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by

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MASTER

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Contents

INTRODUCTION

The theory of Nevanlinna also known as the theory of the distribution of values of a meromorphic function was developed by the Finnish mathematician Rolf Nevanlinna at the turn of the twentieth century.

In the study of the growth of solutions of linear differential equations in the complex domain Nevanlinna's theory plays a crucial role.

The opening and two chapters make up this thesis. The first chapter begins with some basic concepts of Nevanlinna's theory.

The first section begins with a presentation of Jensen's formula, which serves as the foundation for Nevanlinna's theory, the definitions of the functions $m(r, f)$, $N(r, f)$, $T(r, f)$ and some of their properties are then given, the first Nevanlinna theorem, which is a result of Jensen's formula is next discussed.

The second section of this chapter looks at the $[p, q]$-order of growth of a meromorphic and an analytic function and its properties, in the third section we will set out some very important lemmas to finish the first chapter. These will be used in the following steps of this thesis.

The application of Nevanlinna's theory of linear differential equations is the subject of the second chapter. We are particularly interested in the linear differential equations of the form:

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{0.0.1}$$

where $A_i(z)$ are analytic or meromorphic functions in the unit disc $\Delta = \{z : |z| < 1\}$, $i = 0, 1, ..., k-1, k \geq 2$.

Since the 1980s, the theory of complex differential equations in the unit disc has been developed by establishing the definition of the function spaces in the year 2000, Heittokangas [9] examined the growth theory of equation (0.0.1) when the coefficients A_i $(i = 0, 1, ...k-1)$ are analytic and meromorphic functions in the unit disc for the first time. His findings also provided some useful instruments for more research into the meromorphic theory solutions of equations (0.0.1) in the unit disc Δ.

This chapter is devoted to investigating the growth of solutions of equation (0.0.1), it is about the $[p, q]$-order of growth which is the generalisation of the paper of Hamouda [6], who studied the growth of solutions of (0.0.1) by using the concepts of iterated n-order. Also, we will improve the result of the paper of Qin *et al.* [12]. So, the conditions on the coefficients will be expressed by using the $[p, q]$-order instead of the iterated n-order defined in the paper of Hamouda [6] and the paper of Qin *et al.* [12]. Furthermore, the coefficients of equation (0.0.1) are meromorphic functions.

Chapter 1

Nevanlinna's theory

In this chapter, we will present Jensen's formula, define the functions $N(r,f)$, $m(r,f)$ and $T(r,f)$ which are known as the pole count function of f, proximity function and Nevanlinna's characteristic function and discuss their properties. The goal is to obtain Jensen's simplest formula and demonstrate Nevanlinna's first fundamental theorem, which is a result of this formula. Next, the order of growth of meromorphic functions will be defined. Finally, we will give the statement of some lemmas.

1.1 Jensen's formula

Theorem 1.1.1 [14] *Let f be a meromorphic function such that $f(0) \neq 0, \infty$, with zeros $(a_1, a_2, a_3, \ldots)$ and poles $(b_1, b_2, b_3, \ldots)$, each taken into account according to its multiplicity then:*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi - \sum_{|a_j|<r} \log\frac{r}{|a_j|} + \sum_{|b_j|<r} \log\frac{r}{|b_j|}. \tag{1.1.1}$$

Proof. The theorem is demonstrated when there are no zeros or poles on the circle $|z| = r$, consider the following function:

$$g(z) = f(z)\frac{\prod\limits_{|a_j|<r} \frac{r^2-\bar{a}_j z}{r(z-a_j)}}{\prod\limits_{|b_j|<r} \frac{r^2-\bar{b}_j z}{r(z-b_j)}}.$$

Then $g \neq 0, \infty$ and g is an analytic function in disc $|z| \leq r$ consequently $\log(g)$ is analytic in the disc $|z| \leq r$ and its real part is a harmonic function, according to the moyen formula for the harmonic function, we can write

$$\log(|g(0)|) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g(re^{i\varphi})\right| d\varphi. \tag{1.1.2}$$

We have

$$|g(0)| = |f(0)| \frac{\prod\limits_{|a_j|<r} \frac{r}{|a_j|}}{\prod\limits_{|b_j|<r} \frac{r}{|b_j|}},$$

then

$$\log|g(0)| = \log|f(0)| + \sum_{|a_j|<r} \log\frac{r}{|a_j|} - \sum_{|b_j|<r} \log\frac{r}{|b_j|}. \tag{1.1.3}$$

For $z = re^{i\varphi}$, we have for all $c \in \mathbb{C}$

$$\left|\frac{r^2 - \bar{c}z}{r(z-c)}\right| = \left|\frac{r^2 - \bar{c}re^{i\varphi}}{r(re^{i\varphi}-c)}\right| = \left|\frac{r - \bar{c}e^{i\varphi}}{re^{i\varphi}-c}\right| = \left|e^{i\varphi}\right| \left|\frac{re^{-i\varphi} - \bar{c}}{re^{i\varphi}-c}\right| = 1.$$

Then

$$\left|g(re^{i\varphi})\right| = \left|f(re^{i\varphi})\right|. \tag{1.1.4}$$

By substituting (1.1.3) and (1.1.4) into (1.1.2), we obtain

$$\log(|f(0)|) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi - \sum_{|a_j|<r} \log\frac{r}{|a_j|} + \sum_{|b_j|<r} \log\frac{r}{|b_j|}.$$

□

1.2 Poisson's formula

Theorem 1.2.1 [2] *Let f be a meromorphic function in the disc $|\xi| \leq R$. If $z = re^{i\theta}$ and $r < R$, $\theta \in [0, 2\pi[$, then*

$$\mathrm{Re} f(re^{i\theta}) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \mathrm{Re} f(Re^{i\varphi}) d\varphi. \tag{1.2.1}$$

Proof. Let f be an analytic function in the disc $|\xi| \leq R$ according to Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\xi|=R} \frac{f(\xi)}{\xi - z} d\xi. \tag{1.2.2}$$

For $z^* = \frac{R^2}{\bar{z}}$ the symmetric point to z in relation to the circle, according to the cauchy theorem the function $\frac{f(\xi)}{\xi - z^*}$ is analytic in the disc $|\xi| \leq R$ and

$$\frac{1}{2\pi i} \oint\limits_{|\xi|=R} \frac{f(\xi)}{\xi - z^*} d\xi = 0, \text{ (because } z^* \text{ is outside the circle).} \tag{1.2.3}$$

Then (1.2.2) and (1.2.3) gives us

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint\limits_{|\xi|=R}\frac{f(\xi)}{\xi-z}d\xi - \frac{1}{2\pi i}\oint\limits_{|\xi|=R}\frac{f(\xi)}{\xi-z^*}d\xi \\ &= \frac{1}{2\pi i}\oint\limits_{|\xi|=R}\left(\frac{1}{\xi-z}-\frac{1}{\xi-z^*}\right)f(\xi)d\xi. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{\xi-z}-\frac{1}{\xi-z^*} &= \frac{1}{\xi-z}-\frac{1}{\xi-\frac{R^2}{\bar{z}}} \\ &= \frac{1}{\xi-z}-\frac{\bar{z}}{\xi\bar{z}-R^2}. \end{aligned}$$

We put $\xi = \mathrm{Re}^{i\varphi} \Rightarrow d\xi = Rie^{i\varphi}d\varphi$ and $z = re^{i\theta}$, so

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi i}\int\limits_0^{2\pi}\left(\frac{1}{Re^{i\varphi}-re^{i\theta}}-\frac{re^{-i\theta}}{Re^{i\varphi}re^{-i\theta}-R^2}\right)f(Re^{i\varphi})Rie^{i\varphi}d\varphi \\ &= \frac{1}{2\pi}\int\limits_0^{2\pi}\left(\frac{Re^{i\varphi}}{Re^{i\varphi}-re^{i\theta}}-\frac{re^{-i\theta}e^{i\varphi}}{re^{i\varphi}e^{-i\theta}-R}\right)f(Re^{i\varphi})d\varphi. \end{aligned}$$

We have

$$\begin{aligned} \frac{Re^{i\varphi}}{Re^{i\varphi}-re^{i\theta}}-\frac{re^{-i\theta}e^{i\varphi}}{re^{i\varphi}e^{-i\theta}-R} &= \frac{Re^{i\varphi}\times e^{-i\varphi}}{(Re^{i\varphi}-re^{i\theta})\times e^{-i\varphi}}-\frac{re^{-i\theta}e^{i\varphi}}{re^{i\varphi}e^{-i\theta}-R} \\ &= \frac{R}{R-re^{i\theta}e^{-i\varphi}}-\frac{re^{-i\theta}e^{i\varphi}}{re^{i\varphi}e^{-i\theta}-R} \\ &= \frac{R}{R-re^{i(\theta-\varphi)}}-\frac{re^{i(\varphi-\theta)}}{re^{i(\varphi-\theta)}-R} \\ &= \frac{R}{R-re^{i(\theta-\varphi)}}+\frac{re^{i(\varphi-\theta)}}{R-re^{i(\varphi-\theta)}} \\ &= \frac{R\times\left(R-re^{i(\varphi-\theta)}\right)}{\left(R-re^{i(\theta-\varphi)}\right)\times\left(R-re^{i(\varphi-\theta)}\right)} \\ &\quad +\frac{re^{i(\varphi-\theta)}\times\left(R-re^{i(\theta-\varphi)}\right)}{\left(R-re^{i(\varphi-\theta)}\right)\times\left(R-re^{i(\theta-\varphi)}\right)} \\ &= \frac{R^2-Rre^{i(\varphi-\theta)}+Rre^{i(\varphi-\theta)}-r^2}{R^2-Rr(e^{i(\varphi-\theta)}+e^{-i(\varphi-\theta)})+r^2} \\ &= \frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}, \end{aligned}$$

so

$$f(z) = \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}f(Re^{i\varphi})d\varphi.$$

By taking the real part of $f(z)$, we get

$$\mathrm{Re}f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \mathrm{Re}f(Re^{i\varphi})d\varphi.$$

□

1.3 Poisson Jensen's formula

Theorem 1.3.1 [14] *Let f be a meromorphic function such that $f(0) \neq 0, \infty$, with zeros $(a_1, a_2, a_3, ...)$ and poles $(b_1, b_2, b_3, ...)$ each being counted with its order of multiplicity. If $z = re^{i\theta}$ and $r < R$ then*

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left|f(Re^{i\varphi})\right| d\varphi - \sum_{|a_j|<R} \log\left|\frac{R^2 - \bar{a}_j z}{R(z - a_j)}\right| \\ &+ \sum_{|b_j|<R} \log\left|\frac{R^2 - \bar{b}_j z}{R(z - b_j)}\right|. \qquad (1.3.1)\end{aligned}$$

Proof. Set

$$g(z) = f(z)\frac{\prod\limits_{|a_j|<R} \frac{R^2 - \bar{a}_j z}{R(z - a_j)}}{\prod\limits_{|b_j|<R} \frac{R^2 - \bar{b}_j z}{R(z - b_j)}}.$$

Then $g \neq 0, \infty$ and g is an analytic function in disc $|z| \leq R$ consequently $\log(g)$ is analytic in the disc $|z| \leq R$ and its real part is a harmonic function, according to the Poisson formula, we get

$$\log\left|g(re^{i\theta})\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left|g(Re^{i\varphi})\right| d\varphi. \qquad (1.3.2)$$

For $z = \mathrm{R}e^{i\varphi}$ and for all $c \in \mathbb{C}$

$$\left|\frac{R^2 - \bar{c}z}{R(z - c)}\right| = \left|\frac{z\bar{z} - \bar{c}z}{R(z - c)}\right| = \frac{|z|}{R}\left|\frac{\bar{z} - \bar{c}}{z - c}\right| = 1.$$

So

$$\left|g(Re^{i\varphi})\right| = \left|f(Re^{i\varphi})\right| \frac{\prod\limits_{|a_j|<R} \left|\frac{R^2 - \bar{a}_j z}{R(z - a_j)}\right|}{\prod\limits_{|b_j|<R} \left|\frac{R^2 - \bar{b}_j z}{R(z - b_j)}\right|} = \left|f(Re^{i\varphi})\right|, \qquad (1.3.3)$$

because $\left|\frac{R^2 - \bar{a}_j z}{R(z - a_j)}\right| = \left|\frac{R^2 - \bar{b}_j z}{R(z - b_j)}\right| = 1$. So, we have

$$\log|g(z)| = \log|f(z)| + \sum_{|a_j|<R} \log\left|\frac{R^2 - \bar{a}_j z}{R(z - a_j)}\right| - \sum_{|b_j|<R} \log\left|\frac{R^2 - \bar{b}_j z}{R(z - b_j)}\right|. \qquad (1.3.4)$$

By substituting (1.3.3) and (1.3.4) into (1.3.2), we obtain

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}\log\left|f(Re^{i\varphi})\right|d\varphi - \sum_{|a_j|<R}\log\left|\frac{R^2-\bar{a}_j z}{R(z-a_j)}\right| \\ &+ \sum_{|b_j|<R}\log\left|\frac{R^2-\bar{b}_j z}{R(z-b_j)}\right|.\end{aligned}$$

□

1.4 Truncated logarithm function with properties

Definition 1.4.1 [14] *For all $x \geq 0$ we define*

$$\log^+(x) = \max(\log(x), 0) = \begin{cases} \log(x) & \text{if } x > 1 \\ 0 & \text{if } 0 \leq x \leq 1 \end{cases}$$

this function is called truncated logarithm function.

Lemma 1.4.1 [14] *We have the following properties:*

a) $\log(x) \leq \log^+(x)$, if $x \geq 0$;

b) $\log^+(x) \leq \log^+(y)$, if $0 \leq x \leq y$;

c) $\log(x) = \log^+(x) - \log^+(\frac{1}{x})$, if $x > 0$;

d) $|\log(x)| = \log^+(x) + \log^+(\frac{1}{x})$, if $x > 0$;

e) $\log^+(\prod_{i=1}^{n} x_i) \leq \sum_{i=1}^{n}\log^+(x_i)$, if $x_i \geq 0$, $i = 1...n$;

f) $\log^+(\sum_{i=1}^{n} x_i) \leq \sum_{i=1}^{n}\log^+(x_i) + \log(n)$, if $x_i \geq 0$, $i = 1...n$;

g) $\left|\log^+|x| - \log^+|y|\right| \leq \left|\log\left|\frac{x}{y}\right|\right|$, if $x, y \in \mathbb{C}^*$;

h) $\left|\log^+|x| - \log^+|y|\right| \leq \log^+|x-y| + \log 2$, if $x, y \in \mathbb{C}$.

Proof. c) We have for $x > 0$

$$\begin{aligned}\log^+(x) - \log^+(\frac{1}{x}) &= \max(\log(x), 0) - \max(\log(\frac{1}{x}), 0) \\ &= \max(\log(x), 0) - \max(-\log(x), 0) \\ &= \max(\log(x), 0) + \min(\log(x), 0) \\ &= \log(x)\end{aligned}$$

d) We have for $x > 0$

$$\begin{aligned}
\log^+(x) + \log^+(\frac{1}{x}) &= \max(\log(x), 0) + \max(\log(\frac{1}{x}), 0) \\
&= \max(\log(x), 0) + \max(-\log(x), 0) \\
&= \max(\log(x), 0) - \min(\log(x), 0) \\
&= |\log(x)|
\end{aligned}$$

e) We have for $x_i \geq 0$, $i = 1...n$:
if $\prod_{i=1}^{n} x_i \leq 1$ then

$$\log^+(\prod_{i=1}^{n} x_i) = 0 \leq \sum_{i=1}^{n} \log^+(x_i)$$

if $\prod_{i=1}^{n} x_i \geq 1$ then

$$\log^+(\prod_{i=1}^{n} x_i) = \log(\prod_{i=1}^{n} x_i) = \sum_{i=1}^{n} \log(x_i)$$

according to (a) we find

$$\log^+(\prod_{i=1}^{n} x_i) = \sum_{i=1}^{n} \log(x_i) \leq \sum_{i=1}^{n} \log^+(x_i)$$

f) We have for $x_i \geq 0$, $i = 1...n$: according to (b) and (e) we get

$$\begin{aligned}
\log^+(\sum_{i=1}^{n} x_i) &\leq \log^+(n \max_{1 \leq i \leq n} x_i) \\
&\leq \log(n) + \log^+(\max_{1 \leq i \leq n} x_i) \\
&\leq \sum_{i=1}^{n} \log^+(x_i) + \log(n)
\end{aligned}$$

g) For $x, y \in \mathbb{C}^*$ we have

$$\log^+|x| = \log^+\left|\frac{x}{y}y\right| \leq \log^+\left|\frac{x}{y}\right| + \log^+|y| \leq \left|\log\left|\frac{x}{y}\right|\right| + \log^+|y|$$

because

$$|\log(x)| = \log^+(x) + \log^+(\frac{1}{x}) \Rightarrow \begin{cases} \log^+(x) \leq |\log(x)| \\ \log^+(\frac{1}{x}) \leq |\log(x)| \end{cases}$$

then

$$\log^+|x| - \log^+|y| \leq \left|\log\left|\frac{x}{y}\right|\right|.$$

Similarly

$$\log^+|y| = \log^+\left|\frac{y}{x}x\right| \le \log^+\left|\frac{y}{x}\right| + \log^+|x| \le \left|\log\left|\frac{y}{x}\right|\right| + \log^+|x|$$

so

$$-\left(\log^+|x| - \log^+|y|\right) \le \left|\log\left|\frac{x}{y}\right|\right|$$

then

$$\left|\log^+|x| - \log^+|y|\right| \le \left|\log\left|\frac{x}{y}\right|\right|.$$

h) For $x, y \in \mathbb{C}^*$ we have

$$\log^+|x| = \log^+|x-y+y| \le \log^+|x-y| + \log^+|y| + \log 2$$

then

$$\log^+|x| - \log^+|y| \le \log^+|x-y| + \log 2.$$

Similarly

$$\log^+|y| = \log^+|y-x+x| \le \log^+|x-y| + \log^+|x| + \log 2$$

so

$$-\left(\log^+|x| - \log^+|y|\right) \le \log^+|x-y| + \log 2$$

then

$$\left|\log^+|x| - \log^+|y|\right| \le \log^+|x-y| + \log 2.$$

□

1.5 Pole count function , proximity function and Nevanlinna's characteristic function

Definition 1.5.1 [14] *$n(t,a,f)$ denotes the number of zeros in the disc $|z|<t$ of the equation $f(z)=a$ with each racine being counted with its order of multiplicity and $\bar{n}(t,a,f)$ denotes the number of distinct racines in the disc $|z|<t$.*
The number of poles of f in the disc $|z|<t$ counted with their order of multiplicity is denoted by $n(t,\infty,f)$ and the number of poles distinct of f in $|z|<t$ is denoted by $\bar{n}(t,\infty,f)$.

Example 1.5.1 *For $f(z)=e^z$, we have $n(r,0,f)=\bar{n}(r,0,f)=0$ because f has not any zeros and $n(r,\infty,f)=\bar{n}(r,\infty,f)=0$ because f has not any poles.*

Example 1.5.2 *Consider $f(z)=\frac{1}{\cos^2(z)}$, so $\bar{n}(r,\infty,f)=2\left[\frac{2r}{\pi}\right]$ and $n(r,\infty,f)=4\left[\frac{2r}{\pi}\right]$.*

Definition 1.5.2 [14] *Let f be a meromorphic function such that $f\not\equiv a\in\mathbb{C}$, the function a-points of f is defined by*

$$N(r,\frac{1}{f-a}) = N(r,a,f) = \int_0^r \frac{n(t,a,f)-n(0,a,f)}{t}dt + n(0,a,f)\log(r),$$

$$N(r,f) = N(r,\infty,f) = \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt + n(0,\infty,f)\log(r).$$

Similarly, the function a-points distinct of f is defined by

$$\bar{N}(r,a,f) = \int_0^r \frac{\bar{n}(t,a,f) - \bar{n}(0,a,f)}{t} dt + \bar{n}(0,a,f)\log(r),$$

$$\bar{N}(r,\infty,f) = \int_0^r \frac{\bar{n}(t,\infty,f) - \bar{n}(0,\infty,f)}{t} dt + \bar{n}(0,\infty,f)\log(r).$$

Example 1.5.3 *Consider the following function* $f(z) = \frac{\exp(az^n)}{z^p}$, *such that* $n, p \in \mathbb{N}^*$, $a \in \mathbb{C}^*$. *So* $n(t,\infty,f) = n(0,\infty,f) = p$ *then*

$$\begin{aligned} N(r,f) &= \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt + n(0,\infty,f)\log(r) \\ &= \int_0^r \frac{p-p}{t} dt + p\log(r) \\ &= p\log(r). \end{aligned}$$

Also, we have $\bar{n}(t,\infty,f) = \bar{n}(0,\infty,f) = 1$ *then*

$$\begin{aligned} \bar{N}(r,\infty,f) &= \int_0^r \frac{\bar{n}(t,\infty,f) - \bar{n}(0,\infty,f)}{t} dt + \bar{n}(0,\infty,f)\log(r) \\ &= \log(r). \end{aligned}$$

Lemma 1.5.1 [14] *Let f be a meromorphic function with a-points* $a_1, a_2, ..., a_m$ *in the disc* $|z| \le r$ *such that* $0 < |a_1| \le |a_2| \le ... \le |a_m| \le r$ *each being counted with its order of multiplicity, then*

$$\int_0^r \frac{n(t,a,f)}{t} dt = \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t} dt = \sum_{0<|a_j|\le r} \log\frac{r}{|a_j|}.$$

Proof. We put $|a_j| = r_j$, $(1 \le j \le m)$. Then

$$\sum_{0<|a_j|\le r} \log\frac{r}{|a_j|} = \sum_{j=1}^m \log\frac{r}{r_j} = \log\left(\frac{r^m}{r_1 \times r_2 \times ... \times r_m}\right)$$

$$= \log\left(\frac{r_2}{r_1} \times \left(\frac{r_3}{r_2}\right)^2 \times \left(\frac{r_4}{r_3}\right)^3 \times ... \times \left(\frac{r_m}{r_{m-1}}\right)^{m-1} \times \left(\frac{r}{r_m}\right)^m\right)$$

$$= \log\left(\frac{r_2}{r_1}\right) + 2\log\left(\frac{r_3}{r_2}\right) + 3\log\left(\frac{r_4}{r_3}\right)$$

$$+\cdots+(m-1)\log\left(\frac{r_m}{r_{m-1}}\right)+m\log\left(\frac{r}{r_m}\right)$$

$$=\int_0^{r_1}0\cdot\frac{dt}{t}+\int_{r_1}^{r_2}1\cdot\frac{dt}{t}+\int_{r_2}^{r_3}2\frac{dt}{t}+\int_{r_3}^{r_4}3\frac{dt}{t}$$

$$+\cdots+\int_{r_{m-1}}^{r_m}(m-1)\frac{dt}{t}+\int_{r_m}^{r}m\frac{dt}{t}$$

$$=\int_0^r\frac{n(t,a,f)}{t}dt=\int_0^r\frac{n(t,a,f)-n(0,a,f)}{t}dt.$$

□

Proposition 1.5.1 [14] *Let f be a meromorphic function with the development of Laurent around the origin*

$$f(z)=\sum_{j=m}^{+\infty}c_jz^j,\ c_m\neq 0,\ m\in\mathbb{Z},\ z\in\mathbb{C}.$$

Then

$$\log|c_m|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi+N(r,f)-N(r,\frac{1}{f}).$$

Proof. Consider the meromorphic function h defined by

$$h(z)=\frac{f(z)}{z^m},\ m\in\mathbb{Z}$$

then $h(0)\neq 0,\infty$ and $m=n(0,0,f)-n(0,\infty,f)$. In fact, if $m>0$ then $n(0,0,f)=m$ and $n(0,\infty,f)=0$, if $m<0$ then $n(0,0,f)=0$ and $n(0,\infty,f)=-m$, finally, if $m=0$ then $n(0,0,f)=n(0,\infty,f)=0$. So f and h have the same zeros and poles in the disc that is centered on 0 ($0<|z|\leq r$), according to Jensen's formula

$$\log(|h(0)|)=\frac{1}{2\pi}\int_0^{2\pi}\log\left|h(re^{i\varphi})\right|d\varphi-\sum_{0<|a_j|<r}\log\frac{r}{|a_j|}+\sum_{0<|b_j|<r}\log\frac{r}{|b_j|}.$$

By Lemma 1.5.1

$$\begin{aligned}
\log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\varphi})}{(re^{i\varphi})^m}\right|d\varphi-\int_0^r\frac{n(t,0,f)-n(0,0,f)}{t}dt\\
&+\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi-m\log(r)-\int_0^r\frac{n(t,0,f)-n(0,0,f)}{t}dt\\
&+\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi-(n(0,0,f)-n(0,\infty,f))\log(r)\\
&-\int_0^r\frac{n(t,0,f)-n(0,0,f)}{t}dt+\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi-\left(n(0,0,f)\log(r)+\int_0^r\frac{n(t,0,f)-n(0,0,f)}{t}dt\right)\\
&+\left(n(0,\infty,f)\log(r)+\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi+N(r,f)-N(r,\frac{1}{f}).
\end{aligned}$$

□

Definition 1.5.3 [14] *Let f be a meromorphic function such that $f \not\equiv a \in \mathbb{C}$, we define the function of proximity of f by*

$$m(r,a,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\varphi})-a|}d\varphi,$$

and

$$m(r,\infty,f)=m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi.$$

Example 1.5.4 *Let $f(z)=a\in\mathbb{C}^*$, then*
i) We have $n(t,\infty,f)=0$, so $N(r,f)=0$.

ii) We calculate $m(r,f)$

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |a|\, d\varphi = \log^+ |a|\,. \end{aligned}$$

Example 1.5.5 *Consider the following function:* $f(z) = \frac{\exp(az^n)}{z^p}$, *such that* $n, p \in \mathbb{N}^*$, $a \in \mathbb{C}^*$ *so* $a = |a|\, e^{i \arg a}$, $-\pi < \arg a \leq \pi$. *We have*

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{\exp\left(|a|\, r^n e^{i(\arg a + n\varphi)}\right)}{r^p e^{ip\varphi}}\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(\frac{\exp\left(|a|\, r^n \cos\left(\arg a + n\varphi\right)\right)}{r^p}\right) d\varphi. \end{aligned}$$

We put $\psi = \arg a + n\varphi \Rightarrow d\varphi = \frac{d\psi}{n}$, *we obtain*

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi n}\int_{\arg a}^{2\pi n+\arg a} \log^+\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi n}\int_{0}^{2\pi n} \log^+\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi n}\sum_{j=0}^{n-1}\int_{2\pi j}^{2\pi(j+1)} \log^+\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi n} n\int_{0}^{2\pi} \log^+\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log^+\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{\exp\left(|a|\, r^n \cos\psi\right)}{r^p}\right) d\psi \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a|\, r^n \cos\psi d\psi - \frac{p}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log r d\psi \\
&= \frac{|a|\, r^n}{\pi} - \frac{p}{2}\log r.
\end{aligned}$$

Definition 1.5.4 [14] *Let* f *be a meromorphic function, then*

$$T(r,f) = m(r,f) + N(r,f).$$

$T(r,f)$ *is the Nevanlinna's characteristic function.*

Example 1.5.6 *Consider the following function:* $f(z) = \frac{\exp(az^n)}{z^p}$, *such that* $n, p \in \mathbb{N}^*$, $a \in \mathbb{C}^*$. *Then*

$$\begin{aligned}
T(r,f) &= m(r,f) + N(r,f) \\
&= \frac{|a|\, r^n}{\pi} - \frac{p}{2}\log r + p\log r \\
&= \frac{|a|\, r^n}{\pi} + \frac{p}{2}\log r.
\end{aligned}$$

Proposition 1.5.2 [14] *Let* $f_1, f_2, ..., f_n, f$ *be meromorphic functions, then* *a)* $T(r, \prod_{j=1}^{n} f_j) \leq \sum_{j=1}^{n} T(r, f_j)$,

b) $T(r, \sum_{j=1}^{n} f_j) \leq \sum_{j=1}^{n} T(r, f_j) + \log n$,

c) $T(r, f^m) = mT(r, f)$, $m \in \mathbb{N}^*$.

Proof. a) We have

$$m(r, \prod_{j=1}^{n} f_j) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \prod_{j=1}^{n} f_j(re^{i\varphi}) \right| d\varphi.$$

According to the Lemma 1.4.1 (e)

$$\begin{aligned} m(r, \prod_{j=1}^{n} f_j) &\leq \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{n} \log^+ \left| f_j(re^{i\varphi}) \right| d\varphi \\ &= \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f_j(re^{i\varphi}) \right| d\varphi \\ &= \sum_{j=1}^{n} m(r, f_j). \end{aligned}$$

If z_0 is a pole of order $\lambda_j > 0$ for f_j, it is also a pole of order equal at most $\sum_{j=1}^{n} \lambda_j$ for the function $\prod_{j=1}^{n} f_j$. Then

$$N(r, \prod_{j=1}^{n} f_j) \leq \sum_{j=1}^{n} N(r, f_j),$$

so

$$\begin{aligned} T(r, \prod_{j=1}^{n} f_j) &= m(r, \prod_{j=1}^{n} f_j) + N(r, \prod_{j=1}^{n} f_j) \\ &\leq \sum_{j=1}^{n} m(r, f_j) + \sum_{j=1}^{n} N(r, f_j) \\ &= \sum_{j=1}^{n} (m(r, f_j) + N(r, f_j)) \\ &= \sum_{j=1}^{n} T(r, f_j). \end{aligned}$$

b) We have according to the Lemma 1.4.1 (f)

$$\begin{aligned} m(r, \sum_{j=1}^{n} f_j) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{j=1}^{n} f_j(re^{i\varphi}) \right| d\varphi \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} (\sum_{j=1}^{n} \log^+ \left| f_j(re^{i\varphi}) \right| + \log n) d\varphi \\ &\leq \sum_{j=1}^{n} m(r, f_j) + \log n. \end{aligned}$$

If z_0 is a pole of order $\lambda_j > 0$ for f_j, it is also a pole of order equal at most $\max\limits_{1 \leq j \leq n} \lambda_j \leq \sum\limits_{j=1}^{n} \lambda_j$ for the function $\sum\limits_{j=1}^{n} f_j$. Then

$$N(r, \sum_{j=1}^{n} f_j) \leq \sum_{j=1}^{n} N(r, f_j),$$

so

$$\begin{aligned} T(r, \sum_{j=1}^{n} f_j) &= m(r, \sum_{j=1}^{n} f_j) + N(r, \sum_{j=1}^{n} f_j) \\ &\leq \sum_{j=1}^{n} m(r, f_j) + \log n + \sum_{j=1}^{n} N(r, f_j) \\ &= \sum_{j=1}^{n} (m(r, f_j) + N(r, f_j)) + \log n \\ &= \sum_{j=1}^{n} T(r, f_j) + \log n. \end{aligned}$$

c) We notice that, for all $m \in \mathbb{N}^*$

$$|f^m| \leq 1 \Leftrightarrow |f| \leq 1.$$

i) If $|f| \leq 1$ then

$$m(r, f^m) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f^m(re^{i\varphi}) \right| d\varphi = 0$$

and

$$N(r, f^m) = mN(r, f)$$

So

$$T(r, f^m) = 0 + mN(r, f) = m(m(r, f) + N(r, f)) = mT(r, f).$$

ii) If $|f| \geq 1$ then

$$\begin{aligned} m(r, f^m) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f^m(re^{i\varphi})\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log \left|f^m(re^{i\varphi})\right| d\varphi \\ &= m\frac{1}{2\pi}\int_0^{2\pi} \log \left|f(re^{i\varphi})\right| d\varphi \\ &= m\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi \\ &= mm(r, f) \end{aligned}$$

and

$$N(r, f^m) = mN(r, f).$$

So

$$T(r, f^m) = mm(r, f) + mN(r, f) = m(m(r, f) + N(r, f)) = mT(r, f).$$

□

1.6 Nevanlinna's first fundamental theorem

Theorem 1.6.1 [14] *Let f be a meromorphic function such that $f \not\equiv a \in \mathbb{C}$, let the Laurent development of the function $f - a$ around the origin*

$$f(z) - a = \sum_{j=m}^{+\infty} c_j z^j,\ \ c_m \neq 0,\ \ m \in \mathbb{Z},\ \ z \in \mathbb{C}.$$

Then

$$T(r, \frac{1}{f-a}) = T(r, f) - \log|c_m| + \varphi(r, a)$$

with

$$|\varphi(r, a)| \leq \log^+ |a| + \log 2.$$

Proof. We assume that $a = 0$, so according to the Proposition 1.5.1 and the Lemma 1.4.1 (c), we get

$$\begin{aligned} \log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + N(r, f) - N(r, \frac{1}{f}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\varphi})\right| d\varphi - \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{1}{f(re^{i\varphi})}\right| d\varphi + N(r, f) - N(r, \frac{1}{f}) \\ &= m(r, f) - m(r, \frac{1}{f}) + N(r, f) - N(r, \frac{1}{f}) \\ &= T(r, f) - T(r, \frac{1}{f}). \end{aligned} \tag{1.6.1}$$

Then

$$T(r,\frac{1}{f}) = T(r,f) - \log|c_m|$$

in this case

$$\varphi(r,a) \equiv 0.$$

If $a \neq 0$, we pose $g(z) = f(z) - a$, then

$$\begin{cases} N(r,g) = N(r,f), \\ N(r,\frac{1}{g}) = N(r,\frac{1}{f-a}), \\ m(r,\frac{1}{g}) = m(r,\frac{1}{f-a}). \end{cases}$$

We have

$$\begin{aligned} \log^+|g(z)| &= \log^+|f(z) - a| \\ &\leq \log^+|f(z)| + \log^+|a| + \log 2 \end{aligned}$$

and

$$\begin{aligned} \log^+|f(z)| &= \log^+|g(z) + a| \\ &\leq \log^+|g(z)| + \log^+|a| + \log 2, \end{aligned}$$

so

$$\begin{aligned} m(r,g) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|g(re^{i\varphi})\right| d\varphi \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} (\log^+\left|f(re^{i\varphi})\right| + \log^+|a| + \log 2) d\varphi \\ &= m(r,f) + \log^+|a| + \log 2 \end{aligned}$$

and

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\varphi})\right| d\varphi \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} (\log^+\left|g(re^{i\varphi})\right| + \log^+|a| + \log 2) d\varphi \\ &= m(r,g) + \log^+|a| + \log 2. \end{aligned}$$

We put $\varphi(r,a) = m(r,g) - m(r,f)$, then

$$-\left(\log^+|a| + \log 2\right) \leq m(r,g) - m(r,f) \leq \log^+|a| + \log 2$$

so

$$|\varphi(r,a)| \leq \log^+|a| + \log 2.$$

According to (1.6.1) we obtain

$$\begin{aligned} T(r,\frac{1}{f-a}) &= T(r,\frac{1}{g}) = T(r,g) - \log|c_m| = m(r,g) + N(r,g) - \log|c_m| \\ &= m(r,f) + \varphi(r,a) + N(r,f) - \log|c_m| \\ &= T(r,f) + \varphi(r,a) - \log|c_m| \end{aligned}$$

with $|\varphi(r,a)| \leq \log^+|a| + \log 2$. □

Remark 1.6.1 *The first fundamental theorem of Nevanlinna can be expressed as follows*

$$T(r,\frac{1}{f-a})=T(r,f)+O(1),\ r\rightarrow+\infty.$$

Example 1.6.1 *Consider the following function* $g\left(z\right)=\frac{z^{p}}{\exp(az^{n})}$, *such that* $n,p\in\mathbb{N}^{*}$, $a\in\mathbb{C}^{*}$. *According to Example 1.5.6, we have* $g\left(z\right)=\frac{1}{f(z)}$, *we applicate the first fundamental theorem of Nevanlinna, we get*

$$\begin{aligned}T(r,g)&=T(r,f)+O(1)\\&=\frac{|a|\,r^{n}}{\pi}+\frac{p}{2}\log r+O(1).\end{aligned}$$

Proposition 1.6.1 [14] *Let* f *be a meromorphic function and* $g(z)=\frac{af(z)+b}{cf(z)+d}$, *where* $a,b,c,d\in\mathbb{C}$ *with* $ad-bc\neq0$ *and* $f\not\equiv-\frac{d}{c}$. *Then*

$$T(r,g)=T(r,f)+O(1).$$

Proof. If $g=\frac{af+b}{cf+d}$, then $f=\frac{-dg+b}{cg-a}$, it is therefore sufficient to demonstrate that

$$T(r,g)\leq T(r,f)+O(1).$$

$i)$ If $c=0$ so $g(z)=\frac{a}{d}f(z)+\frac{b}{d}$, according to the Proposition 1.5.2

$$\begin{aligned}T(r,g)&=T(r,\frac{a}{d}f+\frac{b}{d})\\&\leq T(r,\frac{a}{d}f)+T(r,\frac{b}{d})+\log2\\&\leq T(r,\frac{a}{d})+T(r,f)+T(r,\frac{b}{d})+\log2\\&=T(r,f)+O(1).\end{aligned}$$

$ii)$ If $c\neq0$ we can write

$$\begin{aligned}g&=\frac{af+b}{cf+d}\\&=\frac{af+b}{c(f+\frac{d}{c})}\\&=\frac{a(f+\frac{d}{c})+\frac{bc}{c}-\frac{ad}{c}}{c(f+\frac{d}{c})}\\&=\frac{a}{c}+\frac{bc-ad}{c^{2}}\frac{1}{f+\frac{d}{c}}.\end{aligned}$$

Then we use the Proposition 1.5.2 and the first fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned} T(r,g) &= T(r,\frac{a}{c}+\frac{bc-ad}{c^2}\frac{1}{f+\frac{d}{c}}) \\ &\leq T(r,\frac{a}{c})+T(r,\frac{bc-ad}{c^2}\frac{1}{f+\frac{d}{c}})+\log 2 \\ &\leq T(r,\frac{a}{c})+T(r,\frac{bc-ad}{c^2})+T(r,\frac{1}{f+\frac{d}{c}})+\log 2 \\ &= T(r,\frac{a}{c})+T(r,\frac{bc-ad}{c^2})+T(r,f)+O(1)+\log 2 \\ &= T(r,f)+O(1). \end{aligned}$$

Thus

$$T(r,g)=T(r,f)+O(1)$$

□

Example 1.6.2 *Consider the following function:* $h(z)=\frac{f(z)+1}{f(z)+i}$, *where* $f(z)=\frac{\exp(az^n)}{z^p}$, *such that* $n,p\in\mathbb{N}^*$, $a\in\mathbb{C}^*$. *We have* $1\times i-1\times 1=-1+i\neq 0$, *by the Proposition 1.6.1, we obtain*

$$\begin{aligned} T(r,h) &= T(r,f)+O(1) \\ &= \frac{|a|\,r^n}{\pi}+\frac{p}{2}\log r+O(1). \end{aligned}$$

Proposition 1.6.2 [2] *Let* f *be a meromorphic function and* $a\in\mathbb{C}^*$, *we have*

a) $|m(r,af)-m(r,f)|\leq|\log|a||$,

b) $|m(r,f+a)-m(r,f)|\leq\log^+|a|+\log 2$.

Proof. a) For $a\in\mathbb{C}^*$

$$\begin{aligned} |m(r,af)-m(r,f)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|af(re^{i\varphi})\right|d\varphi-\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left|af(re^{i\varphi})\right|-\log^+\left|f(re^{i\varphi})\right|\right|d\varphi \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left|f(re^{i\varphi})\right|+\log^+|a|-\log^+\left|f(re^{i\varphi})\right|\right|d\varphi \\ &\leq |\log|a||. \end{aligned}$$

b) For $a \in \mathbb{C}^*$

$$\begin{aligned}
|m(r,f+a)-m(r,f)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})+a\right|d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left|f(re^{i\varphi})+a\right| - \log^+\left|f(re^{i\varphi})\right|\right|d\varphi \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left|f(re^{i\varphi})\right| + \log^+|a| + \log 2 - \log^+\left|f(re^{i\varphi})\right|\right|d\varphi \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+|a| + \log 2 d\varphi \\
&= \log^+|a| + \log 2.
\end{aligned}$$

□

Remark 1.6.2 *The previous proposition gives us*

$$\begin{aligned}
m(r,af) &= m(r,f)+O(1), \\
m(r,f+a) &= m(r,f)+O(1),
\end{aligned}$$

then

$$\begin{aligned}
T(r,af) &= T(r,f)+O(1), \\
T(r,f+a) &= T(r,f)+O(1).
\end{aligned}$$

Definition 1.6.1 [14] *Let f be a meromorphic function then the order of growth ρ of f is defined by*

$$\rho(f) = \lim_{r\to+\infty}\sup\frac{\log T(r,f)}{\log r}.$$

Example 1.6.3 *Consider the following function $f(z) = \frac{\exp(az^n)}{z^p}$, such that $n,p \in \mathbb{N}^*$, $a \in \mathbb{C}^*$. Then*

$$\begin{aligned}
\rho(f) &= \lim_{r\to+\infty}\sup\frac{\log T(r,f)}{\log r} \\
&= \lim_{r\to+\infty}\sup\frac{\log\left(\frac{|a|r^n}{\pi}+\frac{p}{2}\log r\right)}{\log r} \\
&= \lim_{r\to+\infty}\sup\frac{\log\left(\frac{|a|}{\pi}r^n\left(1+\frac{p\pi}{2|a|r^n}\log r\right)\right)}{\log r} \\
&= \lim_{r\to+\infty}\sup\frac{\log(r^n)+\log\left(\frac{|a|}{\pi}\right)+\log\left(1+\frac{p\pi}{2|a|r^n}\log r\right)}{\log r} \\
&= n.
\end{aligned}$$

Proposition 1.6.3 [2] *Let f and g are two non-constant meromorphic functions then*

a) $\rho(\frac{1}{f}) = \rho(f)$, $f \not\equiv 0$;

b) $\rho(af) = \rho(f)$, $a \in \mathbb{C}^*$;

c) $\rho(f+g) \leq \max\{\rho(f), \rho(g)\}$;

d) $\rho(fg) \leq \max\{\rho(f), \rho(g)\}$;

e) And if $\rho(g) < \rho(f)$, then $\rho(f+g) = \rho(f)$.

Proof. a) We have

$$T(r, \frac{1}{f}) = T(r, f) + O(1),$$

so

$$\begin{aligned}
\rho(\frac{1}{f}) &= \lim_{r\to+\infty} \sup \frac{\log T(r, \frac{1}{f})}{\log r} \\
&= \lim_{r\to+\infty} \sup \frac{\log\left(T(r, f) + O(1)\right)}{\log r} \\
&= \lim_{r\to+\infty} \sup \frac{\log T(r, f) + \log(1 + \frac{O(1)}{T(r,f)})}{\log r} \\
&= \lim_{r\to+\infty} \sup \frac{\log T(r, f)}{\log r} \\
&= \rho(f).
\end{aligned}$$

b) We have

$$T(r, af) = T(r, f) + O(1),$$

then

$$\rho(af) = \rho(f).$$

c) Let $\varepsilon > 0$ then for r sufficiently large, we have

$$\begin{cases} T(r, f) \leq r^{\rho(f)+\frac{\varepsilon}{2}}, \\ T(r, g) \leq r^{\rho(g)+\frac{\varepsilon}{2}}, \end{cases}$$

so

$$\begin{aligned}
T(r, f+g) &\leq T(r, f) + T(r, g) + \log 2 \\
&\leq r^{\rho(f)+\frac{\varepsilon}{2}} + r^{\rho(g)+\frac{\varepsilon}{2}} + \log 2 \\
&\leq 2r^{\max\{\rho(f),\rho(g)\}+\frac{\varepsilon}{2}} + \log 2 \\
&\leq r^{\max\{\rho(f),\rho(g)\}+\varepsilon}
\end{aligned}$$

then

$$\rho(f+g) \leq \max\{\rho(f), \rho(g)\} + \varepsilon,$$

for all $\varepsilon > 0$ being arbitrary, so

$$\rho(f+g) \leq \max\{\rho(f), \rho(g)\}.$$

d) We have

$$\begin{aligned} T(r, fg) &\leq T(r,f) + T(r,g) \\ &\leq r^{\rho(f)+\frac{\varepsilon}{2}} + r^{\rho(g)+\frac{\varepsilon}{2}} \\ &\leq 2r^{\max\{\rho(f),\rho(g)\}+\frac{\varepsilon}{2}} \\ &\leq r^{\max\{\rho(f),\rho(g)\}+\varepsilon}, \end{aligned}$$

then

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\} + \varepsilon,$$

for all $\varepsilon > 0$ being arbitrary, so

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\}.$$

e) If $\rho(g) < \rho(f)$ then

$$\begin{aligned} \rho(f+g) &\leq \max\{\rho(f), \rho(g)\} \\ &= \rho(f) \\ \rho(fg) &\leq \max\{\rho(f), \rho(g)\} \\ &= \rho(f). \end{aligned}$$

We have

$$\begin{aligned} \rho(f) &= \rho(f+g-g) \\ &\leq \max\{\rho(f+g), \rho(-g)\} \\ &= \max\{\rho(f+g), \rho(g)\} \\ &= \rho(f+g), \end{aligned}$$

then

$$\rho(f) = \rho(f+g),$$

we have also

$$\begin{aligned} \rho(f) &= \rho(fg\frac{1}{g}) \\ &\leq \max\{\rho(fg), \rho(\frac{1}{g})\} \\ &= \max\{\rho(fg), \rho(g)\} \\ &= \rho(fg) \end{aligned}$$

then

$$\rho(f) = \rho(fg).$$

□

Definition 1.6.2 [14] *Let f be a meromorphic function of order $0<\rho(f)<+\infty$ then the type of f is defined by*

$$\tau_T=\lim_{r\to+\infty}\sup\frac{T(r,f)}{r^{\rho(f)}}.$$

Example 1.6.4 *Consider the following function $f(z)=\frac{\exp(az^n)}{z^p}$, such that $n,p\in\mathbb{N}^*$, $a\in\mathbb{C}^*$. We have $\rho(f)=n$. Then*

$$\begin{aligned}\tau_T &= \lim_{r\to+\infty}\sup\frac{T(r,f)}{r^{\rho(f)}}\\ &= \lim_{r\to+\infty}\sup\frac{\frac{|a|r^n}{\pi}+\frac{p}{2}\log r}{r^n}\\ &= \frac{|a|}{\pi}.\end{aligned}$$

Definition 1.6.3 *For all $r\in\mathbb{R}_+$ we pose $\exp_1 r=e^r$ and $\exp_{n+1} r=\exp\left(\exp_n r\right)$, $n\in\mathbb{N}$, and for all $r\in\mathbb{R}_+^*$, $\log_1 r=\log r$ and $\log_{n+1} r=\log\left(\log_n r\right)$, $n\in\mathbb{N}$.*
We also denote $\exp_0 r=r=\log_0 r$ and $\exp_{-1} r=\log_1 r$.

Proposition 1.6.4 *For all $x\geq 0$, $y\geq 0$ and $n\in\mathbb{N}^*$ we have*
i) $\log_n^+(x+y)\leq\log_n^+ x+\log_n^+ y+O(1)$;
ii) $\log_n^+(xy)\leq\log_n^+ x+\log_n^+ y+O(1)$.

Proof. *i)* For all $x\geq 0$, $y\geq 0$ and $n\in\mathbb{N}^*$, we will demonstrate it by induction. For $n=1$ by Lemma 1.4.1 (f) we have

$$\begin{aligned}\log^+(x+y) &\leq \log^+ x+\log^+ y+\log 2\\ &\leq \log^+ x+\log^+ y+O(1).\end{aligned}$$

We suppose that

$$\log_n^+(x+y)\leq\log_n^+ x+\log_n^+ y+O(1)$$

and we demonstrate that

$$\log_{n+1}^+(x+y)\leq\log_{n+1}^+ x+\log_{n+1}^+ y+O(1)$$

so

$$\begin{aligned}\log_{n+1}^+(x+y) &= \log^+\left(\log_n^+(x+y)\right)\\ &\leq \log^+\left(\log_n^+ x+\log_n^+ y+O(1)\right).\end{aligned}$$

From Lemma 1.4.1 (f) we obtain

$$\begin{aligned}\log_{n+1}^+(x+y) &\leq \log^+\left(\log_n^+ x\right)+\log^+\left(\log_n^+ y\right)+O(1)\\ &= \log_{n+1}^+ x+\log_{n+1}^+ y+O(1).\end{aligned}$$

Then we get

$$\log_n^+(x+y)\leq\log_n^+ x+\log_n^+ y+O(1).$$

ii) For all $x \geq 0$, $y \geq 0$ and $n \in \mathbb{N}^*$, we will demonstrate it by induction. For $n = 1$ by Lemma 1.4.1 (e) we have

$$\log^+ (xy) \leq \log^+ x + \log^+ y.$$

We suppose that

$$\log_n^+ (xy) \leq \log_n^+ x + \log_n^+ y + O(1)$$

and we demonstrate that

$$\log_{n+1}^+ (xy) \leq \log_{n+1}^+ x + \log_{n+1}^+ y + O(1)$$

so

$$\begin{aligned} \log_{n+1}^+ (xy) &= \log^+ \left(\log_n^+ (xy)\right) \\ &\leq \log^+ \left(\log_n^+ x + \log_n^+ y + O(1)\right). \end{aligned}$$

From Lemma 1.4.1 (f) we get

$$\begin{aligned} \log_{n+1}^+ (xy) &\leq \log^+ \left(\log_n^+ x\right) + \log^+ \left(\log_n^+ y\right) + O(1) \\ &= \log_{n+1}^+ x + \log_{n+1}^+ y + O(1), \end{aligned}$$

then we obtain

$$\log_{n+1}^+ (xy) \leq \log_{n+1}^+ x + \log_{n+1}^+ y + O(1).$$

□

Definition 1.6.4 *i*) *Let f be a meromorphic function in Δ. Then the iterated n-order of f is defined by*

$$\rho_n(f) = \limsup_{r \to 1^-} \frac{\log_n^+ T(r,f)}{-\log(1-r)},$$

where $\log_1^+ r = \log^+ r = \max\{\log r, 0\}$, $\log_{n+1}^+ r = \log^+ \left(\log_n^+ r\right)$, $n \in \mathbb{N}$.
ii) *Let f be an analytic function in Δ. Then its iterated n-order is defined by*

$$\rho_{M,n}(f) = \limsup_{r \to 1^-} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.
For $n = 1$, *we have* $\rho_{M,1}(f) = \rho_M(f)$, $\rho_1(f) = \rho(f)$.

Suppose that p and q are integers satisfying $p \geq q \geq 1$. Then $[p,q]$-order is defined as follows.

Definition 1.6.5 [1] *i*) *Let f be a meromorphic function in Δ, then the $[p,q]$-order of f is defined by*

$$\rho_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{-\log_q(1-r)}.$$

ii) Let f be an analytic function in Δ, then its $[p,q]$-order is defined by

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p+1} M(r,f)}{-\log_q(1-r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.
For $q=1$ *we have* $\rho_{[p,q]}(f) = \rho_p(f)$, $\rho_{M,[p,q]}(f) = \rho_{M,p}(f)$.

Remark 1.6.3 [1] *Let p and q be integers such that $p \geq q \geq 1$, and f be an analytic function in Δ, the following tow statements hold:*
i) If $p=q$, then $\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f)+1$.
ii) If $p>q$, then $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f)$.

Proof. *a*) First step we will demonstrate the following inequalities, let $0<r<R<1$

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{1+3r}{1-r} T(\frac{1+r}{2}, f)$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.
a.i) We have f is an analytic function then the first inequality is trivial because f haven't any poles

$$\begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi \\ &\leq \log^+ M(r,f). \end{aligned}$$

a.ii) Now let's show that $\log^+ M(r,f) \leq \frac{1+3r}{1-r} T(\frac{1+r}{2}, f)$. For $z_0 = re^{i\theta}$, $r<R$ and $|f(z_0)| = M(r,f)$, we have the following inequality

$$\left|\frac{R(z_0-a_j)}{R^2-\bar{a}_j z_0}\right| < 1, \text{ (because } r<R).$$

From Poisson-Jensen Formula we get if $|f(z_0)| = M(r,f) > 1$

$$\log M(r,f) = \log|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2} \log\left|f(Re^{i\varphi})\right| d\varphi$$

$$-\sum_{|a_j|<R} \log\left|\frac{R^2-\bar{a}_j z_0}{R(z_0-a_j)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2} \log\left|f(Re^{i\varphi})\right| d\varphi$$

$$+\sum_{|a_j|<R} \log\left|\frac{R(z_0-a_j)}{R^2-\bar{a}_j z_0}\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2} \log^+\left|f(Re^{i\varphi})\right| d\varphi$$

$$+\sum_{|a_j|<R}\log^+\left|\frac{R(z_0-a_j)}{R^2-\bar{a}_j z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}\log^+\left|f(Re^{i\varphi})\right|d\varphi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R-r)(R+r)}{(R-r)^2+2Rr(1-\cos(\theta-\varphi))}\log^+\left|f(Re^{i\varphi})\right|d\varphi$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{(R-r)(R+r)}{(R-r)^2}\log^+\left|f(Re^{i\varphi})\right|d\varphi,\ \text{(because } 1-\cos(\theta-\varphi)>0)$$

$$= \frac{(R+r)}{(R-r)}\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(Re^{i\varphi})\right|d\varphi = \frac{(R+r)}{(R-r)}m(R,f)$$

$$= \frac{(R+r)}{(R-r)}T(R,f).$$

If $|f(z_0)| = M(r,f) \leq 1$

$$\log M(r,f) \leq \log^+ M(r,f) = 0 < \frac{(R+r)}{(R-r)}T(R,f),$$

then

$$\log^+ M(r,f) < \frac{(R+r)}{(R-r)}T(R,f).$$

If $R=\frac{1+r}{2}$ then

$$\log^+ M(r,f) \leq \frac{1+3r}{1-r}T(\frac{1+r}{2},f).$$

b) Second step we demonstrate the previous remark. By the previous inequalities we have:

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{1+3r}{1-r}T(\frac{1+r}{2},f).$$

Then

$$\log_p^+ T(r,f) \leq \log_{p+1}^+ M(r,f) \leq \log_p^+\left(\frac{1+3r}{1-r}T(\frac{1+r}{2},f)\right),$$

there exist a constant $c>0$, such that

$$\log_p^+\left(\frac{1+3r}{1-r}T(\frac{1+r}{2},f)\right) \leq \log_p^+(1+3r) + \log_p^+\left(\frac{1}{1-r}\right)$$

$$+\log_p^+\left(T(\frac{1+r}{2},f)\right)+c,$$

we get

$$\log_p^+ T(r,f) \leq \log_{p+1}^+ M(r,f)$$

$$\leq \log_p^+(1+3r) + \log_p\left(\frac{1}{1-r}\right) + \log_p^+\left(T(\frac{1+r}{2},f)\right)+c.$$

i) If $p = q$ we get

$$\frac{\log_p^+ T(r,f)}{\log_p\left(\frac{1}{1-r}\right)} \leq \frac{\log_{p+1}^+ M(r,f)}{\log_p\left(\frac{1}{1-r}\right)} \leq \frac{\log_p^+(1+3r)}{\log_p\left(\frac{1}{1-r}\right)}$$

$$+\frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{c}{\log_p\left(\frac{1}{1-r}\right)},$$

then

$$\limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log_p\left(\frac{1}{1-r}\right)} \leq \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_p\left(\frac{1}{1-r}\right)}$$

$$\leq \limsup_{r\to 1^-}\left(\frac{\log_p^+(1+3r)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{c}{\log_p\left(\frac{1}{1-r}\right)}\right)$$

$$= \limsup_{r\to 1^-}\left(\frac{\log_p^+(1+3r)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}\frac{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{c}{\log_p\left(\frac{1}{1-r}\right)}\right)$$

$$= \limsup_{r\to 1^-}\left(\frac{\log_p^+(1+3r)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}\frac{\log_p\left(\frac{2}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{c}{\log_p\left(\frac{1}{1-r}\right)}\right)$$

$$= \limsup_{r\to 1^-}\left(\frac{\log_p^+(1+3r)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)} + \frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}\frac{\log_p\left(\frac{1}{1-r}\right)+\log_p 2 + O(1)}{\log_p\left(\frac{1}{1-r}\right)}\right.$$

$$\left.+\frac{c}{\log_p\left(\frac{1}{1-r}\right)}\right) \leq 0 + 1 + \limsup_{r\to 1^-}\frac{\log_p^+ T(\frac{1+r}{2},f)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}(1+0+0)+0$$

$$= \limsup_{r\to 1^-}\frac{\log_p^+ T(\frac{1+r}{2},f)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)} + 1$$

so

$$\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_p\left(\frac{1}{1-r}\right)} = \rho_{[p,p]}(f) \leq \limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_p\left(\frac{1}{1-r}\right)} = \rho_{M,[p,p]}(f),$$

$$\limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_p\left(\frac{1}{1-r}\right)} = \rho_{M,[p,p]}(f) \leq \limsup_{r\to 1^-}\frac{\log_p^+\left(T(\frac{1+r}{2},f)\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)} + 1 = \rho_{[p,p]}(f) + 1,$$

then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f) + 1.$$

ii) If $p > q$, then

$$\limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{1-r}\right)} \leq \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{1-r}\right)}$$

$$\leq \limsup_{r \to 1^-} \left(\frac{\log_p^+ (1+3r)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p^+ \left(T(\frac{1+r}{2}, f)\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)} \right)$$

$$= \limsup_{r \to 1^-} \left(\frac{\log_p^+ (1+3r)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p^+ \left(T(\frac{1+r}{2}, f)\right)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)} \frac{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)} \right)$$

$$= \limsup_{r \to 1^-} \left(\frac{\log_p^+ (1+3r)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p^+ \left(T(\frac{1+r}{2}, f)\right)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)} \frac{\log_q\left(\frac{2}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)} \right)$$

$$= \limsup_{r \to 1^-} \left(\frac{\log_p^+ (1+3r)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_p^+ \left(T(\frac{1+r}{2}, f)\right)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)} \frac{\log_q\left(\frac{1}{1-r}\right) + \log_q 2 + O(1)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)} \right)$$

$$\leq 0 + 0 + \limsup_{r \to 1^-} \frac{\log_p^+ T(\frac{1+r}{2}, f)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)} (1+0+0) + 0$$

so

$$\limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{1-r}\right)} \leq \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{1-r}\right)} \leq \limsup_{r \to 1^-} \frac{\log_p^+ T(\frac{1+r}{2}, f)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)}.$$

Then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f),$$

consequently

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

□

Example 1.6.5 *Consider the following function* $f(z) = \exp_k \frac{1}{(1-z)^{\mu}}$*, such that* $z = re^{i\varphi} \in \Delta$*,* $k \geq 1$ *and* $\mu > 1$*. We have*

$$\left|\exp_k \frac{1}{(1-z)^{\mu}}\right| \leq \exp_k \frac{1}{(1-r)^{\mu}}$$

then

$$M(r,f) = \exp_k \frac{1}{(1-r)^{\mu}},$$

so

$$\begin{aligned}\rho_{M,[p,q]}(f) &= \limsup_{r\to 1^-} \frac{\log^+_{p+1} M(r,f)}{-\log_q(1-r)} \\ &= \limsup_{r\to 1^-} \frac{\log^+_{p+1} \exp_k \frac{1}{(1-r)^{\mu}}}{-\log_q(1-r)}.\end{aligned}$$

When $0 < r < 1$ *we have* $\frac{1}{(1-r)^{\mu}} > 1$*, so the positive logarithmic function becomes the usual logarithmic function.*
a) If $p \geq q = 1$ *, we have three cases:*
Case 1: *If* $p = k$*, we have*

$$\begin{aligned}\rho_{M,[p,1]}(f) &= \rho_{M,p}(f) \\ &= \limsup_{r\to 1^-} \frac{\log \frac{1}{(1-r)^{\mu}}}{\log \frac{1}{(1-r)}} \\ &= \mu.\end{aligned}$$

Case 2: *If* $p > k$*, we have*

$$\begin{aligned}\rho_{M,[p,1]}(f) &= \rho_{M,p}(f) \\ &= \limsup_{r\to 1^-} \frac{\log_{p-k+1} \frac{1}{(1-r)^{\mu}}}{\log \frac{1}{(1-r)}} \\ &= 0.\end{aligned}$$

Case 3: *If* $p < k$*, we have*

$$\begin{aligned}\rho_{M,[p,1]}(f) &= \rho_{M,p}(f) \\ &= \limsup_{r\to 1^-} \frac{\exp_{k-p-1} \frac{1}{(1-r)^{\mu}}}{\log \frac{1}{(1-r)}} \\ &= +\infty.\end{aligned}$$

b) If $p \geq q > 1$*, we have also three cases:*

Case1: *If $p = k$, we have*

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}} = +\infty.$$

Case 2: *If $p > k$, we have*

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p-k+1} \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}}.$$

here, we have two different cases:
i) If $p-k+1=q$, we get

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_q \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}} = 1.$$

ii) If $p-k+1>q$, we got

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p-k+1} \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}} = 0.$$

iii) If $p-k+1<q$, we obtain

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p-k+1} \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}} = +\infty.$$

Case 3: *If $p < k$, we find*

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\exp_{k-p-1} \frac{1}{(1-r)^{\mu}}}{\log_q \frac{1}{(1-r)}} = +\infty.$$

Finally, If $p > q$ we have

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

1.7 Some Lemmas

Lemma 1.7.1 ([5]) *Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If $f(z)$ is a meromorphic function in Δ such that $f^{(j)}(z)$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|),f)\right\}\right]^{k-j}, \ |z| \notin E, \tag{1.7.1}$$

where $E \subset [0,1)$ with finit logarithmic measure $\int\limits_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1-|z|)$. Moreover, if $\rho_{[1,1]}(f) = \rho_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\rho_1(f)+2+\varepsilon)}, \quad |z| \notin E \tag{1.7.2}$$

and if $\rho_{[p,q]}(f) < \infty$, for $p \geq q \geq 1$ and $p \geq 2$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\rho_{[p,q]}(f)+\varepsilon}\right\}, \quad |z| \notin E \tag{1.7.3}$$

Proof. We demonstrate only (1.7.3), so from (1.7.1), we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right]^{k-j}.$$

By the definition of the $[p,q]$-order we have

$$T(s(|z|), f) \leq \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}.$$

Then

$$\begin{aligned}
\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}\right\}\right]^{k-j} \\
&\leq \left[\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}\right]^{k-j} \\
&\leq \left[\exp\left\{(k-j)\exp_{p-2}\left\{\left(\log_{q-1}\left(\frac{1}{d(1-|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}\right\}\right] \\
&\leq \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\rho_{[p,q]}(f)+\varepsilon}\right\}.
\end{aligned}$$

□

Lemma 1.7.2 ([12]) *Let $f : \Delta \to \mathbb{R}$ be a meromorphic function in Δ. If there exists a point w_0 on the boundary $\partial\Delta = \{z : |z| = 1\}$ and a curve $\gamma \subset \Delta$ tending to w_0 such that*

$$\lim_{\substack{z \to w_0 \\ z \in \gamma}} f(z) < 1,$$

then there exists a set $E \subset [0,1)$ with infinite logarithmic measure $\int\limits_E \frac{dr}{1-r} = \infty$ such that we have $f(z) < 1$ for all $|z| \in E$.

Proof. Let $\lim\limits_{\substack{z\to w_0\\ z\in\gamma}} f(z)=a$, $0\le a<1$. By definition, for $\varepsilon=1-a$, there exists $\delta>0$ such that for all $z\in\gamma$ and $0<|z-w_0|<\delta$, we have $|f(z)-f(w_0)|<\varepsilon$, so

$$||f(z)|-|f(w_0)||\le|f(z)-f(w_0)|<\varepsilon,$$

then

$$-\varepsilon<|f(z)|-|f(w_0)|<\varepsilon\Rightarrow-\varepsilon+|f(w_0)|<|f(z)|<\varepsilon+|f(w_0)|,$$

we obtain

$$|f(z)|<\varepsilon+a=1.$$

Let $E=\{|z|:z\in\gamma \text{ and } 0<|z-w_0|<\delta\}$, we get

$$\int\limits_E\frac{dr}{1-r}=\int\limits_{1-\delta}^{1}\frac{dr}{1-r}=+\infty$$

because

$$\delta>|z-w_0|\ge||z|-|w_0||=||z|-1|=|1-|z||,\ (w_0\in\partial\Delta\ )$$

so

$$-\delta<1-|z|<\delta\Rightarrow1-\delta<|z|<1+\delta$$

and we have

$$z\in\Delta\Rightarrow|z|<1$$

then

$$1-\delta<|z|<1.$$

□

Lemma 1.7.3 ([8]) *Let f be a meromorphic function in Δ, and let $k\ge1$ be an integer. Then*

$$m(r,\frac{f^{(k)}}{f})=S(r,f),$$

where $S(r,f)=O\left(\log^+T(r,f)+\log(\frac{1}{1-r})\right)$, possibly outside a set $E\subset[0,1)$ which satisfies $\int\limits_E\frac{dr}{1-r}<\infty$. If $f(z)$ is of finite order, then

$$m(r,\frac{f^{(k)}}{f})=O\left(\log(\frac{1}{1-r})\right).$$

Lemma 1.7.4 ([11]) *Let p and q be integers such that $p\ge q\ge1$, and let $k\ge1$ be an integer and $f(z)$ be a meromorphic function in Δ that satisfies $\rho_{[p,q]}(f)=\rho<+\infty$. Then for any $\varepsilon>0$ and for all $r\notin E\subset[0,1)$, we have*

$$m(r,\frac{f^{(k)}}{f})=O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right),$$

where E has finite logarithmic measure.

Lemma 1.7.5 ([13]) *Let p and q be integers such that $p \geq q \geq 1$. If the coefficient $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ are analytic functions in Δ, then all solutions $f(z)$ of (0.0.1) satisfies*

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, ..., k-1\right\}.$$

Chapter 2

Application of the Nevanlinna's theory

The following complex linear differential equation has numerous applications in various scientific discipline

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{0.0.1}$$

where $A_i(z)$ are analytic or meromorphic in the unit disc $\Delta = \{z : |z| < 1\}$, $i = 0, 1, ..., k-1, k \geq 2$. In this chapter, we apply Nevanlinna's theory to investigate the growth of solutions of the previous differential equation.

Several researchers have investigated the growth of solutions of the linear differential equation (0.0.1).

In [6], Hamouda come up with a new and a good idea, such that A_0 dominates the other coefficients near the unit disc's boundary to find the fast growing solutions of (0.0.1) and he obtained the following two results that improve and generalize the results of Heittokangas *et al.* [9].

Theorem 2.0.1 ([6]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 such that for any constant $\mu > 0$,*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \left(\frac{1}{(1-|z|)^{\mu}} \right) = 0,$$

then every nontrivial solution $f(z)$ of (0.0.1) is of infinite order.

Theorem 2.0.2 ([6]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 such that for any constant $\mu > 0$, such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} = 0,$$

where $n \geq 1$ is an integer, $\lambda > 0$ is a real constant, then every nontrivial solution $f(z)$ of (0.0.1) satisfies $\rho_n(f) = \infty$ and $\rho_{n+1}(f) \geq \mu$.

Recently, in the study of growth of solutions of the equation (0.0.1) Qin *et al.* [12] improved the result of Hamouda [6] and they obtained the following result.

Theorem 2.0.3 ([12]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 satisfying*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\frac{1}{1-|z|} \right)^{\mu} \right\} < 1,$$

where $\mu > 0$ and $\lambda > 0$ are two real constants, then every nontrivial solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Also, Qin *et al.* [12] treated the case where A_s, $s = 1, ..., k-1, k \geq 2$ dominates the other coefficients near the unit disc's boundary and they got this result.

Theorem 2.0.4 ([12]) *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ satisfying*

$$\max\left\{ \rho_{M,[p,q]}(A_i) : i = 1, ..., k-1 \right\} \leq \rho_{M,[p,q]}(A_0) = \mu,$$

where $\mu > 0$. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 such that

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p \left\{ \lambda \left(\frac{1}{1-|z|} \right)^{\mu} \right\} < 1,$$

where $\lambda > 0$ is a real constant, then every nontrivial solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p,q]}(A_0)$.

A natural question is how to characterize the $[p,q]$-order of growth of solutions of (0.0.1) under Hamouda-like conditions. So, in this thesis we will express the conditions by using the $[p,q]$-order instead the iterated n-order defined in the paper of Hamouda [6] and the paper of Qin *et al.* [12]. Furthermore, the coefficients of equation $(0,0,1)$ are meromorphic instead of analytic.
Here, we investigate the issue and arrive at the following conclusions.

Theorem 2.0.5 *Let$A_0(z), A_1(z), ..., A_{k-1}(z)$ be meromorphic functions in Δ. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and two constants $\mu > 0$ and $\lambda > 0$, such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\} < 1 \tag{2.0.1}$$

for all $p, q \in \mathbb{N}$ such that $p \geq q \geq 1$, then every nontrivial meromorphic solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Proof. Assume that f is a nontrivial solution of (0.0.1) with $\rho_{[p,q]}(f) = \rho < \infty$. According to Lemma 1.7.1, for any given $\varepsilon > 0$, there exists a set $E_1 \subset [0, 1)$ with $\int\limits_{E_1} \frac{dr}{1-r} < \infty$ such that for all z satisfying $r \notin E_1$, we have
if $p = q = 1$, $\rho_{[1,1]}(f) = \rho < \infty$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\frac{1}{1-r}\right)^{j(\rho+2+\varepsilon)}, \; j = 1...k \tag{2.0.2}$$

and if $p \geq q > 1$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\left(\rho_{[p,q]}(f)+\varepsilon\right)}\right\}, \; j = 1...k. \tag{2.0.3}$$

By (0.0.1) we have

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)},$$

so

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{2.0.4}$$

Combining (2.0.2) and (2.0.4), we get

$$\begin{aligned} |A_0(z)| &\leq \left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)} + \sum_{i=1}^{k-1}|A_i(z)|\left(\frac{1}{1-r}\right)^{i(\rho+2+\varepsilon)} \\ &\leq (1 + \sum_{i=1}^{k-1}|A_i(z)|) \sup_{1\leq j\leq k}\left\{\left(\frac{1}{1-r}\right)^{j(\rho+2+\varepsilon)}\right\}. \end{aligned}$$

We have $r < 1$ so $\frac{1}{1-r} > 1$, then $\sup\limits_{1\leq j\leq k}\left\{\left(\frac{1}{1-r}\right)^{j(\rho+2+\varepsilon)}\right\} = \left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)}$, which implies

$$|A_0(z)| \leq (1 + \sum_{i=1}^{k-1}|A_i(z)|)\left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)} \tag{2.0.5}$$

and combining (2.0.3) and (2.0.4), we obtain

$$\begin{aligned} |A_0(z)| \leq\; & \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\left(\rho_{[p,q]}(f)+\varepsilon\right)}\right\} \\ & + \sum_{i=1}^{k-1}|A_i(z)|\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\left(\rho_{[p,q]}(f)+\varepsilon\right)}\right\}, \end{aligned}$$

so

$$|A_0(z)| \leq (1 + \sum_{i=1}^{k-1} |A_i(z)|) \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\left(\rho_{[p,q]}(f)+\varepsilon\right)}\right\}. \qquad (2.0.6)$$

According to assumption (2.0.1) and Lemma 1.7.2, for any $\mu > 0$, there exists a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} = \infty$ such that for all z satisfying $|z| = r \in E_2$, we get

$$\frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} < 1.$$

Then for any $z \in \Delta$ satisfying $|z| = r \in E_2$, we have

$$|A_0(z)| > \left(\sum_{j=1}^{k-1} |A_j(z)| + 1\right) \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}. \qquad (2.0.7)$$

If $z \in \{z \in \Delta : |z| = r \in E_2 \diagdown E_1\}$ such that $\int_{E_2 \diagdown E_1} \frac{dr}{1-r} = \infty$ we obtain

$i)$ (2.0.7) contradicts with (2.0.5), in fact

$$\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} > \left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)}.$$

$ii)$ (2.0.7) contradicts with (2.0.6), in fact

$$\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} > \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\left(\rho_{[p,q]}(f)+\varepsilon\right)}\right\}.$$

So

$$\rho_{[p,q]}(f) = \infty.$$

According to Lemma 1.7.1, there exists $E_3 \subset [0,1)$ with $\int_{E_3} \frac{dr}{1-r} < \infty$ such that for all z satisfying $|z| = r \notin E_3$, and for $s(r) = 1 - d(1-r)$, $d \in (0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-r}, T(s(r), f)\right\}\right)^{j}, \ |z| \notin E_3$$

from $\rho_{[p,q]}(f) = \infty$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\frac{1}{1-r}\right)^{j(2+\varepsilon)} [T(s(r), f)]^{j}, \ |z| = r \notin E_3. \qquad (2.0.8)$$

Combining (2.0.4) and (2.0.8), we obtain

$$\begin{aligned} |A_0(z)| &\leq \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(r),f)]^k + \sum_{i=1}^{k-1} |A_i(z)| \left(\frac{1}{1-r}\right)^{i(2+\varepsilon)} [T(s(r),f)]^i \\ &\leq (1+\sum_{i=1}^{k-1} |A_i(z)|) \sup_{1\leq j\leq k} \left\{ \left(\frac{1}{1-r}\right)^{j(2+\varepsilon)} [T(s(r),f)]^j \right\} \end{aligned}$$

we have $|z| = r < 1$ so $\frac{1}{1-r} > 1$, then

$$\sup_{1\leq j\leq k} \left\{ \left(\frac{1}{1-r}\right)^{j(2+\varepsilon)} [T(s(r),f)]^j \right\} = \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(r),f)]^k$$

so

$$|A_0(z)| \leq (1+\sum_{i=1}^{k-1} |A_i(z)|) \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(r),f)]^k \tag{2.0.9}$$

for all $z \in \Delta$ such that $|z| = r \in E_2 \diagdown E_3$, combining (2.0.9) and (2.0.7), we obtain

$$\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-r} \right) \right)^{\mu} \right\} < \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(|z|),f)]^k \tag{2.0.10}$$

we put $s(r) = R$, so $1 - |z| = \frac{1}{d}(1-R)$ and $\int\limits_{E_3} \frac{dR}{1-R} < \infty$, so we have $R \in d(E_2 \diagdown E_3) + 1 - d$ and $\{d(E_2 \diagdown E_3) + 1 - d\}$ is of infinite logarithmic measure, then we can write (2.0.10) in the following form

$$\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \leq \left(\frac{d}{1-R}\right)^{k(2+\varepsilon)} [T(R,f)]^k . \tag{2.0.11}$$

By (2.0.11) if we take $k = 1$, we have

$$\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \leq \left(\frac{d}{1-R}\right)^{2+\varepsilon} T(R,f)$$

$$\Rightarrow \log^+_{p+1} \left[\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \right] \leq \log^+_{p+1} \left[\left(\frac{d}{1-R}\right)^{2+\varepsilon} T(R,f) \right].$$

From Lemma 1.4.1 (a), we have

$$\log_{p+1} \left[\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \right] \leq \log^+_{p+1} \left[\left(\frac{d}{1-R}\right)^{2+\varepsilon} T(R,f) \right]$$

so there exists a constant $c > 0$ such that

$$\log \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \leq \log^+_{p+1} T(R,f) + \log^+_{p+1} \left(\frac{d}{1-R}\right)^{2+\varepsilon} + c$$

$$\Rightarrow \log\lambda + \mu\log_q\left(\frac{d}{1-R}\right) \le \log^+_{p+1} T(R,f) + \log_{p+1}\left(\frac{d}{1-R}\right)^{2+\varepsilon} + c$$

$$\Rightarrow \frac{\log\lambda}{\log_q\left(\frac{1}{1-R}\right)} + \mu\frac{\log_q\left(\frac{d}{1-R}\right)}{\log_q\left(\frac{1}{1-R}\right)} \le \frac{\log^+_{p+1} T(R,f)}{\log_q\left(\frac{1}{1-R}\right)} + \frac{\log_{p+1}\left(\frac{d}{1-R}\right)^{2+\varepsilon}}{\log_q\left(\frac{1}{1-R}\right)} + \frac{c}{\log_q\left(\frac{1}{1-R}\right)}$$

$$\Rightarrow \lim_{R\to 1^-}\sup\left(\frac{\log\lambda}{\log_q\left(\frac{1}{1-R}\right)} + \mu\frac{\log_q\left(\frac{d}{1-R}\right)}{\log_q\left(\frac{1}{1-R}\right)}\right) \le \lim_{R\to 1^-}\sup\left(\frac{\log^+_{p+1} T(R,f)}{\log_q\left(\frac{1}{1-R}\right)} + \frac{\log_{p+1}\left(\frac{d}{1-R}\right)^{2+\varepsilon}}{\log_q\left(\frac{1}{1-R}\right)} + \frac{c}{\log_q\left(\frac{1}{1-R}\right)}\right),$$

then $\rho_{[p+1,q]}(f) \ge \mu$. □

Corollary 2.0.1 *Let$A_0(z), A_1(z), ..., A_{k-1}(z)$ be meromorphic functions in Δ. If there exists $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and some constants $\mu > 0$, $0 \le \beta < \alpha$, for $z \in \gamma$ and $|z| \to 1^-$, such that*

$$|A_0| \ge \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$|A_j| \le \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ j = 1, ..., k-1,$$

then every nontrivial meromorphic solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \ge \mu$.

Proof. In fact, by the assumptions and taking $0 < \lambda < \alpha - \beta$, we obtain

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1}|A_j(z)| + 1}{|A_0(z)|}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$\le \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{(k-1)\exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} + 1}{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}.$$

We pose $X = \left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu} \to +\infty$, $|z| \to 1^-$, so

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1}|A_j(z)| + 1}{|A_0(z)|}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$\leq \lim_{X\to+\infty} \frac{(k-1)\exp_p\{\beta X\}+1}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\}.$$

We have

$$\lim_{X\to+\infty} \frac{(k-1)\exp_p\{\beta X\}+1}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\} = 0,$$

then we get

$$\lim_{\substack{|z|\to1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1}|A_j(z)|+1}{|A_0(z)|} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} = 0 < 1.$$

We applicate the Theorem 2.0.5, we obtain $\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)\geq\mu$. □

Corollary 2.0.2 *Let$A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ. If there exists $w_0\in\partial\Delta$ and a curve $\gamma\subset\Delta$ tending to w_0 and two constants $\mu>0$ and $\lambda>0$, such that*

$$\lim_{\substack{|z|\to1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1}|A_j(z)|+1}{|A_0(z)|} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1$$

for all $p,q\in\mathbb{N}$ such that $p\geq q\geq 1$, then every nontrivial solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)\geq\mu$.

By using similar proof of Theorem 2.0.5, we easily obtain Corollary 2.0.2.

Corollary 2.0.3 *Let$A_0(z), A_1(z), ..., A_{k-1}(z)$ be analytic functions in Δ. If there exists $w_0\in\partial\Delta$ and a curve $\gamma\subset\Delta$ tending to w_0 and some constants $\mu>0$, $0\leq\beta<\alpha$, for $z\in\gamma$ and $|z|\to1^-$, such that*

$$|A_0| \geq \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$|A_j| \leq \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ j=1,...,k-1,$$

then every nontrivial solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)\geq\mu$.

The proof of Corollary 2.0.3 is similar to the one of Corollary 2.0.1.

Theorem 2.0.6 *Let$A_0(z), A_1(z), ..., A_{k-1}(z)$ be meromorphic functions in Δ. If there exists $w_0\in\partial\Delta$, a curve $\gamma\subset\Delta$ tending to w_0 and $\mu\in(0,+\infty)$, such that*

$$\lim_{\substack{|z|\to1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{m(r,A_i)}}{e^{m(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1, \tag{2.0.12}$$

then every nontrivial meromorphic solution $f(z)$ of (0.0.1) satisfies $\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)\geq\mu$.

Proof. Assume that f is a nontrivial solution of (0.0.1) with $\rho_{[p,q]}(f) = \rho < \infty$.

If $p = q = 1$, then according to Lemma 1.7.3, there exists a set $E_4 \subset [0, 1)$ with $\int_{E_4} \frac{dr}{1-r} < \infty$, such that $z \in \Delta$ satisfying $|z| = r \notin E_4$, we have

$$m(r, \frac{f^{(j)}}{f}) = O\left(\log(\frac{1}{1-r})\right), \ j = 1...k. \tag{2.0.13}$$

From (0.0.1) we can write

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}$$

so

$$m(r, -A_0) = m(r, \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}).$$

According to Proposition 1.5.2 and Remark 1.6.2, we obtain

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{i=1}^{k} m(r, \frac{f^{(i)}}{f}) + O(1). \tag{2.0.14}$$

From (2.0.13) and (2.0.14), we have

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + O\left(\log(\frac{1}{1-r})\right), \ \forall |z| = r \notin E_4. \tag{2.0.15}$$

If $p \geq q > 1$, by Lemma 1.7.4, there exists a set $E_5 \subset [0, 1)$ with $\int_{E_5} \frac{dr}{1-r} < \infty$, such that $z \in \Delta$ satisfying $|z| = r \notin E_5$, for any $\varepsilon > 0$ we have

$$m(r, \frac{f^{(j)}}{f}) = O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right), \ \forall j = 1...k. \tag{2.0.16}$$

By (2.0.14) and (2.0.16) , for all $|z| = r \notin E_5$, we get

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right). \tag{2.0.17}$$

According to assumption (2.0.12) and Lemma 1.7.2, there exists a set $E_6 \subset [0, 1)$ with $\int_{E_6} \frac{dr}{1-r} = \infty$, such that $z \in \Delta$ satisfying $|z| = r \in E_6$, for any $\mu > 0$ we have

$$\frac{\prod_{i=1}^{k-1} e^{m(r,A_i)}}{e^{m(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} < 1$$

so

$$\exp\left(\sum_{i=1}^{k-1} m(r,A_i) - m(r,A_0)\right)\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} < 1,\ \forall |z| = r \in E_6$$

then

$$m(r,A_0) - \sum_{i=1}^{k-1} m(r,A_i) > \exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\},\ \forall |z| = r \in E_6. \tag{2.0.18}$$

Thus
i) if $|z| = r \in E_6 \diagdown E_5$ so (2.0.18) contradicts with (2.0.17), in fact

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} > O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)$$

because we have $p \geq q > 1 \Rightarrow -q < -1 \Rightarrow p-q-1 < p-1$.
ii) if $|z| = r \in E_6 \diagdown E_4$ so (2.0.18) contradicts with (2.0.15), in fact

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} > O\left(\log(\frac{1}{1-r})\right).$$

Therefore, $\rho_{[p,q]}(f) = \infty$. After that, from Lemma 1.7.3 there exists a set $E_7 \subset [0,1)$ with $\int_{E_7} \frac{dr}{1-r} < \infty$, such that $z \in \Delta$ satisfying $|z| = r \notin E_7$, we have

$$m(r, \frac{f^{(j)}}{f}) \leq O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right). \tag{2.0.19}$$

According to (2.0.14) and (2.0.19), we have

$$m(r,A_0) \leq \sum_{i=1}^{k-1} m(r,A_i) + O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right),\ \ |z| = r \notin E_7. \tag{2.0.20}$$

If $|z| = r \in E_6 \diagdown E_7$ with $\int_{E_6 \diagdown E_7} \frac{dr}{1-r} = \infty$, combining (2.0.18) and (2.0.20), we get

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq O\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right). \tag{2.0.21}$$

Then by (2.0.21) there exists a constant k, such that

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq k\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right)$$

$$\Rightarrow \log_p^+\left(\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right) \leq \log_p^+\left(k\left(\log^+ T(r,f) + \log(\frac{1}{1-r})\right)\right).$$

According to Lemma 1.4.1 (a)

$$\log_p\left(\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right)\leq\log_p^+\left(k\left(\log^+T(r,f)+\log(\frac{1}{1-r})\right)\right)$$

and by Proposition 1.5.2

$$\log\left(\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right)\leq\log_{p+1}^+T(r,f)+\log_p^+\log(\frac{1}{1-r})+c$$

$$\Rightarrow\log\lambda+\log\left(\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right)\leq\log_{p+1}^+T(r,f)+\log_p^+\log(\frac{1}{1-r})+c$$

$$\Rightarrow\log\lambda+\mu\log_q\left(\frac{1}{1-r}\right)\leq\log_{p+1}^+T(r,f)+\log_p^+\log(\frac{1}{1-r})+c$$

$$\Rightarrow\frac{\log\lambda}{\log_q\left(\frac{1}{1-r}\right)}+\mu\frac{\log_q\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\leq\frac{\log_{p+1}^+T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{\log_{p+1}(\frac{1}{1-r})}{\log_q\left(\frac{1}{1-r}\right)}+\frac{c}{\log_q\left(\frac{1}{1-r}\right)}$$

$$\Rightarrow\lim_{|z|\to1^-}\sup\left(\frac{\log\lambda}{\log_q\left(\frac{1}{1-r}\right)}+\mu\frac{\log_q\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\right)$$

$$\leq\lim_{|z|\to1^-}\sup\left(\frac{\log_{p+1}^+T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{\log_{p+1}(\frac{1}{1-r})}{\log_q\left(\frac{1}{1-r}\right)}+\frac{c}{\log_q\left(\frac{1}{1-r}\right)}\right),$$

then we get $\rho_{[p+1,q]}(f)\geq\mu$. □

Corollary 2.0.4 *Let*$A_0(z),A_1(z),...,A_{k-1}(z)$ *be meromorphic functions in* Δ. *If there exists* $w_0\in\partial\Delta$ *and a curve* $\gamma\subset\Delta$ *tending to* w_0 *and some constants* $0\leq\beta<\alpha$ *and* $\mu\in(0,+\infty)$, *such that*

$$m(r,A_0)\geq\exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$m(r,A_i)\leq\exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ i=1,...,k-1.$$

where $z\in\gamma$ *and* $|z|\to1^-$, *then every nontrivial meromorphic solution* $f(z)$ *of (0.0.1) satisfies* $\rho_{[p,q]}(f)=\infty$ *and* $\rho_{[p+1,q]}(f)\geq\mu$.

Proof. In fact, by the assumptions, we have

$$\lim_{\substack{|z|\to1^-\\ z\in\gamma}}\frac{\prod_{i=1}^{k-1}e^{m(r,A_i)}}{e^{m(r,A_0)}}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\leq$$

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$= \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\left(\exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\right)^{k-1}}{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}.$$

We pose $X=\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\to+\infty$, $|z|\to 1^-$, so

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{m(r,A_i)}}{e^{m(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \leq \lim_{X\to+\infty} \frac{\left(\exp_p\{\beta X\}\right)^{k-1}}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\}$$

if $p=1$, we have for $0<\lambda<\alpha-(k-1)\beta$ and $0\leq (k-1)\beta<\alpha$

$$\begin{aligned}\lim_{X\to+\infty} \frac{\left(\exp\{\beta X\}\right)^{k-1}}{\exp\{\alpha X\}} \exp\{\lambda X\} &= \lim_{X\to+\infty} \frac{\exp\{(k-1)\beta X\}}{\exp\{\alpha X\}} \exp\{\lambda X\} \\ &= \lim_{X\to+\infty} \exp\{(k-1)\beta X+\lambda X-\alpha X\} \\ &= 0\end{aligned}$$

so in this case we get

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{m(r,A_i)}}{e^{m(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} = 0 < 1.$$

If $p\geq 2$, we have for $0<\lambda<\alpha$ and $0\leq\beta<\alpha$

$$\lim_{X\to+\infty} \frac{\left(\exp_p\{\beta X\}\right)^{k-1}}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\} = \lim_{X\to+\infty} \frac{\exp\left\{(k-1)\exp_{p-1}\{\beta X\}\right\}}{\exp\left\{\exp_{p-1}\{\alpha X\}\right\}} \exp\left\{\exp_{p-1}\{\lambda X\}\right\}$$

$$\begin{aligned}&= \lim_{X\to+\infty} \exp\left\{(k-1)\exp_{p-1}\{\beta X\}+\exp_{p-1}\{\lambda X\}-\exp_{p-1}\{\alpha X\}\right\} \\ &= \lim_{X\to+\infty} \exp\left\{\exp_{p-1}\{\alpha X\}\left((k-1)\frac{\exp_{p-1}\{\beta X\}}{\exp_{p-1}\{\alpha X\}}+\frac{\exp_{p-1}\{\lambda X\}}{\exp_{p-1}\{\alpha X\}}-1\right)\right\}=0.\end{aligned}$$

Then, we obtain

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{m(r,A_i)}}{e^{m(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} = 0 < 1.$$

Next, we applicate Theorem 2.0.6 we get Corollary 2.0.4. □

Corollary 2.0.5 *Let*$A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* Δ, *for* $\mu \in (0,+\infty)$ *we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_0) = \mu$$

If there exists $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and a reel constant* $\lambda > 0$ *such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1,$$

then every nontrivial solution $f(z)$ *of (0.0.1) satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \mu$.

By using the same argument as for the proof in the Theorem 2.0.6 and Lemma 1.7.5 we obtain the Corollary 2.0.5, thinking in count that for an analytic function f, we have $T(r, f) = m(r, f)$.

Corollary 2.0.6 *Let*$A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* Δ, *for* $\mu \in (0,+\infty)$ *we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_0) = \mu.$$

If there exists $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and some constants* $0 \leq \beta < \alpha$, *such that*

$$T(r, A_0) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ i = 1, ..., k-1,$$

where $z \in \gamma$ *and* $|z| \to 1^-$, *then every nontrivial solution* $f(z)$ *of (0.0.1) satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \mu$.

By using proof to Corollary 2.0.4, we easily obtain Corollary 2.0.6.
Next, the coefficient $A_0(z)$ is the dominant coefficient in Theorems 2.0.5 and 2.0.6. A natural question is how to characterize the $[p, q]$-order of growth of solutions of the equation (0.0.1) when $A_s(z)$ dominates the other coefficients near the unit disc's boundary. We continue to look into this new subject in this work, looking at the growth of solutions of equation (0.0.1) when the coefficients $A_s(z)$ $(s = 0, 1, ..., k-1)$ is the dominant coefficient, so we obtain the following results.

Theorem 2.0.7 *Let*$A_0(z), A_1(z), ..., A_{k-1}(z)$ *be meromorphic functions in* Δ. *If there exists* $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and some constants* $\lambda > 0$ *and* $\mu > 0$ *such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i\neq s} e^{m(r,A_i)}}{e^{m(r,A_s)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1. \qquad (2.0.22)$$

Then, every nontrivial meromorphic solution $f(z)$ *of (0.0.1), such that* $f^{(n)}(z)$ *has finite many zeros for all* $n < s$, $(n = 0, ..., s-1)$, *satisfies* $\rho_{[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) \geq \mu$.

Proof. Assume that f is a nontrivial solution of (0.0.1) with finite $[p,q]$ −order, $\rho_{[p,q]}(f) = \rho < \infty$.

a) If $p = q = 1$, by Lemma 1.7.1, there exists a set $E_8 \subset [0,1)$, with $\int\limits_{E_8} \frac{dr}{1-r} < \infty$, such that $r \notin E_8$.

a.i) If $s+1 \le j \le k$, for all $\varepsilon > 0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le \left(\frac{1}{1-r}\right)^{(j-s)(\rho+2+\varepsilon)}, \tag{$*$}$$

so

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le O\left(\left(\frac{1}{1-r}\right)^{(j-s)(\rho+2+\varepsilon)}\right).$$

By $(*)$, we have

$$\begin{aligned} m(r,\frac{f^{(j)}}{f^{(s)}}) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f^{(j)}(re^{i\varphi})}{f^{(s)}(re^{i\varphi})}\right| d\varphi \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(\frac{1}{1-r}\right)^{(j-s)(\rho+2+\varepsilon)} d\varphi \\ &\le (j-s)(\rho+2+\varepsilon)\log^+\left(\frac{1}{1-r}\right) \end{aligned}$$

then

$$m(r,\frac{f^{(j)}}{f^{(s)}}) \le O\left(\log^+\left(\frac{1}{1-r}\right)\right),\ r \to 1^-. \tag{2.0.23}$$

a.ii) If $0 \le j \le s-1$, we use the first fundamental theorem of Nevanlinna, such that $f^{(j)}$ has just finite many zeros, we obtain

$$\begin{aligned} T(r,\frac{f^{(j)}}{f^{(s)}}) &= T(r,\frac{f^{(s)}}{f^{(j)}}) + O(1) \\ &= m(r,\frac{f^{(s)}}{f^{(j)}}) + N(r,\frac{f^{(s)}}{f^{(j)}}) + O(1). \end{aligned}$$

According to the definition of the counting function, we have

$$N(r,\frac{f^{(s)}}{f^{(j)}}) = O(1),$$

so

$$\begin{aligned} m(r,\frac{f^{(j)}}{f^{(s)}}) &\le T(r,\frac{f^{(j)}}{f^{(s)}}) = m(r,\frac{f^{(s)}}{f^{(j)}}) + O(1) \\ &\le O\left(\log^+\frac{1}{1-r}\right). \end{aligned} \tag{2.0.24}$$

From (0.0.1)

$$
\begin{aligned}
-A_s(z) &= \frac{f^{(k)}(z)}{f^{(s)}(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(s)}(z)} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}(z)}{f^{(s)}(z)} \\
&\quad + A_{s-1}(z)\frac{f^{(s-1)}(z)}{f^{(s)}(z)} + \cdots + A_0(z)\frac{f'(z)}{f^{(s)}(z)}
\end{aligned}
$$

which implies

$$
m(r,-A_s) = m\left(r, \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_0\frac{f'}{f^{(s)}}\right).
$$

By Lemma 1.4.1 and Proposition 1.5.2

$$
m(r,A_s) \leq \sum_{i\neq s} m(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + O(1). \tag{2.0.25}
$$

Combining (2.0.24), (2.0.25) and (2.0.23), we have

$$
\begin{aligned}
m(r,A_s) &\leq \sum_{i\neq s} m(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + O(1) \\
&\leq \sum_{i\neq s} m(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(s)}}{f^{(j)}}\right) + O(1)
\end{aligned}
$$

which implies

$$
m(r,A_s) \leq \sum_{i\neq s} m(r,A_i) + O\left(\log^+\left(\frac{1}{1-r}\right)\right). \tag{2.0.26}
$$

b) If $p \geq q > 1$ applying Lemma 1.7.4, there exists a set $E_9 \subset [0,1)$, with $\int_{E_9} \frac{dr}{1-r} < \infty$ such that $r \notin E_9$, we have

$$
m(r,\frac{f^{(j)}}{f}) = O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right),\ j=1...k
$$

f just has finite many zeros.

b.i) When $s+1 \leq j \leq k$, by the definition of the counting function , we get

$$
N(r,\frac{f^{(j)}}{f}) = O(1).
$$

So

$$\begin{aligned} m(r,\frac{f^{(j)}}{f^{(s)}}) &\leq m(r,\frac{f^{(j)}}{f})+m(r,\frac{f}{f^{(s)}}) \\ &\leq O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)+T(r,\frac{f^{(s)}}{f}) \\ &\leq O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)+m(r,\frac{f^{(s)}}{f})+N(r,\frac{f^{(s)}}{f}) \\ &\leq O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right), \end{aligned}$$

then

$$m(r,\frac{f^{(j)}}{f^{(s)}})\leq O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right). \qquad (2.0.27)$$

b.ii) When $0 \leq j \leq s-1$, combining (2.0.24), (2.0.25) and (2.0.27), for all $r \notin E_9$, we obtain

$$m(r,A_s)\leq\sum_{i\neq s}m(r,A_i)+O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right). \qquad (2.0.28)$$

From (2.0.22) and Lemma 1.7.2, there exists a set $E_{10}\subset[0,1)$, with $\int_{E_{10}}\frac{dr}{1-r}<\infty$, such that $r\in E_{10}$, and for all $\mu>0$, we have

$$\frac{\prod_{i\neq s}e^{m(r,A_i)}}{e^{m(r,A_s)}}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}<1$$

which implies

$$m(r,A_s)-\sum_{i\neq s}m(r,A_i)>\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \qquad (2.0.29)$$

Thus
i) If $r\in E_{10}\diagdown E_9$, (2.0.29) contradicts with (2.0.28), in fact

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}>O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\rho+\varepsilon}\right)$$

because we have $p\geq q>1\Rightarrow -q<-1\Rightarrow p-q-1<p-1$.
ii) If $r\in E_{10}\diagdown E_8$, (2.0.29) contradicts with (2.0.26), in fact

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}>O\left(\log^{+}\left(\frac{1}{1-r}\right)\right).$$

Then $\rho_{[p,q]}(f)=\infty$. After that, by Lemma 1.7.1, there exists a set $E_{11}\subset[0,1)$, with $\int\limits_{E_{11}}\frac{dr}{1-r}<\infty$, such that $r\notin E_{11}$, we have if $s+1\leq j\leq k$ so for all $\varepsilon>0$ and $d\in(0,1)$, such that $s(r)=1-d(1-r)$, we obtain

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right|\leq\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r},T(s(r),f)\right\}\right)^{j-s},\ |z|=r. \qquad (**)$$

By $(**)$, we have

$$\begin{aligned}
m(r,\frac{f^{(j)}}{f^{(s)}}) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{f^{(j)}(re^{i\varphi})}{f^{(s)}(re^{i\varphi})}\right|d\varphi\\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r},T(s(r),f)\right\}\right)^{j-s}d\varphi\\
&\leq \log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r},T(s(r),f)\right\}\right)^{j-s}
\end{aligned}$$

also, we have $\lim\limits_{r\to1^-}\frac{s(r)}{r}=1$, so $\log\frac{1}{1-r}>1$ when $r\to1^-$ and $T(s(r),f)=T(r,f)$, we obtain

$$\begin{aligned}
m(r,\frac{f^{(j)}}{f^{(s)}}) &\leq \log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\left\{\log\frac{1}{1-r}+T(s(r),f)\right\}\right)^{j-s}\\
&\leq (j-s)\log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\left\{\log\frac{1}{1-r}+T(s(r),f)\right\}\right)\\
&\leq (j-s)\left[\log^+\left(\frac{1}{1-r}\right)^{2+\varepsilon}+\log^+\log\frac{1}{1-r}+\log^+T(s(r),f)+O(1)\right]\\
&\leq (j-s)\left[(2+\varepsilon)\log^+\left(\frac{1}{1-r}\right)+\log_2\frac{1}{1-r}+\log^+T(s(r),f)+O(1)\right]\\
&\leq (j-s)(2+\varepsilon)\left[\log^+\left(\frac{1}{1-r}\right)+\log_2\frac{1}{1-r}+\log^+T(s(r),f)+O(1)\right]\\
&\leq (j-s)(2+\varepsilon)\left[\log^+\left(\frac{1}{1-r}\right)+o\left(\log^+\frac{1}{1-r}\right)+\log^+T(s(r),f)+O(1)\right]\\
&\leq (j-s)(2+\varepsilon)\left[\log^+\left(\frac{1}{1-r}\right)+O\left(\log^+\frac{1}{1-r}\right)+\log^+T(s(r),f)\right]\\
&\leq O\left(\log^+T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right),\ r\to1^-.
\end{aligned}$$

Then

$$m(r,\frac{f^{(j)}}{f^{(s)}})\leq O\left(\log^+T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right),\ r\to1^-. \qquad (2.0.30)$$

If $0 \leq j \leq s-1$, combining (2.0.24), (2.0.25), (2.0.29) and (2.0.30), we get

$$m(r, A_s) \leq \sum_{i \neq s} m(r, A_i) + O\left(\log^+ T(r,f) + \log^+\left(\frac{1}{1-r}\right)\right), \ r \notin E_{12}$$

so

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq O\left(\log^+ T(r,f) + \log^+\left(\frac{1}{1-r}\right)\right), \ r \in E_{11} \diagdown E_{12} \quad (2.0.31)$$

such that $E_{10} \diagdown E_{11}$ is of infinite logarithmic measure, then from (2.0.31), we obtain

$$\rho_{[p+1,q]}(f) = \lim_{r \to 1^-} \sup \frac{\log^+_{p+1} T(r,f)}{\log_q\left(\frac{1}{1-r}\right)} \geq \mu$$

because there exists a constant k, such that

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq k\left(\log^+ T(r,f) + \log^+\left(\frac{1}{1-r}\right)\right)$$

$$\Rightarrow \log^+_p \exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq \log^+_p\left(k\left(\log^+ T(r,f) + \log^+\left(\frac{1}{1-r}\right)\right)\right)$$

according to Lemma 1.4.1 (a) and Proposition 1.6.4

$$\log_p \exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq \log^+_{p+1} T(r,f) + \log^+_{p+1}\left(\frac{1}{1-r}\right) + c$$

such that c is a positive constant, then

$$\log \lambda + \mu \log_q\left(\frac{1}{1-r}\right) \leq \log^+_{p+1} T(r,f) + \log_{p+1}\left(\frac{1}{1-r}\right) + c$$

$$\Rightarrow \frac{\log \lambda}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\mu \log_q\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} \leq \frac{\log^+_{p+1} T(r,f)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_{p+1}\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)}$$

$$\Rightarrow \lim_{r \to 1^-} \sup\left(\frac{\log \lambda}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\mu \log_q\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\right)$$

$$\leq \lim_{|z| \to 1^-} \sup\left(\frac{\log^+_{p+1} T(r,f)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{\log_{p+1}\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)} + \frac{c}{\log_q\left(\frac{1}{1-r}\right)}\right).$$

Therefore $\mu \leq \rho_{[p+1,q]}(f)$. □

Corollary 2.0.7 *Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be meromorphic functions in* Δ. *If there exists* $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and some constants* $0 \leq \beta < \alpha$ *and* $\mu > 0$, *such that*

$$m(r, A_s) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$m(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ i = 1, ..., s-1, s+1, ..., k-1,$$

where $z \in \gamma$ *and* $|z| \to 1^-$,*then every nontrivial meromorphic solution* $f(z)$ *of (0.0.1), such that* $f^{(n)}(z)$ *has finite many zeros for all* $n < s$, $(n = 0, ..., s-1)$, *satisfies* $\rho_{[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) \geq \mu$.

The proof of Corollary 2.0.7 is similar to the one of Corollary 2.0.4.

Corollary 2.0.8 *Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* Δ, *for any constant* $\mu \in (0, +\infty)$, *we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., s-1, s+1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_s) = \mu$$

If there exists $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and a reel constant* $\lambda > 0$, *such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1$$

Then, every nontrivial solution $f(z)$ *of (0.0.1), such that* $f^{(n)}(z)$ *has finite many zeros for all* $n < s$, $(n = 0, ..., s-1)$, *satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \mu$.

By using the same arguments as for the proof of the Theorem 2.0.7 and Lemma 1.7.5, we obtain Corollary 2.0.8, thinking in count that for f an analytic function $T(r, f) = m(r, f)$.

Corollary 2.0.9 *Let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be analytic functions in* Δ,*for any constant* $\mu \in (0, +\infty)$, *we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., s-1, s+1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_s) = \mu.$$

If there exists $w_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* w_0 *and some constants* $0 \leq \beta < \alpha$, *such that*

$$T(r, A_s) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, i = 1, ..., s-1, s+1, ..., k-1,$$

where $z \in \gamma$ *and* $|z| \to 1^-$, *then every nontrivial solution* $f(z)$ *of (0.0.1), such that* $f^{(n)}(z)$ *has finite many zeros for all* $n < s$, $(n = 0, ..., s-1)$, *satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and* $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \mu$.

The proof of Corollary 2.0.9 is similar to the one of Corollary 2.0.4.

2.1 Examples

Example 2.1.1 *The following example shows that the Theorem 2.0.5 is sharp. Consider the following equation*

$$f'' + A_1(z)f' + A_0(z)f = 0, \qquad (\mathcal{F}\ 1)$$

where $A_0(z) = \frac{7}{z}\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$ *and* $A_1(z) = \frac{1}{z}\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$. *In this case, we have* A_0 *dominates* A_1.

Then, for $w_0 = 1 \in \partial\Delta$ *and a curve* $\gamma = \{z \in \Delta : \arg z = 0\} \subset \Delta$ *tending to* $w_0 = 1$ *and two constants* $\mu = 7 > 0$ *and* $\lambda = 1 > 0$ *and for* $p = q = 7$, *we have*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{|A_1(z)|+1}{|A_0(z)|}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$= \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\left|\frac{1}{z}\right|\left|\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right|+1}{7\left|\frac{1}{z}\right|\left|\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right|}\times$$
$$\exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}$$

$$\begin{aligned}
&= \lim_{r\to 1^-} \frac{\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r}{7\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}}\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\} \\
&= \lim_{r\to 1^-} \frac{1}{7}\left(\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r\right)\times \\
&\quad \exp\left\{\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7 - 2\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\} \\
&= \lim_{r\to 1^-} \frac{1}{7}\left(\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r\right)\exp\left\{-\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\} \\
&= \lim_{r\to 1^-} \frac{1}{7}\left(\begin{array}{c}\exp\left\{\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7 - \exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+ \\ r\exp\left\{-\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}\end{array}\right) \\
&= \frac{1}{7}
\end{aligned}$$

So

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{|A_1(z)|+1}{|A_0(z)|}\exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\} = \frac{1}{7} < 1.$$

From Theorem 2.0.5 we obtain every nontrivial meromorphic or analytic solution $f(z)$ of ($\mathcal{F}1$) satisfies $\rho_{[7,7]}(f) = \rho_{M,[7,7]}(f) = \infty$ and $\rho_{[8,7]}(f) = \rho_{M,[8,7]}(f) \geq 7$.

Example 2.1.2 *The following example shows that the Corollary 2.0.1 is sharp. Consider the following equation:*

$$f'' + A_1(z)f' + A_0(z)f = 0, \qquad (\mathcal{F}2)$$

where $A_0(z) = 7K_0(z)\exp_7\left\{2\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$ and $A_1(z) = K_1(z)\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$, such that K_0, K_1 are meromorphic functions that satisfy

$$\begin{cases} |K_1| \leq 1 \\ |K_0| \geq 1 \end{cases}$$

In this case, we have A_0 dominates A_1.
Then, for $w_0 = 1 \in \partial\Delta$ and a curve $\gamma = \{z \in \Delta : \arg z = 0\} \subset \Delta$ tending to $w_0 = 1$ and two constants $\mu = 7 > 0$ and $\lambda = 1 > 0$ and for $p = q = 7$, we have

$$\begin{aligned} |A_0| &= 7|K_0(z)|\left|\exp_7\left\{2\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right| \\ &\geq \exp_7\left\{2\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\} \end{aligned}$$

and

$$\begin{aligned} |A_1| &= |K_1(z)|\left|\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right| \\ &\leq \exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}. \end{aligned}$$

From Corollary 2.0.1 (when $\alpha = 2$ and $\beta = 1$) we obtain every nontrivial meromorphic or analytic solution $f(z)$ of ($\mathcal{F}2$) satisfies $\rho_{[7,7]}(f) = \rho_{M,[7,7]}(f) = \infty$ and $\rho_{[8,7]}(f) = \rho_{M,[8,7]}(f) \geq 7$.

CONCLUSION

In this thesis, we confirmed the usefulness of powerful Nevanlinna theory tools such as the characteristic function and the first fundamental theorem of Nevanlinna. These techniques helped us to improve on several results obtained by other researchers concerning the following linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_i(z)$ are analytic or meromorphic in the unit disc $\Delta = \{z : |z| < 1\}$, $i = 0, 1, ..., k-1, k \geq 2$.
In the first instance, when A_0 dominates the other coefficients near a point on the boundary of Δ, we gave the statement of theorems of Hamouda.
Secondly, we investigated the growth of solutions of differential linear equations of $[p, q]$-order.
In the final stages of this project, we considered generalizing some of the above-mentioned results by assuming A_s dominates the other coefficients near a point on the boundary of Δ.
A natural question: Is it possible to generalize the results of previous theorems if the equation is non-homogeneous? If we look at the linear differential equations of the following form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = K(z),$$

where $A_i(z), K(z)$ are analytic or meromorphic functions in the unit disc $\Delta = \{z : |z| < 1\}$, $i = 0, 1, ..., k-1, k \geq 2$.
Is it possible to use the same approach as in this thesis and Hamouda's paper, namely that one coefficient dominates the other coefficients near a point on the boundary of Δ?
Can we improve the results found in this thesis when the coefficients $A_i(z)$ are entire functions?

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