be the unique solution of (4.1).

By integration of (4.7), we obtain

$$
\begin{aligned}
&\Big|u(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds \\
&-\sum_{i=1}^{\infty}\nu_i\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}h_i(\tau,u(\tau))d\tau\Big)ds \\
&-\Big[\frac{\Lambda_1 t^2+\psi_1 t+\Delta_1}{\varphi}\Big]\Big[\kappa_3\int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_3\nu_i\int_0^\eta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-2)} \\
&\times\int_0^1 (1-s)^{\alpha-3}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] - \Big[\frac{\Lambda_2 t^2+\psi_2 t+\Delta_2}{\varphi}\Big] \\
&\Big[\kappa_2\int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_2\nu_i\int_0^\theta \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha-1)} \\
&\times\int_0^1 (1-s)^{\alpha-2}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big] - \Big[\frac{\Lambda_3 t^2+\psi_3 t+\Delta_3}{\varphi}\Big] \\
&\Big[\kappa_1\int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}[\delta g(\tau,u(\tau),D^\gamma u(\tau)) - \lambda f(u(\tau),u^{''}(\tau))]d\tau ds \\
&+\sum_{i=1}^{\infty}\kappa_1\nu_i\int_0^\xi \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\Phi_i(\tau)\frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-\chi)^{\alpha-1}h_i(\chi,u(\chi))d\chi\Big)d\tau ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[\delta g(s,u(s),D^\gamma u(s)) - \lambda f(u(s),u^{''}(s))]ds - \sum_{i=1}^{\infty}\frac{\nu_i}{\Gamma(\alpha)} \\
&\times\int_0^1 (1-s)^{\alpha-1}\Big(\Phi_i(s)\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}H_i(\tau,u(\tau))d\tau\Big)ds\Big]\Big| \le \frac{\varepsilon}{\Gamma(\alpha+1)}.
\end{aligned} \tag{4.8}
$$

By using (4.7) and (4.8), we can write

$$
\begin{aligned}
\|u-v\|_\infty \leq\ & \frac{\varepsilon}{\Gamma(\alpha+1)}+\left[\frac{M|\delta|+N|\lambda|}{\Gamma(\alpha+1)}+\frac{O}{\Gamma(2\alpha+1)}\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_1|+|\psi_1|+|\Delta_1|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_3|\eta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-1)}\right)\right. \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_3|\eta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha-1)}\right)\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_2|+|\psi_2|+|\Delta_2|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_2|\theta^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha)}\right)\right. \qquad (4.9) \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_2|\theta^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha)}\right)\right]\|u-v\|_X \\
&+\left[\frac{|\Lambda_3|+|\psi_3|+|\Delta_3|}{|\varphi|}\right]\left[\left(M|\delta|+N|\lambda|\right)\left(\frac{|\kappa_1|\xi^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\right)\right. \\
&\left.+\left(O\sum_{i=1}^{\infty}\|\nu_i\Phi_i\|_\infty\right)\left(\frac{|\kappa_1|\xi^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{1}{\Gamma(2\alpha+1)}\right)\right]\|u-v\|_X.
\end{aligned}
$$

So

$$\|u-v\|_\infty \leq \frac{\varepsilon}{\Gamma(\alpha+1)}+\Upsilon_1\|u-v\|_\infty,$$

Therefore, we have

$$\|u-v\|_\infty \ \leq \ \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_1)} \leq \varepsilon\ \Xi_1.$$

On the other hand, we have

$$\|D^\gamma(u-v)\|_\infty \ \leq \ \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_2)} \leq \varepsilon\ \Xi_2.$$

Also, we have

$$\|u''-v''\|_\infty \ \leq \ \frac{\varepsilon}{\Gamma(\alpha+1)(1-\Upsilon_3)} \leq \varepsilon\ \Xi_3.$$

Thus,

$$\|u-v\|_X \leq \varepsilon\big(\ \Xi_1+\Xi_2+\Xi_3\big).$$

Thus, (4.1) has the Ulam Hyers stability.

Remark 4.4.1

In the case $\Omega(\varepsilon)=\varepsilon\big(\ \Xi_1+\Xi_2+\Xi_3\big)$, we obtain the generalised Ulam Hyers stability for (4.1).

Remark And Example 4.4.1

The above two examples are UH stable since they fulfill the conditions of Theorem 4.1. In particular, in both cases, we have proved that there is a solution v, such that for each $\varepsilon>0, t\in]0,1]$ and for each $u\in X$ solution of inequality (4.7), we can write, for the first example:

$$\|u-v\|_\infty \leq 0.7849\varepsilon, \quad \|D^{\frac{3}{2}}(u-v)\|_\infty \leq 0.8902\varepsilon, \quad \|u'' - v''\|_\infty \leq 0.7834\varepsilon.$$

Thus,

$$\|u-v\|_X \leq 2.4585\varepsilon.$$

However, for the second example, we can write

$$\|u-v\|_\infty \leq 1.0675\varepsilon, \quad \|D^{1.2}(u-v)\|_\infty \leq 1.5568\varepsilon, \quad \|u'' - v''\|_\infty \leq 0.9986\varepsilon.$$

Thus,

$$\|u-v\|_X \leq 3.6229\varepsilon.$$

4.5 Numerical simulations

In this paragraph, we apply an effective numerical approach to Riemann-Liouville integral and Caputo derivative. We need to recall the approximation theorems of the papers [22, 44]. Based on Caputo derivative approximation, we investigate, for some given parameters, the behavior of the considered problem by studying one of the two proposed examples with a parameter α. In order to do this, we should initially obtain a reduced fractional differential system which can be equivalent to the considered problem. The numerical simulations are then used a Runge-Kutta method.

Théorème 4.3 *[59] Let $y \in \mathcal{C}^1([0,1],\mathbb{R})$. Then, we have*

$$J^{\alpha}y(t_i) \simeq \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{i} y(t_j)\ \sigma_j(\alpha), \qquad i = 0, \ldots, n+1,$$

where,

$$\sigma_j(\alpha) = \begin{cases} (n+2-j)^{(\alpha+1)} + (n-j)^{(\alpha+1)} - 2(n-j+1)^{(\alpha+1)},\ j = 1 \ldots i-1. \\ (n)^{(\alpha+1)} - (n-\alpha)(n+1)^{\alpha},\ j = 0, \mathit{and}\ 1,\ j = i. \end{cases}$$

Théorème 4.4 *[59] Let $y \in \mathcal{C}^1([0,1],\mathbb{R})$ and $0 < \alpha \leq 1$. Then, we get:*

$$D^{\alpha}y(t_i) \simeq \frac{h^{1-\alpha}}{\Gamma(1-\alpha+2)} \sum_{j=0}^{i} y^{(j)}(t_j)\sigma_j(1-\alpha), \qquad i = 0, \ldots, n,$$

where,

$$y^{(j)} = \left\{ \frac{y_1 - y_0}{h},\ j = 0,\quad \frac{y_{j+1} - y_{j-1}}{2h},\ j = 1 \ldots i-1,\quad \frac{y_i - y_{i-1}}{h},\ j = i. \right.$$

Remark 4.5.1 *The problem (4.1) can be reduced to the formula below:*

$$\begin{aligned} D^1u(t) &= v(t) = f_1(t, u(t), v(t), w(t)) \\ D^1v(t) &= w(t) = f_2(t, u(t), v(t), w(t)) \\ D^1w(t) &= D^{3-\alpha}\Big(-\lambda f(u(t), w(t)) + \delta g(t, u(t), D^{\gamma}u(t)) + \sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^{\alpha} h_i(t, u(t)) \Big) \\ &= f_3(t, u(t), v(t), w(t)) \end{aligned}$$

and

$$\begin{aligned} w(0) + w(1) &= \kappa_1 \int_0^{\xi} u(s)ds, \quad 0 < \xi < 1, \\ v(0) + v(1) &= \kappa_2 \int_0^{\theta} u(s)ds, \quad 0 < \theta < 1, \\ u(0) + u(1) &= \kappa_3 \int_0^{\eta} u(s)ds, \quad 0 < \eta < 1, \end{aligned}$$

where

$$f_3(t,u(t),v(t),w(t)) = \frac{h^{\alpha-2}}{\Gamma(\alpha)} \sum_{j=0}^{i} \sigma_j(\alpha-2) \Big[-\lambda f(u(t_j),w(t_j)) + \delta g(t_j,u(t_j),D^{\gamma}u(t_j))$$
$$+ \sum_{i=1}^{\infty} \nu_i \Phi_i(t) I^{\alpha} h_i(t_j,u(t_j))\Big)\Big]$$

The complete numerical scheme for the computations is

$$\begin{cases} K_1 &= & f_1(t_i,u_i,v_i,w_i) & K_2 &= & f_1(t_i+\frac{h}{2},u_i+\frac{h\ K_1}{2},v_i,w_i), \\ K_3 &= & f_1(t_i+\frac{h}{2},u_i+\frac{h\ K_2}{2},v_i,w_i) & K_4 &= & f_1(t_i+\frac{h}{2},u_i+h\ K_3,v_i,w_i), \\ P_1 &= & f_2(t_i,u_i,v_i,w_i) & P_2 &= & f_2(t_i+\frac{h}{2},u_i,v_i+\frac{h\ P_1}{2},w_i), \\ P_3 &= & f_2(t_i+\frac{h}{2},u_i,v_i+\frac{h\ P_2}{2},w_i) & P_4 &= & f_2(t_i+\frac{h}{2},u_i,v_i+h\ P_3,w_i), \\ L_1 &= & f_2(t_i,u_i,v_i,w_i) & L_2 &= & f_2(t_i+\frac{h}{2},u_i,v_i,w_i+\frac{h\ L_1}{2}), \\ L_3 &= & f_2(t_i+\frac{h}{2},u_i,v_i,w_i+\frac{h\ L_2}{2}) & L_4 &= & f_2(t_i+\frac{h}{2},u_i,v_i,w_i+h\ L_3), \end{cases}$$

$$\begin{cases} t_{i+1} = t_0 + ih, & & t_0 = 0, \\ u_{i+1} & = & u_i + h\psi_1 \\ v_{i+1} & = & v_i + h\psi_2 \\ w_{i+1} & = & w_i + h\psi_3 \end{cases}$$

Where

$$\psi_1 := \frac{K_1+2K_2+2K_3+K_4}{6} \quad \psi_2 := \frac{P_1+2P_2+2P_3+P_4}{6}, \quad \psi_3 := \frac{L_1+2L_2+2L_3+L_4}{6}$$

Through numerical simulations achieved by a combination of Caputo approach and the fourth-order Runge-Kutta method on the first example, we obtain:

Figure 4.1: Solution for the first example, on the plan u-w, for four values of α.

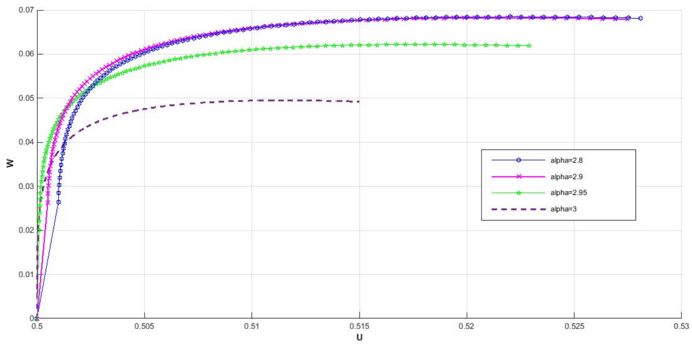


Figure 4.2: Behavior of the dynamics of the first example, on the plan v-w, for four values of α.

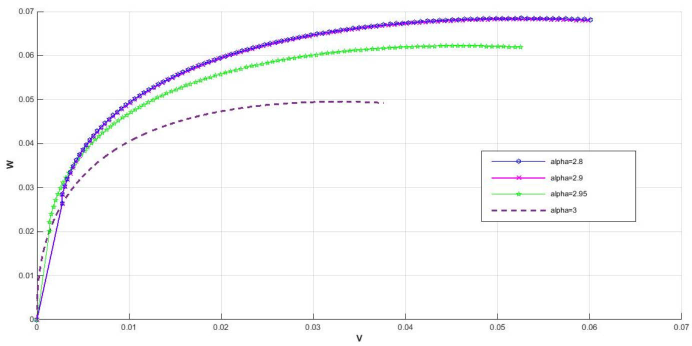


Figure 4.3: Behavior of the solution for the first example, on the plan u-v, for different values of α.

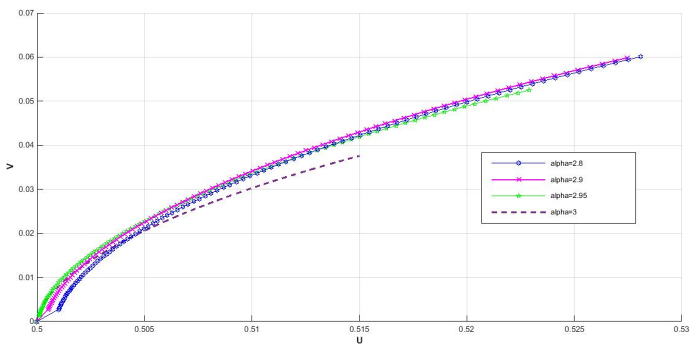
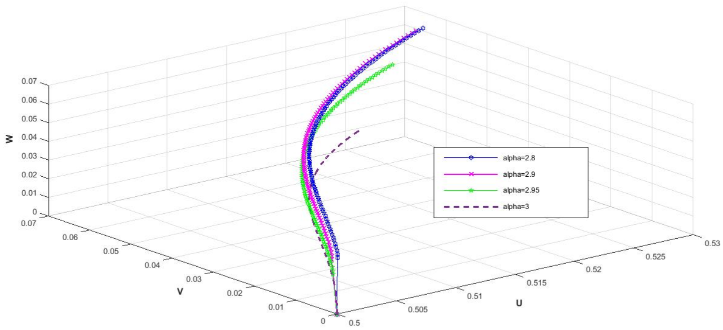


Figure 4.4: 3D representation for the solution of the first example, for different values of α.



Remark 4.5.2

- *Numerical simulation accounts for the effect of fractional order on reduced systems.*
- *The comparison of the numerical simulations made it possible to establish a significant correlation between specific parameters. Unfortunately, it differs in other cases.*
- *Thanks to the continuous evaluation, we observe the influence of the approximations on the display of the behaviors in the simulations for particular cases (for example when $\alpha \leq 2.75$ the solution loses the shape of the curvature).*
- *It seems that we are in perfect harmony between the numerical simulations and the result of for $\alpha \to 3$.*

Conclusion and Perspectives

In our thesis project, we have studied two classes of differential equations with singularities. In the first class, we have been concerned with singular differential equations that are supposed with n sequential Caputo derivatives, this sequentiality does not satisfy the semi group and commutativity properties. We have first presented and proved the unique integral representation of the studied class. Then, using the integral inequality theory presented in the second chapter, we have proved a first existence and uniqueness result. Another main result has been then proved and some conditions on the data of the studied problem have been imposed. Several examples have also been discussed in details. In this project, we have also been concerned with another class of BVPs with time singularity that involves series. For this class, we have studied the uniqness of solutions which has allowed us to pass to study the UH stability of solutions. Some examples have been studied for the UH stability results. At the end, some numerical simulations have been discussed; they have concerned the approximation of Caputo derivatives for the problem.

As perspective of this thesis, we propose to study the sequential cases with series and singularities in time and space. We think, it is an important problem to be deal with in the future.

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