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Abbreviations

CFLT	:	Conformable fractional Laplace transform
CFST	:	Conformable fractional Sumudu transform
LTI	:	Linear time-invariant.
1D	:	Unidimensional system.
2D	:	Bidimensional system.
CRSD	:	2D Roesser causal recursive separable denominato.
IIR	:	Infinite impulse response.
FIR	:	Finite impulse response.

Notations

δ_{i0}	:	Delta Kronecker.
$\sigma(A_1)$	:	The degree of superstability of matrix A_1.
δ	:	Dirac Delta function.
$\mathbf{T}^{\alpha}$	:	Conformable derivative.
$\mathbf{D}_c^{\alpha}$	:	Caputo derivative.
$\mathbf{D}_{\mathrm{RL}}^{\alpha}$	:	Riemann-Liouville derivative.
$\mathscr{L}_{\alpha}$	:	Laplace transform.
S_{α}	:	Sumudu transform.
$\mathscr{Z}$	:	$\mathscr{Z}$-transform.
$\|G\|_{\mathscr{H}_{\infty}}$	:	H_{∞}-norm of transfer function G.
$\mathbb{N}^*$	:	Set of non-zero natural numbers $\{1,2,3,...\}$.
$\mathbb{N}$	:	Set of natural numbers $\{0,1,2,3,...\}$.
$\mathbb{Z}$	:	Set of integer numbers.
$\mathbb{R}$	:	Field of real numbers.
$\mathbb{C}$	:	Field of complex numbers.
$\mathrm{R}e$	:	Real part of a complex number.
$\mathscr{M}_n$	:	Space of $n \times n$ Metzler matrices.
$\mathbb{R}^{n\times m}$	:	Space of $n \times m$ real matrices.
$\mathbb{R}^n$	:	Space of n-dimensional real vectors.
$\mathbb{C}^n$	:	Space of n-dimensional complex vectors.
$\mathbb{C}^{n\times m}$	:	Space $n \times m$ complexmatrices.
I	:	Identity matrix.
$\det(.)$	:	Determinant of a matrix.
A^*	:	Conjugate transpose of matrix A.
$A \geq 0$	:	Non-negative matrix A.
$A > 0$	:	Positive matrix A.
$A \gg 0$	:	Strictly positive matrix A.
$diag(A)$	:	The diagonal matrix entries A.
W_c	:	Controllability Gramian matrix.
W_o	:	Observability Gramian matrix.

Introduction

Many real-life phenomena, regardless of their nature (chemical, physical, biological, electromechanical, or economic), can be described by mathematical models. A dynamical system describes the evolution over time of the investigated mathematical model. This can be compared to a set of finite equations that produce different mathematical representations, which are given by ordinary differential equations, partial differential equations, or difference equations.
The interaction between a system and its environment is a key concept in systems theory. It is common practice to process mathematical models of input dynamic systems to produce outputs. The aim is to bring the system from a given initial state to a certain final state with respect to certain criteria. Various classes of dynamic systems can be identified. Systems with discrete dynamics are represented by a difference equation, where the state variables only change at a discrete set of points in time. For instance, population models (such as populations of rabbits or microorganisms) are examples of systems with discrete dynamics. Systems with continuous dynamics have state variables that change continuously across time, such as the amount of water that flows through a dam. These systems are represented by a differential equation. Finally, systems with continuous and discrete dynamics (hybrid) involve interactions between continuous and discrete processes. This dynamic involves switching behaviors frequently seen in electronic systems or robotic manipulation systems that can impact the dynamics of the system in several industrial applications.

Fractional systems have generated a great deal of interest in many areas of applied science, engineering, and control theory [55, 56, 81, 91]. The objective of fractional calculus is to generalize classical, integer-order derivatives to a non-integer order. Fractional order derivatives are used to model various phenomena across numerous domains [91], such as:

- Fractional derivatives are frequently used in the mathematical representation of material viscoelasticity.
- A fractional-order dynamic can be observed in several financial systems in eco-

nomics.

- It has been established in biology that the membranes of biological cells exhibit electrical conductivity of fractional order, which is then categorized into a group of non-integer order models.

In control theory, for instance, the state of fractional continuous-time systems was discussed in [52, 55, 56]. Various methods, including integral transformations like Laplace transform, Mellin transform, and Sumudu transform [3, 4, 5, 41, 48, 67, 73, 72, 96, 97, 99], have been proposed for resolving these systems [20, 56, 62, 63]. Therefore, the first part of this thesis focuses on the solution of a fractional one-dimensional (1D) state-space system. We propose to solve singular and standard linear continuous-time systems with a new fractional derivative using the Sumudu transform, which has many interesting and attractive advantages over other integral transforms, specifically the unity it provides by ensuring convergence when solving differential equations and the resolvability of problems without resorting to a new frequency domain [1, 99]. We obtained the expression of the state of our system thanks to some properties and formulas of the fractional Sumudu transform that we have established and proved. On the other hand, we will be interested in the analysis of this new fractional singular system, such as controllability and observability, positivity, stability, and super-stability.
In the same frame of our study, we also focus on two-dimensional (2D) digital filters which have attracted considerable interest in numerous applications, including image processing, edge extraction, pattern identification via matched filtering, and restoration of linearly deteriorated images [45]. Recursive filters are a crucial component of these systems because they have the potential to reduce computation time and memory costs [44, 45, 66]. Different state-space models for 2D systems have been proposed by a number of authors like Attasi [6], Fornasini-Marchesini [36], and Roesser [88]. These models can be used to simulate recursive filters [77]. We are particularly interested in this second part in the computation of the H_∞ norm of two-dimensional digital filters modeled by (2D) Roesser system since in control theory the H_∞ approach is used to synthesize controllers to achieve stabilization with guaranteed performance. These tools have the advantage over classical control techniques in that H_∞ methods are readily applicable to problems involving multivariate systems. To use H_∞ methods, a control designer expresses the control problem as a mathematical optimization problem and then finds the controller that solves this optimization. Thus, the design of 2D control systems is an interesting and challenging problem, and it has received considerable attention [32, 33, 74].
This manuscript, which focuses on a theoretical study followed by digital applications, is made up of 5 chapters.

- In the first chapter, the most important mathematical background used in this work is presented. We will start by recalling some definitions and properties of the fractional derivative operator. Then, we will present the definitions of some integral transforms and some particular matrices are also drawn. Finally, the notion of Schur complement are given.

- In the second chapter, we introduce and establish the resolution of singular continuous-time linear systems of order α with conformable and Caputo derivative by using the Sumudu transform method. Furthermore, we discuss the solution of regular continuous-time linear systems with two derivatives. We also focus on numerical examples to demonstrate the advantages and effectiveness of our approach using a Matlab code. Finally, in the last section, we draw some conclusions and comparisons between the two systems.

- In the third chapter, we continue our investigation into the controllability and observability of the new system. We present new findings supported by academic examples.

- In the fourth chapter, our focus is on the analysis of the new fractional singular system. We begin by introducing the definitions and properties of positive singular systems with this new derivative. Then, we establish different concepts of stability and super-stability as an extension of the analysis tools of singular systems.

- The final chapter focuses on the evaluation of the H_∞ norm of a particular class of two-dimensional systems. We also provide a numerical illustration to demonstrate the benefits and effectiveness of our approach.

Chapter 1

Definitions and basic concepts

1 Introduction

The aim of this chapter is to provide the necessary foundational knowledge to understand the technical progress presented in subsequent chapters. The first section covers the essential tools of fractional calculus relevant to this thesis. In the second section, we discuss two important integral transforms used in continuous functions (Laplace and Sumudu) and discrete functions ($\mathscr{Z}$ -transform). The fourth section presents specific matrices that will be used later in the thesis. Finally, the last section introduces the definitions and characteristics of the Schur complement.

2 Fractional calculus

For more than 300 years, several mathematicians such as Riemann, Liouville, and Caputo have shown interest in fractional calculus [27, 84, 90]. Fractional-order systems have generated considerable interest in many fields of applied sciences, engineering, and control theory [55, 56, 81, 91]. However, a new derivative operator called the conformable derivative operator has been proposed by Khalil et al. *Khalil et al.* [71] and has been used in several areas including engineering, finance, biology, medicine, physics, and applied mathematics [7, 8, 9, 34, 46, 102]. In fact, various problems have been solved, methods and resolutions have been developed and improved, and other definitions of the conformable derivative operator have been exploited in [71]. For example, fractional partial differential equations [102], time-fractional one-dimensional cable differential equations [101, 104], fractional Cauchy problems [103], linear/nonlinear differential equations [105], and other applications.

2.1 Some basic concepts of special functions

In this subsection, we define some special functions which are plays an important role in solution of the fractional differential equations.

Definition 2.1 *[56] Let Γ be a given function described by the following formula*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0, \tag{1.1}$$

*where, Γ is called the **Euler's Gamma function**.*

Proposition 2.2 *[56] The **Euler's Gamma function** Γ verifies the following properties*

1. $$\Gamma(n) = (n-1)!,\ n \in \mathbb{N}^*; \tag{1.2}$$

2. $$\Gamma\left(\frac{1}{2}\right) = \pi; \tag{1.3}$$

3. $$\Gamma(z+1) = z\Gamma(z),\ z \in \mathbb{C},\ \mathrm{Re}(z) > 0. \tag{1.4}$$

In the following definition, we introduce the **Mittag-Leffler Function**, which is a generalization of the exponential function $e^{s_i t}$.

Definition 2.3 *[56] A function of the complex variable z defined by*

$$\mathrm{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{1.5}$$

*is called the one parameter **Mittag-Leffler Function**.*

Example 2.4 *[56] For $\alpha = 1$ we obtain the classical exponential function described by*

$$\mathrm{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{1.6}$$

2.2 Fractional derivatives

Conformabel derivative

Definition 2.5 *[71] Given a function $x : [0, +\infty) \to \mathbb{R}$. Then, the conformable derivative of the function x of order α, with $\alpha \in (0,1]$ is defined by*

$$\mathbf{T}^{\alpha}(x)(t) = \lim_{\epsilon \to 0} \frac{x\left(t + \epsilon t^{1-\alpha}\right) - x(t)}{\epsilon}, \quad \forall\, t > 0.$$

If the conformable derivative of the function x of order α for all $t > 0$ exists, then, we simply say x is α-differentiable.

Theorem 2.6 *[71] Let $\alpha \in (0,1]$ and $x_1,\ x_2 : \mathbb{R}_+ \to \mathbb{R}$ be α-differentiable functions. Then, $\forall t > 0$*

(*a*) $\mathbf{T}^\alpha(ax_1(t) + bx_2(t)) = a\mathbf{T}^\alpha(x_1)(t) + b\mathbf{T}^\alpha(x_2)(t)$, *for all $a, b \in \mathbb{R}$;*

(*b*) $\mathbf{T}^\alpha(t^p) = pt^{p-\alpha}$, *for all $p \in \mathbb{R}$;*

(*c*) $\mathbf{T}^\alpha(\lambda) = 0$, *for all constant function $x_1(t) = \lambda$;*

(*d*) $\mathbf{T}^\alpha(x_1(t)x_2(t)) = x_1(t)\mathbf{T}^\alpha(x_2)(t) + x_2(t)\mathbf{T}^\alpha(x_1)(t)$;

(*e*) $\mathbf{T}^\alpha\left(\dfrac{x_1(t)}{x_2(t)}\right) = \dfrac{x_2(t)\mathbf{T}^\alpha(x_1)(t) + x_1(t)\mathbf{T}^\alpha(x_2)(t)}{x_2^2(t)}$;

(*f*) *If x_1 is differentiable, then,* $\mathbf{T}^\alpha(x_1)(t) = t^{1-\alpha}\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t}$.

Riemann-Liouville derivative

Definition 2.7 *[79, 92] Let's define the fractional derivative of the continuous function x as following*

$$\mathbf{D}^\alpha_{\mathrm{RL}} x(t) = \frac{1}{\Gamma(\mathrm{N}-\alpha)} \frac{d^{\mathrm{N}}}{dt^{\mathrm{N}}} \int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha+1-\mathrm{N}}} d\tau, \tag{1.7}$$

where $\mathbf{D}^\alpha_{\mathrm{RL}}$ is called the Riemann-Liouville fractional derivative of ordor α with $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$.

Theorem 2.8 *[79, 92] The Riemann-Liouville operator is linear such that*

$$\mathbf{D}^\alpha_{\mathrm{RL}}\left[\lambda x_1(t) + \mu x_2(t)\right] = \lambda \mathbf{D}^\alpha_{\mathrm{RL}} x_1(t) + \mu \mathbf{D}^\alpha_{\mathrm{RL}} x_2(t). \tag{1.8}$$

Caputo derivative

Definition 2.9 *[79, 85] The function defined by*

$$\mathbf{D}^\alpha_c x(t) = \frac{1}{\Gamma(\mathrm{N}-\alpha)} \int_0^t \frac{x^{(\mathrm{N})}(\tau)}{(t-\tau)^{\alpha+1-\mathrm{N}}} d\tau,\ x^{(\mathrm{N})}(\tau) = \frac{d^{\mathrm{N}} x(\tau)}{d\tau^{\mathrm{N}}}, \tag{1.9}$$

is called the Caputo derivative-integral, where $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$.

Remark 2.10 *[56, 79] From definition (2.9) it follows that the Caputo derivative of constant is equal to zero.*

Theorem 2.11 *[85] The Caputo derivative-integral operator is linear and satisfying the relation*

$$\mathbf{D}_c^{\alpha}\left[\lambda x_1(t)+\mu x_2(t)\right]=\lambda\mathbf{D}_c^{\alpha}x_1(t)+\mu\mathbf{D}_c^{\alpha}x_2(t). \tag{1.10}$$

2.3 Interesting conclusions and contrasts

The relationship between fractional derivatives in the Riemann Liouville sense and in the Caputo sense is given by the following theorem

Theorem 2.12 *[79] Let* $\mathrm{N}-1<\alpha\leq\mathrm{N}, \mathrm{N}\in\mathbb{N}^*$ *and* $x\in\mathrm{C}^n([a,b])$. *Then*

$$\mathbf{D}_c^{\alpha}x(t)=\mathbf{D}_{\mathrm{RL}}^{\alpha}\left[x(t)-\sum_{i=0}^{n-1}\frac{(t-a)^i}{i!}x^{(i)}(a)\right]. \tag{1.11}$$

The main advantage of Caputo's definition of a fractional derivative over Riemann Liouville's definition is that Caputo's definition allows for the consideration of initial conditions that are commonly used in the resolution of fractional differential linear equations. Additionally, the fractional derivative Riemann-Liouville of a constant is not bounded in $t=0$.

In the following and based on [95], we will give some advantages of the conformable derivative over the other fractional derivatives

- Conformable derivative performs well in product rule and chain rule while complicated formulas appear in the case of the usual fractional calculation.
- Contrary to Riemann fractional derivatives, the conformable derivative of a constant function is zero.
- As a generalization of exponential functions, Mittag-Leffler functions are fundamental in fractional calculus, and in the case of conformable calculus, the fractional exponential function $x(t)=e^{\frac{t^{\alpha}}{\alpha}}$ arises.
- Some functions in traditional calculus need Taylor power series representations at specific points, but in the theory of conformable, they do.
- The conformable derivative preserves the properties of the usual exact derivatives such as: quotient, product, chain rules, Rolle's theorem, and mean-value theorem.
- Conformable derivative does not contain any integral terms, that make it much more easier to apply on the fractional differential equations.

3 Integral transforms

In literature, different integral transforms have been proposed to solve differential equations and control engineering problems, for instance the Laplace transform, the Sumudu transform, the Naturelle transform, and the Mellin transform, the most important characterization of them is the possibility to manipulate numerous problems by changing the domain of the equation [3, 4, 5, 25, 73, 72, 96, 97, 99, 106].

More recently, the fractional integral transforms has received much attention of many researches, due to its importance and efficiency to solve the fractional differential equations, which has many applications in physics, electric circuit, engineering ....ect [41, 48, 67]. In this section, we will present a list of interesting rules and properties of Laplace and Sumudu transform of a continuous function, then these integrals transforms in the fractional case will be presented (conformable and Caputo derivative). Furthermore we will give the relationship between this transforms. Moreover, we are interested in the study of the one and two dimensional discrete integral $\mathcal{Z}$-transform [31, 93] which will be useful in the fifth chapter.

3.1 Laplace transform

In this section, the Laplace transform will be introduced, this transform is the most classic method and is widely used in several domains. We will begin by recalling some needed definitions and theorems on this transform. Then the fractional Laplace transform will be given.

Direct Laplace transform

Definition 3.1 *[29] A function of variable t is said to be* ***causal*** *if it is zero for $t < 0$.*

Definition 3.2 *[83] A function x has exponential order a if there exist a constant $\mathrm{M} > 0$. Then*

$$|x(t)| \leq \mathrm{M}e^{at}, \quad \forall t > \mathrm{T}. \tag{1.12}$$

Example 3.3 *[83] Consider the function x such as*

$$x(t) = t^2,$$

we have

$$\left|t^2\right| = t^2 < e^{3t}, \quad \forall t > 0, \tag{1.13}$$

then $x(t)$ has exponential order 3.

Definition 3.4 *[31, 83] Let $x : [0, +\infty) \to \mathbb{R}$ be a **causal** function. Then the Laplace transform of x is*

$$\begin{aligned} \mathscr{L}[x(t)](s) &= \mathrm{X}(s), \\ &= \int_0^{\infty} e^{-st} x(t) \mathrm{d}t. \end{aligned} \tag{1.14}$$

The Laplace transform of a function $x(t)$ does exist only if the above integral converges.

The following theorem gives the conditions of the existence of Laplace transform.

Theorem 3.5 *[31, 83] If x is piecewise continuous function on $[0, \infty[$ and of exponential order, then, the Laplace transform $\mathscr{L}[x]$ exists for $\mathrm{Re}(s) > a$ and converges absolutely.*

Example 3.6 *Consider the unit step function defined in [56] by the following formula*

$$x(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t \geq 0. \end{cases} \tag{1.15}$$

Now we will compute the Laplace transform of unit function (1.15), thus

$$\begin{aligned} \mathscr{L}\big[x(t)\big](s) &= \int_0^{+\infty} e^{-st} dt, \\ &= -\frac{1}{s} e^{-st} \Big|_0^{+\infty}, \end{aligned}$$

we have $s = x + iy$, then

$$\begin{aligned} -\frac{1}{s} e^{-st} \Big|_0^{+\infty} &= -\frac{1}{s} e^{xt} e^{-ity} \Big|_0^{+\infty}, \\ &= \frac{1}{s}. \end{aligned}$$

The Laplace transform of the function x exists since $\left|e^{-iyt}\right| = 1$, thus, the convergence of the integral depends only on $\mathrm{Re}(s)$ which is strictly positive. Therefore the Laplace transform of unit function (1.15) is

$$\mathrm{X}(v) = \frac{1}{v}. \tag{1.16}$$

Certain major Laplace transforms properties

Theorem 3.7 *[31, 83] For $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$, $a \in \mathbb{R}^*$ and for x, x_1 and x_2 are causal functions, we describe the important and useful properties of the Laplace transform.*

- ***Linearity***

$$\mathscr{L}[\alpha x_1 + \beta x_2](s) \quad = \quad \lambda X_1(s) + \beta X_2(s), \quad s > 0; \tag{1.17}$$

- ***Integration***

$$\mathscr{L}\left[\int_0^t x(\tau)d\tau\right] = \frac{1}{s}X(s); \tag{1.18}$$

- ***Convolution***

$$\mathscr{L}[(x_1 \star x_2)](s) = \mathscr{L}[x_1](s)\mathscr{L}[x_2](s) = X_1(s)X_2(s), \quad s > 0, \tag{1.19}$$

with

$$(x_1 \star x_2)(t) = \int_0^t x_1(\tau)x_2(t-\tau)d\tau;$$

- ***Dirac impulse***

$$\mathscr{L}[\delta(t)](s) = 1; \tag{1.20}$$

- ***Multiplication by a scalar***

$$\mathscr{L}[x(at)](s) = \frac{1}{a}X\left(\frac{s}{a}\right). \tag{1.21}$$

The boundary properties are given in the following theorem.

Theorem 3.8 *[29] Let x be a function admitting a Laplace transform* X*. Then,*

1.

$$\lim_{t\to+\infty} x(t) = \lim_{s\to 0} sX(s); \tag{1.22}$$

2.

$$\lim_{t\to 0^+} x(t) = \lim_{s\to+\infty} sX(s). \tag{1.23}$$

the envisaged limits exist.

Inverse of Laplace transform

Definition 3.9 *[31] The function x is the inverse Laplace transform of* X*, denoted by* $\mathscr{L}^{-1}$*, is provided by*

$$\begin{aligned}\mathscr{L}^{-1}\left[\mathrm{X}(s)\right] &= x(t),\\ &= \frac{1}{2j\pi}\int_{c+j\infty}^{c-j\infty}\mathrm{X}(s)e^{st}ds, c>0.\end{aligned}$$

Example 3.10 *We consider the function* X *defined by*

$$\mathrm{X}(s) = \frac{1}{s},$$

we obtain the inverse Laplace transform which is the unit function (1.15) *by using definition* (3.9).

3.2 Fractional Laplace transform

In this subsection, we shall mention some interesting Laplace of conformable and Caputo derivative properties of a continuous function.

Conformable Laplace transform

Definition 3.11 *[1] Let* $x:[0,+\infty)\to\mathbb{R}$ *be a* ***causal*** *function and* $0<\alpha\leq 1$. *Then the conformable fractional Laplace transform (CFLT) of x is*

$$\begin{aligned}\mathscr{L}_{\alpha}\left[x(t)\right](s) &= \mathrm{X}_{\alpha}(s),\\ &= \int_{0}^{\infty}e^{s\frac{-t^{\alpha}}{\alpha}}x(t)\mathrm{d}t^{\alpha}.\end{aligned} \tag{1.24}$$

The conformable Laplace transform of a function $x(t)$ does exist only if the above integral converges.

Theorem 3.12 *[1] Let* $x:[0,+\infty)\to\mathbb{R}$ *be a* ***causal*** *function and* $0<\alpha\leq 1$. *Then the conformable fractional Laplace transform of conformable derivative of x is given by*

$$\mathscr{L}_{\alpha}\left[\mathbf{T}^{\alpha}x(t)\right](s) = s\mathrm{X}_{\alpha}(s) - x(0), \quad s>0. \tag{1.25}$$

In the following theorem, we give the relationship between the conformable Laplace transform and Laplace transform.

Theorem 3.13 *[1] Let* $x:[0,+\infty)\to\mathbb{R}$ *be a* ***causal*** *function and* $0<\alpha\leq 1$. *Thus*

$$\mathrm{X}_{\alpha}(s) = \mathscr{L}\left[x(\alpha t)^{\frac{1}{\alpha}}\right](s). \tag{1.26}$$

The following theorem describes the Laplace transform of the some usual functions.

Theorem 3.14 *[1] Consider c, a, $p \in \mathbb{R}$ and for $0 < \alpha \leq 1$. Thus*

1. $$\mathscr{L}_\alpha[c](s) = \frac{c}{s}, \quad s > 0; \tag{1.27}$$

2. $$\mathscr{L}_\alpha\left[t^p\right](s) = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{s^{1+\frac{p}{\alpha}}}, \quad s > 0; \tag{1.28}$$

3. $$\mathscr{L}_\alpha\left[e^{a\frac{t^\alpha}{\alpha}}\right](s) = \frac{1}{s-a}, \quad s > a. \tag{1.29}$$

Caputo Laplace transform

Definition 3.15 *[56] The Laplace transform of the fractional Caputo derivative of the function x is defined by*

$$\mathscr{L}\left[\mathbf{D}_c^\alpha x(t)\right](s) = s^\alpha \mathrm{X}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \, x^{(k)}(t)\Big|_{t=0}, \tag{1.30}$$

where $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$, X is the Laplace transform of x and $x^{(k)}(t)\big|_{t=0}$ is the derivative of order k of the function x at the point $t = 0$.

Some interesting properties of the Caputo Laplace transform are described in the following proposition.

Proposition 3.16 *[56] For any $a \in \mathbb{R}_+^*$ and $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$, we have*

1. $\mathscr{L}\left[\dfrac{t^a}{\Gamma(a+1)}\right](s) = s^{-a+1}$.
2. $\mathscr{L}\left[\mathbf{D}^\alpha \delta(t)\right](s) = s^\alpha$, *where δ is the Dirac delta function.*

3.3 Sumudu transform

Sumudu transform has many interesting and attractive advantages over other integral transforms specifically the unity by providing the convergence when solving differential equations and also the resolvability of problems without resorting to a new frequency domain [3, 4, 5, 99]. In this subsection, several terms and theorems related to the Sumudu transform will be presented. Following that, the fractional Sumudu transform will be introduced.

Direct Sumudu transform

Definition 3.17 *[99] We take into account functions with exponential order in the set $\mathscr{A}$, defined by*

$$\mathscr{A} = \left\{x(t) | \exists \mathrm{M}, \tau_1, \tau_2 > 0, |x(t)| < \mathrm{M}e^{-\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\right\},$$

the Sumudu transform X *of a continuous function x, is represent by*

$$\mathrm{S}\left[x(t)\right](v) = \mathrm{X}(v) = \int_0^{\infty} x(vt)\, e^{-t}\, dt, \quad v \in (-\tau_1, \tau_2), \tag{1.31}$$

or a similar alternative

$$\mathrm{S}\left[x(t)\right](v) = \mathrm{X}(v) = \frac{1}{v}\int_0^{\infty} x(t)\, e^{-\frac{t}{v}}\, dt, \quad v > 0. \tag{1.32}$$

The duality of Sumudu transforms with Laplace transform is provided by the succeeding theorem

Theorem 3.18 *[47] Consider $x \in \mathscr{A}$ a continuous function and* $\mathrm{X}_1, \mathrm{X}_2$ *their integral transforms Laplace and Sumudu respectively, Then*

$$\mathrm{X}_2(v) = \frac{\mathrm{X}_1\left(\frac{1}{v}\right)}{v}, \quad \forall v > 0, \tag{1.33}$$

or

$$\mathrm{X}_1(s) = \frac{\mathrm{X}_2\left(\frac{1}{s}\right)}{s}, \quad s > 0. \tag{1.34}$$

Example 3.19 *[99] The Sumudu transfom of the unit function (1.15) is given by*

$$\mathrm{X}(v) = 1. \tag{1.35}$$

Certain major Sumudu transforms properties

Theorem 3.20 *[47] For $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$, $a \in \mathbb{R}$ and for x, x_1 and x_2 a given functions, we represent the fundamental properties of the Sumudu transform.*

- ***Linearity property***

$$\mathrm{S}\left[\alpha x_1 + \beta x_2\right](v) \;=\; \lambda \mathrm{X}_1(v) + \beta \mathrm{X}_2(v), \quad v \in (-\tau_1, \tau_2); \tag{1.36}$$

- ***Integral function***

$$\mathrm{S}\left[\int_0^t x(\tau)d\tau\right] = v\mathrm{X}(v); \tag{1.37}$$

- ***Convolution product***

$$\mathrm{S}\left[(x_1 \star x_2)\right](v) = v\mathrm{S}\left[x_1\right](v)\mathrm{S}\left[x_2\right](v), \quad v \in (-\tau_1, \tau_2), \tag{1.38}$$

where

$$(x_1 \star x_2)(t) = \int_0^t x_1(\tau)x_2(t-\tau)d\tau;$$

- ***Dirac impulsion***

$$\mathrm{S}\left[\delta(t)\right](v) = v^{-1};$$

- ***Multiplication by a scalar***

$$\mathrm{S}\left[x(at)\right](v) = \mathrm{X}(av). \tag{1.39}$$

In the following theorem, the boundary properties are presented.

Theorem 3.21 *[47, 99] Let* X *The Sumudu transform of function* x *which admit limits in the neighborhood of* 0 *and* ∞. *Then,*

1. $$\lim_{t\to 0} x(t) = \lim_{v\to 0} \mathrm{X}(v); \tag{1.40}$$

2. $$\lim_{t\to\infty} x(t) = \lim_{v\to\infty} \mathrm{X}(v). \tag{1.41}$$

Inverse of Sumudu transforms

Theorem 3.22 *[15, 47] Let* $\mathrm{X}(v)$ *be the Sumudu transform of* $x(t)$ *and we consider the following statements*

1. $v\mathrm{X}(v)$ *is a meromorphic function, having singularities* $\mathrm{Re}\left(\frac{1}{v}\right) < \gamma$.

2. *There exists a circular region* Γ *of radius* r *such that*

$$\| v\mathrm{X}(v) \| < \mathrm{M}r^{-k}, \tag{1.42}$$

where r *and* k *are positive constants.*

Therefore, the function x is represented by

$$\mathrm{S}^{-1}\left[\mathrm{X}(v)\right](t) = x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} -\frac{1}{v}e^{\frac{t}{v}}\mathrm{X}(v)\,dv. \tag{1.43}$$

Example 3.23 *[15, 47] Let be* $\mathrm{X}(v)$ *the Sumudu transform, such that*

$$\mathrm{X}(v) = \begin{cases} 0 & \text{if} \quad v < 0, \\ 1 & \text{if} \quad v \geq 0, \end{cases}$$

then

$$\begin{aligned} x(t) &= \mathrm{S}^{-1}\left[\mathrm{X}(v)\right](t), \\ &= -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{v}e^{\frac{t}{v}}\mathrm{X}(v)\,dv, \end{aligned}$$

therefore, the inverse of Sumudu transform is given by

$$x(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t \geq 0. \end{cases}$$

3.4 Fractional Sumudu transform

Conformable Sumudu transform

Definition 3.24 *[1] Over the following set of function*

$$\mathscr{A}_\alpha = \left\{ x(t) : \exists \mathrm{M}, \tau_1, \tau_2 > 0, |x(t)| < \mathrm{M}e^{\left|\frac{t^\alpha}{\alpha\tau_j}\right|}, \text{ if } t^\alpha \in (-1)^j \times [0,\infty),\ j = 1,2 \right\},$$

then, the conformable fractional Sumudu transform (CFST) of the function x is defined by

$$\begin{aligned} \mathrm{S}_\alpha[x(t)](v) &= \mathrm{X}_\alpha(v), \\ &= \frac{1}{v}\int_0^\infty e^{\frac{-t^\alpha}{\alpha v}} x(t)\,\mathrm{d}t^\alpha, \quad v > 0, \end{aligned} \tag{1.44}$$

where $\mathrm{d}t^\alpha = t^{\alpha-1}\mathrm{d}t$ *and* $\alpha \in]0,1]$.

Theorem 3.25 *[1] Let $x : [0,+\infty) \to \mathbb{R}$ be a given functions, $0 < \alpha \leq 1$. Then, we have the following property*

$$\mathrm{S}_\alpha\left[\mathbf{T}^\alpha x(t)\right](v) = \frac{1}{v}\left[\mathrm{S}_\alpha[x(t)](v) - x(0)\right], \quad \forall t > 0, \tag{1.45}$$

Theorem 3.26 *[1] Let $x : [0,+\infty) \to \mathbb{R}$ be an n-differentiable function and α such that,* $0 <$

$\alpha \leq 1$. *Then,*

$$\mathrm{S}_\alpha\left[\mathbf{T}^{n\alpha}x(t)\right](v) = \frac{\mathrm{S}_\alpha\left[x(t)\right](v) - x(0)}{v^n}, \quad \forall n \in \mathbb{N} \text{ and } \forall v > 0, \tag{1.46}$$

and as in [95], $\mathbf{T}^{n\alpha}$ *is known as the conformable derivative operator of order* n.

The following theorem describes the Sumudu transform of the some usual functions.

Theorem 3.27 *[1] Let* c *and* $a \in \mathbb{R}$ *and* $0 < \alpha \leq 1$. *Then*

1. $$\mathrm{S}_\alpha\left[e^{-a\frac{t^\alpha}{\alpha}}x(t)\right] = \frac{\mathrm{S}_\alpha[x(t)]\left(\frac{1}{v}+a\right)}{v} \quad v > 0; \tag{1.47}$$

2. $$\mathrm{S}_\alpha[c](v) = c; \tag{1.48}$$

3. $$\mathrm{S}_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\right](v) = \frac{\Gamma(1+n)}{v}^n, \quad v > 0; \tag{1.49}$$

4. $$\mathrm{S}_\alpha\left[e^{\frac{at^\alpha}{\alpha}}\right](v) = \frac{1}{1-av}, \quad v > \frac{1}{a}. \tag{1.50}$$

Caputo Sumudu transform

Definition 3.28 *The Sumudu transform of the fractional Caputo derivative* (1.9) *for* $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$, *has the following form [68]*

$$\mathrm{S}\left[\mathbf{D}_c^\alpha x(t)\right](v) = v^{-\alpha}\left(\mathrm{X}(v) - \sum_{k=1}^{n} v^{k-1}\, x^{(k-1)}(t)\Big|_{t=0}\right). \tag{1.51}$$

Some interesting properties of the Sumudu transform are described in the following proposition.

Proposition 3.29 [14, 68, 99] *We consider* $a \in \mathbb{R}_+^*$ *and* $\mathrm{N}-1 < \alpha \leq \mathrm{N}$, $\mathrm{N} \in \mathbb{N}^*$, *then*

1. $\mathrm{S}\left[\dfrac{t^a}{\Gamma(a+1)}\right](v) = v^a$.
2. $\mathrm{S}\left[\mathbf{D}_c^\alpha \delta(t)\right](v) = v^{-\alpha-1}$, *where* δ *is the Dirac delta function.*

Remark 3.30 *The fractional Laplace and Sumudu transform preserves all the properties of linearity, integral and convolution product of Laplace and Sumudu transforms.*

3.5 Comparison and discussion

In this subsection we will compare between the two transforms presented previously by specifying their advantages.

- From the theorem (3.5) and the definition (3.17), we notice that the Sumudu have fewer conditions of existence when compared to the Laplace transform.
- From the properties (1.21) and (1.39), we deduce that the Laplace transform change definitely when multiplying by a scalar however the Sumudu transformation remains unchanged.
- From the boundary properties in the theorem (3.8) and (3.21) of Laplace and Sumudu transform respectively, we can notice that the neighborhoods in Sumudu transform do not change when passing to the limit, unlike the Laplace transform.
- The unit function does not change its expression under the effect of the Sumudu transform (1.35) contrary to Laplace transform (1.16).

Finally, we conclude that the most advantageous integral transforms is that of Sumudu. For this reason, in the following chapter, we have opted to use the Sumudu transform.

3.6 $\mathcal{Z}$-transforms

The $\mathcal{Z}$-transform is the discrete transform equivalent of the Laplace transform which is a tool for automatic and signal processing.

Definition 3.31 *[31, 93] The unilateral $\mathcal{Z}$-transform of a discrete time function $x(n)$ is defined by the following formula*

$$\mathrm{X}(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n) z^{-n}. \tag{1.52}$$

where $x(n) = 0$ for $n < 0$.

Remark 3.32 *[31] Any $\mathcal{Z}$-transform must be accompanied by the region for which it converges. To determine the convergence region, the Cauchy criterion is used on the following series*

$$\sum_{n=0}^{+\infty} u_n = u_0 + u_1 + u_2 + \ldots, \tag{1.53}$$

which converges if

$$\lim_{n\to\infty} |u_n|^{\frac{1}{n}} < 1. \tag{1.54}$$

Some important properties will be given in the following [31] and [93].

- **Linearity**: Consider that $(U_n)_{n\in\mathbb{N}}$, $(W_n)_{n\in\mathbb{N}}$ admitting z-transforms , $\alpha, \beta \in \mathbb{R}$, then

$$\mathcal{Z}[\alpha U_n + \beta W_n](z) = \alpha\mathcal{Z}[U_n](z) + \beta\mathcal{Z}[W_n](z); \tag{1.55}$$

- **Derivation**: we have

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n},$$

and

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} (-n)x(n)z^{-n-1},$$

therefore

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} nx(n)z^{-n};$$

- **Convolution** : If $y(n)$ is obtained by convolution of $x(n)$ et $g(n)$, we only have

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)g(n-m), \tag{1.56}$$

thus

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{+\infty} y(n)z^{-n} = \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} x(m)g(n-m)z^{-n}, \\ &= \Big[\sum_{m=-\infty}^{+\infty} x(m)z^{-m}\Big]\Big[\sum_{n=-\infty}^{+\infty} g(n-m)z^{-(n-m)}\Big], \\ &= X(z)G(z). \end{aligned}$$

Inverse of $\mathcal{Z}$-transforms

In [31], we obtain inverse of $\mathcal{Z}$-transforms by using the definition of the $\mathcal{Z}$-transform provided by (1.52), multiplying the two members by z^{k-1} and integrating along a contour surrounding the origin and belonging to the convergence domain, we find

$$\begin{aligned} \oint_{\Gamma} X(z)z^{k-1}dz &= \oint_{\Gamma}\sum_{n=-\infty}^{+\infty} x(n)z^{-n+k-1}dz, \\ &= x(n)\oint_{\Gamma}\sum_{n=-\infty}^{+\infty} z^{-n+k-1}dz. \end{aligned}$$

Finally, by applying Cauchy's theorem, we have

$$x(n) = \frac{1}{2\pi i}\oint_{\Gamma}\sum_{n=-\infty}^{+\infty} z^{-n+k-1}dz.$$

3.7 $\mathcal{Z}$-transforms bidimentional

Theorem 3.33 *[56] Consider the bidimentional $\mathcal{Z}$-transform $\mathrm{X}(z_1, z_2)$ of the discrete function x_{ij} represented by*

$$\mathcal{Z}\left[x_{ij}\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}, \tag{1.57}$$

then, we have the following property

1. $\mathcal{Z}\left[x_{i+1,j+1}\right] = z_1 z_2 \left[\mathrm{X}(z_1,z_2) - \mathrm{X}(z_1,0) - \mathrm{X}(0,z_2) + x_{00}\right]$;
2. $\mathcal{Z}\left[x_{i-k,j+1}\right] = z_1^{-k} z_2 \left[\mathrm{X}(z_1,z_2) - \mathrm{X}(z_1,0)\right]$;
3. $\mathcal{Z}\left[x_{i+1,j-l}\right] = z_1 z_2^{-l} \left[\mathrm{X}(z_1,z_2) - \mathrm{X}(z_1,0) - \mathrm{X}(0,z_2)\right]$;
4. $\mathcal{Z}\left[x_{i-k,j-l}\right] = z_1^{-k} z_2^{-l} \left[\mathrm{X}(z_1,z_2)\right]$;
5. $\mathcal{Z}\left[x_{i+1,j}\right] = z_1 \left[\mathrm{X}(z_1,z_2) - \mathrm{X}(0,z_2)\right]$;
6. $\mathcal{Z}\left[x_{i,j+1}\right] = z_2 \left[\mathrm{X}(z_1,z_2) - \mathrm{X}(z_1,0)\right]$;

with

$$\mathrm{X}(z_1,0) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i}, \mathrm{X}(0,z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j}. \tag{1.58}$$

4 Particular matrices

In this section, we recall some needed definitions and characterizations of non-negative, positive, monomial and Metzler matrices, these matrices are used for analyzing the positivity problem in the four chapter . There are a large number of references on this notions, we focus principally on the following references [13, 18, 51, 76, 82].

Definition 4.1 *[13, 51] Let $\mathrm{A} \in \mathbb{R}^{n\times m}$ be a **non-negative** matrix if $\forall i \in n, \forall j \in m : a_{ij} \geq 0$ i.e. if all its coefficients are non-negative, we denote this matrix by $\mathrm{A} \geq 0$, or $\mathrm{A} \in \mathbb{R}_+^{n\times m}$.*

Example 4.2

$$\mathrm{A} = \begin{pmatrix} 0 & 7 & 2 \\ 9 & 8 & 1 \\ 3 & 11 & 0 \end{pmatrix}, \tag{1.59}$$

A *is a **non-negative** matrix.*

Definition 4.3 *[51]* A *is a **positive** matrix if* A *is non-negative and $\exists k \in \bar{n}, \exists l \in \bar{m} : a_{kl} > 0$ i.e. all these non-negative coefficients with at least one strictly positive coefficient, we will note such a matrix $\mathrm{A} > 0$.*

Example 4.4

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 \\ 0 & 3 & 6 & 8 \end{pmatrix}, \tag{1.60}$$

A *is a* ***positive*** *matrix.*

Definition 4.5 *[51]* A *is a* ***strictly positive*** *matrix if* $\forall i \in n$, $\forall j \in m$ *with* $a_{ij} > 0$ *i.e. all these coefficients are strictly positive, we will note such a matrix by* $A \gg 0$.

Definition 4.6 *[51] The matrix* $A \in \mathbb{R}_+^{n\times n}$ *is called* ***monomial*** *if in each row and column only one entry is positive and the remaining entries are zero.*

Theorem 4.7 *[51] Let* $A \in \mathbb{R}_+^{n\times n}$ *is* ***monomial*** *matrix if and only if* $A^{-1} \in \mathbb{R}_+^{n\times n}$.

Example 4.8 *Let A be a* ***monomial*** *matrix.*

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 5 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \in \mathbb{R}_+^{3\times 3},$$

then, the inverse of this matrix is

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \in \mathbb{R}_+^{3\times 3}.$$

Definition 4.9 *[51] A real square matrix* $A = [a_{ij}]_{i,j=1\cdots n}$ *is called* ***Metzler*** *matrix if its off diagonal entries are non-negative, i.e.* $a_{ij} \geq 0$ *for* $i \neq j$.

Lemma 4.10 *[60] Let* $A \in \mathcal{M}_n$ *if and only if* $e^{A\frac{t^\alpha}{\alpha}} \in \mathbb{R}_+^{n\times n}$ *for* $t \geq 0$ *and* $0 < \alpha \leq 1$.

Example 4.11

$$A = \begin{pmatrix} -1 & 1 & 4 & 3 \\ 5 & -2 & 1 & 2 \\ 6 & 3 & 5 & 1 \\ 2 & 1 & 3 & -1 \end{pmatrix}, \tag{1.61}$$

A is a ***Metzler*** *matrix.*

5 Schur complement

In this section, we provide some details and definitions of Schur complement [42] which will be useful throughout the last chapter.

Let M be a matrix block of dimension $(p+q)\times(p+q)$ such that

$$\mathrm{M}=\begin{bmatrix}\mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix}, \tag{1.62}$$

where $\mathrm{A}\in\mathbb{R}^{p\times p}$, $\mathrm{B}\in\mathbb{R}^{p\times q}$, $\mathrm{C}\in\mathbb{R}^{q\times p}$ and $\mathrm{D}\in\mathbb{R}^{q\times q}$.

Definition 5.1 *[42] Consider the matrix* A *is invertible, then the Schur complement of the matrix* A *in* M *is*

$$Sch(\mathrm{A},\mathrm{M})=\mathrm{D}-\mathrm{C}\mathrm{A}^{-1}\mathrm{B}. \tag{1.63}$$

Theorem 5.2 *[42] Let* M *be a square matrix given by the formula* (1.62)*, then*

$$det\,(\mathrm{M}/\mathrm{A})=det\mathrm{M}/det\mathrm{A}, \tag{1.64}$$

where A is nonsingular matrix.

Theorem 5.3 *[42] Consider* M, A, *and* E *are square nonsingular matrices. Then*

$$\mathrm{M}=\begin{bmatrix}\mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix}\quad and\quad \mathrm{A}=\begin{bmatrix}\mathrm{E}_1 & \mathrm{F}_1\\ \mathrm{G}_1 & \mathrm{H}_1\end{bmatrix}. \tag{1.65}$$

Where A/E_1 *is a nonsingular principal submatrix of* M/E_1 *such that*

$$\mathrm{M}/\mathrm{A}=(\mathrm{M}/\mathrm{E}_1)\,(\mathrm{A}/\mathrm{E}_1)\,. \tag{1.66}$$

6 Conclusion

In this chapter, we recalled some fundamental notions and definitions of special functions and some basic concepts of matrix theory. The different definitions of the fractional derivatives and two most important integral transforms (Lapace and Sumudu) with their properties are also presented. After examining their characteristics, the differences between them are identified, as a result, we have determined the most useful fractional derivative applied in the following three chapters and the powerful method that play a very important role in resolving fractional linear dynamic systems in subsequent chapters. We also discuss the $\mathcal{Z}$-transform, which is necessary in order to calculate the transfer function of a certain class of one-dimensional and two-dimensional systems. The final section discusses certain Schur complement properties that will be very important in the fifth chapter.

Chapter 2

Fractional continuous-time linear systems

1 Introduction

Recently, the concept of fractional calculus has been successfully used in control systems, with many applications in various areas of science such as chemistry, engineering, and electrical circuits [19, 20, 56]. In this chapter, we introduce a new class of fractional linear systems based on the conformable derivative. The regular linear continuous-time system with conformable derivative in unidimensional (1D) and two dimensional (2D) models has received much attention in the last two years [16, 86, 89]. Our objective is to solve this system using a recent and efficient method called the Sumudu transform, which is useful for resolving fractional linear dynamical systems. Furthermore, we provide a solution to a fractional linear system with Caputo derivative, and compare it to the conformable derivative solution. Our focus is primarily on the following references [35, 62, 63].

2 State equations of fractional continuous-time linear systems

In recent years, the behavior of actual systems in numerous fields of science and biology, engineering, electrochemistry and much more, have been modeled through fractional differential equations [52, 55, 56, 60]. Linear time-invariant dynamic systems (LTI) of fractional order can be described using the following fractional equations.

$$\sum_{j=0}^{k} b_j \mathbf{D}^{\alpha_j} y(t) = \sum_{i=0}^{l} a_i \mathbf{D}^{\beta_i} u(t), \tag{2.1}$$

with α_j and β_i for $j = 0, \cdots, k$ and $i = 0, \cdots, l$ represent the fractional order derivatives of the output $y \in \mathbb{R}^{p_1}$ and input $u \in \mathbb{R}^{m_1}$ and b_j and a_i for $j = 0, \cdots, k$ and $i = 0, \cdots, l$ are real coefficients.
Under some assumptions, the state representation for the system (2.1) can be expressed as the following fractional continuous-times linear systems

$$\mathrm{E}\mathbf{D}^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \tag{2.2}$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \tag{2.3}$$

where $\mathbf{D}^{\alpha}$ presents the fractional derivative operator of order α with $0 < \alpha \leq 1$, $x \in \mathbb{R}^{n_1}$, $u \in \mathbb{R}^{m_1}$ and $y \in \mathbb{R}^{p_1}$ are, respectively, the state, the control, and the output of the system. $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{n_1 \times n_1}$, $\mathrm{B} \in \mathbb{R}^{n_1 \times m_1}$, $\mathrm{C} \in \mathbb{R}^{p_1 \times n_1}$ and $\mathrm{D} \in \mathbb{R}^{p_1 \times m_1}$. The boundary condition of the system is given by $x(0) = x_0$ and $u(0) = 0$.

Definition 2.1 *[51, 19] If the matrix* E *of the system equations* (2.2) *and* (2.3) *is non-invertible i.e* $det\mathrm{E} = 0$, *then, the systems is called singular or descriptor system, Otherwise, If the matrix* E *is invertible i.e* $det\mathrm{E} \neq 0$, *the systems of equations* (2.2) *and* (2.3) *is called standard, in addition if* $\mathrm{E} = I_n$ *the system of equations is called standard or explicit.*

2.1 Illustrative example

Let us consider the electrical circuit presented in [56] by figure 2.1, with $0 < \alpha \leq 1$

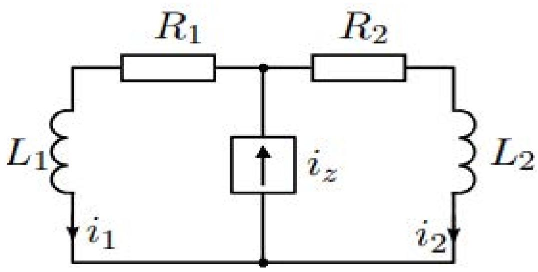

Figure 2.1: Fractional electrical circuit [56].

Using Kirchhoff's laws we can write the equations

$$\mathrm{L}_1 \frac{\mathrm{d}^{\alpha} i_1}{\mathrm{d}t^{\alpha}} + \mathrm{R}_1 i_1 = \mathrm{L}_2 \frac{\mathrm{d}^{\alpha} i_2}{\mathrm{d}t^{\alpha}} + \mathrm{R}_2 i_2, \tag{2.4}$$

$$i_z = i_1 + i_2. \tag{2.5}$$

$\mathrm{R}_1, \mathrm{R}_2$ are the resistances, $\mathrm{L}_1, \mathrm{L}_2$ are the inductances and i_z, which represents the control $u(t)$, is the source current, then, using the conformable derivative, the system becomes

$$\mathbf{T}^{\alpha}\mathrm{E}x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \tag{2.6}$$

with

$$x(t) = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \mathrm{E} = \begin{pmatrix} \mathrm{L}_1 & -\mathrm{L}2 \\ 0 & 0 \end{pmatrix}, \mathrm{A} = \begin{pmatrix} -\mathrm{R}_1 & \mathrm{R}_2 \\ -1 & -1 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the initial condition

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

3 Solvability of fractional dynamic linear systems with Caputo derivative

The purpose of this section is to present the solvability of singular and standard fractional dynamic linear systems with Caputo derivative using the Sumudu transform which is discussed in [62, 63].

Consider the following fractional continuous-times linear systems

$$\mathrm{E}\mathbf{D}_c^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \tag{2.7}$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \tag{2.8}$$

where $\mathbf{D}^{\alpha}$ presents the Caputo derivative of order α with $0 < \alpha \le 1$, $x \in \mathbb{R}^{n_1}$, $u \in \mathbb{R}^{m_1}$ and $y \in \mathbb{R}^{p_1}$ are, respectively, the state, the control, and the output of the system. $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{n_1 \times n_1}$, $\mathrm{B} \in \mathbb{R}^{n_1 \times m_1}$, $\mathrm{C} \in \mathbb{R}^{p_1 \times n_1}$ and $\mathrm{D} \in \mathbb{R}^{p_1 \times m_1}$. The boundary condition of the system is given by $x(0) = x_0$.

3.1 Solvability of singular fractional dynamic linear systems with Caputo derivative

Consider the system of equations (2.7) and (2.8), we have the following statement in [63]

1. $det(\mathrm{E}) = 0$;
2. $v^{-i\alpha}\mathrm{E}x(0)$ exists for $i = 1 \cdots \mu$, $0 < \alpha \le 1$ and $v \in (-\tau_1, \tau_2)$;
3. $u(0) = 0$ and $u(t)$ is provided;
4. The pencil of the pair (E, A) is regular i.e

$$det(\mathrm{E} - v^{\alpha}\mathrm{A})^{-1} \neq 0, \quad v \in \mathbb{C}. \tag{2.9}$$

Theorem 3.1 *[63] The state response of the singular implicit fractional dynamical system with Caputo derivative (2.7) is given by*

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty}\phi_i\left(B\frac{1}{\Gamma((i+1)\alpha)}\int_0^t (t-\tau)^{(i+1)\alpha-1}u(\tau)d\tau + E\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}x(0)\right) \\ &\quad + \sum_{i=1}^{\mu}\phi_{-i}\left(\mathbf{D}_c^{(i-1)\alpha}Bu(t) + E\mathbf{D}_c^{i\alpha-1}\delta(t)x(0)\right), \end{aligned} \tag{2.10}$$

where α *is the fractional order of Caputo derivative,* ϕ_i, $i \in \mathbb{N}$ *is the fundamental matrices,* δ *is the Dirac delta function and* μ *represent the index of nilpotency of* $(v^{-\alpha}E - A)$.

Proof. Using the Sumudu transform in formulas (1.31) and (1.51) the system of equations (2.7) become

$$S\left[E\mathbf{D}_c^{\alpha}x(t)\right](v) = S\left[Ax(t) + Bu(t)\right](v).$$

Using the definition (3.28) and the linearity property , we obtain

$$X(v) = (E - v^{\alpha}A)^{-1}\left(v^{\alpha}BU(v) + Ex(0)\right).$$

As $\det E = 0$ (non invertible matrix) and $\det(E - v^{\alpha}A) \neq 0$, thus, there exists a Laurent series expansion [78] and [80] about zero, which is given

$$(E - v^{\alpha}A)^{-1} = \sum_{i=-\mu}^{\infty}\phi_i\, v^{i\alpha}, \tag{2.11}$$

with $\mu = \mathrm{rg}(E) - \deg\left(\det(v^{-\alpha}E - A)\right) + 1$ is the index of nilpotency of $(v^{-\alpha}E - A)$ and ϕ_i are the fundamental matrices. By applying the Laurent series expansion to the equation, thus

$$\begin{aligned} X(v) &= \left(\sum_{i=-\mu}^{\infty}\phi_i\, v^{i\alpha}\right)\left(v^{\alpha}BU(v)\right) + \left(\sum_{i=-\mu}^{\infty}\phi_i\, v^{i\alpha}\right)Ex(0), \\ &= v\sum_{i=0}^{\infty}\phi_i\, v^{(i+1)\alpha-1}BU(v) + \sum_{i=1}^{\mu}\phi_{-i}\, v^{(1-i)\alpha}BU(v) \\ &\quad + \sum_{i=0}^{\infty}E\phi_i\, v^{i\alpha}x(0) + \sum_{i=1}^{\mu}\phi_{-i}\, E\, v^{-i\alpha}x(0). \end{aligned}$$

Finally, by using the inverse Sumudu transform and the convolution product, we obtain the appropriate result. ■

For the case $\alpha = 1$, we find the same result in [19, 51].

Corollary 3.2 *For $\alpha = 1$, the state of the singular implicit fractional dynamical system of equations (2.7) with Caputo derivative is given by the following formula*

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} \phi_i \left(\mathrm{B} \frac{1}{\Gamma(i+1)} \int_0^t (t-\tau)^i u(\tau) d\tau + \mathrm{E} \frac{t^i}{\Gamma(i+1)} x(0) \right) \\ &+ \sum_{i=1}^{\mu} \phi_{-i} \left(\mathrm{B} u^{(i-1)}(t) + \mathrm{E}\, \delta^{(i-1)}(t)\, x(0) \right). \end{aligned} \tag{2.12}$$

3.2 Solvability of standard fractional dynamic linear systems with Caputo derivative

Now, we consider the system of equations (2.7) and (2.8) and we suggest the following statement given in [62]

1. $det(\mathrm{E}) \neq 0$;
2. $v^{i\alpha} \mathrm{A}^i x(0)$ exists $\forall i \in \mathbb{N}$, $0 < \alpha \leq 1$ and $v \in (-\tau_1, \tau_2)$;
3. $u(t)$ is given;
4. The pencil of the pair (I_n, A) is regular i.e

$$det(I_n - v^{\alpha}\mathrm{A})^{-1} \neq 0, \quad v \in \mathbb{C}. \tag{2.13}$$

Proposition 3.3 *[56] Let $\mathrm{A} \in \mathbb{R}^{n_1 \times n_1}$ be a matrix, for $0 < \alpha \leq 1$. Thus, the Laurent series is given by*

$$\left(I_n - v^{\alpha}\mathrm{A}\right)^{-1} = \sum_{i=0}^{\infty} \mathrm{A}^i v^{i\alpha}. \tag{2.14}$$

Theorem 3.4 *[62] The state response of implicit standard fractional dynamical system with Caputo derivative (2.7) is given by*

$$x(t) = \sum_{i=0}^{\infty} \frac{\mathrm{A}^i \mathrm{B}}{\Gamma((i+1)\alpha)} \int_0^t (t-\tau)^{(i+1)\alpha - 1} u(\tau) d\tau + \sum_{i=0}^{\infty} \mathrm{A}^i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} x(0), \tag{2.15}$$

where α and Γ is respectively, the fractional order, the standard Gamma function.

Proof. Using formulas (1.31) and (1.51), the system of equations (2.7) become

$$\mathrm{S}\left[\mathbf{D}_c^{\alpha} x(t)\right](v) = \mathrm{S}\left[\mathrm{A}x(t) + \mathrm{B}u(t)\right](v).$$

Using the definition (3.28) and the linearity property of fractional Sumudu , we obtain

$$\mathrm{X}(v) = (I_n - v^{\alpha}\mathrm{A})^{-1} \left(v^{\alpha}\mathrm{BU}(v) + x(0)\right).$$

By applying the Laurent series expansion of proposition (2.11) to the equation, thus

$$\mathrm{X}(v) \quad = \quad \sum_{i=0}^{\infty} \mathrm{A}^{i} \mathrm{B}\, v^{(i+1)\alpha-1} \mathrm{U}(v) + \sum_{i=0}^{\infty} \mathrm{A}^{i}\, v^{i\alpha} x(0),$$

Finally, by using the inverse Sumudu transform and the convolution product, we obtain the solution. ■

For $\alpha = 1$, we get the following result that is the same one as in [19, 56].

Corollary 3.5 *For $\alpha = 1$, the solution of the standard implicit fractional dynamical system of equations (2.7) with Caputo derivative is given by the following formula*

$$x(t) \quad = \quad \sum_{i=0}^{\infty} \frac{\mathrm{A}^{i}\mathrm{B}}{\Gamma(i+1)} \int_{0}^{t} (t-\tau)^{i} u(\tau) d\tau + \sum_{i=0}^{\infty} \mathrm{A}^{i} \frac{t^{i}}{\Gamma(i+1)} x(0), \tag{2.16}$$

where Γ is the standard Gamma function.

4 Solvability of fractional dynamic linear systems with conformable derivative

The objective of this section is the application of the Sumudu transform for solving singular and standard continuous-time linear systems based on the conformable derivative operator.

We will consider the following fractional continuous-times linear systems

$$\mathrm{E}\mathbf{T}^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \tag{2.17}$$

$$y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \tag{2.18}$$

where $\mathbf{T}^{\alpha}$ presents the fractional conformable derivative operator of order α with $0 < \alpha \leq 1$, $x \in \mathbb{R}^{n_1}$, $u \in \mathbb{R}^{m_1}$ and $y \in \mathbb{R}^{p_1}$ are, respectively, the state, the control, and the output of the system. $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{n_1 \times n_1}$, $\mathrm{B} \in \mathbb{R}^{n_1 \times m_1}$, $\mathrm{C} \in \mathbb{R}^{p_1 \times n_1}$ and $\mathrm{D} \in \mathbb{R}^{p_1 \times m_1}$. The boundary condition of the system is given by $x(0) = x_0$.

Lemma 4.1 *Let $x_1, x_2 : [0, +\infty) \to \mathbb{R}$ be a given functions. Then, the conformable Sumudu transform of the convolution product of x_1 and x_2 is defined by*

$$\mathrm{S}_{\alpha}\left[(x_1 \star x_2)(t)\right](v) = v\mathrm{S}_{\alpha}[x_1(t^{\alpha})](v)\mathrm{S}_{\alpha}[x_2(t)](v), \quad v > 0,$$

where

$$(x_1 \star x_2)(t) = \int_{0}^{t} x_1\left(t^{\alpha} - \tau^{\alpha}\right) x_2(\tau) \mathrm{d}\tau^{\alpha}.$$

Proof. Using the relationship between conformable Sumudu transform and conformable Laplace transform [1], we get

$$\begin{aligned} \mathrm{S}_\alpha\left[(x_1 \star x_2)(t)\right](v) &= \frac{\mathscr{L}_\alpha[(x_1 \star x_2)(t)](s)}{v}, \quad s \to \frac{1}{v}, \\ &= \frac{(\mathscr{L}_\alpha[x_1(t^\alpha)]\mathscr{L}_\alpha[x_2(t)])(s)}{v}, \quad s \to \frac{1}{v}, \\ &= v\mathrm{S}_\alpha[x_1(t^\alpha)](v)\mathrm{S}_\alpha[x_2(t)](v), \end{aligned} \tag{2.19}$$

where, $\mathscr{L}_\alpha$ is the conformable Laplace transform [46]. ■

4.1 Solvability of singular fractional dynamic linear systems with conformable derivative

This subsection is devoted to present our main results. For this purpose, we will consider the system of equations (2.17) and (2.18).

We take into account the following hypotheses which implies that the solution is impulse free:

(i) $\mathrm{E}x(0)$ and $v^{-i}\mathrm{E}x(0)$ exist for $i = \overline{1,\mu}$ and $v \in (-\tau_1, \tau_2)$;

(ii) $u(t)$ is specified for $t \geq 0$;

(iii) The pencil $\left(\frac{1}{v}\mathrm{E} - \mathrm{A}\right)$ is regular for all $v \in \mathbb{C}$.

Proposition 4.2 *Let* $\alpha \in]0,1]$ *and for all* $v > 0$, *the conformable Sumudu transform of the conformable derivative of order* $(n-1)$ *of the function* $t^{1-\alpha}\delta(t)$ *is given by*

$$\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^{n-1}}\mathrm{S}_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^n}, \quad \forall n \in \mathbb{N}^*. \tag{2.20}$$

Proof. To proof formula (2.20), we will proceed by induction and we will use the properties of the function δ given in [43].

1. First step: for $n = 1$, we get

$$\begin{aligned} \mathrm{S}_\alpha\left[t^{1-\alpha}\delta(t)\right](v) &= \frac{1}{v}\int_0^\infty t^{1-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t \\ &= \frac{1}{v}\int_0^\infty \delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t, \end{aligned}$$

   using the property of δ function, yields

$$\mathrm{S}_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}e^0,$$

finally,

$$\mathrm{S}_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}.$$

2. Second step: we assume that the expression (2.20) is true up to the order $n-2$ and we proof that it stays true at the order $n-1$.

   For $\alpha \in (0,1]$ and all $v > 0$, we have

$$\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}\int_0^\infty \mathbf{T}^{(n-1)\alpha}\left[t^{1-\alpha}\delta(t)\right] e^{-\frac{t^\alpha}{v\alpha}} t^{\alpha-1}\mathrm{d}t,$$

   applying the definition of $\mathbf{T}^{n\alpha}$, we get

$$\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}\int_0^\infty \mathbf{T}^{(n-2)\alpha}\left[\mathbf{T}^{\alpha}(t^{1-\alpha}\delta(t))\right] e^{-\frac{t^\alpha}{v\alpha}} t^{\alpha-1}\mathrm{d}t,$$

   as the formula (2.20) is true for $n-2$, we obtain

$$\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^{n-1}}\int_0^\infty \mathbf{T}^{\alpha}\left[t^{1-\alpha}\delta(t)\right] e^{-\frac{t^\alpha}{v\alpha}} t^{\alpha-1}\mathrm{d}t,$$

   by the use of the definition of $\mathbf{T}^{\alpha}$, we find

$$\begin{aligned}\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) &= \frac{1}{v^{n-1}}\int_0^\infty t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{1-\alpha}\delta(t)\right] e^{-\frac{t^\alpha}{v\alpha}} t^{\alpha-1}\mathrm{d}t\\ &= \frac{1}{v^{n-1}}\left[\int_0^\infty (1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right.\\ &\quad\left.+\int_0^\infty t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)\right]e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right],\end{aligned}$$

   using the property of the function δ, it follows

$$\begin{aligned}\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) =& \frac{1}{v^{n-1}}\left[\int_0^\infty (1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right.\\ &\left.+\frac{1}{v}\int_0^\infty \delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t-\int_0^\infty (1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right],\end{aligned}$$

   finally, we obtain

$$\mathrm{S}_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^{n-1}}\mathrm{S}_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^n}, \quad \forall n\in\mathbb{N}^*.$$

■

By the extension of the series of Laurent [78] we find the following proposition.

Proposition 4.3 *Let* $\mathrm{A},\mathrm{E}\in\mathbb{R}^{n_1\times n_1}$ *be a real matrices with* $\det\mathrm{E}=0$*, then, we have*

$$\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right)^{-1} = \sum_{i=-\mu}^{\infty}\phi_i v^{i+1}, \quad v>0, \tag{2.21}$$

with $\mu = rg(\mathrm{E}) - deg\left(\det\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right)\right)+1$ *represents the index of nilpotency of* $\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right)$ *and* ϕ_i *are the fundamental matrices, which depend on the regularity of* E *and satisfy*

$$\phi_i = (\phi_0 \mathrm{A})^i \phi_0, \quad \forall i \in \mathbb{N}, \tag{2.22}$$

and

$$\phi_i \mathrm{E} - \phi_{i-1}\mathrm{A} = \delta_{i0} I = \mathrm{E}\phi_i - \mathrm{A}\phi_{i-1}, \tag{2.23}$$

where δ_{i0} *is the Kronecker delta.*

In the following, we denote X_α and U_α the conformable Sumudu transform of x and u respectively.

Theorem 4.4 *The solution of the singular dynamical system of order* α *described by the equation* (2.17) *is given by*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} \phi_i \left(\frac{t^{\alpha i}}{\alpha^i i!} \mathrm{E}x(0) + \int_0^t \frac{(t^\alpha - \tau^\alpha)^i}{\alpha^i i!} \mathrm{B}u(\tau)\mathrm{d}\tau^\alpha \right) \\ &+ \sum_{i=1}^{\mu} \phi_{-i} \left(\mathrm{B}\mathbf{T}^{\alpha(i-1)} u(t) + \mathrm{E}\mathbf{T}^{\alpha(i-1)} t^{1-\alpha}\delta(t)x(0) \right), \end{aligned} \tag{2.24}$$

where $\mu = rg(\mathrm{E}) - deg\left(\det\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right)\right)+1$ *represents the index of nilpotency of* $\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right)$, ϕ_i *are the fundamental matrices defined in proposition* 4.3, *and* δ *is the Dirac delta function.*

Proof. Applying the conformable Sumudu transform to the equation (2.17), we obtain

$$\mathrm{S}_\alpha \left[\mathrm{E}\mathbf{T}^\alpha x(t)\right](v) = \mathrm{S}_\alpha \left[\mathrm{A}x(t) + \mathrm{B}u(t)\right](v), \quad v > 0.$$

The use of the linearity property of conformable Sumudu transform together with the first property of the theorem 3.25, yields

$$\mathrm{E}\left(\frac{\mathrm{X}_\alpha(v) - x(0)}{v}\right) = \mathrm{A}\mathrm{X}_\alpha(v) + \mathrm{B}\mathrm{U}_\alpha(v),$$

which is equivalent to

$$\left[\frac{1}{v}\mathrm{E} - \mathrm{A}\right]\mathrm{X}_\alpha(v) = \frac{1}{v}\mathrm{E}x(0) + \mathrm{B}\mathrm{U}_\alpha(v).$$

As the pencil (E, A) is regular, so

$$\mathrm{X}_\alpha(v) = \left[\frac{1}{v}\mathrm{E} - \mathrm{A}\right]^{-1}\left[\frac{1}{v}\mathrm{E}x(0) + \mathrm{B}\mathrm{U}_\alpha(v)\right]. \tag{2.25}$$

Thanks to the formula (2.21), the relation (2.25) becomes

$$\mathrm{X}_\alpha(v) = \sum_{i=-\mu}^{\infty} \phi_i v^i \mathrm{E}x(0) + \sum_{i=-\mu}^{\infty} \phi_i v^{i+1}\mathrm{B}\mathrm{U}_\alpha(v),$$

by dividing the sum we get

$$
\begin{aligned}
X_\alpha(v) = &\sum_{i=0}^{\infty} \phi_i v^i E x(0) + \sum_{i=0}^{\infty} \phi_i v^{i+1} B U_\alpha(v) \\
&+ \sum_{i=1}^{\mu} \phi_{-i} v^{-i} E x(0) + \sum_{i=1}^{\mu} \phi_{-i} v^{-i+1} B U_\alpha(v).
\end{aligned} \tag{2.26}
$$

Finally, by the use of the inverse conformable Sumudu transform and convolution product, we obtain the theorem which represents the first result of this chapter. ■

Theorem 4.4 can be expressed using the exponential expression and the formula (2.22) as follow

Corollary 4.5 *The state of the singular dynamical system of order* α *described by the equation* (2.17) *is given by*

$$
\begin{aligned}
x(t) = &e^{\phi_0 A \frac{t^\alpha}{\alpha}} \phi_0 E x(0) + \int_0^t e^{\phi_0 A \frac{t^\alpha - \tau^\alpha}{\alpha}} \phi_0 B u(\tau) d\tau^\alpha \\
&+ \sum_{i=1}^{\mu} \phi_{-i} \left(B \mathbf{T}^{\alpha(i-1)} u(t) + E \mathbf{T}^{\alpha(i-1)} t^{1-\alpha} \delta(t) x(0) \right),
\end{aligned} \tag{2.27}
$$

where $\mu = rg(E) - deg\left(\det\left(\frac{1}{v}E - A\right)\right) + 1$ *represents the index of nilpotency of* $\left(\frac{1}{v}E - A\right)$*, and* ϕ_i *are the fundamental matrices defined in proposition* 4.3*, and* δ *is the Dirac delta function.*

Remark 4.6 *If* $\alpha = 1$*, we find the state response of the singular dynamical system defined in [30]*

$$
\begin{aligned}
x(t) = &e^{\phi_0 A t} \phi_0 E x(0) + \int_0^t e^{\phi_0 A (t-\tau)} \phi_0 B u(\tau) d\tau \\
&+ \sum_{i=1}^{\mu} \phi_{-i} \left(B u^{(i-1)}(t) + E \delta^{(i-1)}(t) x(0) \right).
\end{aligned} \tag{2.28}
$$

where $\mu = rg(E) - deg\left(\det\left(\frac{1}{v}E - A\right)\right) + 1$ *represents the index of nilpotency of* $\left(\frac{1}{v}E - A\right)$*, and* ϕ_i *are the fundamental matrices defined in proposition* 4.3*, and* δ *is the Dirac delta function.*

4.2 Solvability of standard fractional dynamic linear systems with conformable derivative

Let us, now, discuss the case where E is a regular matrix, i.e., $\det E \neq 0$. For this case, we assume that $[E^{-1}A]^i v^i x(0)$ exist for all $i \in \mathbb{N}$ and $v \in (-\tau_1, \tau_2)$. Hence, if $\det E \neq 0$, the Laurent series which is the extension of [78] are described by the following proposition

Proposition 4.7 *Let* $\mathrm{A}, \mathrm{E} \in \mathbb{R}^{n_1 \times n_1}$ *be a real matrices with* $\det \mathrm{E} \neq 0$*, then, we have*

$$\left(\frac{1}{v}\mathrm{E} - \mathrm{A}\right)^{-1} = \sum_{i=0}^{\infty} \phi_i v^{i+1}, \quad v > 0, \tag{2.29}$$

with ϕ_i *are the fundamental matrices, which depend on the regularity of* E *and satisfy*

$$\phi_i = \left(\mathrm{E}^{-1}\mathrm{A}\right)^i \mathrm{E}^{-1}. \tag{2.30}$$

Theorem 4.8 *The solution of the implicit dynamical system of order* α *given by the equation* (2.17) *is*

$$x(t) = \sum_{i=0}^{\infty} \left[\mathrm{E}^{-1}\mathrm{A}\right]^i \frac{t^{\alpha i}}{\alpha^i i!} x(0) + \int_0^t \sum_{i=0}^{\infty} \left[\mathrm{E}^{-1}\mathrm{A}\right]^i \mathrm{E}^{-1} \frac{(t^\alpha - \tau^\alpha)^i}{\alpha^i i!} \mathrm{B}u(\tau) d\tau^\alpha. \tag{2.31}$$

Therefore, by using the exponential expression, we obtain

$$x(t) = e^{[\mathrm{E}^{-1}\mathrm{A}]\frac{t^\alpha}{\alpha}} x(0) + \int_0^t e^{[\mathrm{E}^{-1}\mathrm{A}]\frac{t^\alpha - \tau^\alpha}{\alpha}} \mathrm{E}^{-1}\mathrm{B}u(\tau) d\tau^\alpha.$$

Proof. Thanks to the formula (2.29), the relation (2.25) becomes

$$\mathrm{X}(v) = \sum_{i=0}^{\infty} \phi_i v^i \mathrm{E}x(0) + \sum_{i=0}^{\infty} \phi_i v^{i+1} \mathrm{B}\mathrm{U}_\alpha(v),$$

it follows that

$$\mathrm{X}_\alpha(v) = \sum_{i=0}^{\infty} \left[\mathrm{E}^{-1}\mathrm{A}\right]^i v^i x(0) + \sum_{i=0}^{\infty} \left[\mathrm{E}^{-1}\mathrm{A}\right]^i \mathrm{E}^{-1}\mathrm{B} v^{i+1} \mathrm{U}_\alpha(v).$$

Finally by applying the inverse of conformable Sumudu transform and the convolution product, we obtain the solution. ■

Remark 4.9 *If* $\mathrm{E} = \mathrm{I}$*, we obtain the standard dynamical system of order* α *and the state is*

$$x(t) = e^{\mathrm{A}\frac{t^\alpha}{\alpha}} x(0) + \int_0^t e^{\mathrm{A}\frac{t^\alpha - \tau^\alpha}{\alpha}} \mathrm{B}u(\tau) d\tau^\alpha.$$

Furthermore, if $\alpha = 1$*, the state of the standard dynamical system is*

$$x(t) = e^{\mathrm{A}t} x(0) + \int_0^t e^{\mathrm{A}(t-\tau)} \mathrm{B}u(\tau) d\tau.$$

5 Experimental results

In this section, we present some illustrative academic and real examples in order to show the efficiency and the accuracy of our approach. It must be emphasized that all examples were already discussed in [56, 63]. we always consider the fact that $u(0) = 0$.

Example 5.1 *Let us consider, for* $\alpha \in]0,1]$, *the following system of electrical circuit given in [56]*

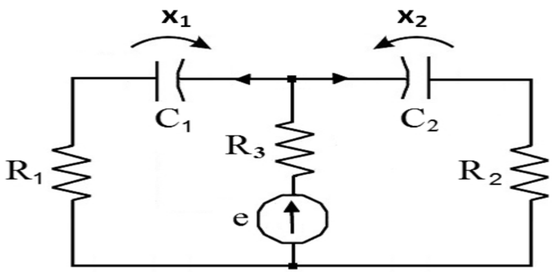

Figure 2.2: Electrical circuit [56].

R_1, R_2, R_3 *represent resistances,* C_1, C_2 *the capacitances, and* e *the source voltage (the control* $u(t) = e$*). Using Kirchhoff's laws, we can write the equations*

$$e = R_1C_1\frac{d^\alpha x_1}{dt^\alpha} + x_1 + R_3\left(C_1\frac{d^\alpha x_1}{dt^\alpha} + C_2\frac{d^\alpha x_2}{dt^\alpha}\right), \tag{2.32}$$

$$e = R_3\left(C_1\frac{d^\alpha x_1}{dt^\alpha} + C_2\frac{d^\alpha x_2}{dt^\alpha}\right) + R_2C_2\frac{d^\alpha x_2}{dt^\alpha} + x_2, \tag{2.33}$$

which are equivalent to

$$\begin{bmatrix} (R_1+R_3)C_1 & R_3C_2 \\ R_3C_1 & (R_2+R_3)C_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e. \tag{2.34}$$

The general expression of the system (2.34) *is*

$$E\mathbf{T}^\alpha x(t) = Ax(t) + Bu(t), \tag{2.35}$$

with boundary condition $x_0 = 0_{\mathbb{R}^2}$ *and*

$$E = \begin{pmatrix} (R_1+R_3)C_1 & R_3C_2 \\ R_3C_1 & (R_2+R_3)C_2 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

as $\det E = [R_1(R_2+R_3) + R_2R_3]C_1C_2 \neq 0$, *then,*

$$E^{-1} = \frac{1}{\det E}\begin{pmatrix} (R_2+R_3)C_2 & -R_3C_2 \\ -R_3C_1 & (R_1+R_3)C_1 \end{pmatrix},$$

$$E^{-1}A = \frac{1}{\det E}\begin{pmatrix} -(R_2+R_3)C_2 & R_3C2 \\ R_3C_1 & -(R_1+R_3)C_1 \end{pmatrix} \mathit{and} E^{-1}B = \frac{1}{\det E}\begin{pmatrix} R_2C_2 \\ R_1C_1 \end{pmatrix}.$$

For $e = 1\mathrm{V}$, the solution of the electrical circuit is

$$x(t) = \int_0^t e^{E^{-1}A\frac{(t^\alpha - \tau^\alpha)}{\alpha}} E^{-1}Bd\tau^\alpha, \tag{2.36}$$

which is the same one as in [60].

The solution with Caputo derivative is

$$\tilde{x}(t) = \sum_{k=0}^{\infty}\left(A^k \int_0^t \frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}d\tau\right)B. \tag{2.37}$$

To show the efficiency of our method we will plot, in the following figures, both solutions together with the exact solution for different values of α. *We assume that* $R_1 = R_2 = 10\Omega$, $R_3 = 20\Omega$, $C_1 = C_2 = 100m\mathrm{F}$ *ant the input* $u(t) = e = 1\mathrm{V}$.

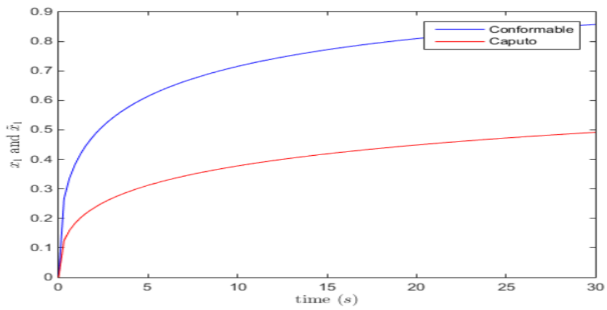

Figure 2.3: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.4$.

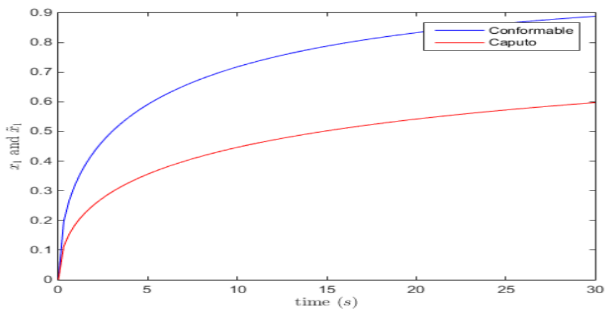

Figure 2.4: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.5$.

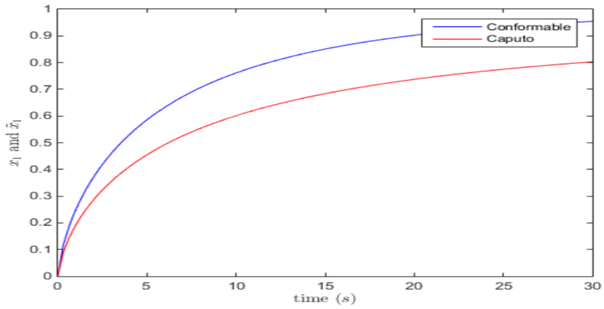

Figure 2.5: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.7$.

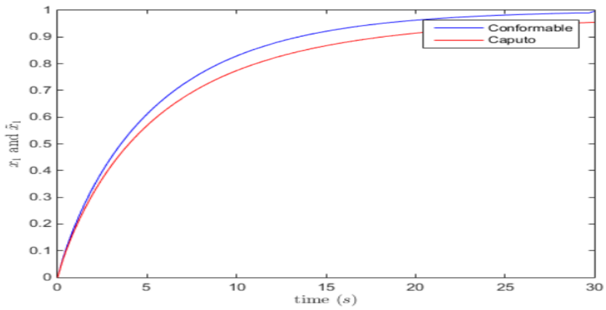

Figure 2.6: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.9$.

Example 5.2 *Let* $0 < \alpha \leq 1$ *and the following singular system*

$$\mathbf{T}^{\alpha} \mathrm{E} x(t) = \mathrm{A} x(t) + \mathrm{B} u(t), \tag{2.38}$$

with

$$\mathrm{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and the initial condition

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

Since

$$\det\left(\frac{1}{v}\mathrm{E} - \mathrm{A}\right) = \frac{2 + 2v}{v} \neq 0, \quad \forall v > 0,$$

and $\mu = 1$*, it follows*

$$\phi_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \phi_{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi_{2m+1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall m \in \mathbb{N}.$$

The state of the system (2.38) *is given by*

$$x(t) = \begin{pmatrix} e^{\frac{-t^{\alpha}}{\alpha}} x_{0,1} + \int_0^t e^{-\frac{t^{\alpha} - \tau^{\alpha}}{\alpha}} u(\tau) d\tau^{\alpha} \\ u(t) \end{pmatrix}. \tag{2.39}$$

However, with the Caputo derivative, we find

$$\tilde{x}(t) = \begin{pmatrix} \sum_{i=0}^{\infty}(-1)^{i}\left[\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}x_{0,1} + \frac{1}{\Gamma((i+1)\alpha)}\int_{0}^{t}(t-\tau)^{(i+1)\alpha-1}u(\tau)\mathrm{d}\tau\right] \\ u(t) \end{pmatrix}. \tag{2.40}$$

For different values of α, $u(t) = 1$, $x_{0,1} = 3$, *and* $x_{0,2} = 0$, *the comparison of the states between conformable derivative* $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, *Caputo derivative* $\tilde{x}(t) = [\tilde{x}_1(t), \tilde{x}_2(t)]^{\mathrm{T}}$ *is plotted in figures* 2.7, 2.8, *and* 2.9.

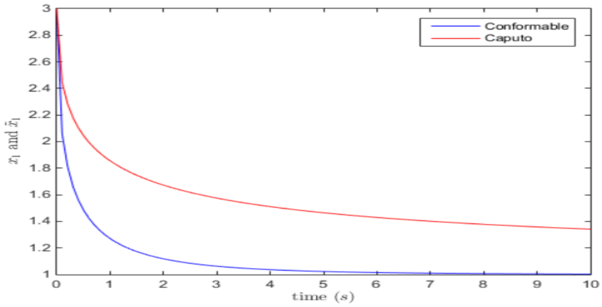

Figure 2.7: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.5$.

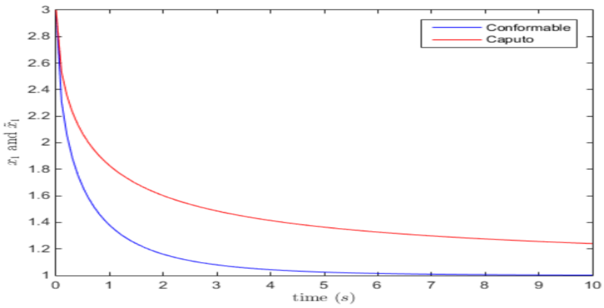

Figure 2.8: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.6$.

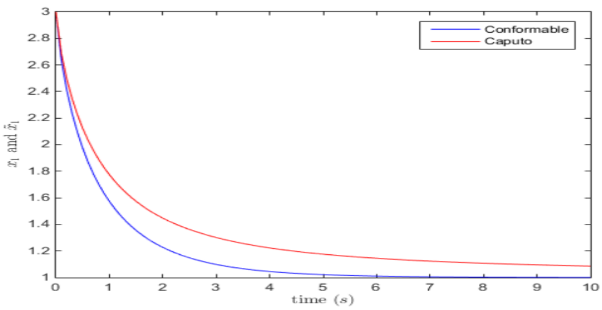

Figure 2.9: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.8$.

6 Concluding Remarks

In this section, the continuous-time linear systems based on the conformable derivatives operator are introduced where another approach to compute there solutions are presented. The main idea behind this approach consists on using the conformable Sumudu transform which is recognized by its important properties. The singular and regular cases are discussed and the method can be used for several practical applications as for instance the electrical circuit. Through the numerical examples presented the final section, it easy to see that the solution of dynamical systems with conformable derivative is consistent to the classical derivative. More then that, it has been shown in [60] that for the conformable derivative, the electrical circuit could be reach its steady state in a shorter time.

Chapter 3

Controllability and observability

1 Introduction

In this chapter, we focus on the concepts of controllability and observability, which are common terms in control theory. To make the analysis of controllability and observability more straightforward, we will use the Weierstrass decomposition method. This chapter is organized as follows: First, we apply the Weierstrass-Kronecker decomposition method to the singular dynamical conformable linear time-invariant system. Then, we focus on establishing the solution, controllability, and observability properties of this system. Finally, we conclude this chapter.

2 Weierstrass-kronecker decomposition method

Several authors have attempted to define the Weierstrass-Kronecker Decomposition Method [30, 51, 56, 61]. In this section, we will show the Weierstrass decomposition of a singular dynamical conformable linear time-invariant system in order to simplify the study of various concepts such as positivity, stability, super-stability, controllability, and observability. Assume that the system is regular, thus there exists a pair of nonsingular matrices P, $Q \in \mathbb{R}^{n_1 \times n_1}$ as follows

$$\mathrm{PEQ} = diag(I_{\bar{n}_1}, \mathrm{N}),\ \mathrm{PAQ} = diag(\mathrm{A}_1, I_{\bar{n}_2}),\ \mathrm{PB} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix},\ \mathrm{CQ} = \begin{bmatrix} \mathrm{C}_1 \\ \mathrm{C}_2 \end{bmatrix},\ \mathrm{Q}^{-1}x = \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}. \quad (3.1)$$

Where $\bar{x}_1 \in \mathbb{R}^{\bar{n}_1}$ and $\bar{x}_2 \in \mathbb{R}^{\bar{n}_2}$ with $n_1 = \bar{n}_1 + \bar{n}_2$, $u \in \mathbb{R}_1^m$, $\mathrm{A}_1 \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}$, $\mathrm{B}_1 \in \mathbb{R}^{\bar{n}_1 \times m_1}$, $\mathrm{B}_2 \in \mathbb{R}^{\bar{n}_2 \times m_1}$ and $\mathrm{N} \in \mathbb{R}^{\bar{n}_2 \times \bar{n}_2}$ is a nilpotent matrix of the nilpotency index μ, i.e. $\mathrm{N}^{\mu-1} \neq 0$ and $\mathrm{N}^{\mu} = 0$. Premultiplying the equation (2.17) of the singular dynamical conformable linear time-invariant system by the matrix P and using the transformations (3.1), the system can be divided into two following subsystems

1. The conformable slow subsystem

$$\begin{cases} \mathbf{T}^{\alpha}\bar{x}_1(t) &= \mathrm{A}_1\bar{x}_1(t)+\mathrm{B}_1u(t), \\ \bar{x}_1(0) &= \bar{x}_{10}, \end{cases} \tag{3.2}$$

2. The conformable fast subsystem

$$\begin{cases} \mathrm{N}\mathbf{T}^{\alpha}\bar{x}_2(t) &= \bar{x}_2(t)+\mathrm{B}_2u(t), \\ \bar{x}_2(0) &= \bar{x}_{20}, \end{cases} \tag{3.3}$$

and the output of the system become

$$y(t)=\mathrm{C}_1\bar{x}_1(t)+\mathrm{C}_2\bar{x}_2(t)+\mathrm{D}u(t). \tag{3.4}$$

The solution conformable slow subsystem (3.2) is given in the following theorem which is the same one given in the previous section and in [60] of standard conformable slow subsystem.

Theorem 2.1 *The solution of the conformable slow subsystem* (3.2), *for initial condition* $\bar{x}_{10}\in\mathbb{R}^{\bar{n}_1}$ *and admissible input* $u(t)\in\mathbb{R}^{m_1}$ *is given by*

$$\bar{x}_1(t)=e^{\mathrm{A}_1\frac{t^{\alpha}}{\alpha}}\bar{x}_{10}+\int_0^t e^{\mathrm{A}_1\frac{t^{\alpha}-\tau^{\alpha}}{\alpha}}\mathrm{B}_1u(\tau)\mathrm{d}\tau^{\alpha}. \tag{3.5}$$

Theorem 2.2 *The state of the conformable fast subsystem* (3.3) *for consistent initial condition* $\bar{x}_{20}\in\mathbb{R}^{\bar{n}_2}$ *and admissible input* $u(t)\in\mathrm{U}$ *is given by*

$$\bar{x}_2(t)=-\sum_{i=0}^{\mu-1}\mathrm{N}^i\left(\mathrm{B}_2\mathbf{T}^{i\alpha}u(t)+\mathrm{N}\mathbf{T}^{i\alpha}t^{1-\alpha}\delta(t)\bar{x}_2(0)\right), \tag{3.6}$$

where μ *represents the index of nilpotency of* N, *and* δ *is the Dirac delta function.*

Proof. Applying the conformable Sumudu transform, we obtain

$$\mathrm{N}\left(\frac{\bar{\mathrm{X}}_{2\alpha}(v)-\bar{x}_2(0)}{v}\right)=\bar{\mathrm{X}}_{2\alpha}+\mathrm{B}_2\mathrm{U}_{\alpha}(v),$$

which is equivalent to

$$\left[\frac{1}{v}\mathrm{N}-I_{n2}\right]\bar{\mathrm{X}}_{2\alpha}(v)=\frac{1}{v}\mathrm{N}\bar{x}_2(0)+\mathrm{B}_2\mathrm{U}_{\alpha}(v),$$

and

$$\mathrm{X}_{2\alpha}(v)=\left[\frac{1}{v}\mathrm{N}-I_{n2}\right]^{-1}\left[\frac{1}{v}\mathrm{N}\bar{x}_2(0)+\mathrm{B}_2\mathrm{U}_{\alpha}(v)\right], \tag{3.7}$$

Note that
$$\left(\frac{1}{\upsilon}\mathrm{N}-I_{n2}\right)^{-1}=-\sum_{i=0}^{\mu-1}\mathrm{N}^{i}\upsilon^{-i},\tag{3.8}$$
according to the formula (3.8), the relation (3.7) becomes

$$\mathrm{X}_{2\alpha}(\upsilon)=-\sum_{i=0}^{\mu-1}\mathrm{N}^{i+1}\upsilon^{-(i+1)}\mathrm{N}\bar{x}_2(0)-\sum_{i=0}^{\mu-1}\mathrm{N}^{i}\upsilon^{i}\mathrm{B}_2\mathrm{U}_{\alpha}(\upsilon).$$

Finally, by the use of the inverse conformable Sumudu transform and convolution product, we find the state response of the fast subsystem (3.3). ■

Theorem 2.3 *The non-impluse solution of the fast system* (3.3) *with admissible input* $u(t)\in$ U *and consistent initial condition* $\bar{x}_{20}=-\sum_{i=0}^{\mu-1}\mathrm{N}^{i}\mathrm{B}_2\mathbf{T}^{i\alpha}u(0)$. *has the form*

$$\bar{x}_2(t)=-\sum_{i=0}^{\mu-1}\mathrm{N}^{i}\mathrm{B}_2\mathbf{T}^{i\alpha}u(t).\tag{3.9}$$

Example 2.4 *Consider the singular dynamical conformable linear time-invariant system*

$$\mathrm{E}=\begin{bmatrix}-0.4&0&-0.5&0\\-0.2&0&0&0\\0.4&1&0.5&0\\0.2&0&0&0\end{bmatrix},\mathrm{A}=\begin{bmatrix}-0.2&1.8&0.5&0\\0.4&0.4&0&0\\0.2&-1.8&-0.5&0.5\\-0.4&0.6&0&0\end{bmatrix},\mathrm{B}=\begin{bmatrix}-1&-3.6\\0&-0.8\\-1&2.6\\0&-0.2\end{bmatrix},\tag{3.10}$$

and the input is given by

$$u(t)=\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix}=\begin{bmatrix}\dfrac{t^{2\alpha}}{2\alpha}\\ sin\left(\dfrac{t^{\alpha}}{\alpha}\right)+2\dfrac{t^{2\alpha}}{2\alpha}\end{bmatrix},\tag{3.11}$$

we have the pencil
$$\det\left[\mathrm{E}\frac{1}{\upsilon}-\mathrm{A}\right]=-0.05\left(\frac{1}{\upsilon}+1\right)\left(\frac{1}{\upsilon}+2\right)\neq0,$$
is regular. then there exists a pair of nonsingular matrices $\mathrm{P},\mathrm{Q}\in\mathbb{R}^{n_1\times n_1}$ *described by*

$$\mathrm{P}=\begin{bmatrix}-1&3&0&1\\0&-3&0&2\\1&0&1&0\\0&1&0&1\end{bmatrix}\;and\;\mathrm{Q}=\begin{bmatrix}0&1&0&0\\0&0&0&1\\2&0&0&0\\0&0&2&0\end{bmatrix},\tag{3.12}$$

such that

$$\mathrm{PEQ} = \begin{bmatrix} I_2 & 0 \\ 0 & \mathrm{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.13}$$

$$\mathrm{PAQ} = \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.14}$$

and

$$\mathrm{PB} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0- & 1 \end{bmatrix}. \tag{3.15}$$

Therefore the slow and fast subsystem is represented by the following systems

$$\begin{cases} \mathbf{T}^{\alpha}\bar{x}_1(t) & = & \mathrm{A}_1\bar{x}_1(t) + \mathrm{B}_1 u(t), \\ \bar{x}_1(0) & = & \bar{x}_{10}, \end{cases} \tag{3.16}$$

with

$$\mathrm{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \mathrm{B}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \tag{3.17}$$

and

$$\begin{cases} \mathrm{N}\mathbf{T}^{\alpha}\bar{x}_2(t) & = & \bar{x}_2(t) + \mathrm{B}_2 u(t), \\ \bar{x}_2(0) & = & \bar{x}_{20}, \end{cases} \tag{3.18}$$

with

$$\mathrm{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{B}_2 = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}. \tag{3.19}$$

The state of slow subsystem (3.16) *for initial condition* $\bar{x}_1(0) = \bar{x}_{10} = [\bar{x}_{101} \quad \bar{x}_{102}]^{\mathrm{T}}$ *is given by*

$$\bar{x}_1(t) = \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \end{bmatrix}, \quad t \geq 0, \tag{3.20}$$

where

$$\bar{x}_{11}(t) = e^{-\frac{t^{\alpha}}{\alpha}}\bar{x}_{101} + e^{-\frac{t^{\alpha}}{\alpha}}\bar{x}_{102} - e^{-2\frac{t^{\alpha}}{\alpha}}\bar{x}_{102} + 5\frac{t^{2\alpha}}{2\alpha} - (6\alpha+2)\frac{t^{\alpha}}{\alpha} + \left(-7\alpha - \frac{3}{2}\right)e^{-\frac{t^{\alpha}}{\alpha}} + \left(\frac{1}{10} + \frac{\alpha}{2}\right)e^{-2\frac{t^{\alpha}}{\alpha}}$$
$$+ \frac{7}{10}\sin\left(\frac{t^{\alpha}}{\alpha}\right) - \frac{11}{10}\cos\left(\frac{t^{\alpha}}{\alpha}\right) + \left(\frac{13}{2}\alpha + \frac{5}{2}\right), \tag{3.21}$$

and

$$\bar{x}_{12}(t) = e^{-2\frac{t^{\alpha}}{\alpha}}\bar{x}_{102} + \frac{t^{2\alpha}}{\alpha} - (\alpha+1)\frac{t^{\alpha}}{\alpha} - \left(\frac{1}{10}+\frac{\alpha}{2}\right)e^{-2\frac{t^{\alpha}}{\alpha}} + \frac{4}{5}\sin\left(\frac{t^{\alpha}}{\alpha}\right) - \frac{2}{5}\cos\left(\frac{t^{\alpha}}{\alpha}\right) + \left(\frac{1}{2}\alpha+\frac{\alpha}{2}\right). \tag{3.22}$$

The state of fast subsystem (3.18) *for initial condition* $\bar{x}_2(0) = \bar{x}_{20} = [1 \quad 0]^{\mathrm{T}}$ *is given by*

$$\bar{x}_2(t) = \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \end{bmatrix}, \quad t \geq 0, \tag{3.23}$$

where

$$\bar{x}_2(t) = \begin{bmatrix} 2\frac{t^{2\alpha}}{\alpha} + 2t^{\alpha} + \cos\left(\frac{t^{\alpha}}{\alpha}\right) + \sin\left(\frac{t^{\alpha}}{\alpha}\right) \\ \frac{t^{2\alpha}}{\alpha} + \sin\left(\frac{t^{\alpha}}{\alpha}\right) \end{bmatrix}, \quad t \geq 0. \tag{3.24}$$

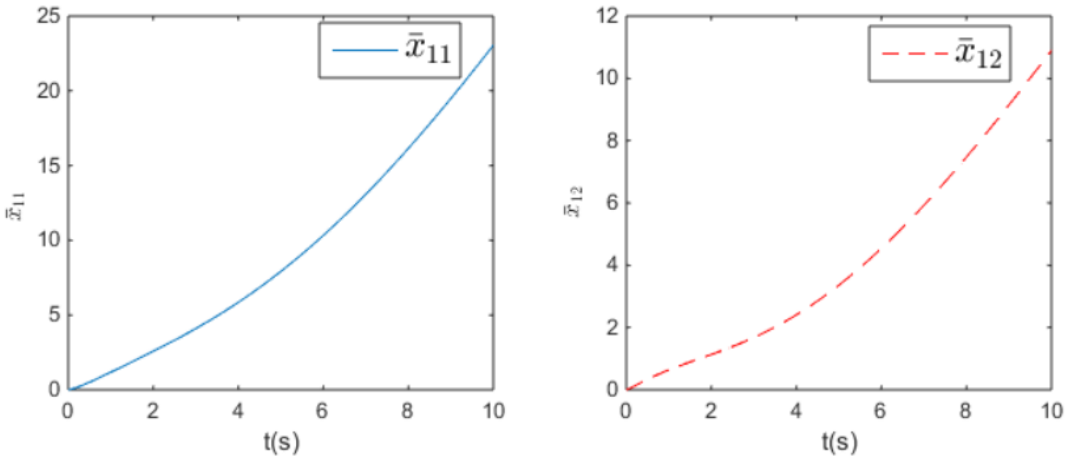

Figure 3.1: The solutions of slow subsystem $\bar{x}_{11}$ and $\bar{x}_{12}$ for $\alpha = 0.5$.

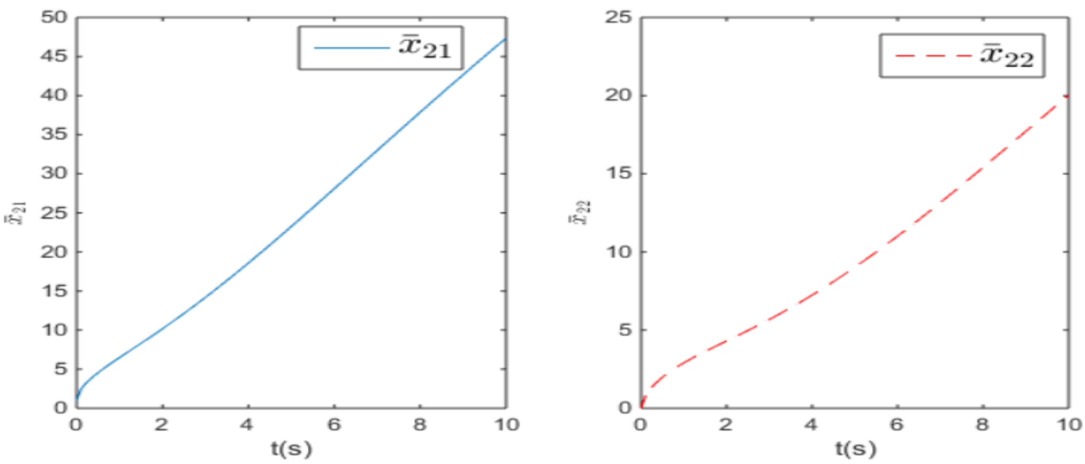

Figure 3.2: The solutions of fast subsystem $\bar{x}_{21}$ and $\bar{x}_{22}$ for $\alpha = 0.5$.

3 Controllability of singular dynamical conformable linear time-invariant system

In 1960, Kalman introduced the concept of controllability which is of great importance in the analysis and design of control systems [64, 65]. In recent years, the controllability of fractional differential systems has received a great deal of attention [11, 12, 24, 100, 107]. The controllability of conformable differential standard systems is discussed in [2, 98]. Throughout this section, various concepts of controllability for singular dynamical conformable linear time-invariant system will be established.

The following definition is an extension of definition of singular system given in [51, 61].

Definition 3.1 *A singular dynamical conformable linear time-invariant system is called controllable on* $[0, \mathrm{T}]$ *if for any state* $x_0, x_{t_1} \in \mathbb{R}_1^n$ *there exists a control input* $u(t) : [0, \mathrm{T}] \rightarrow \mathbb{R}_1^m$*, then, we have the solution of the system satisfies* $x(0) = x_0$ *and* $x(t_1) = x_{t_1}$ *, such that* $t_1 \in [0, \mathrm{T}]$.

Based on [2] and [98] we obtain the following theorem.

Theorem 3.2 *The conformable slow subsystem* (3.2) *is controllable on* $[0, t_1]$ *if and only if the following controllability Gramian matrix*

$$\mathrm{W}_c(0, t_1) := \int_0^{t_1} e^{\mathrm{A}_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} \mathrm{B}_1 \mathrm{B}_1^{\mathrm{T}} e^{\mathrm{A}_1^{\mathrm{T}} \frac{t_1^\alpha - \tau^\alpha}{\alpha}} d^\alpha \tau, \tag{3.25}$$

is nonsingular.

Proof. Sufficiency. As $\mathrm{W}_c[0, t_1]$ is nonsingular, therefore its inverse exists. For any initial condition $x_1(0) = x_{10} \neq 0$, we define the control as

$$u(t) = \mathrm{B}_1^{\mathrm{T}} e^{\mathrm{A}_1^{\mathrm{T}} \frac{t^\alpha - \tau^\alpha}{\alpha}} \mathrm{W}_c^{-1}(0, t) \left[x_{t_1} - e^{\mathrm{A}_1 \frac{t^\alpha}{\alpha}} \bar{x}_{10} \right]. \tag{3.26}$$

From the solution (3.5) we obtain

$$\begin{aligned}
x(t_1) =& e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} + \int_0^{t_1} e^{\mathrm{A}_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} \mathrm{B}_1 u(\tau) \mathrm{d}\tau^\alpha, \\
=& e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} + \int_0^{t_1} e^{\mathrm{A}_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} \mathrm{B}_1 \mathrm{B}_1^{\mathrm{T}} e^{\mathrm{A}_1^{\mathrm{T}} \frac{t_1^\alpha - \tau^\alpha}{\alpha}} \mathrm{W}_c^{-1}(0, t_1) \left[x_{t_1} - e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} \right] \mathrm{d}\tau^\alpha, \\
=& e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} + \int_0^{t_1} e^{\mathrm{A}_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} \mathrm{B}_1 \mathrm{B}_1^{\mathrm{T}} e^{\mathrm{A}_1^{\mathrm{T}} \frac{t_1^\alpha - \tau^\alpha}{\alpha}} \mathrm{d}\tau^\alpha \mathrm{W}_c^{-1}(0, t_1) \left[x_{t_1} - e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} \right], \\
=& e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} + \mathrm{W}_c(0, t_1) \mathrm{W}_c^{-1}(0, t_1) \left[x_{t_1} - e^{\mathrm{A}_1 \frac{t_1{}^\alpha}{\alpha}} \bar{x}_{10} \right], \\
=& x_{t_1}.
\end{aligned} \tag{3.27}$$

Thus, the system (3.5) is controllable on $[0, t_1]$.

Necessity. We assume that the system (3.2) is controllable on $[0, t_1]$ and the matrix $W_c(0, t_1)$ is singular. then there exists an vector $v \in \mathbb{R}^{\bar{n}_1^*}$ such that

$$\begin{aligned} 0 = & v^T W_c(0, t_1) v = \int_0^{t_1} v^T e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 B_1^T e^{A_1^T \frac{t_1^\alpha - \tau^\alpha}{\alpha}} v d^\alpha \tau, \\ & = \int_0^{t_1} \| v^T e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 \|^2 d^\alpha \tau, \end{aligned} \tag{3.28}$$

this yields

$$v^T e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 = 0, \quad \forall \tau \in [0, t_1]. \tag{3.29}$$

As the systems is controllable, then, there exist an input such that the initial state $x_1(0) = x_{10}$ can be transformed to $x_1(t_1) = 0$, we choose

$$x_{10} = -e^{-A_1 \frac{t_1{}^\alpha}{\alpha}} v, \tag{3.30}$$

then

$$x_1(t_1) = -v + \int_0^{t_1} e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 u(\tau) d^\alpha \tau = 0, \tag{3.31}$$

which implies

$$v = \int_0^{t_1} e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 u(\tau) d^\alpha \tau, \tag{3.32}$$

pre-multiplying the equation (3.32) by v^T, we get

$$v^T v = \int_0^{t_1} v^T e^{A_1 \frac{t_1{}^\alpha - \tau^\alpha}{\alpha}} B_1 u(\tau) d^\alpha \tau = 0, \tag{3.33}$$

which contradicts the fact that $v \neq 0$, therefore the matrix $W_c(0, t_1)$ is non-singular. ■

On the other hand, we rewrite the solution of the fast subsystem given by

$$\bar{x}_2(t) = -\sum_{i=0}^{\mu-1} N^i \left(B_2 \mathbf{T}^{i\alpha} u(t) + N \mathbf{T}^{i\alpha} t^{1-\alpha} \delta(t) \bar{x}_2(0) \right), \tag{3.34}$$

in the following form

$$\bar{x}_2(t) = \psi(t) x_{20} - W U^\alpha(t), \tag{3.35}$$

where

$$\psi(t) = -\sum_{i=0}^{\mu-1} N^{i+1} \mathbf{T}^{i\alpha} t^{1-\alpha} \delta(t), \tag{3.36}$$

$$W = \begin{bmatrix} B_2 & NB_2 & \cdots & N^{\mu-1} B_2 \end{bmatrix}, \tag{3.37}$$

and

$$U^\alpha(t) = \begin{bmatrix} u(t) & T^\alpha u(t) & \cdots & T^{(\mu-1)\alpha} \end{bmatrix}. \tag{3.38}$$

Theorem 3.3 *The conformable fast subsystem* (3.3) *is controllable on* $[0, t_1]$ *if the matrix* W *is row full rank.*

Proof. We consider W^+ is the pseudo inverse of the matrix W, we suppose that $U^\alpha(t) = W^+\left(\psi(t)x_{20} + x_{2t_1}\right)$, then

$$x_2(t_1) = \psi(t)x_{20} - WW^+\left(\psi(t)x_{20} + x_{2t_1}\right) = x_{2t_1}, \tag{3.39}$$

by choosing $U^\alpha(t)$ the system is able to transfer x_{20} to x_{2t_1}, moreover, form $U^\alpha(t)$ we can deduce the control $u(t)$. This is implies that the conformable fast subsystem is controllable on $[0, t_1]$. ∎

The following theorem gives us conditions on controllability of conformable singular linear system that are the same in [30].

Theorem 3.4 1. *The conformable slow subsystem* (3.2) *is controllable if and only if*

$$rank[sE - AB] = n_1, \forall s \in \mathbb{C}.$$

2. *The following statements are equivalent*

   (a) *The conformable fast subsystem* (3.3) *is controllable on* $[0, t_1]$*;*

   (b)
   $$rank\begin{bmatrix} B_2 & NB_2 & \cdots & N^{\mu-1}B_2 \end{bmatrix} = \bar{n}_2; \tag{3.40}$$

   (c)
   $$rank\begin{bmatrix} N & B_2 \end{bmatrix} = \bar{n}_2; \tag{3.41}$$

   (d)
   $$rank\begin{bmatrix} E & B \end{bmatrix} = n_1. \tag{3.42}$$

3. *The statements listed below are equivalent*

   (a) *The singular dynamical conformable linear time-invariant system is controllable on* $[0, t_1]$*;*

   (b) *the slow* (3.2) *and fast* (3.3) *conformable subsystem are both controllable on* $[0, t_1]$*;*

   (c)
   $$rank\begin{bmatrix} B_1 & AB_1 & \cdots & A_1^{\bar{n}_1-1}B_1 \end{bmatrix} = \bar{n}_1, \tag{3.43}$$

   and

   $$rank\begin{bmatrix} B_2 & NB_2 & \cdots & N^{\mu-1}B_2 \end{bmatrix} = \bar{n}_2; \tag{3.44}$$

(d)

$$rank\begin{bmatrix} s\mathrm{E}-\mathrm{A} & \mathrm{B}\end{bmatrix} = n_1 \quad \forall s \in \mathbb{C}, \tag{3.45}$$

and

$$rank\begin{bmatrix} \mathrm{E} & \mathrm{B}\end{bmatrix} = n_1; \tag{3.46}$$

(e) The following matrix $\mathrm{D} \in \mathbb{R}^{n_1^2 \times (n_1+m_1-1)n_1}$ *such that*

$$\mathrm{D} = \begin{bmatrix} -\mathrm{A} & \cdots & \cdots & \cdots & \mathrm{B} & \cdots & \cdots & \cdots & \cdots \\ \mathrm{E} & -\mathrm{A} & \cdots & \cdots & \cdots & \mathrm{B} & \cdots & \cdots & \cdots \\ \cdots & \mathrm{E} & \ddots & \cdots & \cdots & \cdots & \mathrm{B} & \cdots & \cdots \\ \cdots & \cdots & \ddots & -\mathrm{A} & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \mathrm{E} & \cdots & \cdots & \cdots & \cdots & \mathrm{B} \end{bmatrix}, \tag{3.47}$$

has full row rank.

Example 3.5 *Consider the singular dynamical conformable linear time-invariant system described by the equations* (2.17) *and* (2.18) *and*

$$\mathrm{E} = \begin{bmatrix} 0.5 & -0.375 & 0 \\ 0 & 0.25 & 0 \\ -0.5 & -0.125 & 0 \end{bmatrix}, \mathrm{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix}, \tag{3.48}$$

the pencil (E, A) *is regular since* $\det\left(\frac{1}{v}\mathrm{E}-\mathrm{A}\right) = 0.25\left(\frac{1}{v}+2\right)\left(\frac{1}{v}+4\right) \neq 0$, *then there exist two matrices* P *and* Q *described by*

$$\mathrm{P} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \mathrm{Q} = \begin{bmatrix} 1 & 0.75 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \tag{3.49}$$

if we choose the transformation (3.49) *we obtain*

$$\mathrm{PEQ} = \begin{bmatrix} I_2 & 0 \\ 0 & \mathrm{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.50}$$

$$\mathrm{PAQ} = \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} -2 & 2.5 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.51}$$

and

$$\mathrm{PB} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ -1 & 0 \end{bmatrix}. \tag{3.52}$$

We obtain the following systems

$$\begin{cases} \mathbf{T}^{\alpha}\bar{x}_1 &= \begin{bmatrix} -2 & 2.5 \\ 0 & -4 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} u(t), \\ 0 &= \bar{x}_2(t) + \begin{bmatrix} -1 & 0 \end{bmatrix} u(t). \end{cases} \tag{3.53}$$

We define the Gramian matrix theorem of controllability on $t \in [0,1]$ *and we find that*

$$\begin{aligned} \mathrm{W}_c(0,1) :&= \int_0^1 e^{\mathrm{A}_1 \frac{1-\tau^{\alpha}}{\alpha}} \mathrm{B}_1 \mathrm{B}_1^{\mathrm{T}} e^{\mathrm{A}_1^{\mathrm{T}} \frac{1-\tau^{\alpha}}{\alpha}} d^{\alpha}\tau, \\ &= \int_0^1 \begin{bmatrix} e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & 1.25\left(e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - e^{-4\left(\frac{1-\tau^{\alpha}}{\alpha}\right)}\right) \\ 0 & 4e^{-4\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \\ &\quad \begin{bmatrix} e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & 0 \\ 1.25\left(e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - e^{-4\left(\frac{1-\tau^{\alpha}}{\alpha}\right)}\right) & 4e^{-4\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} d^{\alpha}\tau, \\ &= \int_0^1 \begin{bmatrix} 29e^{-4\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - 50e^{-6\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + 25e^{-8\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & 20e^{-6\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - 20e^{-8\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \\ 20e^{-6\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - 20e^{-8\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & 16e^{-8\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} d^{\alpha}\tau, \\ &= \begin{bmatrix} \frac{392}{192} - \frac{29}{4}e^{-4\left(\frac{1}{\alpha}\right)} + \frac{50}{6}e^{-6\left(\frac{1}{\alpha}\right)} - \frac{25}{8}e^{-8\left(\frac{1}{\alpha}\right)} & \frac{40}{48} - \frac{20}{6}e^{-6\left(\frac{1}{\alpha}\right)} + \frac{20}{8}e^{-8\left(\frac{1}{\alpha}\right)} \\ \frac{40}{48} - \frac{20}{6}e^{-6\left(\frac{1}{\alpha}\right)} + \frac{20}{8}e^{-8\left(\frac{1}{\alpha}\right)} & 2 - 2e^{-8\left(\frac{1}{\alpha}\right)} \end{bmatrix}, \end{aligned} \tag{3.54}$$

for $\alpha = 0.5$ *we have*

$$\det\left(\mathrm{W}_c(0,1)\right) = 3.3841 \neq 0. \tag{3.55}$$

Therefor the slow subsystem is controllable since the Gramian matrix is nonsingular. On the other hand, in the following the controllability of fast subsystem , will be examined.

We have $\mu = 1$*, then*

$$rank(\mathrm{W}) = rank(\mathrm{B}_2) = rank\left(\begin{bmatrix} -1 & 0 \end{bmatrix}\right) = 1. \tag{3.56}$$

For consequent the singular dynamical conformable linear time-invariant system is controllable on $[0,1]$*;*

4 Observability of singular dynamical conformable linear time-invariant system

In this section, we focus on the notion of observability, which consists in finding and rebuilding the initial state of a given system from its output data. The observability of non homogeneous systems and standard conformable linear time-invariant control systems was addressed in [2] and [10] respectively. In this section we will introduce the observability of singular dynamical conformable linear time-invariant system.

Consider the following singular dynamical conformable linear time-invariant system

$$\begin{aligned} \mathrm{E}\mathbf{T}^{\alpha}x(t) &= \mathrm{A}x(t)+\mathrm{B}u(t), && (3.57)\\ \tilde{y}(t) &= \mathrm{C}x(t). && (3.58) \end{aligned}$$

The solution of this system is composed of two part, as follow

$$x(t,u,x_0)=x_i(t,x_0)+x_u(t,u,\mathbf{T}^{k\alpha}u),$$

where $x_i(t,x_0,\mathrm{T}^{k\alpha}u)$ depends on the initial condition x_0 and $x_u(t,u,\mathrm{T}^{k\alpha}u)$ is determined by the input $u(t)$ and its conformable derivatives,$t\in[0,\infty)$, then, the output can be expressed by the following form

$$\tilde{y}(t)=\mathrm{C}x_i(t,x_0)+\mathrm{C}x_u(t,u,\mathbf{T}^{k\alpha}u).$$

For convenience, we introduce the output as

$$y(t)=\tilde{y}(t)-\mathrm{C}x_u(t,u,\mathbf{T}^{k\alpha}u)=\mathrm{C}x_i(t,x_0),$$

therefore, the output y(t) is designed by the system input and output data of system of equations (3.57) and (3.58). In addition, we introduce the the following system without control, when $x_i(t,x_0)$ and $y(t)$ are the state response and the output, respectively

$$\begin{aligned} \mathrm{E}\mathbf{T}^{\alpha}x(t) &= \mathrm{A}x(t), && (3.59)\\ y(t) &= \mathrm{C}x(t). && (3.60) \end{aligned}$$

In this section, we present the observability of the system of equations (3.59) and (3.60) , the problem is to reconstruction of the state $x_i(t,x_0)$ according to the output data $y(t)$ which is equivalent to the problem of reconstructing the state $x(t,u,x_0,\mathrm{T}^{k\alpha}u)$ of the system of equations (3.57) and (3.58) from its input and output data.
As in the previous section, we use the Weierstrass decomposition, and the system of equa-

tions (3.59) and (3.60) will become

1. The conformable slow subsystem with initial condition $\bar{x}_1(0) = \bar{x}_{10}$

$$\begin{cases} \mathbf{T}^{\alpha}\bar{x}_1(t) &= \mathrm{A}_1\bar{x}_1(t), \\ \bar{y}_1(t) &= \mathrm{C}_1\bar{x}_1(t), \end{cases} \tag{3.61}$$

2. The conformable fast subsystem with initial condition $\bar{x}_2(0) = \bar{x}_{20}$

$$\begin{cases} \mathrm{N}\mathbf{T}^{\alpha}\bar{x}_2(t) &= \bar{x}_2(t), \\ \bar{y}_2(t) &= \mathrm{C}_2\bar{x}_2(t), \end{cases} \tag{3.62}$$

Definition 4.1 *The system described by the equations* (3.59) *and* (3.60) *is observable if the initial condition* x_0 *may be uniquely determined by the output data* $y(t)$, $t \in [0,\infty)$.

The above theorem is an extension of the theorem of linear independence in [40].

Theorem 4.2 *Let* μ *be a positive integer, and let* $x(t)$ *be a continuous function that does not equal to zero. As a result, the functions* $x(t), \delta(t), \mathbf{T}^{i\alpha}\left(t^{1-\alpha}\delta(t)\right)$, *with* $i = 1\cdots\mu-1$ *are linearly independent.*

Based on [2] and [10], we obtain the following theorems.

Theorem 4.3 *The conformable slow subsystem* (3.61) *is observable on* $[0;t_1]$ *if and only if the observability matrix for the matrix pair* (A,C) *by*

$$rank\begin{bmatrix} \mathrm{C}_1 \\ \mathrm{C}_1\mathrm{A}_1 \\ \vdots \\ \mathrm{C}_1\mathrm{A}_1^{\bar{n}_1-1} \end{bmatrix} = \bar{n}_1. \tag{3.63}$$

Theorem 4.4 *The conformable slow subsystem* (3.61) *is observable on* $[0,t_1]$ *if and only if the observability Gramian matrix*

$$\mathrm{W}_o(0,t_1) := \int_0^{t_1} e^{\mathrm{A}_1{}^{\mathrm{T}}\frac{t_1{}^{\alpha}-\tau^{\alpha}}{\alpha}}\mathrm{C}_1^{\mathrm{T}}\mathrm{C}_1 e^{\mathrm{A}_1\frac{t_1^{\alpha}-\tau^{\alpha}}{\alpha}}d^{\alpha}\tau, \tag{3.64}$$

is nonsingular.

According to [40] we will obtain the following results.

Theorem 4.5 *Let the conformable fast subsystem* (3.62), *then,* $y_2(t) \equiv 0$, $t \geq 0$ *if and only if*

$$\bar{x}_{20} \in Ker \begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{\mu-1} \end{bmatrix}. \tag{3.65}$$

Proof. The solution of the conformable fast subsystem (3.62) is given by

$$\bar{x}_2(t) \quad = \quad -\sum_{i=0}^{\mu-1} N^{i+1}\mathbf{T}^{i\alpha}[t^{1-\alpha}\delta(t)]\bar{x}_2(0), \tag{3.66}$$

for instance, $y_2(t) = C_2\bar{x}_2(t)$, thus, the output has the following form

$$y_2(t) = C_2\bar{x}_2(t) = -C_2\sum_{i=0}^{\mu-1} N^{i+1}\mathbf{T}^{i\alpha}[t^{1-\alpha}\delta(t)]\bar{x}_2(0) = 0. \tag{3.67}$$

Applying the theorem (4.2), we obtain $y_2(t) = 0$ if and only if

$$C_2N^i = 0, \quad i = 0, 1, 2, \cdots, \mu - 1.$$

this is equivalent to

$$\begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{\mu-1} \end{bmatrix} \bar{x}_{20} = 0, \tag{3.68}$$

therefore

$$\bar{x}_{20} \in Ker \begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{\mu-1} \end{bmatrix}.$$

■

based on [40] we have the following lemma.

Lemma 4.6 *The conformable fast subsystem* (3.62) *is observable if and only if*

$$rank \begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{\mu-1} \end{bmatrix} = \bar{n}_2. \tag{3.69}$$

The following theorem represent the different conditions of the observability of conformable singular linear system which are the same theorem in [30].

Theorem 4.7 *1. The conformable slow subsystem* (3.61) *is observable if and only if*

$$rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n_1, \quad s \in \mathbb{C} \text{ and } s \text{ finite}; \tag{3.70}$$

2. The following properties are equivalent

(a) The conformable fast subsystem (3.62) *is observable;*

(b)

$$rank \begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{\mu-1} \end{bmatrix} = \bar{n}_2; \tag{3.71}$$

(c)

$$Ker \begin{bmatrix} N \\ C_2 \end{bmatrix} = 0; \tag{3.72}$$

(d)

$$rank \begin{bmatrix} N \\ C_2 \end{bmatrix} = \bar{n}_2; \tag{3.73}$$

(e)

$$rank \begin{bmatrix} E \\ C \end{bmatrix} = n_1; \tag{3.74}$$

3. The following statements are equivalents

(a) The conformable singular system of equations (3.59) *and* (3.60) *is observable;*

(b) Both its conformable slow and fast subsystems (3.61)*,* (3.62) *are observable;*

(c) $rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n_1, \forall s \in \mathbb{C}$*, s finite and* $rank \begin{bmatrix} E \\ C \end{bmatrix} = n_1;$

(d) *The following matrix*

$$\begin{bmatrix} -A & E & \cdots & \cdots & \cdots \\ \cdots & -A & E & \cdots & \cdots \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & -A & E \\ C & \cdots & \cdots & \cdots & \cdots \\ \cdots & C & \cdots & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & C \end{bmatrix} \tag{3.75}$$

is of full column rank n_1^2.

Example 4.8 *Consider the singular dynamical conformable linear time-invariant system represented by the equations* (3.59) *and* (3.60)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \tag{3.76}$$

since the pencil (E, A) *is regular, then there exist two matrices* P *and* Q *described by*

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.77}$$

and the system of equations (3.59) *and* (3.60) *becomes*

$$\begin{cases} \mathbf{T}^{\alpha}\bar{x}_1 &= A_1\bar{x}_1(t), \\ N\mathbf{T}^{\alpha}\bar{x}_2(t) &= \bar{x}_2(t), \\ y(t) &= C_1\bar{x}_1(t) + C_2\bar{x}_2(t), \end{cases} \tag{3.78}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, N = 0, C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, C_2 = 1. \tag{3.79}$$

We compute the Gramian observability matrix on $[0,1]$*, this yields*

$$
\begin{aligned}
W_0(0,1) :&= \int_0^1 e^{A_1{}^T \frac{t_1{}^{\alpha}-\tau^{\alpha}}{\alpha}} C_1^T C_1 e^{A_1 \frac{t_1^{\alpha}-\tau^{\alpha}}{\alpha}} d^{\alpha}\tau, \\
&= \int_0^1 \begin{bmatrix} \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \\ \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&\begin{bmatrix} \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \\ \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{2}e^{\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + \frac{1}{2}e^{-\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} d^{\alpha}\tau \\
&= \int_0^1 \begin{bmatrix} \frac{1}{4}e^{2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{2} + \frac{1}{4}e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{4}e^{2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{4}e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \\ \frac{1}{4}e^{2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} - \frac{1}{4}e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} & \frac{1}{4}e^{2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} + \frac{1}{2} + \frac{1}{4}e^{-2\left(\frac{1-\tau^{\alpha}}{\alpha}\right)} \end{bmatrix} d^{\alpha}\tau, \\
&= \begin{bmatrix} -\frac{1}{2\alpha} + \frac{1}{8}e^{\frac{2}{\alpha}} - \frac{1}{8}e^{-\frac{2}{\alpha}} & -\frac{1}{4} + \frac{1}{8}e^{\frac{2}{\alpha}} + \frac{1}{8}e^{-\frac{2}{\alpha}} \\ -\frac{1}{4} + \frac{1}{8}e^{\frac{2}{\alpha}} + \frac{1}{8}e^{-\frac{2}{\alpha}} & \frac{1}{2\alpha} + \frac{1}{8}e^{\frac{2}{\alpha}} - \frac{1}{8}e^{-\frac{2}{\alpha}} \end{bmatrix},
\end{aligned} \tag{3.80}
$$

and

$$
\begin{aligned}
\det(W_0(0,1)) &= \left[\frac{1}{8}e^{\frac{2}{\alpha}} - \frac{1}{8}e^{-\frac{2}{\alpha}}\right]^2 - \frac{1}{4\alpha^2} - \left[\frac{1}{8}e^{\frac{2}{\alpha}} + \frac{1}{8}e^{-\frac{2}{\alpha}} - \frac{1}{4}\right]^2 \\
&= -\frac{1}{4\alpha^2} + \frac{1}{16}e^{\frac{2}{\alpha}} + \frac{1}{16}e^{-\frac{2}{\alpha}}
\end{aligned} \tag{3.81}
$$

for $\alpha = 0.5$ *we have*

$$
\det(W_0(0,1)) = -1 + \frac{1}{16}e^{4} + + \frac{1}{16}e^{-4} = 2.4135 \neq 0. \tag{3.82}
$$

Since the Gramian matrix is nonsingular, then the conformable slow subsystem is observable. However, the observability of the conformable fast subsystem will be examined in the following

Since $\mu = 1$*, then*

$$
rank(W) = rank(C_2) = 1. \tag{3.83}
$$

For consequent the singular dynamical conformable linear time-invariant system is observable on $[0,1]$.

5 Conclusion

In this chapter, we have applied the Weierstrass-Kronecker theorem on singular conformable continuous-time linear invariant system as an extension of the decomposition of the regular pencil. For computing the solution, we propose using the Sumudu transform. New conditions for controllability and observability were established. The discussion was illustrated by some academics examples.

Chapter 4

Positivity, stability and super-stability

1 Introduction

For many years, positive systems have seen a dynamic evolution. The essential characteristic of these systems is that the state trajectory is entirely in the non-negative orthant if the initial state is positive (or at least non-negative). Moreover, one of the most crucial components of dynamical systems is stability, which specifies the reaction behavior of the system at infinity with regard to disturbances in the initial conditions. Kaczorek examined positive standard and singular systems, and also their stability in [49, 50, 51]. The stability of fractional positive standard and singular systems is addressed in [53, 54, 55, 56, 57, 58]. In the literature, the notion of superstability is also an important components in control theory [59, 61, 87]. Superstable systems, in which the state vector's norm monotonically decreases to zero, are a specific kind of stable systems with more constrained dynamics requirements. This chapter focuses on the study of the positivity of conformable singular continuous-time linear systems, after that, we are interested in the notions of the stability and superstability of this systems.

2 Positivity of conformable linear time-invariant system

Positive linear systems are commonly referred to be the active research area of mathematics due to its application in various field of engineering, management science, economics, social sciences, biology, and medicine. In this section, the positivity of conformable linear time-invariant system will be dealt with in more detail as an extension of the positive linear continuous-time system that can be found in [19, 51, 56, 60, 61].

2.1 External positivity

Firstly, we start with the definitions of external positivity of conformable linear time-invariant system.

Definition 2.1 *The standard implicit conformable linear time-invariant system of equations (2.17) and (2.18) is called externally positive if and only if the output corresponding to the null initial state is non-negative for each non-negative input, i.e. for $x_0 = 0$ and $u(t) \in \mathbb{R}_+^{m_1}$ for $t \in [0, \infty)$, the output $y(t) \in \mathbb{R}_+^{p_1}$, $t \in [0, \infty)$.*

Theorem 2.2 *The standard implicit conformable linear time-invariant system of equations (2.17) and (2.18) is externally positive if and only if its matrix of impulse response is non-negative, i.e. $g_\alpha(t) \in \mathbb{R}_+^{p_1 \times m_1}$ for $t \geq 0$ with g_α is defined in the following.*

The output of the implicit dynamical conformable linear time-invariant system described by the equation (2.18) is given by

$$y(t) = Cx(t) + Du(t), \tag{4.1}$$

substituting the solution (4.9) in (4.1) we obtain

$$y(t) = Ce^{E^{-1}A\frac{t^\alpha}{\alpha}}x(0) + \int_0^t Ce^{E^{-1}A\frac{t^\alpha - \tau^\alpha}{\alpha}}E^{-1}Bu(\tau)d\tau^\alpha + Du(t). \tag{4.2}$$

We replace $x_0 = 0$ and $u(t) = \delta(t)$ in the output expression (4.2), we obtain the impulse response $g(t)$ of the system of equations (2.17) and (2.18) as following

$$g_\alpha(t) = \begin{cases} Ce^{E^{-1}A\frac{t^\alpha}{\alpha}}E^{-1}B & \text{for} \quad t > 0, \\ D\delta(t) & \text{for} \quad t = 0. \end{cases} \tag{4.3}$$

Definition 2.3 *The singular conformable linear time-invariant system of equations (2.17) and (2.18) is called externally positive if for $x_0 = 0$ and any non-negative admissible control $u(t) \geq 0$ with $\mathbf{T}^{\alpha(k-1)}u(t) \in \mathbb{R}_+^{m_1}$, $k = 1, \cdots, \mu$, $t \in [0, \infty)$, the output is also non-negative i.e. $y(t) \geq 0$ for $t > 0$.*

Theorem 2.4 *The singular conformable linear time-invariant system of equations (2.17) and (2.18) with $D = 0$ is said to be externally positive, if and only if, its matrix of impulse response $g_\alpha(t)$, is non-negative for $t \geq 0$, i.e., $g_\alpha(t) \in \mathbb{R}_+^{p_1 \times m_1}$ which is defined by*

Substituting the solution (2.27) in (2.18) we obtain

$$\begin{aligned} y(t) = & \mathrm{C}e^{\phi_0 \mathrm{A}\frac{t^\alpha}{\alpha}}\phi_0 \mathrm{E}x(0) + \int_0^t \mathrm{C}e^{\phi_0 \mathrm{A}\frac{t^\alpha - \tau^\alpha}{\alpha}}\phi_0 \mathrm{B}u(\tau)d\tau^\alpha \\ & + \sum_{i=1}^{\mu} \mathrm{C}\phi_{-i}\left(\mathbf{B}\mathbf{T}^{\alpha(i-1)}u(t) + \mathrm{E}\mathbf{T}^{\alpha(i-1)}t^{1-\alpha}\delta(t)x(0)\right) + \mathrm{D}u(t), \end{aligned} \tag{4.4}$$

We replace $x_0 = 0$ and $u(t) = \delta(t)$ in the output expression (4.4), we obtain the impulse response $g(t)$ of the system of equations (2.17) and (2.18) as follows

$$g(t) = \begin{cases} \mathrm{C}e^{\phi_0 \mathrm{A}\frac{t^\alpha}{\alpha}}\phi_0 \mathrm{B} & \textit{for} \quad t > 0, \\ \mathrm{C}e^{\phi_0 \mathrm{A}\frac{t^\alpha}{\alpha}}\phi_0 \mathrm{B} + \sum_{i=1}^{\mu} \mathrm{C}\phi_{-i}\left(\mathbf{B}\mathbf{T}^{\alpha(i-1)}\delta(t)\right) + \mathrm{D}\delta(t) & \textit{for} \quad t = 0. \end{cases} \tag{4.5}$$

2.2 Internal positivity

Secondly and in this part, we'll discuss the internal positivity of the conformable linear time-invariant system.

Definition 2.5 *[60] The standard implicit dynamical conformable linear time-invariant system of equations (2.17) and (2.18) is called internally positive if and only if for any initial condition $x_0 \in \mathbb{R}_+^{n_1}$ and all admissible inputs $u(t) \in \mathbb{R}_+^{m_1}$, $t \in [0,\infty)$ we have $x(t) \in \mathbb{R}_+^{n_1}$ and $y(t) \in \mathbb{R}_+^{p_1}$, $t \in [0,\infty)$.*

Theorem 2.6 *[60] The standard implicit dynamical conformable linear time-invariant system of equations (2.17) and (2.18) is internally positive if and only if*

$$\mathrm{A} \in \mathscr{M}_{n_1}, \mathrm{B} \in \mathbb{R}_+^{n_1 \times m_1}, \mathrm{C} \in \mathbb{R}_+^{p_1 \times n_1}, \mathrm{D} \in \mathbb{R}_+^{p_1 \times m_1}.$$

Definition 2.7 *The singular conformable linear time-invariant system of equations (2.17) and (2.18) is called (internally) positive if for any consistent initial condition $x_0 \in \mathrm{X}_0 \subset \mathbb{R}_+^{n_1}$ and all admissible inputs $u(t) \in \mathrm{U} \subset \mathbb{R}_+^{m_1}$, $t \in [0,\infty)$ such that $\mathbf{T}^{\alpha(k-1)}u(t) \in \mathbb{R}_+^{m_1}$, $k = 1, ..., \mu$, $t \in [0,\infty)$ we have $x(t) \in \mathbb{R}_+^{n_1}$ and $y(t) \in \mathbb{R}_+^{p_1}$, $t \in [0,\infty)$.*

Definition 2.8 *The singular conformable linear time-invariant system of equations (2.17) and (2.18) is weakly positive if and only if*

$$\mathrm{A} \in \mathscr{M}_{n_1}, \mathrm{E} \in \mathbb{R}_+^{n_1 \times n_1}, \mathrm{B} \in \mathbb{R}_+^{n_1 \times m_1}, \mathrm{C} \in \mathbb{R}_+^{p_1 \times n_1}, \mathrm{D} \in \mathbb{R}_+^{p_1 \times m_1}.$$

Remark 2.9 *Internal positivity implies external positivity but the reverse implication does not hold.*

Now, we consider the non-impulse solution of the singular conformable linear time-invariant system of equations (2.17) and (2.18), since in several application of physically systems the Dirac impulse and its derivatives does not appear because it does not impact on the trajectory just in $t=0$ [61], for this reason we can neglect this part of solution when $t>0$ and assume the consistant initial condition $x_0 \in \mathrm{X}_0$. Accordingly, we will present the following theorem.

Theorem 2.10 *The non-impulse solution of the singular dynamical conformable linear time-invariant system of equations* (2.17) *and* (2.18) *for initial condition $x_0 \in \mathrm{X}_0$ and admissible input $u(t) \in \mathrm{U}$ has the following form*

$$\begin{aligned} x(t) = & e^{\phi_0 \mathrm{A}\frac{t^\alpha}{\alpha}} \phi_0 \mathrm{E} x_0 + \int_0^t e^{\phi_0 \mathrm{A}\left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)} \phi_0 \mathrm{B} u(\tau) d\tau^\alpha \\ & + \sum_{i=1}^{\mu} \phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i-1)} u(t), \end{aligned} \tag{4.6}$$

and the consistant initial conditions are provided by

$$x_0 = \phi_0 \mathrm{E} c + \sum_{i=1}^{\mu} \phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i-1)} u(0), \tag{4.7}$$

where, $c \in \mathbb{R}^{n_1}$ is an arbitrary vector and ϕ_k, $k=-\mu,\cdots,-1$ are the fundamental matrices.

Proof. Let's derive the solution (4.6) and pre-multiplying by E and using the exponential expression, we obtain

$$\begin{aligned} \mathrm{E}\mathbf{T}^\alpha x(t) = & \sum_{i=1}^{\infty} \left(\mathrm{E}\phi_i \frac{t^{\alpha(i-1)}}{\alpha^{(i-1)}(i-1)!} \mathrm{E} x_0 + \int_0^t \mathrm{E}\phi_i \frac{(t^\alpha - \tau^\alpha)^{i-1}}{\alpha^{(i-1)}(i-1)!} \mathrm{B} u(\tau) d\tau^\alpha \right) \\ & + \mathrm{E}\phi_0 \mathrm{B} u(t) + \sum_{i=1}^{\mu} \mathrm{E}\phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i)} u(t), \end{aligned} \tag{4.8}$$

Alternatively, we have

$$\begin{aligned} \mathrm{A} x(t) = & \sum_{i=0}^{\infty} \mathrm{A}\phi_i \left(\frac{t^{\alpha i}}{\alpha^i i!} \mathrm{E} x_0 + \int_0^t \frac{(t^\alpha - \tau^\alpha)^i}{\alpha^i i!} \mathrm{B} u(\tau) d\tau^\alpha \right) \\ & + \sum_{i=1}^{\mu} \mathrm{A}\phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i-1)} u(t). \end{aligned} \tag{4.9}$$

According to the (4.3) we have

$$\phi_i \mathrm{E} - \phi_{i-1} \mathrm{A} = 0 = \mathrm{E}\phi_i - \mathrm{A}\phi_{i-1}, \tag{4.10}$$

where $i \neq 0$, then

$$\mathrm{E}\mathbf{T}^\alpha x(t) - \mathrm{A} x(t) = \mathrm{E}\phi_0 \mathrm{B} u(t) + \sum_{i=1}^{\mu} \mathrm{E}\phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i)} u(t) - \sum_{i=1}^{\mu} \mathrm{A}\phi_{-i} \mathrm{B} \mathbf{T}^{\alpha(i-1)} u(t) = \mathrm{B} u(t), \tag{4.11}$$

as $E\phi_{-\mu}=0$ and $E\phi_0 - A\phi_{-1} = I_n$, then, the non impulse solution (4.6) verifies the equation (2.18). ■

Theorem 2.11 *The singular dynamical conformable linear time-invariant system of equations* (2.17) *and* (2.18) *is (internally) positive if*

$$\begin{gathered}\phi_0 A \in M_{n_1}, \phi_0 E \in \mathbb{R}_+^{n_1\times n_1}, \phi_i B \in \mathbb{R}_+^{n_1\times m_1},\ i=-\mu,\dots,0,\\ C\in\mathbb{R}_+^{p_1\times n_1},\ D\in\mathbb{R}_+^{p_1\times m_1},\end{gathered} \tag{4.12}$$

where ϕ_i, $i=-\mu,\dots,0$ *are the fundamental matrices given in proposition* (4.3).

Proof. Based on Definition 2.7, we have $x_0\in X_0\subset\mathbb{R}_+^{n_1}$ and $\mathbb{T}^{i\alpha}u(t)\in U\subset\mathbb{R}_+^{m_1}$ for $i=0\dots\mu-1$, $t\in[0,\infty)$ and the lemma (4.10). This implies that, $x(t)\in\mathbb{R}_+^{n_1}$ if $\phi_0A\in M_{n_1}$, $\phi_0E\in\mathbb{R}_+^{n_1\times n_1}$, and $\phi_iB\in\mathbb{R}_+^{n_1\times m_1}$, $i=-\mu,\dots,0$. On the other hand, by substituting (4.6) in (2.18) we obtain

$$y(t)=C\left(e^{\phi_0A\frac{t^\alpha}{\alpha}}\phi_0Ex(0)+\int_0^t e^{\phi_0A\frac{t^\alpha-\tau^\alpha}{\alpha}}\phi_0Bu(\tau)d\tau^\alpha+\sum_{i=1}^{\mu}\phi_{-i}B\mathbf{T}^{\alpha(i-1)}u(t)\right)+Du(t). \tag{4.13}$$

Therefore, $y(t)\in\mathbb{R}_+^{p_1}$ for $t\in[0,\infty)$ if $C\in\mathbb{R}_+^{p_1\times n_1}$ and $D\in\mathbb{R}_+^{p_1\times m_1}$. ■

Example 2.12 *Let's consider the example* (5.2) *and from* (4.12), *it follows that*

$$\begin{gathered}\phi_0A=\begin{bmatrix}-1&0\\0&0\end{bmatrix}\in M_{n_1},\ \phi_0E=\begin{bmatrix}1&0\\0&0\end{bmatrix}\in\mathbb{R}_+^{n_1\times n_1},\\ \phi_0B=\begin{bmatrix}1\\0\end{bmatrix}\in\mathbb{R}_+^{n_1\times m_1},\ \phi_{-1}B=\begin{bmatrix}0\\1\end{bmatrix}\in\mathbb{R}_+^{n_1\times m_1}.\end{gathered} \tag{4.14}$$

Thus the singular dynamical conformable linear time-invariant system of example (5.2) *is positive.*

In this part, we will extend the notions of positivity on the subsystems (3.2) and (3.3). A sufficiently and necessary conditions for positivity of singular dynamical conformable linear time-invariant system are provided by the following theorem.

Theorem 2.13 *Consider the decomposition* (3.1) *for a monomial matrix* $Q\in\mathbb{R}_+^{n_1\times n_1}$, *then, singular dynamical conformable linear time-invariant system of order* α *is positive if and only if*

$$A_1\in\mathscr{M}_{\bar n_1},\ B_1\in\mathbb{R}_+^{\bar n_1\times m_1},\ C_1\in\mathbb{R}_+^{p_1\times\bar n_1},\ C_2\in\mathbb{R}_+^{p_1\times\bar n_2},$$

$$D\in\mathbb{R}_+^{p_1\times m_1},\ -N^iB_2\in\mathbb{R}_+^{\bar n_2\times m_1},\ i=0,1,\dots,\mu-1.$$

Proof. Based on lemma (4.10) , if $A_1\in\mathscr{M}_{\bar n_1}$ and $B_1\in\mathbb{R}_+^{\bar n_1\times\bar n_1}$, we have $x(t)\in\mathbb{R}_+^{n_1}$ if $\bar x(t)\in\mathbb{R}_+^{\bar n_1}$ and as from the theorem (4.7) $\bar x_{10}=Q^{-1}x_{10}\in\mathbb{R}_+^{\bar n_1}$ and the definition (2.5) $u(t)\in\mathbb{R}_+^{m_1}$. On the other hand for the second equation, if $-N^iB_2\in\mathbb{R}_+^{\bar n_2\times m}$, $i=0,1,\dots,\mu-1$, we find that

$\bar{x}_2 \in \mathbb{R}_+^{\bar{n}_2}$, since from definition (2.7) $\mathbf{T}^{i\alpha}u(t)$, $i = 0, 1, ..., \mu - 1$, therefore $x(t) \in \mathbb{R}_+^{n_1}$. For $\mathrm{C} \in \mathbb{R}_+^{p_1\times n_1}$ and $\mathrm{D} \in \mathbb{R}_+^{p_1\times n_1}$ we obtain $y(t) \in \mathbb{R}_+^{p_1}$ since from $x(t) \in \mathbb{R}_+^{n_1}$ and $u(t) \in \mathbb{R}_+^{m_1}$. ■

Example 2.14 *We will consider the example* (2.4) *of the preceding section*

$$\mathrm{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \mathrm{B}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \mathrm{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{B}_2 = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}, \tag{4.15}$$

and

$$-\mathrm{B}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, -\mathrm{NB}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{4.16}$$

given that the matrix Q *in* (3.12) *is monomial, for positive initial condition* $\bar{x}_{10}$ *and* $x_{20} \in \mathbb{R}_+^{2\times 1}$ *and positive input* $u(t) \in \mathbb{R}_+^{2\times 1}$*, we have*

$$\mathrm{A}_1 \in \mathcal{M}_2, \mathrm{B}_1 \in \mathbb{R}_+^{2\times 2}, -\mathrm{B}_2 \in \mathbb{R}_+^{2\times 2} \textit{ and } -\mathrm{NB}_2 \in \mathbb{R}_+^{2\times 2}.$$

Therefore, the singular dynamical conformable linear time-invariant system is internally positive since both the slow and fast subsystem (3.16) *and* (3.18) *are positive.*

3 Stability of positive singular dynamical conformable linear time-invariant system

In this section, the stability of positive singular dynamical conformable linear time-invariant system will be investigated.

Consider the positive singular dynamical conformable linear time-invariant system of order α without control i.e $(u(t) = 0)$. Notice that $\bar{x}_2(t) = 0$ and the stability of the positive conformable system of equation (2.17) only depends on the stability of the conformable slow subsystem (3.2) represented by the following equation

$$\mathbf{T}^\alpha \bar{x}_1(t) = \mathrm{A}_1\bar{x}_1(t), \quad \bar{x}_1 \in \mathbb{R}_+^{\bar{n}_1}, \mathrm{A}_1 \in \mathcal{M}_{\bar{n}_1}. \tag{4.17}$$

Definition 3.1 *[60] The positive singular dynamical conformable linear time-invariant system of order* α *is called asymptotically stable if*

$$\lim_{t\to+\infty} \bar{x}_1(t) = 0 \textit{ for any } \bar{x}_{10} \in \mathbb{R}_+^{\bar{n}_1} \textit{ and } u(t) = 0. \tag{4.18}$$

The stability requirement of the positive standard conformable system given in Kaczoreck (2018) [60] is represented by the following theorems.

Theorem 3.2 *The positive singular dynamical conformable linear time-invariant system of order* α *is asymptotically stable if and only if one of the following equivalent conditions is verified.*

1. *There exists a vector* $v^T = [v_1 \cdots v_{\bar{n}_1}]$, $v_k > 0$, $k = 1, ..., \bar{n}_1$ *that is strictly positive such that*
$$A_1 v < 0. \tag{4.19}$$

2. *The coefficients of the following characteristic polynomial of the matrix* A_1
$$det[I_{\bar{n}_1} s - A_1] = s^{\bar{n}_1} + a_{\bar{n}_1 - 1} s^{\bar{n}_1 - 1} + \cdots + a_1 s + a_0, \tag{4.20}$$
are positive, i.e., $a_k > 0$ *for* $k = 0, 1, \cdots, \bar{n}_1 - 1$.

3. *The principal minors of the matrix*
$$\bar{A}_1 = -A_1 = \begin{bmatrix} \bar{a}_{11} & \cdots & \bar{a}_{1\bar{n}_1} \\ \vdots & \ddots & \vdots \\ \bar{a}_{n_1 1} & \cdots & \bar{a}_{\bar{n}_1 \bar{n}_1} \end{bmatrix}, \tag{4.21}$$
are positives, i.e.,
$$\bar{a}_{11} > 0, \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} > 0, \cdots, \det[-A_1] > 0.$$

Example 3.3 *For the same example above, the positive system* (4.17) *is asymptotically stable since the Metzler matrix* A_1 *is stable with eigenvalues* $s_1 = -1$ *and* $s_2 = -2$.

1. *There exists a strictly positive vector* $v^T = [2 \quad 1] > 0$ *such that*
$$A_1 v = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} < 0. \tag{4.22}$$

2. *The coefficients of the characteristic polynomial of the matrix A*
$$det[I_{\bar{n}_1} s - A_1] = s^2 + 3s + 2, \tag{4.23}$$
are positive, i.e., $a_k > 0$ *for* $k = 0, 1$.

3. *The principal minors of the matrix*
$$\bar{A}_1 = -A_1 = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}, \tag{4.24}$$

are positives, i.e.,

$$\bar{a}_{11} = 1 > 0, \det[-A_1] = 2 > 0.$$

4 Superstability of positive singular dynamical conformable linear time-invariant system

This section stands for the superstability of positive singular dynamical conformable linear time-invariant system.

As in last section, the super-stability of positive singular dynamical conformable linear time-invariant system of order α depends only on the superstability of the slow subsystem described by the following equation

$$\mathbf{T}^{\alpha}\bar{x}_1(t) = A_1\bar{x}_1(t), \quad \bar{x}_1 \in \mathbb{R}_+^{\bar{n}_1}, A_1 \in \mathcal{M}_{\bar{n}_1}. \tag{4.25}$$

Based on [61] we will obtain the following results

Definition 4.1 *[61] Let $\bar{x}_1 \in \mathbb{R}_+^{\bar{n}_1}$, the ∞-norm of a positive vector $\bar{x}_1$ has the following form*

$$\|\bar{x}_1\|_\infty = \max_{1\le i\le \bar{n}_1} |\bar{x}_{1i}|. \tag{4.26}$$

Definition 4.2 *[61] The 1-norm of a matrix $A_1 = [a_{ij}]$ is given by*

$$\|A_1\|_1 = \max_{1\le i\le \bar{n}_1}\left(\sum_{j=1}^{\bar{n}_1} |a_{ij}|\right). \tag{4.27}$$

Definition 4.3 *[61] The matrix $A_1 \in \mathcal{M}_{\bar{n}_1}$ of the positive singular dynamical conformable linear time-invariant system of order α is called superstable if*

$$\sigma(A_1) = \sigma = \min_{1\le i\le \bar{n}_1}\left(-a_{ii} - \sum_{j=1, j\ne i}^{\bar{n}_1} |a_{ij}|\right) > 0, \tag{4.28}$$

where $\sigma(A_1)$ denotes the degree of superstability of the matrix A_1. If the matrix is superstable, it must also be stable, but the converse is not true.

Lemma 4.4 *If A_1 is a superstable matrix, we have*

$$\|e^{A_1\frac{t^\alpha}{\alpha}}\|_1 \le e^{-\sigma\frac{t^\alpha}{\alpha}}. \tag{4.29}$$

Theorem 4.5 *If the conformable system is superstable, then*

$$\|x_1(t)\|_\infty \le \|x_0\|_\infty e^{-\sigma\frac{t^\alpha}{\alpha}}, \quad t \ge 0. \tag{4.30}$$

Although the superstability guarantees a monotonic decline in the state vector's norm, some state variables may oscillate. The key distinction is that the equation (4.30) for asymptotic stable systems is replaced by

$$\|x_1(t)\|_\infty \leq b(\mathrm{A}_1, v_1)\|x_0\|_\infty e^{-v_1 \frac{t^\alpha}{\alpha}}, \quad 0 < v_1 < \min_{1\leq i\leq \bar{n}_1}\{-\mathrm{Re}s_i\},\ t \geq 0, \tag{4.31}$$

where the initial state vector of the trajectory allows the constant $b(\mathrm{A}_1, v_1)$ to take on significant values. Such undesired "peaks" do not exist in superstable systems [61].

Example 4.6 *Let us consider the following positive conformable system*

$$\mathbb{T}^\alpha \bar{x}_1(t) = \mathrm{A}_1\bar{x}_1(t), \quad 0 < \alpha \leq 1, \tag{4.32}$$

where $\mathrm{A}_1 = \begin{bmatrix} -6 & 3 \\ 0 & -4 \end{bmatrix} \in \mathcal{M}_2$ *and* $\bar{x}_1 \in \mathbb{R}^2_+$.

The eigenvalues of the Metzler matrix A_1 *are* $s_1 = -6$ *and* $s_2 = -4$, *then the system (4.32) is asymptotically stable. We have* $\sigma = 3 > 0$ *and the system is also superstable.*
The solution $\bar{x}_{11}$ *and* $\bar{x}_{12}$ *and the norm of the solution* $\|\bar{x}_1\|$ *of the positive stable conformable system (4.32) for initial conditions* $\bar{x}_{10} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$ *is represented in the following figures*

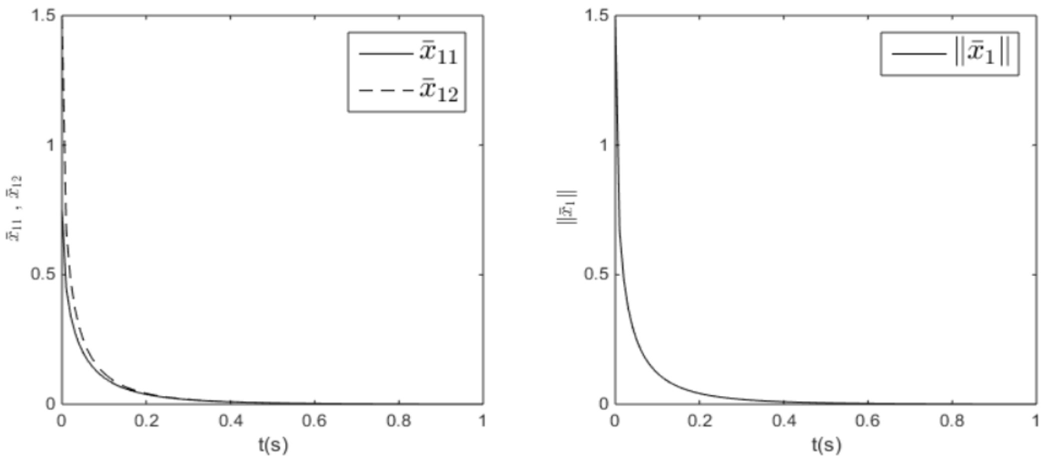

Figure 4.1: The solutions and norm of state vector of the positive stable conformable system (4.32) $\bar{x}_{11}$ and $\bar{x}_{12}$ for $\alpha = 0.5$.

Example 4.7 *Now we consider the following positive conformable system*

$$\mathbb{T}^\alpha \bar{x}_1(t) = \mathrm{A}_1\bar{x}_1(t), \quad 0 < \alpha \leq 1, \tag{4.33}$$

with $\mathrm{A}_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$, *the eigenvalues of the Metzler matrix* A_1 *are* $s_1 = -6$ *and* $s_2 = -4$, *then the system (4.33) is asymptotically stable. We have* $\sigma = 0$, *then the system is not superstable.*

the following figure represents the solution $\bar{x}_{11}$ and $\bar{x}_{12}$ and the norm of the solution $\|\bar{x}_1\|$ of the positive stable conformable system (4.33) for initial conditions $\bar{x}^1_{10} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

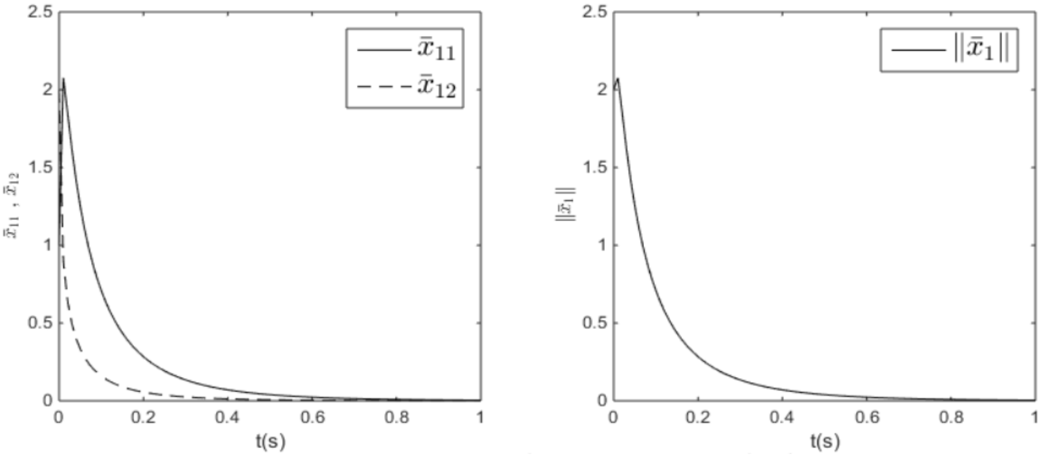

Figure 4.2: The solutions the and norm of state vector of positive stable conformable system (4.33) $\bar{x}_{21}$ and $\bar{x}_{22}$ for $\alpha = 0.5$.

The difference between the two examples appeared in figures (4.1) and (4.2). From figure (4.1) it is clear that the state vector's norm declines monotonically for $t \to +\infty$, however as can be seen in figure (4.2), the state variable $\bar{x}_{10}$ increases highly above the initial condition, and the state vector's norm does not decrement monotonically for $t \to +\infty$.

5 Conclusion

This chapter was devoted to the presentation of several specific definitions of positivity which was extended in singular conformable continuous-time-linear invariant system. We have established a new conditions for the positivity. Firstly, by using a non-impulse solution of this system that has been given and proven and also by using the Weierstrass-Kronecker method. Then, sufficient and necessary conditions of asymptotic stability and super-stability of this positive system have been proposed. A numerical examples are given for approved the results obtained.

Chapter 5

H_∞ Norm of 2D digital filters

1 Introduction

Over the past few decades, bidimensional digital signal processing applications have expanded significantly. The signal is exposed to the unwanted parts, as random noise. For this reason, we need to introduce the concept of filters. Their main objective is to remove undesirable components from a signal. For instance, it is possible to enhance a wideband noise-damaged image without introducing edge blur [75]. The analysis and design of a 2D digital filter can often be greatly simplified when the filter is separable in the denominator [45, 70, 77]. In this case, the analysis and design of the 2D filter reduce to those of 1-D filters; thus, the well-established techniques for 1-D filters (e.g., stability analysis, realization, H_∞ control, etc.) can be applied [94].

The H_∞ norm of a stable transfer function is appeared in [23, 37]. There are different methods for calculating the H_∞-norm for the 1D system [21, 23, 26, 38, 39]. In this chapter we propose a practical algorithm to compute the H_∞ norm of 2D separable recursive causality filters modeled by the Roesser Models as an extension of the work in [22].

2 Transfer function

The dynamic system receives actions (assimilated to commands or controls) and sends back information, thus, it can be defined as a mathematical relationship between its input and output data. In control theory, the transfer function of a system is a widely used concept, it is a model of input/output behavior which is obtained from the linear differential equation with constant coefficient [51].

2.1 Transfer function of unidimensional system (1D)

Consider the dynamical discrete-time linear system represented by the following equations

$$x_{i+1} = Ax_i + Bu_i, \tag{5.1}$$

$$y_i = Cx_i + Du_i, \tag{5.2}$$

where $x_i \in \mathbb{R}^{n_1}$, $u_i \in \mathbb{R}^{m_1}$ and $y_i \in \mathbb{R}^{p_1}$ are, respectively, the state, the control, and the output of the system. $A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times m_1}$, $C \in \mathbb{R}^{p_1 \times n_1}$ and $D \in \mathbb{R}^{p_1 \times m_1}$. Furthermore the system is regular i.e $(\det(zI - A)) \neq 0$. By applying the Z-transform to the discrete system of equations (5.1) and (5.2) we transform the difference equation to the algebraic equation, so in order to find y as a function of u, we have to find Y as a function of U.

$$zX(z) = AX(z) + BU(z), \tag{5.3}$$

$$Y(z) = CX(z) + DU(z). \tag{5.4}$$

We need to eliminate $X(z)$ from the output equation in order to find the relationship between $Y(z)$ and $U(z)$.
First, we resolve the state equation (5.3) since $(\det\left(zI_{n_1} - A\right)) \neq 0$, we get

$$X(z) = \left(zI_{n_1} - A\right)^{-1} BU(z),$$

when the expression for $X(z)$ is substituted into the output equation (5.4), we obtain

$$Y(z) = C\left(\left(zI_{n_1} - A\right)^{-1} B + D\right) U(z).$$

Therefore, the transfer function of discrete system of equations (5.1) and (5.2) is given by

$$\begin{aligned} G(z) &= \frac{Y(z)}{U(z)}, \\ &= C\left(I_{n_1} z - A\right)^{-1} B + D. \end{aligned}$$

2.2 Transfer function of bi-dimensional system (2D)

Consider the 2D Roesser models represented by the following state space equations

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \mathrm{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{B}u(i,j), \tag{5.5}$$

$$y(i,j) = \mathrm{C} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{D}u(i,j), \tag{5.6}$$

where

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix}, \mathrm{B} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C}_2 \end{bmatrix},$$

where $x^h(i,j) \in \mathbb{R}^{n_1}$ and $x^v(i,j) \in \mathbb{R}^{n_2}$ are the horizontal and vertical state vectors at $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $u(i,j) \in \mathbb{R}^m$ and $y(i,j) \in \mathbb{R}^p$ are the input and the output vectors, respectively and $\mathrm{A}_{i,j} \in \mathbb{R}^{n_i \times n_j}$ $i,j = 1,2$, $\mathrm{B}_i \in \mathbb{R}^{n_i \times m}$ $i = 1,2$, $\mathrm{C}_i \in \mathbb{R}^{p \times n_i}$, $\mathrm{D} \in \mathbb{R}^{p \times m}$.
We suppose that the system of equations (5.5) and (5.6) is regular and applying the bi-dimensional Z-transform on this system, we obtain

$$z_1 \mathrm{X}^h(z_1,z_2) = \mathrm{A}_{11}\mathrm{X}^h(z_1,z_2) + \mathrm{A}_{12}\mathrm{X}^v(z_1,z_2) + \mathrm{B}_1\mathrm{U}(z_1,z_2), \tag{5.7}$$

$$z_2 \mathrm{X}^v(z_1,z_2) = \mathrm{A}_{21}\mathrm{X}^h(z_1,z_2) + \mathrm{A}_{22}\mathrm{X}^v(z_1,z_2) + \mathrm{B}_2\mathrm{U}(z_1,z_2), \tag{5.8}$$

$$\mathrm{Y}(z_1,z_2) = \mathrm{C}_1\mathrm{X}^h(z_1,z_2) + \mathrm{C}_2\mathrm{X}^v(z_1,z_2) + \mathrm{DU}(z), \tag{5.9}$$

which implies that

$$\left(z_1 I_{n_1} - \mathrm{A}_{11}\right)\mathrm{X}^h(z_1,z_2) - \mathrm{A}_{12}\mathrm{X}^v(z_1,z_2) = \mathrm{B}_1\mathrm{U}(z_1,z_2), \tag{5.10}$$

$$\left(z_2 I_{n_2} - \mathrm{A}_{22}\right)\mathrm{X}^v(z_1,z_2) - \mathrm{A}_{21}\mathrm{X}^h(z_1,z_2) = \mathrm{B}_2\mathrm{U}(z_1,z_2), \tag{5.11}$$

and

$$\mathrm{Y}(z_1,z_2) = \mathrm{C}_1\mathrm{X}^h(z_1,z_2) + \mathrm{C}_2\mathrm{X}^v(z_1,z_2) + \mathrm{DU}(z_1,z_2),$$

we obtain

$$\begin{bmatrix} z_1 I_{n_1} - \mathrm{A}_{11} & -\mathrm{A}_{12} \\ -\mathrm{A}_{21} & z_2 I_{n_2} - \mathrm{A}_{22} \end{bmatrix} \begin{bmatrix} \mathrm{X}^h(z_1,z_2) \\ \mathrm{X}^v(z_1,z_2) \end{bmatrix} = \mathrm{BU}(z_1,z_2), \tag{5.12}$$

$$\tag{5.13}$$

and

$$\mathrm{Y}(z_1,z_2) = \mathrm{C} \begin{bmatrix} \mathrm{X}^h(z_1,z_2) \\ \mathrm{X}^v(z_1,z_2) \end{bmatrix} + \mathrm{DU}(z_1,z_2). \tag{5.14}$$

In the beginning, we solve the state equation (5.12) since the system of equations (5.5) and (5.6) is regular, then

$$\begin{bmatrix} X^h(z_1,z_2) \\ X^v(z_1,z_2) \end{bmatrix} = \begin{bmatrix} z_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & z_2 I_{n_2} - A_{22} \end{bmatrix}^{-1} BU(z_1,z_2), \tag{5.15}$$

by substituting the expression of the state equation (5.15) in the output expression (5.14), we get

$$Y(z_1,z_2) = C \begin{bmatrix} z_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & z_2 I_{n_2} - A_{22} \end{bmatrix}^{-1} BU(z_1,z_2) + DU(z_1,z_2), \tag{5.16}$$

and

$$Y(z_1,z_2) = \left(C \begin{bmatrix} z_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & z_2 I_{n_2} - A_{22} \end{bmatrix}^{-1} B + D \right) U(z_1,z_2). \tag{5.17}$$

As a result, the transfer function of discrete system of equations (5.5) and (5.6) is provided by

$$\begin{aligned} G(z_1,z_2) &= \frac{Y(z_1,z_2)}{U(z_1,z_2)}, \\ &= C \begin{bmatrix} z_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & z_2 I_{n_2} - A_{22} \end{bmatrix}^{-1} B + D. \end{aligned}$$

3 2D Roesser causal recursive separable denominator models (CRSD)

2D filters can be divided into two classes non-recursive filters (finite impulse response (FIR)) and recursive filters (infinite impulse response (IIR)). For the recursive processing, the output is a function of all previous outputs and the current and previous inputs [45] and [75]. This filters can be represented in the state space model, as with digital 1D filters. To describe the behavior of filters, a combination of internal signals identified as the variables of state, the main advantage is that digital filters can be described in terms of matrices, which makes it easy to manipulate. A number of authors have presented models for 2D digital filtering in state space model, including Attasi [6], Roesser [88] and Fornasini-Marchesini [36] and others. In this section we consider the 2D Roesser models described by the following state space model

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \mathrm{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{B}u(i,j), \tag{5.18}$$

$$y(i,j) = \mathrm{C} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{D}u(i,j), \tag{5.19}$$

where

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 \\ \mathrm{A}_3 & \mathrm{A}_4 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C}_2 \end{bmatrix},$$

and $x^h(i,j) \in \mathbb{R}^m$, $x^v(i,j) \in \mathbb{R}^m$ are the horizontal and vertical state vectors, respectively, $u(i,j) \in \mathbb{R}^p$ is the input vector, $y(i,j) \in \mathbb{R}^q$ is the output vector and the matrices A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, and D are real matrices of compatible size.

Definition 3.1 *[45] A causal system is a system whose output signal depends only on the internal values (past or present).*

Example 3.2 *The system of equations (5.18) and (5.19) is causal since the output in the equation (5.19) depends only on the present value* $\left(x^h(i,j) \text{ and } x^v(i,j)\right)$.

Theorem 3.3 *[94] The 2D filter represented by equations (5.18) and (5.19) is minimally separable if and only if one of the following two sets of conditions holds*

1. $\mathrm{A}_3 = 0$ *and*
$$rank \begin{bmatrix} -\mathrm{A}_2 & \mathrm{B}_1 \\ -\mathrm{C}_2 & \mathrm{D} \end{bmatrix} = p;$$

2. $\mathrm{A}_2 = 0$ *and*
$$rank \begin{bmatrix} -\mathrm{A}_3 & \mathrm{B}_2 \\ -\mathrm{C}_1 & \mathrm{D} \end{bmatrix} = p.$$

Theorem 3.4 *[94] The transfer matrix of the system of equations (5.18) and (5.19) is given by*

$$\mathrm{G}(z_1,z_2) = \begin{bmatrix} \mathrm{C}_1 & \mathrm{C}_2 \end{bmatrix} \begin{bmatrix} I_1 z_1 - \mathrm{A}_1 & -\mathrm{A}_2 \\ -\mathrm{A}_3 & I_4 z_2 - \mathrm{A}_4 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} + \mathrm{D}. \tag{5.20}$$

If the condition (1) is verified, our system can be written as

$$\mathrm{G}(z_1,z_2) = \mathrm{G}_1(z_1)\mathrm{G}_2(z_2), \tag{5.21}$$

such that

$$\mathrm{G}_1(z_1) = \mathrm{C}_1(z_1 I_1 - \mathrm{A}_1)^{-1}\tilde{\mathrm{B}}_1 + \tilde{\mathrm{D}}_1, \tag{5.22}$$

$$\mathrm{G}_2(z_2) = \tilde{\mathrm{C}}_2(z_2 I_4 - \mathrm{A}_4)^{-1}\mathrm{B}_2 + \tilde{\mathrm{D}}_2, \tag{5.23}$$

with the following matrix factorization

$$\begin{bmatrix} A_2 & B_1 \\ C_2 & D \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_1 \end{bmatrix} \begin{bmatrix} \tilde{C}_2 & \tilde{D}_2 \end{bmatrix}. \tag{5.24}$$

If the condition (2) *is verified, the system of equations* (5.18) *and* (5.19) *can be written as*

$$G(z_1, z_2) = G_2(z_2)G_1(z_1), \tag{5.25}$$

where

$$G_1(z_1) = \tilde{C}_1(z_1 I_1 - A_1)^{-1}B_1 + \tilde{D}_1, \tag{5.26}$$

$$G_2(z_2) = C_2(z_2 I_4 - A_4)^{-1}\tilde{B}_2 + \tilde{D}_2, \tag{5.27}$$

with the following matrix factorization

$$\begin{bmatrix} A_3 & B_2 \\ C_1 & D \end{bmatrix} = \begin{bmatrix} \tilde{B}_2 \\ \tilde{D}_2 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_1 \end{bmatrix}. \tag{5.28}$$

4 H_∞ norm

This section focuses on the computation of H_∞ norm 2D Roesser CRSD models based on a simple process of the singular values of a stable rational transfer function matrix and the parahermitian matrix function of 1-D filters.
The H_∞-norm of a rational transfer function $G(z)$, $\gamma^* = \|G\|_{\mathcal{H}_\infty}$ is bounded if and only if it is stable [28].

Definition 4.1 *[39] Let $\phi : \mathbb{C} \longrightarrow \mathbb{C}^{n\times n}$ be a matrix function maps the complex variable s to a complex matrix $\phi(z)$. We define the para-conjugate transpose of this function with respect to a particular curve in the complex plane by*

$$\phi_*(z) := \phi^*\left(\frac{1}{z}\right), \quad \Gamma = e^{j\mathbb{R}},$$

where $\phi^*(.)$ is the para conjugate transpose of $\phi(.)$, it is called parahermitian matrix function, if

$$\phi_*(z) = \phi(z).$$

Definition 4.2 *[32] The H_∞ norm of a stable rational transfer function matrix $G(z_1, z_2)$ is equal to the maximum of the largest singular value of the transfer function $G(e^{j\omega_1}, e^{j\omega_2})$ evaluated on the unit circle $e^{j\omega_1}, e^{j\omega_2}$*

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega_1,\omega_2\in[-\pi,\pi]} \sigma_{max} G(e^{j\omega_1}, e^{j\omega_2}), \tag{5.29}$$

where

$$\sigma_{max} G(e^{j\omega_1}, e^{j\omega_2}) := \max_i \sqrt{\lambda i \left(G(e^{j\omega_1}, e^{j\omega_2})[G(e^{j\omega_1}, e^{j\omega_2})]^* \right)}. \tag{5.30}$$

Remark 4.3 *Note that the optimum ω_1^* and ω_2^* of the H_∞ norm of a stable rational transfer function matrix $G_1(z_1)$ and $G_2(z_2)$ are in the neighborhood of the points which verifies the maximum of the largest singular value of the transfer function $G(z_1, z_2)$ evaluated on the unit circle $e^{j\omega_1}$, $e^{j\omega_2}$ and $\sigma_{max} G(e^{j\omega_1}, e^{j\omega_2}) \le \sigma_{max} G_1\left(e^{j\omega_1}\right) \sigma_{max} G_2\left(e^{j\omega_2}\right)$.*

For $\gamma > 0$, we define the para-hermitian matrix function of transfer function $G_i(e^{j\omega_i})$ as

$$\phi_{G_i}(\gamma_i, e^{j\omega_i}) := \gamma_i^2 I - G_i\left(e^{j\omega_i}\right)\left[G_i\left(e^{j\omega_i}\right)\right]^*, \quad i = 1, 2, \tag{5.31}$$

which is hermitian for every point $z_i = e^{j\omega_i}$ since $G_i^*(e^{-j\omega_i}) = \left[G_i\left(e^{j\omega_i}\right)\right]^*$, then, we deduce that

$$\|G_i\|_{\mathcal{H}_\infty} := \inf_{\gamma_i \in \mathbb{R}} \left\{ \phi_{G_i}(\gamma_i, e^{j\omega_i}) \succ 0, \ \forall \omega_i \in [-\pi, \pi] \right\}. \tag{5.32}$$

For consequent, if the para-hermitian matrix $\phi_{G_i}(\gamma_i, e^{j\omega_i}) \succ 0$ for all $\omega_i \in [-\pi, \pi]$, then, $\gamma_i > \sigma_{max} G_i(e^{j\omega_i})$ for all $\omega_i \in [-\pi, \pi]$. The resolution of this problem is equivalent to solving a parahermitian generalized eigenvalue problem [39].

Based on [22], we assume that the given quadruple $\{A_1, B_1, \tilde{C}_1, \tilde{D}_1\}$ and $\{A_4, \tilde{B}_2, C_2, \tilde{D}_2\}$ are realization of a stable transfer matrices functions $G_1(z_1)$ and $G_2(z_2)$ respectively, such that

$$G_1(z_1) = \tilde{C}_1 (I_1 z_1 - A_1)^{-1} B_1 + \tilde{D}_1, \quad G_2(z_2) = C_2 (I_4 z_2 - A_4)^{-1} \tilde{B}_2 + \tilde{D}_2, \tag{5.33}$$

the transfer matrices functions (5.33) are the Schur complement of

$$S_{G_1}(z_1) = \left[\begin{array}{c|c} A_1 - I_1 z_1 & B_1 \\ \hline \tilde{C}_1 & \tilde{D}_1 \end{array}\right], \quad S_{G_2}(z_2) = \left[\begin{array}{c|c} A_4 - I_4 z_2 & \tilde{B}_2 \\ \hline C_2 & \tilde{D}_2 \end{array}\right], \tag{5.34}$$

and the so-called para-conjugate transfer matrices functions of (5.33) are

$$G_1^*(z_1) = z_1 B_1^T \left(I_1 - z_1 A_1^T\right)^{-1} \tilde{C}_1^T + \tilde{D}_1^T, \quad G_2^*(z_2) = z_2 \tilde{B}_2^T \left(I_4 - z_2 A_4^T\right)^{-1} C_2^T + \tilde{D}_2^T, \tag{5.35}$$

which are also the Schur complement of the corresponding system matrices (5.35) is

$$S_{G_1^*}(z_1) = \left[\begin{array}{c|c} z_1 A_1^T - I_1 & \tilde{C}_1^T \\ \hline z_1 B_1^T & \tilde{D}_1^T \end{array}\right], \quad S_{G_2^*}(z_2) = \left[\begin{array}{c|c} z_2 A_4^T - I_4 & C_2^T \\ \hline z_2 \tilde{B}_2^T & \tilde{D}_2^T \end{array}\right]. \tag{5.36}$$

$\phi_1(\gamma_1, e^{j\omega_1})$ is the Schur complement of the following matrix function

$$S_{\phi_1}(\gamma_1, e^{j\omega_1}) = \left[\begin{array}{cc|c} 0 & A_1 - I_1 e^{j\omega_1} & B_1 \\ e^{j\omega_1} A_1^T - I_1 & -\tilde{C}_1^T \tilde{C}_1 & -\tilde{C}_1^T \tilde{D}_1 \\ \hline e^{j\omega_1} B_1^T & -\tilde{D}_1^T \tilde{C}_1 & \gamma_1^2 I - \tilde{D}_1^T \tilde{D}_1 \end{array}\right], \tag{5.37}$$

we can represent (5.37) as a pencil of the form

$$A_h^1 - e^{j\omega_1} F_h^1 = \left[\begin{array}{cc|c} 0 & A_1 & B_1 \\ -I_1 & -\tilde{C}_1^T \tilde{C}_1 & -\tilde{C}_1^T \tilde{D}_1 \\ \hline 0 & -\tilde{D}_1^T \tilde{C}_1 & \gamma_1^2 I - \tilde{D}_1^T \tilde{D}_1 \end{array}\right] - e^{j\omega_1} \left[\begin{array}{cc|c} 0 & I_1 & 0 \\ -A_1^T & 0 & 0 \\ \hline -B_1^T & -0 & 0 \end{array}\right]. \tag{5.38}$$

$\phi_2(\gamma_2, e^{j\omega_2})$ is the Schur complement of the following matrix function

$$S_{\phi_2}(\gamma_2, e^{j\omega_2}) = \left[\begin{array}{cc|c} 0 & A_4 - I_4 e^{j\omega_2} & \tilde{B}_2 \\ e^{j\omega_2} A_4^T - I_4 & -C_2^T C_2 & -C_2^T \tilde{D}_2 2 \\ \hline e^{j\omega_2} \tilde{B}_2^T & -\tilde{D}_2^T C_2 & \gamma_2^2 I - \tilde{D}_2^T \tilde{D}_2 \end{array}\right], \tag{5.39}$$

which are represented by the following pencil

$$A_h^2 - e^{j\omega_2} F_h^2 = \left[\begin{array}{cc|c} 0 & A_4 & \tilde{B}_2 \\ -I_4 & -C_2^T C_2 & -C_2^T \tilde{D}_2 \\ \hline 0 & -\tilde{D}_2^T C_2 & \gamma_2^2 I - \tilde{D}_2^T \tilde{D}_2 \end{array}\right] - e^{j\omega_2} \left[\begin{array}{cc|c} 0 & I_4 & 0 \\ -A_4^T & 0 & 0 \\ \hline -\tilde{B}_2^T & -0 & 0 \end{array}\right]. \tag{5.40}$$

For every fixed value γ_i, the pencil (5.38) and (5.40) are Hermitian in the variable ω_1 and ω_2, respectively. That follows, its generalized eigenvalues are real analytical function of the real variable ω_1 and ω_2 [28] and [69]. The zeros of the corresponding pencils located on the $e^{j\omega_i}$ are the point at which $\sigma_{\max} G_i(e^{j\omega_i}) = \gamma_i$, $(i = 1, 2)$. The H_∞ norm can be computed in several different ways [23, 26, 38, 39]. The corresponding methods can be obtained by using the information of the eigenvalues function. Start from a point γ_i^0 which intersects the singular values of $G_i(e^{j\omega_i^0})$, and obtain from this the intervals for which $\sigma_{\max} G_i(e^{j\omega_i}) \geq \gamma_i^0$ (these are called the level sets for γ_i^0) and compute the $\sigma_{\max} G_i(e^{j\omega_i})$ as the piecewise analytic function that is maximal at each frequency ω_i, this method produce a quadratically convergence.

5 Numerical example

In this section, we present an illustrative real example in order to the show the efficiency of our method.

Example 5.1 *Consider the* 2D *Roesser models described by the state space equations*

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \mathrm{A}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{B}u(i,j), \tag{5.41}$$

$$y(i,j) = \mathrm{C}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathrm{D}u(i,j), \tag{5.42}$$

where

$$\mathrm{A} = \begin{bmatrix} 0.1 & 1 & -4 \\ 0 & 0.4 & -0.2 \\ 0 & 1 & 0.1 \end{bmatrix}, \quad \mathrm{C} = \begin{bmatrix} -3 & 2 & -8 \\ 7 & 0 & 1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0.2 & 0.6 \\ 0.7 & -2 \\ -0.5 & 0.1 \end{bmatrix}, \quad \mathrm{D} = \begin{bmatrix} 0.4 & 1.2 \\ 1 & 0 \end{bmatrix}. \tag{5.43}$$

The transfer matrix of the system of equations (5.41) *and* (5.42) *is*

$$\left[\begin{matrix} \dfrac{-32.0146e^{2j\omega_2} - 415.9881e^{j\omega_2} - 352.1996e^{j\omega_1} + 20.0083e^{j\omega_1}e^{2j\omega_2} + 259.9801e^{j\omega_2}e^{j\omega_1} + 563.5254}{-5e^{2j\omega_2} + 2.5e^{j\omega_2} + 12e^{j\omega_1} + 50e^{j\omega_1}e^{2j\omega_2} - 25e^{j\omega_2}e^{j\omega_1} - 1.2} \\ \dfrac{65.0316e^{2j\omega_2} + 914.9749e^{j\omega_2} + 57.0001e^{j\omega_1} + 49.9988e^{j\omega_1}e^{2j\omega_2} - 49.9988e^{j\omega_2}e^{j\omega_1} - 1238.4}{-5e^{2j\omega_2} + 2.5e^{j\omega_2} + 12e^{j\omega_1} + 50e^{j\omega_1}e^{2j\omega_2} - 25e^{j\omega_2}e^{j\omega_1} - 1.2} \end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{-96.0323e^{2j\omega_2} + 431.9905e^{j\omega_2} + 848.4066e^{j\omega_1} + 60.0195e^{j\omega_1}e^{2j\omega_2} - 269.9910e^{j\omega_2}e^{j\omega_1} - 1357.5}{-5e^{2j\omega_2} + 2.5e^{j\omega_2} + 12e^{j\omega_1} + 50e^{j\omega_1}e^{2j\omega_2} - 25e^{j\omega_2}e^{j\omega_1} - 1.2} \\ \dfrac{210.0710e^{2j\omega_2} - 945.4796e^{j\omega_2} - 102.0012e^{j\omega_1} + 4.9981e^{j\omega_2}e^{j\omega_1} + 2979.6}{-5e^{2j\omega_2} + 2.5e^{j\omega_2} + 12e^{j\omega_1} + 50e^{j\omega_1}e^{2j\omega_2} - 25e^{j\omega_2}e^{j\omega_1} - 1.2} \end{matrix}\right],$$

we have $\mathrm{A}_3 = 0_{\mathbb{R}^2}$ *and*

$$rank\begin{bmatrix} -\mathrm{A}_2 & \mathrm{B}_1 \\ -\mathrm{C}_2 & \mathrm{D} \end{bmatrix} = rank\begin{bmatrix} -1 & 4 & 0.2 & 0.6 \\ -2 & 8 & 0.4 & 1.2 \\ 0 & -1 & 1 & 0 \end{bmatrix} = 2. \tag{5.44}$$

Therefore, the corresponding system is minimally separable, from theorem (3.4) *there exist two 1-D rational matrices* G_1 *and* G_2 *as*

$$\mathrm{G}_1(e^{j\omega_1}) = \begin{bmatrix} \dfrac{-8.3421e^{j\omega_1} + 13.3475}{e^{j\omega_1} - 0.1} & \dfrac{0.0948e^{j\omega_1} - 0.1517}{e^{j\omega_1} - 0.1} \\ \dfrac{0.9232e^{j\omega_1} - 29.29}{e^{j\omega_1} - 0.1} & \dfrac{1.0713e^{j\omega_1} + 0.2247}{e^{j\omega_1} - 0.1} \end{bmatrix},$$

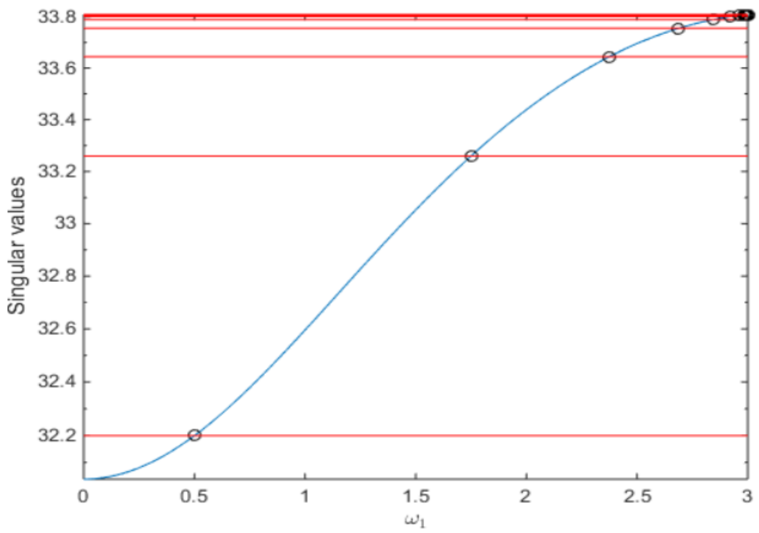

Figure 5.1: Maximum of singular values of transfer function G_1 with points ω_1 and levels γ_1.

The maximizing frequency is $\omega_1^* = 2.9999$ *and we obtain* $\gamma_1 = 33.8091$*, and*

$$G_2(e^{j\omega_2}) = \begin{bmatrix} \dfrac{42.4089 - 31.389e^{j\omega_2} - 1.85e^{2j\omega_2}}{50e^{2j\omega_2} - 25e^{j\omega_2} + 12} & \dfrac{-7.125e^{2j\omega_2} + 32.1035e^{j\omega_2} - 101.787}{50e^{2j\omega_2} - 25e^{j\omega_2} + 12} \\ \dfrac{48.265e^{2j\omega_2} - 19.6215e^{j\omega_2} + 16.6605}{50e^{2j\omega_2} - 25e^{j\omega_2} + 12} & \dfrac{6.14e^{2j\omega_2} - 23e^{j\omega_2} - 7.4970}{50e^{2j\omega_2} - 25e^{j\omega_2} + 12} \end{bmatrix},$$

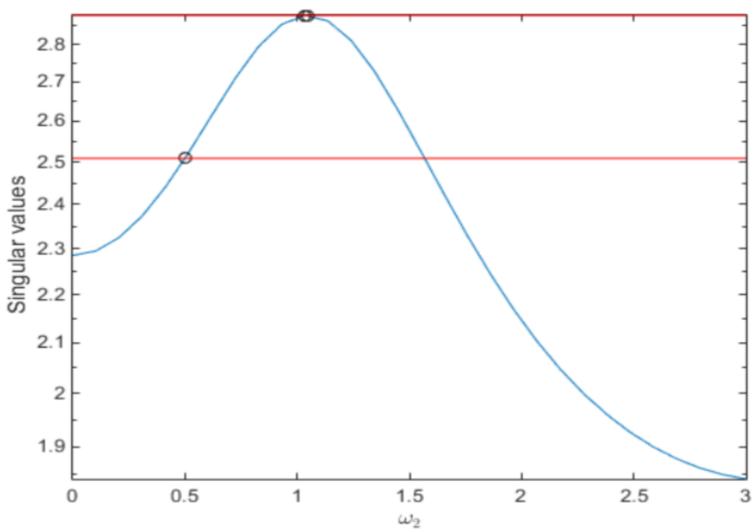

Figure 5.2: Maximum of singular values of transfer function G_2 with points ω_2 and levels γ_2.

The maximizing frequency is $\omega_2^* = 1.0467$ *and we obtain* $\gamma_2 = 2.8796$.

As consequent, from corollary 3.4 *the* H_∞ *norm of the system of equations* (2.2) *and* (2.3) *is* $\gamma^* = 95.1394$ *for the optimum frequency* $\omega_1 = 2.8788$ *and* $\omega_2 = 1.0909$ *in the neighbourhood of* ω_1^* *and* ω_2^* *such that* $\|G(\omega_1^*, \omega_2^*)\|_{H_\infty} = 94.9813$ *and* $\gamma \leq \gamma_1\gamma_2 = 97.3575$.

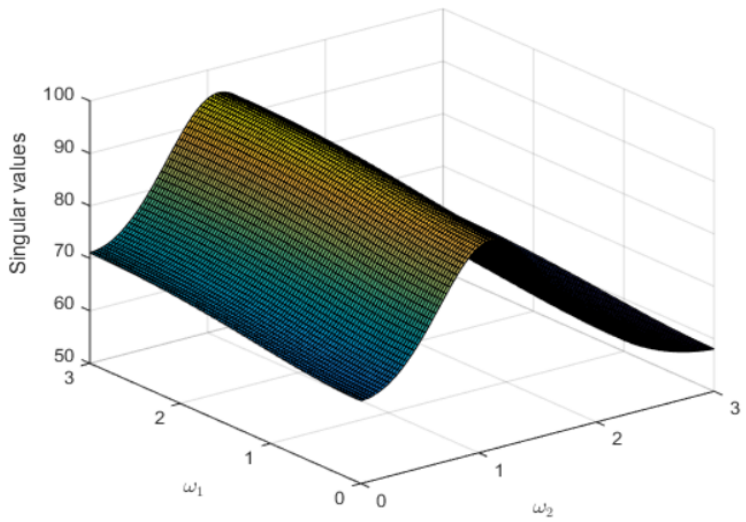

Figure 5.3: Maximum of singular values of transfer function G.

In the first step, we start from a level γ_i^0 and find the intervals $I_{i1}......I_{il}$ by computing the real zeros of (5.38) and (5.40) corresponding of this level (γ_i^0). Secondly, using midpoints (ω_i^{k+1}) of the previous intervals, we obtain the next level $\gamma_i^{k+1} = \sigma_{max} G(e^{j\omega_i^{k+1}})$, $i = 1, 2$. We took into account the algorithm in [23] with tolerance 10^{-5}. The method converge in 15 step for G_1 and 3 step for G_2 as showing in figure (5.1) and (5.2). Finally, the maximum singular values of the system of equations (2.2) and (2.3) is bounded $\sigma_{max} G(e^{j\omega_1^*}, e^{j\omega_2^*}) \leq \gamma_1^* \gamma_2^*$.

6 Conclusion

This chapter presents an accurate and economical algorithm for solving a two-dimensional (2D) digital recursive filters, based on the computation of level sets of the maximum singular value of the transfer function. The findings of this study can be used for several practical image processing applications, including image ...etc, due to their minimization of cost. An illustrative example is introduced to prove the accuracy of the proposed approach.

Conclusion

The methodologies developed in this thesis are dedicated to the analysis and synthesis of control laws for linear systems in the one and two dimension forms where fractional-order singular models and also models with conformable derivatives are considered. Their institutions appeal exclusively to the Sumudu transform method, the parahermitian matrix function and to the midpoint approach. This work belongs to one of the axes of the control theory of linear complex systems, the complexity being in the non-integer order of the derivation of the differential equations describing a class of singular models. The results reported in this dissertation can be viewed as extensions of some existing results in the literature of linear singular systems to their homologous of fractional-order and exclusively to a new class of two dimensional models. The study we have conducted is organized in two parts: In the first part, we focus on the solvability of a new class of singular system with conformable derivative by the use of the Sumudu transform, and also, the case of standard conformable system has been treated; subsequently, we have conducted a comparative analysis between the different resolution of fractional linear dynamical systems, since in the last few years, different phenomena in various domains have been modeled by fractional dynamical systems.
The conformable derivative is useful for modeling many physical problems, note that the differential equations with conformable fractional derivatives are simpler to solve computationally compared to those using Caputo or Riemann-Liouville fractional derivatives. Various applications of the conformable fractional derivative have investigated in the literature in many areas.
The Laplace transform is an extremely efficient mathematical method used in many fields of research and engineering. The Sumudu transform, although less known than the Laplace transform, has several interesting properties compared to other integral transforms, in particular the "unity" aspect, which can be useful for solving differential equations. The novelty of our contributions is to use this approache to compute the solution of these new fractional models. We have then established the analysis of these new systems, extending the main results concerning the class of singular and fractional one-dimensional systems such as controllability, observability, positivity, stability and superstability. The second part relates to the computation of the H infinity norm for a class of two-dimensional stan-

dard Roesser type model. We have extracted important results on the computation of the H infinity norm based on the work of BOUAGADA et al. Our results can be extracted from the fractional two-dimensional systems given by special forms of the discrete Roesser. The main achievements are summarized and future research topics are then discussed.

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تحكم و مراقبة النماذج الكسرية و تطبيقها

الملخص : الهدف من هذه الأطروحة هو طرح مجموعة متنوعة من التقنيات المتعلقة بقوانين المراقبة و التحكم للأنظمة الكسرية ذات البعد الواحد حيث تتم مناقشة نماذج جديده ذات مشتقة قابلة للتوافق و نماذج ذات رتبة كسرية (فردية/ قياسية) و كذلك قمنا بتطوير منهجية التحكم H_∞ على نظام ثنائي الأبعاد. في الجزء الأول من هذه الأطروحة، نقترح تطبيق تحويل سمودو لحل الأنظمة الخطية الفردية ذات الزمن المستمر و المشتقة التوافقية, وهذا بفضل الخصائص المثيره للاهتمام لتحويل سمودو التي أنشأنها, تم تطوير نهج جديد. من خلال الأمثلة الأكاديمية و التطبيقية؛ تتم مقارنة طريقتنا بالأمثلة المتواجده, حيث تعرض قابلية التطبيق و دقة العملية المطوره. يتم أيضا تقديم منهج آخر باستخدام طريقة تفكيك وايستراس كرونيكر حيث يتم إيجاد حل لهذا النظام بمقاربة جديده. كما تم إنشاء تحليل مختلف لمفاهيم التحكم, المراقبة, الايجابية, الاستقرار و فائق الاستقرار لهذه الأنظمة الجديده. بالإضافة إلى ذلك, في الجزء الثاني من هذه الأطروحة, قمنا بتطبيق خوارزمية فعالة لحساب معيار H_∞ لمرشحات رقمية ثنائية الأبعاد الممثلة بنماذج روسر حيث تم الحصول عليها كامتداد لمقال بوعقاده و فان دورن باستخدام دالة مصفوفة بار هيرميتية و طرق تعيين المستوى للقيمة الفردية القصوى لدالة التحويل. تتقارب هذه الطريقة بشكل تربيعي في بضع خطوات نحو التردد ω_1 و ω_2 كما قدمنا كذلك أمثلة توضيحية لعرض كفاءه و دقة منهجنا.

الكلمات المفتاحية. أنظمة فردية, مشتقة قابلة للتوافق, مشتق كابوتو, تحويل سمودو, ايجابية, استقرار, فائق الاستقرار, إمكانية التحكم, قابلية الملاحظة, مرشحات رقمية ثنائية الأبعاد, مقام قابل للفصل, معيار H_∞.

Control and observation of fractional models and applications

Abstract : The aim of this thesis is to describe a variety of synthesis techniques concerning the observation and control laws of fractional one dimensional systems when a new models with conformable derivative and fractional-order (singular/ standard) models are discussed and also we developed a approach of H_∞ control on two-dimension system. In the first part of this thesis, we propose the application of the Sumudu transform for solving singular continuous-time linear systems based on the conformable derivative operator. Thanks to the interesting properties of the conformable Sumudu transform that we have established, a new approach is developed. Through academic and real examples, our method is compared to the existing ones, where the applicability and the accuracy of the developed process are shown. An alternative approach using the Weierstrass-Kronecker decomposition method is also given when the solution is provided with a different form. The analysis of different concepts of controllability, observability, positivity, stability and superstability of this new system are established. Additionally, In the second part of this thesis, an efficient algorithm to calculate the H_∞ norm of two-dimensional digital filters described by Roesser models is derived as an extension of the work of BOUAGADA et al. by using a para-hermitian matrix function and level sets methods of maximum singular value of the transfer function, this method converges quadratically in a few steps towards the frequency ω_1 and ω_2. We present an illustrative examples in order to show the efficiency and the accuracy of our approach.

Key Words. Singular systems, Conformable derivative operator, Caputo derivative, Conformable fractional Sumudu transform, Controllability, Observability, Positivity, Stability, Superstability, 2D digital Filters, Causal recursive separable denominator, H_∞ norm.

Contrôle et observation de modèle fractionnaires et applications

Résumé : L'objective de cette thèse est de décrire une variété de techniques de synthèse concernant les lois d'observation et de contrôle des systèmes unidimensionnels fractionnaires lorsqu'un nouveau modèle avec la dérivés conformes et des modèles (singuliers/standard) fractionnaires sont discutés et nous avons développé aussi une approche de contrôle H_∞ sur un système à deux dimensions. Dans la première partie de cette thèse, nous proposons l'application de la transformée de Sumudu pour résoudre des systèmes linéaires singuliers à temps continu basés sur l'opérateur dérivé conforme. Grâce aux propriétés intéressantes de la transformée de Sumudu conformable que nous avons établies, une nouvelle approche est développée. Par le biais d'exemples théoriques et réels, notre méthode est comparée aux méthodes existantes, ce qui démontre l'applicabilité et la précision du processus développé. Une autre approche en utilisant la méthode de décomposition Weierstrass-Kronecker est également donnée lorsque la solution est fournie avec une autre forme. L'analyse des différents concepts de contrôlabilité, observabilité, positivité, stabilité et superstabilité de ces nouveaux systèmes est établie. En outre, dans la deuxième partie de ce projet, un algorithme efficace pour calculer la norme H_∞ de filtres numériques bidimensionnels décrits par des modèles de Roesser est obtenu comme une extension du travail de BOUAGADA et al. en utilisant une fonction de matrice para-hermitienne et des méthodes d'ensembles de niveaux de la valeur singulière maximale de la fonction de transfert, cette méthode converge quadratiquement en quelques étapes vers la fréquence ω_1 et ω_2. Nous présentons des exemples illustratifs afin de montrer l'efficacité et la précision de notre approche.

Mots-Clés. Systèmes singuliers, Dérivé conforme, Dérivé de Caputo, Transformée de Sumudu fractionnaire conformable, Contrôlabilité, Observabilité, Positivité, Stabilité, Superstabilité, Filtres numériques 2D, Dénominateur séparable récursif causal, Norme H_∞.