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MASTER

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INTRODUCTION

In 1925, the Nevanlinna's theory or values distribution theory of meromorphic functions founded by the famous mathematician Rolf Nevanlinna, has a very important role in studying the growth of solutions of linear differential equations in the complex plane.

Bernal [7] was the first to define the concept of iterated order to express the growth of solutions of the complex linear differential equation $(k \geqslant 2)$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{0.0.1}$$

where $A_i \not\equiv 0$ $(i = 0, 1, ..., k-1)$ are analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. After that, the iterated order of solution of higher order equation was investigated by Cao in [8], he extend the results of Yang [11], Belaïdi [3] on $\mathbb{C}$ and obtained some results concerns equations of the form

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0. \tag{0.0.2}$$

In addition, Cao and Yi [9] obtained several precise theorems about the hyper order, the hyper convergence exponent of zero points and fixed points of solutions of homogeneous linear differential equations in D.

Recently, Chen *et al.* in [10] utilize iteration to investigated the growth and fixed points of solutions and their arbitrary-order of higher-order linear differential equations (0.0.1) and (0.0.2) in D.

Many results on $[p, q]$-order of solutions have been found by different researchers in D.

This thesis consists of an introduction, three chapters and conclusion.

The first chapter presents an introduction to the theory of Nevanlinna, in which we give some fundamental notions, notations, definitions and results that we will need in the other chapters.

In the second chapter we will prove some results concerning the study of the growth of $[p, q]$-order solutions of linear differential equations in the unit disc with analytic coefficients.

In the last chapter we will prove some results concerning the study of the $[p, q]$-exponent of convergence of fixed point solutions and their arbitrary-order of differential equations studied in the second chapter.

Chapter 1

Some Elements of Nevanlinna's Theory

The theory of Nevanlinna is the main tool used throughout this thesis. This provides a way to analyze the meromorphic functions. For this reason, in this chapter, we will give the Jensen, Poisson and Poisson-Jensen formulas. Next, we define the functions $N(r,a,f), m(r,a,f), T(r,a,f)$ (for $a=\infty$ and $a\in\mathbb{C}$) and mention their properties. After that, we're going to state the first fundamental theorem of Nevanlinna which is a consequence of Jensen's formula and we give some necessary results as well as Cartan's theorem and its corollaries. We conclude this chapter by giving a definition of the $[p,q]$-order, the $[p,q]$-exponent of convergence of a function and the density of a set.

For more details, consult the references $([2],[15],[16],[18],[20],[21])$.

1.1 Jensen formula

Theorem 1.1.1 ([18]) *Let f be a meromorphic function such that $f(0)\neq 0,\infty$ and $a_1, a_{2,\ldots}$(resp. $b_1, b_{2,\ldots}$), its zeros (resp. its poles), each taken into account according to its multiplicity. Then*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\varphi}\right)\right|d\varphi + \sum_{|b_j|<r}\log\frac{r}{|b_j|} - \sum_{|a_j|<r}\log\frac{r}{|a_j|}.$$

Proof. We give the proof for the case that f has no zeros and no poles on the circle $|z|=r$.

Consider the function

$$g(z)=f(z)\frac{\prod\limits_{|a_j|<r}\dfrac{r^2-\overline{a_j}z}{r(z-a_j)}}{\prod\limits_{|b_j|<r}\dfrac{r^2-\overline{b_j}z}{r(z-b_j)}}.$$

Then, $g\neq 0,\infty$ in the disc $|z|\leqslant r$, hence $\ln|g(z)|$ is a harmonic function. By the mean formula of harmonic functions, we have

$$\ln|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}\ln\left|g\left(re^{i\varphi}\right)\right|d\varphi. \tag{1.1.1}$$

On the other hand,

$$|g(0)|=|f(0)|\frac{\prod\limits_{|a_j|<r}\dfrac{r}{|a_j|}}{\prod\limits_{|b_j|<r}\dfrac{r}{|b_j|}},$$

from which

$$\log|g(0)|=\log|f(0)|+\sum_{|a_j|<r}\log\frac{r}{|a_j|}-\sum_{|b_j|<r}\log\frac{r}{|b_j|}. \tag{1.1.2}$$

For $z=re^{i\varphi}$, we have for all a_j and b_j

$$\left|\frac{r^2-\overline{a_j}z}{r(z-a_j)}\right|=\left|\frac{z\bar{z}-\overline{a_j}z}{r(z-a_j)}\right|=\left|\frac{z\left(\overline{z-a_j}\right)}{r(z-a_j)}\right|=1=\left|\frac{r^2-\overline{b_j}z}{r(z-b_j)}\right|.$$

Hence

$$\left|g\left(re^{i\varphi}\right)\right|=\left|f\left(re^{i\varphi}\right)\right|. \tag{1.1.3}$$

Applying (1.1.2) and (1.1.3) to (1.1.1), we obtain the Jensen formula. □

Theorem 1.1.2 *(Poisson formula) Let f be an analytic function in the disc $|\xi|\leqslant R$ $(0<R<\infty)$. Then, for $z=re^{i\theta}$ and $r<R$, $\theta\in[0,2\pi]$, we have*

$$\mathrm{Re}f\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}\mathrm{Re}f\left(Re^{i\varphi}\right)d\varphi.$$

Proof. Let f be an analytic function in the disc $|\xi|<R$. According to Cauchy's integral formula, we have

$$f(z)=\frac{1}{2\pi i}\oint\limits_{|\xi|=R}\frac{f(\xi)}{\xi-z}d\xi.$$

Let $z^* = \frac{R^2}{\bar{z}}$ be the symmetric point to z with respect to the circle $|\xi| = R$. From the Cauchy formula, the function $\dfrac{f(\xi)}{\xi - z^*}$ is analytic in $|\xi| \leqslant R$ and $\dfrac{1}{2\pi i} \oint\limits_{|\xi|=R} \dfrac{f(\xi)}{\xi - z^*} d\xi = 0$, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint\limits_{|\xi|=R} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint\limits_{|\xi|=R} \frac{f(\xi)}{\xi - z^*} d\xi \\ &= \frac{1}{2\pi i} \oint\limits_{|\xi|=R} \left(\frac{1}{\xi - z} - \frac{1}{\xi - z^*} \right) f(\xi)\, d\xi. \end{aligned}$$

We have

$$\frac{1}{\xi - z} - \frac{1}{\xi - z^*} = \frac{1}{\xi - z} - \frac{1}{\xi - \dfrac{R^2}{\bar{z}}} = \frac{1}{\xi - z} - \frac{\bar{z}}{\xi \bar{z} - R^2}.$$

Pose $\xi = Re^{i\varphi} \Rightarrow d\xi = Rie^{i\varphi} d\varphi$, so

$$\begin{aligned} f\left(re^{i\theta}\right) &= \frac{1}{2\pi i} \int_0^{2\pi} \left[\frac{1}{Re^{i\varphi} - re^{i\theta}} - \frac{re^{-i\theta}}{Re^{i\varphi} re^{-i\theta} - R^2} \right] f\left(Re^{i\varphi}\right) Rie^{i\varphi} d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{Re^{i\varphi}}{Re^{i\varphi} - re^{i\theta}} - \frac{re^{i(\varphi-\theta)}}{re^{i(\varphi-\theta)} - R} \right] f\left(Re^{i\varphi}\right) d\varphi, \end{aligned}$$

we have

$$\begin{aligned} \frac{Re^{i\varphi}}{Re^{i\varphi} - re^{i\theta}} - \frac{re^{i(\varphi-\theta)}}{re^{i(\varphi-\theta)} - R} &= \frac{R}{R - re^{-i(\varphi-\theta)}} - \frac{re^{i(\varphi-\theta)}}{re^{i(\varphi-\theta)} - R} \\ &= \frac{R}{R - re^{-i(\varphi-\theta)}} + \frac{re^{i(\varphi-\theta)}}{R - re^{i(\varphi-\theta)}} \\ &= \frac{R^2 - rRe^{i(\varphi-\theta)} + rRe^{i(\varphi-\theta)} - r^2}{R^2 - rR\left(e^{i(\varphi-\theta)} + e^{-i(\varphi-\theta)}\right) + r^2} \\ &= \frac{R^2 - r^2}{R^2 - 2rR\cos(\varphi - \theta) + r^2}. \end{aligned}$$

Therefore

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} f\left(Re^{i\varphi}\right) d\varphi,$$

by taking the real part of $f(z)$, we obtain

$$\mathrm{Re} f\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \mathrm{Re} f\left(Re^{i\varphi}\right) d\varphi.$$

$\square$

Theorem 1.1.3 ([21]) *(Poisson-Jensen formula) Let f be a meromorphic function in the disc $|\xi| \leqslant R$ $(0 < R < \infty)$, such that $f(0) \neq 0, \infty$ and $a_1, a_{2,\ldots}$(resp. $b_1, b_{2,\ldots}$), its zeros (resp. its poles), each taken into account according to its multiplicity. Then, for $z = re^{i\theta}$ and $r < R$, we have*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left|f\left(Re^{i\varphi}\right)\right| d\varphi + \sum_{|a_j|<R} \log\left|\frac{R(z - a_j)}{R^2 - \overline{a_j}z}\right| - \sum_{|b_j|<R} \log\left|\frac{R(z - b_j)}{R^2 - \overline{b_j}z}\right|.$$

Proof. Consider the function

$$g(z) = f(z)\frac{\prod\limits_{|b_j|<R} \frac{R(z - b_j)}{R^2 - \overline{b_j}z}}{\prod\limits_{|a_j|<R} \frac{R(z - a_j)}{R^2 - \overline{a_j}z}}.$$

Then, $g \not\equiv 0, \infty$ and $\log g$ is an analytic function in the disc $|z| \leq R$, therefore its real part $\log|g|$ is harmonic function. By applying the Poisson formula, we get

$$\log|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left|g\left(Re^{i\varphi}\right)\right| d\varphi. \tag{1.1.4}$$

For $z = re^{i\varphi}$, we have for all $c \in \mathbb{C}$

$$\left|\frac{R(z - c)}{R^2 - \bar{c}z}\right| = \left|\frac{R(z - c)}{z\bar{z} - \bar{c}z}\right| = \left|\frac{R(z - c)}{z\left(\overline{z - c}\right)}\right| = 1.$$

Hence

$$\left|g\left(Re^{i\varphi}\right)\right| = \left|f\left(Re^{i\varphi}\right)\right|. \tag{1.1.5}$$

On the other hand, we have

$$\log|g(z)| = \log|f(z)| + \sum_{|b_j|<R} \log\left|\frac{R(z - b_j)}{R^2 - \overline{b_j}z}\right| - \sum_{|a_j|<R} \log\left|\frac{R(z - a_j)}{R^2 - \overline{a_j}z}\right|, \tag{1.1.6}$$

substituting (1.1.5) and (1.1.6) into (1.1.4), we obtain

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left|f\left(Re^{i\varphi}\right)\right| d\varphi + \sum_{|a_j|<R} \log\left|\frac{R(z - a_j)}{R^2 - \overline{a_j}z}\right| \\ &\quad - \sum_{|b_j|<R} \log\left|\frac{R(z - b_j)}{R^2 - \overline{b_j}z}\right|.\end{aligned}$$

□

1.2 Characteristic function of Nevanlinna

Definition 1.2.1 ([18]) *Let x be a positive real number. The truncated logarithm* $\log^+$ *is defined by*

$$\log^+ x \quad = \quad \max\{\log x \ , 0\} \quad = \quad \begin{cases} \log x & if \quad x > 1. \\ 0 & if \quad 0 \leqslant x \leqslant 1. \end{cases}$$

Notice that the truncated logarithm defined above is a continuous function and nonnegative on $(0, \infty)$.

Lemma 1.2.1 ([18]) *We have the following properties:*

(**a**) $\log x \leqslant \log^+ x, \qquad (x > 0).$

(**b**) $\log^+ x \leqslant \log^+ y, \qquad (0 < x \leqslant y).$

(**c**) $\log x = \log^+ x - \log^+ \frac{1}{x}, \qquad (x > 0).$

(**d**) $|\log x| = \log^+ x + \log^+ \frac{1}{x}, \qquad (x > 0).$

(**e**) $\log^+ \left(\prod\limits_{i=1}^{n} x_i \right) \leqslant \sum\limits_{i=1}^{n} \log^+ x_i, \qquad (x_i \geqslant 0 \ , 1 \leqslant i \leqslant n).$

(**f**) $\log^+ \left(\sum\limits_{i=1}^{n} x_i \right) \leqslant \sum\limits_{i=1}^{n} \log^+ x_i + \log n, \qquad (x_i \geqslant 0 \ , 1 \leqslant i \leqslant n).$

Proof. (**c**) We have

$$\begin{aligned} \log^+ x - \log^+ \frac{1}{x} \quad &= \quad \max\{\log x, 0\} - \max\left\{\log \frac{1}{x}, 0\right\} \\ &= \quad \max\{\log x, 0\} + \min\left\{-\log \frac{1}{x}, 0\right\} \\ &= \quad \max\{\log x, 0\} + \min\{\log x, 0\} \\ &= \quad \log x. \end{aligned}$$

(d) We have

$$\begin{aligned}\log^+ x + \log^+ \frac{1}{x} &= \max\{\log x, 0\} + \max\left\{\log\frac{1}{x}, 0\right\} \\ &= \max\{\log x, 0\} + \max\{-\log x, 0\} \\ &= \max\{\log x, 0\} - \min\{\log x, 0\} \\ &= |\log x| .\end{aligned}$$

(e) • If $\prod_{i=1}^{n} x_i \leqslant 1$, then the inequality holds trivially.

• If $\prod_{i=1}^{n} x_i > 1$, then

$$\log^+\left(\prod_{i=1}^{n} x_i\right) = \log\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \log x_i \underset{(by\ (a))}{\leqslant} \sum_{i=1}^{n} \log^+ x_i.$$

(f) By (b) and (e) above

$$\log^+\left(\sum_{i=1}^{n} x_i\right) \leqslant \log^+\left(n \max_{1\leqslant i\leqslant n} x_i\right) \leqslant \log^+ n + \log^+\left(\max_{1\leqslant i\leqslant n} x_i\right) \leqslant \log^+ n + \sum_{i=1}^{n} \log^+ x_i.$$

□

Definition 1.2.2 ([18]) *(Unintegrated counting function) Let f be a meromorphic function, not being identically equal to $a \in \mathbb{C}$. We denote by $n(r, a, f)$ the number of the roots of $f(z) = a$ in the disc $|z| < r$, each root according to its multiplicity. Similarly $\bar{n}(r, a, f)$ counts the number of the distincts roots of $f(z) = a$ in the disc $|z| < r$. And we denote by $n(r, \infty, f)$ the number of the poles of f in the disc $|z| < r$, each pole according to its multiplicity. Similarly $\bar{n}(r, \infty, f)$ counts the number of the distincts poles of f in the disc $|z| < r$.*

Example 1.2.1 *Let $f(z) = \cosh z$, we have*

$$n(r, \infty, f) = \bar{n}(r, \infty, f) = 0,$$

because f has no poles. And we have

$$n\left(r,0,f\right)=\bar{n}\left(r,0,f\right)=2\left[\frac{r}{\pi}\right].$$

Example 1.2.2 *Let $f\left(z\right)=\frac{1}{\cosh^{2}z}$, we have*

$$\bar{n}\left(r,\infty,f\right)=2\left[\tfrac{r}{\pi}\right]\quad \textit{and}\quad n\left(r,\infty,f\right)=4\left[\tfrac{r}{\pi}\right],$$

because f has a poles of order 2 at $z_k=i\left(\frac{\pi}{2}+k\pi\right),\forall k\in\mathbb{Z}$.

Definition 1.2.3 ([18]) *Let f be a meromorphic function, we define the a-point function of f by*

$$N(r,a,f)=N\left(r,\frac{1}{f-a}\right):=\int_0^r\frac{n(t,a,f)-n(0,a,f)}{t}dt+n(0,a,f)\log r$$

if $f\not\equiv a\in\mathbb{C}$ and

$$N(r,\infty,f)=N\left(r,f\right):=\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt+n(0,\infty,f)\log r.$$

Similary, we define the a-point distinct function of f by

$$\bar{N}(r,a,f)=\bar{N}\left(r,\frac{1}{f-a}\right):=\int_0^r\frac{\bar{n}(t,a,f)-\bar{n}(0,a,f)}{t}dt+\bar{n}(0,a,f)\log r$$

if $f\not\equiv a\in\mathbb{C}$ and

$$\bar{N}(r,\infty,f)=\bar{N}\left(r,f\right):=\int_0^r\frac{\bar{n}(t,\infty,f)-\bar{n}(0,\infty,f)}{t}dt+\bar{n}(0,\infty,f)\log r.$$

Example 1.2.3 *Let $f\left(z\right)=\frac{\exp(z^n)}{z^2}$, we have*

$$n\left(t,\infty,f\right)=n(0,\infty,f)=2\quad \textit{and}\quad \bar{n}\left(t,\infty,f\right)=\bar{n}(0,\infty,f)=1,$$

then

$$\begin{aligned} N\left(r,f\right) &= \int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt+n(0,\infty,f)\log r\\ &= 2\log r, \end{aligned}$$

and

$$\begin{aligned} \bar{N}\left(r,f\right) &= \int_0^r\frac{\bar{n}(t,\infty,f)-\bar{n}(0,\infty,f)}{t}dt+\bar{n}(0,\infty,f)\log r\\ &= \log r. \end{aligned}$$

Remark 1.2.1 *If f is an analytic or entire function, then $N(r,f) = \bar{N}(r,f) = 0$.*

Example 1.2.4 *Let* $f(z) = \exp\left\{-\frac{i}{1-z}\right\}$ $(|z|<1)$. *f is an analytic function in $|z|<1$, then $N(r,f) = 0$.*

Lemma 1.2.2 ([18]) *Let f be a meromorphic function with a-points $\alpha_1, \alpha_2, ..., \alpha_m$ in the disc $|z| \leqslant r$ such that $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant ... \leqslant |\alpha_m| \leqslant r$, each counted according to its multiplicity. Then*

$$\int_0^r \frac{n(t,a,f)}{t}dt = \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t}dt = \sum_{0<|\alpha_j|\leqslant r} \log\frac{r}{|\alpha_j|}.$$

Proof. Denoting $r_j = |\alpha_j|$ $(j = 1, 2, ..., m)$. Then, we have

$$\begin{aligned}
\sum_{0<|\alpha_j|\leqslant r} \log\frac{r}{|\alpha_j|} &= \sum_{j=1}^{m} \log\frac{r}{r_j} \\
&= \log\left(\frac{r^m}{r_1 \times ... \times r_m}\right) \\
&= \log\left(\frac{r_2}{r_1} \times \frac{r_3^2}{r_2^2} \times ... \times \frac{r_m^{m-1}}{r_{m-1}^{m-1}} \times \frac{r^m}{r_m^m}\right) \\
&= \sum_{j=1}^{m-1} j\left(\log r_{j+1} - \log r_j\right) + m\left(\log r - \log r_n\right) \\
&= \sum_{j=1}^{m-1} j\int_{r_j}^{r_{j+1}} \frac{dt}{t} + m\int_{r_m}^{r} \frac{dt}{t} = \int_0^r \frac{n(t,a,f)}{t}dt.
\end{aligned}$$

□

Proposition 1.2.1 ([18]) *Let f be a meromorphic function with the Laurent expansion at the origin*

$$f(z) = \sum_{i=m}^{+\infty} c_i z^i, \qquad c_m \in \mathbb{C}^*, m \in \mathbb{Z}.$$

Then

$$\log|c_m| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + N(r,f) - N\left(r, \frac{1}{f}\right).$$

Proof. Consider the meromorphic function h, defined by

$$h(z) := f(z)\, z^{-m}, \qquad z \in \mathbb{C}.$$

it is clear that $m = n(0,0,f) - n(0,\infty,f)$ and $h(0) \neq 0, \infty$. Indeed,

If $m > 0$, then $n(0,\infty,f) = 0$ and $m = n(0,0,f)$.

If $m < 0$, then $n(0,0,f) = 0$ and $n(0,\infty,f) = -m$.

If $m = 0$, then $n(0,0,f) = n(0,\infty,f) = 0$.

Hence, the functions h and f have the same poles and zeros in $0 < |z| \leqslant r$. from Jensen's formula and lemma 1.2.2, we have

$$\begin{aligned}
\log|c_m| &= \log|h(0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right) r^{-m}\right| d\varphi + \sum_{0<|b_j|\leqslant r} \log\frac{r}{|b_j|} - \sum_{0<|a_j|\leqslant r} \log\frac{r}{|a_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi - \left[n(0,0,f) - n(0,\infty,f)\right]\log r \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right).
\end{aligned}$$

□

Definition 1.2.4 ([18]) *Let f be a meromorphic function, we define the proximity function of f by*

$$m(r,a,f) = m\left(r,\frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\left|f\left(re^{i\varphi}\right) - a\right|} d\varphi \quad \text{if } f \not\equiv a \in \mathbb{C},$$

and

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi.$$

Example 1.2.5 *Let* $f(z) = \exp\left\{-\frac{i}{1-z}\right\}$ $(|z| < 1)$. *Then, for* $z = re^{i\varphi}\,(r < 1, \varphi \in [0,2\pi])$, *we have*

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left\{-\frac{i}{1-re^{i\varphi}}\right\}\right| d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left\{-\frac{i\left(1-re^{-i\varphi}\right)}{\left|1-re^{i\varphi}\right|^2}\right\}\right| d\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left\{\frac{r\sin\varphi}{1+r^2-2r\cos\varphi} - i\frac{(1-r\cos\varphi)}{1+r^2-2r\cos\varphi}\right\}\right| d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(\exp\left\{\frac{r\sin\varphi}{1+r^2-2r\cos\varphi}\right\}\right) d\varphi \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{r\sin\varphi}{1+r^2-2r\cos\varphi} d\varphi \\
&= \frac{1}{2\pi}\cdot\frac{1}{2}\int_0^{\pi} \frac{d\left(1+r^2-2r\cos\varphi\right)}{1+r^2-2r\cos\varphi} \\
&= \frac{1}{4\pi}\log\left(1+r^2-2r\cos\varphi\right)\Big|_0^{\pi} \\
&= \frac{1}{4\pi}\log\left(\frac{1+r}{1-r}\right)^2 \\
&= \frac{1}{2\pi}\log\left(\frac{1+r}{1-r}\right).
\end{aligned}
$$

Definition 1.2.5 ([18]**)** *Let f be a meromorphic function, the characteristic function of Nevanlinna of f will be defined as*

$$T(r,f) := m(r,f) + N(r,f).$$

Example 1.2.6 *Let $f(z) = \cosh z$. f is an entire function, then*

$$
\begin{aligned}
T(r,\cosh z) &= m(r,\cosh z) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\cosh\left(re^{i\varphi}\right)\right| d\varphi,
\end{aligned}
$$

on the other hand, we have

$$\frac{\exp\left(|\operatorname{Re} z|\right)-1}{2} \leqslant |\cosh z| \leqslant \exp\left(|\operatorname{Re} z|\right),$$

then, $\log^{+}|\cosh z| = |\mathrm{Re}\, z| + O(1)$. *Hence*

$$\begin{aligned} T(r, \cosh z) &= \frac{1}{2\pi}\int_0^{2\pi} (|r\cos\varphi| + O(1))\, d\varphi \\ &= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{3\pi}{2}} |r\cos\varphi|\, d\varphi + O(1) \\ &= \frac{r}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\varphi d\varphi - \frac{r}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\varphi d\varphi + O(1) \\ &= \frac{r}{2\pi}\left[\sin\varphi|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \sin\varphi|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] + O(1) \\ &= \frac{2r}{\pi} + O(1). \end{aligned}$$

Example 1.2.7 *Let* $f(z) = \exp\left\{-\frac{i}{1-z}\right\}$ $(|z| < 1)$. f *is an analytic function in* $|z| < 1$, *then*

$$T(r, f) = m(r, f) = \frac{1}{2\pi}\log\left(\frac{1+r}{1-r}\right).$$

Proposition 1.2.2 ([18]) *Let* $f_1, ..., f_n, f$ *be a meromorphic functions and* $a \in \mathbb{C}^*$, *then*

(1) $m\left(r, \prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} m(r, f_i)$, $(n \in \mathbb{N}^*)$,

(2) $m\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} m(r, f_i) + \log n$, $(n \in \mathbb{N}^*)$,

(3) $T\left(r, \prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} T(r, f_i)$, $(n \in \mathbb{N}^*)$,

(4) $T\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} T(r, f_i) + \log n$, $(n \in \mathbb{N}^*)$,

(5) $T(r, f^n) = nT(r, f)$, $(n \in \mathbb{N}^*)$,

(6) $m(r, a + f) = m(r, f) + O(1)$ *and* $m(r, af) = m(r, f) + O(1)$,

(7) $T(r, a + f) = T(r, f) + O(1)$ *and* $T(r, af) = T(r, f) + O(1)$.

Proof. **(1), (3)** We have

$$
\begin{aligned}
m\left(r,\prod_{i=1}^{n} f_i\right) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\prod_{i=1}^{n} f_i\left(re^{i\varphi}\right)\right| d\varphi \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{n}\log^+\left|f_i\left(re^{i\varphi}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi}\log^+\left|f_i\left(re^{i\varphi}\right)\right| d\varphi \\
&= \sum_{i=1}^{n} m\left(r,f_i\right).
\end{aligned}
$$

If f_i has a pole of order $\lambda_i \geqslant 0$ at z_0, then it is a pole of order equal at most to $\sum_{i=1}^{n}\lambda_i$ for the function $\prod_{i=1}^{n} f_i$. Hence

$$
N\left(r,\prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} N\left(r,f_i\right),
$$

therefore

$$
\begin{aligned}
T\left(r,\prod_{i=1}^{n} f_i\right) &= m\left(r,\prod_{i=1}^{n} f_i\right) + N\left(r,\prod_{i=1}^{n} f_i\right) \\
&\leqslant \sum_{i=1}^{n} m\left(r,f_i\right) + \sum_{i=1}^{n} N\left(r,f_i\right) = \sum_{i=1}^{n} T\left(r,f_i\right).
\end{aligned}
$$

(2), (4) We have

$$
\begin{aligned}
m\left(r,\sum_{i=1}^{n} f_i\right) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\sum_{i=1}^{n} f_i\left(re^{i\varphi}\right)\right| d\varphi \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{n}\log^+\left|f_i\left(re^{i\varphi}\right)\right| + \log n\right) d\varphi \\
&= \sum_{i=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f_i\left(re^{i\varphi}\right)\right| d\varphi + \log n \\
&= \sum_{i=1}^{n} m\left(r,f_i\right) + \log n.
\end{aligned}
$$

If f_i has a pole of order $\lambda_i \geqslant 0$ at z_0, then it is a pole of order equal at most to $\max\limits_{1\leqslant i\leqslant n}\lambda_i \leqslant \sum_{i=1}^{n}\lambda_i$

for the function $\prod_{i=1}^{n} f_i$. Hence

$$N\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} N\left(r, f_i\right),$$

therefore

$$\begin{aligned} T\left(r, \sum_{i=1}^{n} f_i\right) &= m\left(r, \sum_{i=1}^{n} f_i\right) + N\left(r, \sum_{i=1}^{n} f_i\right) \\ &\leqslant \sum_{i=1}^{n} m\left(r, f_i\right) + \log n + \sum_{i=1}^{n} N\left(r, f_i\right) = \sum_{i=1}^{n} T\left(r, f_i\right) + \log n. \end{aligned}$$

(5) We have : $|f^n| = |f|^n \leqslant 1 \Longleftrightarrow |f| \leqslant 1$.

• If $|f| \leqslant 1$, then

$$T\left(r, f^n\right) = N\left(r, f^n\right) = nN\left(r, f\right) = nT\left(r, f\right).$$

• If $|f| > 1$, then

$$\begin{aligned} m\left(r, f^n\right) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f^n\left(re^{i\varphi}\right)\right| d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left|f^n\left(re^{i\varphi}\right)\right| d\varphi \\ &= \frac{1}{2\pi} \cdot n \int_0^{2\pi} \log \left|f\left(re^{i\varphi}\right)\right| d\varphi \\ &= n \cdot \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi \\ &= nm\left(r, f\right). \end{aligned}$$

Hence

$$\begin{aligned} T\left(r, f^n\right) &= m\left(r, f^n\right) + N\left(r, f^n\right) \\ &= nm\left(r, f\right) + nN\left(r, f\right) \\ &= nT\left(r, f\right). \end{aligned}$$

(6) We have

$$\begin{aligned} \left|m\left(r, a+f\right) - m\left(r, f\right)\right| &= \left|\frac{1}{2\pi} \int_0^{2\pi} \left(\log^+ \left|f\left(re^{i\varphi}\right) + a\right| - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right) d\varphi\right| \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|\log^+ \left(\left|f\left(re^{i\varphi}\right)\right| + |a|\right) - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right| d\varphi \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left|\log^+ |a| + \log 2\right| d\varphi \leqslant \log^+ |a| + \log 2, \end{aligned}$$

and

$$\begin{aligned}
|m(r,af)-m(r,f)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|af\left(re^{i\varphi}\right)\right|-\log^+\left|f\left(re^{i\varphi}\right)\right|\right)d\varphi\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left(|a|\left|f\left(re^{i\varphi}\right)\right|\right)-\log^+\left|f\left(re^{i\varphi}\right)\right|\right|d\varphi \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+|a|\right|d\varphi=\log^+|a|=|\log|a||-\log^+\frac{1}{|a|} \\
&\leqslant |\log|a||.
\end{aligned}$$

Hence,

$$m(r,a+f)=m(r,f)+O(1) \quad \text{and} \quad m(r,af)=m(r,f)+O(1).$$

(7) From (6), we get

$$\begin{aligned}
T(r,a+f) &= N(r,a+f)+m(r,a+f) \\
&= N(r,f)+m(r,f)+O(1) \\
&= T(r,f)+O(1),
\end{aligned}$$

and

$$\begin{aligned}
T(r,af) &= N(r,af)+m(r,af) \\
&= N(r,f)+m(r,f)+O(1) \\
&= T(r,f)+O(1).
\end{aligned}$$

□

Theorem 1.2.1 ([18]**)** *(First Fundamental Theorem of Nevanlinna) Let f be a meromorphic function, let $a\in\mathbb{C}$ and let*

$$f(z)-a=\sum_{i=m}^{+\infty}c_iz^i, \qquad c_m\in\mathbb{C}^*,\ m\in\mathbb{Z},$$

be the Laurent expansion of $f-a$ at the origin. Then

$$T(r,a,f)=T\left(r,\frac{1}{f-a}\right)=T(r,f)-\log|c_m|+\varphi(r,a),$$

where $|\varphi(r,a)|\leqslant\log 2+\log^+|a|.$

Proof. Assume first $a = 0$, then by the Proposition1.2.1 and Lemma1.2.1 (c), we have

$$\begin{aligned}
\log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\varphi}\right)\right|d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\varphi}\right)\right|d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{\left|f\left(re^{i\varphi}\right)\right|}d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right)\\
&= m(r,f) - m\left(r,\frac{1}{f}\right) + N(r,f) - N\left(r,\frac{1}{f}\right)\\
&= T(r,f) - T\left(r,\frac{1}{f}\right),
\end{aligned}$$

hence

$$T\left(r,\frac{1}{f}\right) = T(r,f) - \log|c_m|, \qquad where \quad \varphi(r,0)\equiv 0. \tag{1.2.1}$$

Proceeding now to the general case $a \neq 0$, we pose $h := f - a$. Then

$$N\left(r,\frac{1}{h}\right) = N\left(r,\frac{1}{f-a}\right), \quad N(r,f) = N(r,h) \quad et \quad m\left(r,\frac{1}{h}\right) = m\left(r,\frac{1}{f-a}\right).$$

Moreover

$$\begin{aligned}
\log^+|h| &= \log^+|f-a| \leqslant \log^+|f| + \log^+|a| + \log 2,\\
\log^+|f| &= \log^+|h+a| \leqslant \log^+|h| + \log^+|a| + \log 2.
\end{aligned}$$

By integrating these two inequalities, we find that

$$\begin{aligned}
m(r,h) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|h\left(re^{i\varphi}\right)\right|d\varphi\\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|f\left(re^{i\varphi}\right)\right| + \log^+|a| + \log 2\right)d\varphi\\
&= m(r,f) + \log^+|a| + \log 2,
\end{aligned}$$

and

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\varphi}\right)\right|d\varphi\\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|h\left(re^{i\varphi}\right)\right| + \log^+|a| + \log 2\right)d\varphi\\
&= m(r,h) + \log^+|a| + \log 2.
\end{aligned}$$

We pose $\varphi(r,a) := m(r,h) - m(r,f)$. Then

$$-\left[\log^+|a| + \log 2\right] \leqslant m(r,h) - m(r,f) \leqslant \log^+|a| + \log 2 \Longleftrightarrow |\varphi(r,a)| \leqslant \log^+|a| + \log 2.$$

Applying $(1, 2, 1)$ for h, we obtain

$$
\begin{aligned}
T\left(r, \frac{1}{h}\right) &= T(r, h) - \log|c_m| \\
&= m(r, h) + N(r, h) - \log|c_m| \\
&= m(r, f) + \varphi(r, a) + N(r, f) - \log|c_m| \\
&= T(r, f) - \log|c_m| + \varphi(r, a).
\end{aligned}
$$

□

Remark 1.2.2 *The first fundamental theorem may be expressed as: for all $a \in \mathbb{C}$, we have*

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1), \quad r \to +\infty.$$

Theorem 1.2.2 ([18]**)** *Let f be an entire function and assume that $0 < r < R < +\infty$. Then*

$$T(r, f) \leqslant \log^+ M(r, f) \leqslant \frac{R+r}{R-r} T(R, f),$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Proof. *The first inequality is trivial. Indeed, f being an entire function, then*

$$
\begin{aligned}
T(r, f) &= m(r, f) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\varphi}\right) \right| d\varphi \\
&\leqslant \frac{1}{2\pi} \int_0^{2\pi} \log^+ M(r, f)\, d\varphi = \log^+ M(r, f).
\end{aligned}
$$

- *If $M(r, f) \leqslant 1$, then $\log^+ M(r, f) = 0 \leqslant \dfrac{R+r}{R-r} T(R, f)$.*
- *Suppose that $M(r, f) > 1$, we then take z_0 such that $z_0 = re^{i\theta}$ and $|f(z_0)| = M(r, f)$. Since*

$$\left| \frac{R(z - a_j)}{R^2 - \overline{a_j} z} \right| < 1, \quad \textit{for} \quad |z| < R,$$

then by the Poisson-Jensen formula, we obtain

$$
\begin{aligned}
\log^+ M(r, f) &= \log M(r, f) = \log|f(z_0)| \\
&\leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} \log\left| f\left(Re^{i\varphi}\right) \right| d\varphi \\
&\leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{(R-r)^2 + 2rR(1 - \cos(\theta - \varphi))} \log^+ \left| f\left(Re^{i\varphi}\right) \right| d\varphi \\
&\leqslant \frac{R^2 - r^2}{(R-r)^2} \left(\int_0^{2\pi} \log^+ \left| f\left(Re^{i\varphi}\right) \right| d\varphi \right) = \frac{R+r}{R-r} m(R, f) = \frac{R+r}{R-r} T(R, f).
\end{aligned}
$$

□

The following lemma is needed to present an identity of H. Cartan that will be used to discuss further properties of characteristic functions.

Lemma 1.2.3 *For all $w \in \mathbb{C}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|w - e^{i\theta}\right| d\theta = \log^+ |w|,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\log\left|w - e^{i\theta}\right|\right| d\theta \leqslant \log^+ |w| + 2\log 2.$$

Proof. By applying Jensen formula for $f(z) = z - w$ and $r = 1$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|w - e^{i\theta}\right| d\theta = \begin{cases} \log|w| & if \quad |w| > 1, \\ 0 & if \quad |w| \leqslant 1, \end{cases}$$

hence

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|w - e^{i\theta}\right| d\theta = \log^+ |w|. \tag{1.2.2}$$

By using the formulas (c) and (d) of Lemma 1.2.1, we have

$$\left|\log\left|w - e^{i\theta}\right|\right| = 2\log^+\left|w - e^{i\theta}\right| - \log\left|w - e^{i\theta}\right|,$$

hence from $(1.2.2)$, we get

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \left|\log\left|w - e^{i\theta}\right|\right| d\theta &= \frac{2}{2\pi}\int_0^{2\pi} \log^+\left|w - e^{i\theta}\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\left|w - e^{i\theta}\right| d\theta \\ &\leqslant \frac{1}{\pi}\int_0^{2\pi} \left(\log^+|w| + \log 2\right) d\theta - \log^+|w| = \log^+|w| + 2\log 2. \end{aligned}$$

□

Theorem 1.2.3 ([21]) *(Cartan theorem) Let f be a meromorphic function such that $f(0) = \infty$, then*

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta + \log^+ |f(0)|.$$

Proof. By applying the Jensen formula for $f(z) - e^{i\theta}$, we get

$$\log\left|f(0) - e^{i\theta}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right) - e^{i\theta}\right| d\varphi + N(r, \infty, f) - N\left(r, e^{i\theta}, f\right),$$

then

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(0)-e^{i\theta}\right|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\varphi}\right)-e^{i\theta}\right|d\varphi\right)d\theta$$
$$+N(r,\infty,f)-\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta. \qquad (1.2.3)$$

To be able to use the Fubini theorem and invert the order of integration, we must prove the convergence of the integral

$$I=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|f\left(re^{i\varphi}\right)-e^{i\theta}\right|\right|d\theta\right)d\varphi.$$

By Lemma 1.2.3, we obtain

$$I=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|f\left(re^{i\varphi}\right)-e^{i\theta}\right|\right|d\theta\right)d\varphi$$
$$\leqslant\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|f\left(re^{i\varphi}\right)\right|+2\log 2\right)d\varphi=m(r,f)+2\log 2<\infty.$$

Therefore, by using (1.2.2) and from (1.3.1), we get

$$\begin{aligned}\log^+|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\varphi}\right)-e^{i\theta}\right|d\varphi\right)d\theta\\ &\quad+N(r,\infty,f)-\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\varphi}\right)\right|d\varphi+N(r,\infty,f)-\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta,\end{aligned}$$

hence

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta+\log^+|f(0)|.$$

□

Corollary 1.2.1 ([21]) *$T(r,f)$ is an increasing function of r.*

Proof. Since $N\left(r,e^{i\theta},f\right)$ is an increasing function for all $\theta\in[0,2\pi]$, then $T(r,f)$ is an increasing function. □

1.3 The $[p, q]$-order and the $[p, q]$-exponent of convergence of a meromorphic function

First, we need to define the following expressions :

• For $r \in \mathbb{R}$, we have : $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right),\ p \in \mathbb{N}$. We also define

• For all r sufficiently large in $(0, +\infty)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$.

• Moreover, we denote $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$.

Proposition 1.3.1 *Let $x_i \in \mathbb{R}$ such that $x_i > 1$ and $i = 1, ..., n$, then*

(i) $\log_p \left(\sum\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log_p x_i + O(1),$

(ii) $\log_p \left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log_p x_i + O(1).$

Proof. For the proof, we use the principle of mathematical induction.

(i) • For $p = 1$, we have $\log \left(\sum\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log x_i + O(1)$.

•We suppose that $\log_p \left(\sum\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log_p x_i + O(1)$, holds and we prove that it holds at order $p+1$. We have

$$\begin{aligned} \log_{p+1}\left(\sum_{i=1}^{n} x_i\right) &= \log\left(\log_p\left(\sum_{i=1}^{n} x_i\right)\right) \\ &\leqslant \log\left(\sum_{i=1}^{n} \log_p x_i + O(1)\right) \\ &\leqslant \sum_{i=1}^{n} \log_{p+1} x_i + O(1). \end{aligned}$$

Hence, $\log_p \left(\sum\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log_p x_i + O(1)$.

(ii) • For $p = 1$, we have $\log \left(\prod\limits_{i=1}^{n} x_i\right) = \sum\limits_{i=1}^{n} \log x_i$, then $\log \left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log x_i + O(1)$.

•We suppose that $\log_p \left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n} \log_p x_i + O(1)$, holds and we prove that it holds at order

$p+1$. We have

$$\begin{aligned}\log_{p+1}\left(\prod_{i=1}^{n}x_i\right) &= \log\left(\log_p\left(\prod_{i=1}^{n}x_i\right)\right)\\ &\leqslant \log\left(\sum_{i=1}^{n}\log_p x_i + O(1)\right)\\ &\leqslant \sum_{i=1}^{n}\log_{p+1}x_i + O(1).\end{aligned}$$

Hence, $\log_p\left(\prod\limits_{i=1}^{n}x_i\right) \leqslant \sum\limits_{i=1}^{n}\log_p x_i + O(1)$.

□

Now, we introduce the concept of $[p,q]$-order of meromorphic and analytic functions in the unit disc $D=\{z\in \mathbf{C}:|z|<1\}$.

1.3.1 The $[p,q]$-order of a meromorphic function

Definition 1.3.1 **(**[4], [5], [6]**)** *Let $p\geqslant q\geqslant 1$ be integers. Let f be meromorphic function in D, the $[p,q]$-order of $f(z)$ is defined by*

$$\sigma_{[p,q]}(f) = \limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

For $p=q=1$, this notation is called order $(\sigma_1(f)=\sigma(f))$, for $p=2$ and $q=1$ hyper-order and for $q=1$ iterated $p-$order. For an analytic function f in D, we also define

$$\sigma_{M,[p,q]}(f) = \limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{1-r}\right)},$$

where $M(r,f)=\max\limits_{|z|=r}|f(z)|$.

Example 1.3.1 *Let $f(z)=\exp_3\left(\cosh\frac{1}{(1-z)^2}\right)$ is an analytic function in D such that*

$$M(r,f) = \max_{|z|=r}|f(z)| = \exp_3\left\{\cosh\left(\frac{1}{(1-r)^2}\right)\right\}.$$

Then

$$\begin{aligned}\sigma_{M,[2,1]}(f) &= \limsup_{r\to 1^-}\frac{\log_3^+ M(r,f)}{\log\left(\frac{1}{1-r}\right)}\\ &= \limsup_{r\to 1^-}\frac{\log_3\left(\exp_3\left\{\cosh\left(\frac{1}{(1-r)^2}\right)\right\}\right)}{\log\left(\frac{1}{1-r}\right)}\\ &= \limsup_{r\to 1^-}\frac{\cosh\left(\frac{1}{(1-r)^2}\right)}{\log\left(\frac{1}{1-r}\right)} = +\infty.\end{aligned}$$

Remark 1.3.1 *It is easy to see that* $0 \leqslant \sigma_{[p,q]}(f) \leqslant \infty$.

Proposition 1.3.2 **(**[4], [5], [6]**)** *Let* $p \geqslant q \geqslant 1$ *be integers, and let* f *be analytic function in* D *of* $[p,q]$*-order. The following two statements hold:*

(i) *If* $p = q$, *then*

$$\sigma_{[p,q]}(f) \leqslant \sigma_{M,[p,q]}(f) \leqslant \sigma_{[p,q]}(f) + 1.$$

(ii) *If* $p > q$, *then*

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f).$$

Proof. By the standard inequalities (see [15]), we obtain

$$T(r,f) \leqslant \log^+ M(r,f) \leqslant \frac{1+3r}{1-r}T\left(\frac{1+r}{2},f\right),$$

It follows that

$$\log_p T(r,f) \leqslant \log_{p+1}^+ M(r,f) \leqslant \log_p\left(\frac{1+3r}{1-r}T\left(\frac{1+r}{2},f\right)\right)$$

$$\leqslant \log_p(1+3r) + \log_p\left(\frac{1}{1-r}\right) + \log_p T\left(\frac{1+r}{2},f\right) + C,$$

$C > 0$ is a constant.

• If $p=q$, then

$$\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_p\left(\frac{1}{1-r}\right)}=\sigma_{[p,p]}(f)\leqslant \limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_p\left(\frac{1}{1-r}\right)}=\sigma_{M,[p,p]}(f)$$

$$\leqslant \limsup_{r\to 1^-}\left(\frac{\log_p(1+3r)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{C}{\log_p\left(\frac{1}{1-r}\right)}\right)$$

$$= \limsup_{r\to 1^-}\left(\frac{\log_p(1+3r)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{\log_p\left(\frac{1}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}\cdot\frac{\log_p\left(\frac{2}{1-r}\right)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{C}{\log_p\left(\frac{1}{1-r}\right)}\right)$$

$$\leqslant \limsup_{r\to 1^-}\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_p\left(\frac{1}{1-\frac{1+r}{2}}\right)}+1=\sigma_{[p,p]}(f)+1.$$

Hence

$$\sigma_{[p,q]}(f)\leqslant \sigma_{M,[p,q]}(f)\leqslant \sigma_{[p,q]}(f)+1.$$

• If $p>q$, then

$$\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}=\sigma_{[p,q]}(f)\leqslant \limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{1-r}\right)}=\sigma_{M,[p,q]}(f)$$

$$\leqslant \limsup_{r\to 1^-}\left(\frac{\log_p(1+3r)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{C}{\log_q\left(\frac{1}{1-r}\right)}\right)$$

$$= \limsup_{r\to 1^-}\left(\frac{\log_p(1+3r)}{\log_p\left(\frac{1}{1-r}\right)}+\frac{\log_p\left(\frac{1}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)}\cdot\frac{\log_q\left(\frac{2}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}+\frac{C}{\log_q\left(\frac{1}{1-r}\right)}\right)$$

$$\leqslant \limsup_{r\to 1^-}\frac{\log_p T\left(\frac{1+r}{2},f\right)}{\log_q\left(\frac{1}{1-\frac{1+r}{2}}\right)}=\sigma_{[p,q]}(f).$$

So

$$\sigma_{M,[p,q]}(f)=\sigma_{[p,q]}(f).$$

□

1.3.2 The $[p,q]$-exponent of convergence of a meromorphic function

Definition 1.3.2 ([5]) *Let $p \geqslant q \geqslant 1$ be integers. Let f be a meromorphic function in D. Then, the $[p,q]$-exponent of convergence of the sequence of zeros in D of $f(z)$ is defined by*

$$\lambda_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\bar{\lambda}_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ \bar{N}\left(r, \frac{1}{f}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.3.3 ([5]) *Let $p \geqslant q \geqslant 1$ be integers. Let f be a meromorphic function in D. Then, the $[p,q]$-exponent of convergence of the sequence of fixed points in D of $f(z)$ is defined by*

$$\lambda_{[p,q]}(f-z) = \limsup_{r\to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f-z}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\bar{\lambda}_{[p,q]}(f-z) = \limsup_{r\to 1^-} \frac{\log_p^+ \bar{N}\left(r, \frac{1}{f-z}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

1.4 The density of a set

Definition 1.4.1 ([5], [6]) *For a mesurable set $E \subset [0,1)$, the upper and lower densities of E are defined by*

$$\overline{dens}_D E = \limsup_{r\to 1^-} \frac{m(E\cap[0,r))}{m([0,r))} \qquad \textit{and} \qquad \underline{dens}_D E = \liminf_{r\to 1^-} \frac{m(E\cap[0,r))}{m([0,r))},$$

respectively, where $m(F) = \int_F \frac{dt}{1-t}$ for $F \subset [0,1)$. It is clear that $0 \leqslant \underline{dens}_D E \leqslant \overline{dens}_D E \leqslant 1$ for any measurable set $E \subset [0,1)$.

Example 1.4.1 *The upper and lower densities of the set $F = \left[0, \frac{1}{2}\right] \subset [0,1)$ are*

$$\underline{dens}_D F = \overline{dens}_D F = 0.$$

Proposition 1.4.1 ([10]) *If a set E satisfies $\overline{dens}_D E > 0$, then $m(E) = \int_E \frac{dt}{1-t} = +\infty$.*

Proof. Suppose that $m(E) = \int_E \frac{dt}{1-t} = \delta < \infty$. We have

$$m([0,r)) = \int_0^r \frac{dt}{1-t} = -\log(1-t)|_0^r = -\log(1-r).$$

Since $m(E \cap [0,r)) \leqslant m(E)$, then

$$\overline{dens}_D E = \limsup_{r \to 1^-} \frac{m(E \cap [0,r))}{m([0,r))} \leqslant \limsup_{r \to 1^-} \frac{m(E)}{m([0,r))} = \limsup_{r \to 1^-} \frac{\delta}{-\log(1-r)} = 0.$$

Hence

$$\overline{dens}_D E > 0 \Longrightarrow m(E) = \int_E \frac{dt}{1-t} = +\infty.$$

□

Chapter 2

The [p,q]-Order of Growth of Solutions of Linear Differential Equations in the Unit Disc

2.1 Introduction and Some Results

Consider for $k \geq 2$ the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \qquad (2.1.1)$$

where $A_0, ..., A_{k-1}, F$ are analytic functions in the unit disc D not being identically equal to 0.

In 2012, Belaïdi in [5] and [6] studied the $[p, q]-$order of the growth of solutions of linear differential equations denoted by (0.0.1) and (2.1.1) in which the coefficients are analytic functions in D, and he obtained for equation (0.0.1) the following results.

Theorem A ([5]) *Let $p \geqslant q \geqslant 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that for real constants α, β where $0 \leqslant \beta < \alpha$, we have*

$$|A_0(z)| \geqslant \exp_{p+1}\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\},$$

and

$$|A_i(z)| \leqslant \exp_{p+1}\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\}, \qquad i = 1, ..., k-1,$$

as $|z| \to 1^-$ for $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \alpha$.

Theorem B ([6]) *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0(z), ..., A_{k-1}(z)$ *be analytic functions in the unit disc* D *such that for the real constants* α, β *where* $0 \leqslant \beta < \alpha$, *we have*

$$T(r, A_0) \geqslant \exp_p \left\{ \alpha \log_q \left(\frac{1}{1-|z|} \right) \right\},$$

and

$$T(r, A_i) \leqslant \exp_p \left\{ \beta \log_q \left(\frac{1}{1-|z|} \right) \right\}, \quad i = 1, ..., k-1,$$

as $|z| = r \to 1^-$ for $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \alpha$.

After that in 2021, Chen *et al.* [10] have investigated the growth of solutions of equations (0.0.1) and (0.0.2) in D by using the iterated order, and they got the following results

Theorem C (see [10]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$. *Let* $A_0, A_1, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max \{\sigma_{M,n}(A_i) : i = 1, 2, ..., k-1\} \leqslant \sigma_{M,n}(A_0) = \mu \, (0 < \mu < \infty),$$

and for a constant $\alpha \geqslant 0$, *we have*

$$\liminf_{|z| \to 1^-, z \in H} \left((1-|z|)^{\mu} \log_n |A_0(z)| \right) > \alpha,$$

and

$$|A_i(z)| \leqslant \exp_n \left\{ \alpha \left(\frac{1}{1-|z|} \right)^{\mu} \right\}, \qquad i = 1, 2, ..., k-1,$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every solution* $f \not\equiv 0$ *of* (0.0.1) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) = \sigma_{M,n}(A_0) = \mu$.

Theorem D (see [10]) *Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$. *Let* $A_0, A_1, ..., A_k$ *be analytic functions in the unit disc* D, *and for some constants* $\alpha \geqslant 0$ *and*

$\mu > 0$, *we have*

$$\liminf_{|z|\to 1^-, z\in H} \left((1-|z|)^{\mu} \log_{n-1} T(r, A_0)\right) > \alpha,$$

and

$$T(r, A_i) \leqslant \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}, \qquad i = 1, 2, ..., k,$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of* (0.0.2) *satisfies* $\sigma_n(f) = \infty$ *and* $\sigma_{n+1}(f) \geqslant \mu$.

In this thesis, we improve and generalize the recent results of Chen *et al.*[10] by using $[p, q]$ −order instead iterated order with less control constant. At the same time, our work improve some results of Belaïdi in [5] and [6].

To be specific, we will decrease the control constants of the coefficients' modulus or characteristic functions and obtain results which extend those of Chen *et al.* Here, we study the problem and get the following results.

Theorem 2.1.1 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0, ..., A_k$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$ *and for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$|A_0(z)| \geqslant \exp_{p+1}\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \tag{2.1.2}$$

and

$$|A_i(z)| \leqslant \exp_{p+1}\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \quad i = 1, ..., k, \tag{2.1.3}$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

Theorem 2.1.2 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0, ..., A_k$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$ *and for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$T(|z|, A_0) \geqslant \exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \tag{2.1.4}$$

and

$$T\left(|z|, A_i\right) \leqslant \exp_p\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \quad i = 1, ..., k, \tag{2.1.5}$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

Theorem 2.1.3 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu\,(0 < \mu < +\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$\liminf_{|z|\to 1^-, z\in H} \frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)} > \mu - \varepsilon, \tag{2.1.6}$$

and

$$|A_i(z)| \leqslant \exp_{p+1}\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \quad i = 1, ..., k-1, \tag{2.1.7}$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Example 2.1.1 *Consider the following equation*

$$f'' + K_1(z)\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\} f' + K_0(z)\exp_4\left\{\left(2 - \frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\} f = 0, \tag{2.1.8}$$

where K_0 *and* K_1 *are analytic functions in the unit disc* D *such that*

$$\begin{cases} \sigma_{M,[3,2]}(K_0) > 2 \quad \textit{and} \quad |K_0| > 1. \\ \sigma_{M,[3,2]}(K_1) < 1 \quad \textit{and} \quad |K_1| < 1. \end{cases}$$

Let $H = \{z \in \mathbb{C} : |z| = r < 1$ *and* $\arg z = 0\} \subset D$ *a set of complex numbers satisfying* $\overline{dens}_D\{|z| : z \in H\} = 1 > 0$.

In the equation (2.1.8) *we have for all* ε $(0 < \varepsilon < 1)$ *sufficiently small :*

$$A_0(z) = K_0(z)\exp_4\left\{\left(2 - \tfrac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}, \; A_1(z) = K_1(z)\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\},$$

then we get

1/

$$\begin{aligned}
&\sigma_{M,[3,2]}\left(\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right)\\
&= \limsup_{r\to1^-}\frac{\log_4^+\left(\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-r}\right)\right\}\right)}{\log_2\left(\frac{1}{1-r}\right)}\\
&= \limsup_{r\to1^-}\frac{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-r}\right)}{\log_2\left(\frac{1}{1-r}\right)}\\
&= 2-\frac{\varepsilon}{2}.
\end{aligned}$$

- *If* $\sigma_{M,[3,2]}\left(\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right)>\sigma_{M,[3,2]}\left(K_0\right)$, *then*

$$\sigma_{M,[3,2]}\left(A_0\right)=\sigma_{M,[3,2]}\left(\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right)=2-\frac{\varepsilon}{2}.$$

- *If* $\sigma_{M,[3,2]}\left(K_0\right)>\sigma_{M,[3,2]}\left(\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right)$, *then*

$$\sigma_{M,[3,2]}\left(A_0\right)=\sigma_{M,[3,2]}\left(K_0\right)>2.$$

2/

$$\begin{aligned}
\sigma_{M,[3,2]}\left(\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right) &= \limsup_{r\to1^-,z\in H}\frac{\log_4^+\left(\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-r}\right)\right\}\right)}{\log_2\left(\frac{1}{1-r}\right)}\\
&= \limsup_{r\to1^-}\frac{(2-2\varepsilon)\log_2\left(\frac{1}{1-r}\right)}{\log_2\left(\frac{1}{1-r}\right)}\\
&= 2-2\varepsilon.
\end{aligned}$$

- *If* $\sigma_{M,[3,2]}\left(\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right)>\sigma_{M,[3,2]}\left(K_1\right)$, *then*

$$\sigma_{M,[3,2]}\left(A_1\right)=\sigma_{M,[3,2]}\left(\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right)=2-2\varepsilon.$$

- *If* $\sigma_{M,[3,2]}(K_1) > \sigma_{M,[3,2]}\left(\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right)$, *then*

$$\sigma_{M,[3,2]}(A_1) = \sigma_{M,[3,2]}(K_1) < 1.$$

Hence, by 1/ and 2/ we deduce that $\sigma_{M,[3,2]}(A_1) < \sigma_{M,[3,2]}(A_0)$.

In the other hand

$$\begin{aligned} |A_0(z)| &= |K_0(z)|\left|\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right| \\ &> \exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-r}\right)\right\}, \end{aligned}$$

then

$$\frac{\log_4|A_0(z)|}{\log_2\left(\frac{1}{1-r}\right)} > 2-\frac{\varepsilon}{2} \Longrightarrow \liminf_{r\to 1^-, z\in H}\frac{\log_4|A_0(z)|}{\log_2\left(\frac{1}{1-r}\right)} \geqslant 2-\frac{\varepsilon}{2} > 2-\varepsilon,$$

and

$$\begin{aligned} |A_1(z)| &= |K_1(z)|\left|\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right| \\ &< \exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-r}\right)\right\}, \end{aligned}$$

az $r \to 1^-$ *for* $z \in H$.

It is clear that the conditions of Theorem 2.1.3 hold with $\mu = 2$, $p = 3$ *and* $q = 2$ *on the set* H *such that* $\overline{dens}_D\{|z| : z \in H\} > 0$.

By Theorem 2.1.3, every solution $f \not\equiv 0$ *of equation* (2.1.8) *satisfies*

$$\sigma_{[3,2]}(f) = \infty \quad \text{and} \quad \sigma_{[4,2]}(f) = 2.$$

Theorem 2.1.4 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu\,(0 < \mu < +\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$\limsup_{|z|\to 1^-, z\in H}\frac{\log_{p+1}|A_i(z)|}{\log_q\left(\frac{1}{1-|z|}\right)} < \liminf_{|z|\to 1^-, z\in H}\frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)}, \qquad i = 1, ..., k-1, \tag{2.1.9}$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Theorem 2.1.5 *Let $p \geqslant q \geqslant 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu \, (0 < \mu < +\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ sufficiently small, we have

$$\liminf_{|z| \to 1^-, z \in H} \frac{\log_p T(|z|, A_0)}{\log_q \left(\frac{1}{1-|z|}\right)} > \mu - \varepsilon, \tag{2.1.10}$$

and

$$T(|z|, A_i) \leq \exp_p \left\{(\mu - 2\varepsilon) \log_q \left(\frac{1}{1-|z|}\right)\right\}, \quad i = 1, ..., k-1, \tag{2.1.11}$$

as $|z| \to 1^-$ for $z \in H$.Then every solution $f \not\equiv 0$ of equation (0.0.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Theorem 2.1.6 *Let $p \geqslant q \geqslant 1$ be integers. Let H be a set of complexe numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu \, (0 < \mu < +\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ sufficiently small, we have

$$\limsup_{|z| \to 1^-, z \in H} \frac{\log_p T(|z|, A_i)}{\log_q \left(\frac{1}{1-|z|}\right)} < \liminf_{|z| \to 1^-, z \in H} \frac{\log_p T(|z|, A_0)}{\log_q \left(\frac{1}{1-|z|}\right)}, \quad i = 1, ..., k-1, \tag{2.1.12}$$

as $|z| \to 1^-$ for $z \in H$ .Then every solution $f \not\equiv 0$ of equation (0.0.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

The following corollaries can be easily obtained from Theorem 2.1.3 to Theorem 2.1.6.

Corollary 2.1.1 *Let H be a set of complexe numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu \, (0 < \mu < +\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$|A_0(z)| \geqslant \exp_{p+1}\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\},$$

and

$$|A_i(z)| \leqslant \exp_{p+1}\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \qquad i=1,...,k-1,$$

as $|z| \to 1^-$ *for* $z \in H$ *.Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Proof. We can get the conclusion of Corollary 2.1.1, by using a similar proof as in Theorem 2.1.3 or Theorem 2.1.4.

Corollary 2.1.2 *Let* H *be a set of complexe numbers satisfying* $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i=1,2,...,k-1\right\} \leqslant \sigma_{M,[p,q]}(A_0) = \mu\,(0<\mu<+\infty)$$

and for all ε $(0 < 2\varepsilon < \mu)$ *sufficiently small, we have*

$$T(|z|, A_0) \geqslant \exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\},$$

and

$$T(|z|, A_i) \leqslant \exp_p\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}, \qquad i=1,...,k-1,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Proof. We can get the conclusion of Corollary 2.1.2, by using a similar proof as in Theorem 2.1.5 or Theorem 2.1.6.

For equation (0.0.2) , we generalize Theorem A and Theorem C to Theorem 2.1.7 and Theorem 2.1.8, also Theorem B and Theorem D to Theorem 2.1.9 and Theorem 2.1.10 as follows.

Theorem 2.1.7 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0(z), ..., A_k(z)$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$, *we have for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small (2.1.5) and*

$$|A_i(z)| \leqslant \exp_{p+1} \left\{ (\mu - 2\varepsilon) \log_q \left(\frac{1}{1 - |z|} \right) \right\}, \qquad i = 1, ..., k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

Theorem 2.1.8 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0(z), ..., A_k(z)$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$ *we have, for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small*

$$\limsup_{|z| \to 1^-, z \in H} \frac{\log_{p+1} |A_i(z)|}{\log_q \left(\frac{1}{1-|z|} \right)} < \liminf_{|z| \to 1^-, z \in H} \frac{\log_{p+1} |A_0(z)|}{\log_q \left(\frac{1}{1-|z|} \right)}, \qquad i = 1, ..., k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

Theorem 2.1.9 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0(z), ..., A_k(z)$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$ *we have for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small (2.1.10) and*

$$T(|z|, A_i) \leqslant \exp_p \left\{ (\mu - 2\varepsilon) \log_q \left(\frac{1}{1 - |z|} \right) \right\}, \qquad i = 1, ..., k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

Theorem 2.1.10 *Let* $p \geqslant q \geqslant 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, *and let* $A_0(z), ..., A_k(z)$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$ *we have, for all* ε $(0 < 2\varepsilon < \mu)$ *sufficiently small*

$$\limsup_{|z| \to 1^-, z \in H} \frac{\log_p T(|z|, A_i)}{\log_q \left(\frac{1}{1-|z|} \right)} < \liminf_{|z| \to 1^-, z \in H} \frac{\log_p T(|z|, A_0)}{\log_q \left(\frac{1}{1-|z|} \right)}, \qquad i = 1, ..., k,$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (0.0.2) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu$.

2.2 Preliminary lemmas

In this section of chapter 2, we present some necessary lemmas which are used in the proofs of the theorems of this chapter and chapter 3.

Lemma 2.2.1 ([14], *Theorem* 3.1) *Let k and j be integers satisfying $k > j \geqslant 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If f is a meromorphic function in D such that $f^{(j)}$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right); T\left(s\left(|z|\right), f\right)\right\}\right]^{k-j}, \quad |z| \notin E,$$

where $E \subset [0,1)$ is a set with $\int_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1-|z|)$.

Lemma 2.2.2 ([16]) *Let f be a meromorphic function in the unit disc D, and let $k \geqslant 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$.

Lemma 2.2.3 ([1]) *Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leqslant h(r)$ holds outside of an exceptional set $E \subset [0,1)$ for which $\int_E \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leqslant h(s(r))$ for all $r \in [0,1)$.*

Lemma 2.2.4 ([4]) *Let $p \geqslant q \geqslant 1$ be integers. If $A_0(z), ..., A_{k-1}(z)$ are analytic functions of $[p,q]-$order in the unit disc D, then every solution $f \not\equiv 0$ of (0.0.1) satisfies*

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leqslant \max\left\{\sigma_{M,[p,q]}(A_j) : j = 0, 1, ..., k-1\right\}.$$

Lemma 2.2.5 ([5]) *Let $p \geqslant q \geqslant 1$ be integers. If f and g are meromorphic functions of $[p,q]-$order in D, then we have*

(i) $\sigma_{[p,q]}(f) = \sigma_{[p,q]}\left(\frac{1}{f}\right)$, $\sigma_{[p,q]}(af) = \sigma_{[p,q]}(f)$ *and* $\sigma_{[p,q]}(f+a) = \sigma_{[p,q]}(f)$, $\quad (a \in \mathbb{C}^*)$.

(ii) $\sigma_{[p,q]}(f') = \sigma_{[p,q]}(f)$.

(iii) $\sigma_{[p,q]}(f+g) \leqslant \max\{\sigma_{[p,q]}(f), \sigma_{[p,q]}(g)\}$.

(iv) $\sigma_{[p,q]}(fg) \leqslant \max\{\sigma_{[p,q]}(f), \sigma_{[p,q]}(g)\}$,

(v) *if* $\sigma_{[p,q]}(f) > \sigma_{[p,q]}(g)$, *then we obtain* $\sigma_{[p,q]}(f+g) = \sigma_{[p,q]}(fg) = \sigma_{[p,q]}(f)$.

Lemma 2.2.6 ([5]) *Let* $p \geqslant q \geqslant 1$ *be integers. Let* $A_0, ..., A_{k-1}$ *and* $F \not\equiv 0$ *be finite* $[p,q]-$*order analytic functions in the unit disc D. If f is a solution with* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma < \infty$ *of* (2.1.1), *then*

$$\bar{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f) = +\infty,$$

$$\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma.$$

2.3 Proofs of Theorem 2.1.1 and 2.1.2

Proof of Theorem 2.1.1. Suppose that every solution f of equation (0.0.2) not being identically equal to 0.

From the conditions of Theorem 2.1.1, there exist *a set* H of complex numbers satisfying $\overline{dens}_D H_1 > 0$, where $H_1 = \{r = |z| : z \in H \subseteq D\}$. Then H_1 is a set with $\int_{H_1} \frac{dr}{1-r} = +\infty$, such that for $z \in H$ we have (2.1.2) and (2.1.3) az $|z| \to 1^-$.

By Lemma 2.2.1, there exist $s(|z|) = 1 - d(1-|z|)$, $d \in (0,1)$ and a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$ such that for $|z| \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right), T(s(|z|), f)\right\}\right]^j, (j = 1, ..., k). \tag{2.3.1}$$

From (0.0.2), we get

$$|A_0(z)| \leqslant |A_k(z)| \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{2.3.2}$$

Applying (2.1.2), (2.1.3) and (2.3.1) into (2.3.2), we obtain

$$\exp_{p+1}\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \leqslant |A_0(z)| \leqslant$$

$$k\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right), T(s(|z|), f)\right\}\right]^k \times \exp_{p+1}\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\},$$

for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$, where $E_1 \subset [0,1)$ is a set with $\int_{E_1} \frac{dr}{1-r} < \infty$. Noting that $(\mu - \varepsilon) > (\mu - 2\varepsilon)$, by the last inequality, we have

$$\exp\left((1-o(1))\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}\right) \leqslant$$

$$k\left(\frac{1}{1-|z|}\right)^{k(2+\varepsilon)} T^k\left(s\left(|z|\right), f\right), \tag{2.3.3}$$

for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Then, by (2.3.3) and combining with Lemma 2.2.3, we get for all $r = |z| \in H_1$

$$\exp\left((1-o(1))\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}\right) \leqslant k\left(\frac{1}{1-s\left(|z|\right)}\right)^{k(2+\varepsilon)} T^k\left(s_1\left(|z|\right), f\right),$$

where $s_1(|z|) = s(s(r)) = 1 - d^2(1-|z|)$ with $d \in (0,1)$. Then, we get for $|z| \in H_1$

$$\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \leqslant$$

$$\frac{\log k}{1-o(1)} + \frac{k(2+\varepsilon)}{1-o(1)}\log\left(\frac{1}{1-s\left(|z|\right)}\right) + \frac{k}{1-o(1)}\log^+ T\left(s_1\left(|z|\right), f\right),$$

$$\Longrightarrow \quad \frac{\exp\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} \leqslant \frac{O(1)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_p\left(\frac{1}{1-s(|z|)}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_p^+ T\left(s_1\left(|z|\right), f\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)},$$

$$\Longrightarrow \quad \limsup_{s_1(r)\to 1^-} \frac{\exp\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} \leqslant$$

$$\limsup_{s_1(r)\to 1^-}\left(\frac{O(1)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_p\left(\frac{1}{1-s(|z|)}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_p^+ T\left(s_1\left(|z|\right), f\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)}\right),$$

$$\Longrightarrow \quad +\infty \leqslant 0 + 0 + \limsup_{s_1(r)\to 1^-} \frac{\log_{p+1}^+ T\left(s_1\left(|z|\right),\ f\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} = \sigma_{[p,q]}(f).$$

Therefore, we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$, and we have

$$\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \leq$$

$$\frac{\log k}{1-o(1)} + \frac{k(2+\varepsilon)}{1-o(1)}\log\left(\frac{1}{1-s\left(|z|\right)}\right) + \frac{k}{1-o(1)}\log^+ T\left(s_1\left(|z|\right), f\right),$$

$$\Longrightarrow \quad (\mu-\varepsilon)\frac{\log_q\left(\frac{1}{1-|z|}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} \leqslant \frac{O(1)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_{p+1}\left(\frac{1}{1-s(|z|)}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log^+_{p+1} T(s_1(|z|), f)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)},$$

$$\Longrightarrow \quad \limsup_{s_1(r)\to 1^-}\left((\mu-\varepsilon)\frac{\log_q\left(\frac{1}{1-|z|}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)}\right) = \mu-\varepsilon \leqslant$$

$$\limsup_{s_1(r)\to 1^-}\left(\frac{O(1)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log_{p+1}\left(\frac{1}{1-s(|z|)}\right)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} + \frac{\log^+_{p+1} T(s_1(|z|), f)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)}\right),$$

$$\leqslant 0+0+\limsup_{s_1(r)\to 1^-}\frac{\log^+_{p+1} T(s_1(|z|),\ f)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} = \sigma_{[p+1,q]}(f).$$

Since $\varepsilon>0$ is arbitrary we deduce that

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \limsup_{s_1(r)\to 1^-}\frac{\log^+_{p+1} T(s_1(|z|),\ f)}{\log_q\left(\frac{1}{1-s_1(|z|)}\right)} \geqslant \mu.$$

Proof of Theorem 2.1.2. Suppose that every solution f of equation (0.0.2) not being identically equal to 0. It follows from (0.0.2) that

$$-A_0(z) = A_k(z)\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f}. \tag{2.3.4}$$

From the assumptions of Theorem 2.1.2, there exist *a set* H of complex numbers satisfying $\overline{dens}_D H_1 > 0$, where $H_1 = \{r=|z| : z\in H\subseteq D\}$. Then H_1 is a set with $\int_{H_1}\frac{dr}{1-r} = +\infty$, such that for $z\in H$ we have (2.1.4) and (2.1.5) az $r\to 1^-$. From the assumption (2.1.5), we get by using (2.3.4) and Lemma 2.2.2 that

$$m(r, A_0) \leqslant \sum_{i=1}^{k} m(r, A_i) + \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + O(1)$$

$$\leqslant k\exp_p\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\} + S(r,f), \tag{2.3.5}$$

holds for all z satisfying $r\in H_1$ as $r\to 1^-$ outside a set $E_1\subset[0,1)$ with $\int_{E_1}\frac{dr}{1-r}<\infty$, where $S(r,f) = O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$. By Lemma 2.2.3 and (2.3.5), we have for all $r\in H_1$

$$m(r, A_0) \leqslant k\exp_p\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-s(r)}\right)\right\}$$

$$+ O\left(\log^+ T(s(r), f) + \log\left(\frac{1}{1-s(r)}\right)\right), \tag{2.3.6}$$

as $r \to 1^-$. The assumption (2.1.4) gives us

$$m(r, A_0) = T(r, A_0) \geqslant \exp_p\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}. \tag{2.3.7}$$

By (2.3.6) and (2.3.7), we obtain

$$(1 - o(1))\exp_p\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}$$
$$\leqslant O\left(\log^+ T(s(r), f) + \log\left(\frac{1}{1-s(r)}\right)\right), \tag{2.3.8}$$

as $r \to 1^-$. Hence, by (2.3.8), we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \limsup_{s(r)\to 1^-} \frac{\log_{p+1}^+ T(s(r),\ f)}{\log_q\left(\frac{1}{1-s(r)}\right)} \geqslant \mu,$$

because $\varepsilon > 0$ is arbitrary.

2.4 Proofs of Theorems 2.1.3 to 2.1.10

Proof of Theorem 2.1.3 Suppose that every solution f of equation (0.0.1) not being identically equal to 0. By (2.1.6), we know that

$$\exists(\lambda - \varepsilon) \in \mathbb{R}: \qquad \liminf_{|z|\to 1^-, z\in H} \frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)} > \lambda - \varepsilon > \mu - \varepsilon,$$

Obviously

$$\frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)} > \mu - \varepsilon, \tag{2.4.1}$$

az $|z| \to 1^-$ for $z \in H$. By (2.1.7) and (2.4.1), we obtain

$$|A_0(z)| > \exp_{p+1}\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$
$$> \exp_{p+1}\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant |A_i(z)|, \tag{2.4.2}$$

az $|z| \to 1^-$ for $z \in H$ $(i = 1, 2, ..., k-1)$. By (2.4.2) and applying Theorem 2.1.1 $(A_k(z) \equiv 1)$, we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \quad \textit{and} \quad \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu. \tag{2.4.3}$$

By Lemma 2.2.4, we get

$$\begin{aligned}\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leqslant \max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, ..., k-1\right\} \\ = \sigma_{M,[p,q]}(A_0) = \mu.\end{aligned} \tag{2.4.4}$$

Therefore, by (2.4.3) and (2.4.4), we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0) = \mu$.

Proof of Theorem 2.1.4 Set

$$\alpha_0 = \liminf_{|z|\to 1^-, z\in H} \frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)},$$

$$\alpha_i = \limsup_{|z|\to 1^-, z\in H} \frac{\log_{p+1}|A_i(z)|}{\log_q\left(\frac{1}{1-|z|}\right)}, \qquad (i = 1, 2, ..., k-1).$$

By (2.1.9), there exist real numbers α, β such that $\alpha_i < \beta < \alpha < \alpha_0$, $i = 1, 2, ..., k-1$. It yields

$$\frac{\log_{p+1}|A_i(z)|}{\log_q\left(\frac{1}{1-|z|}\right)} < \beta < \alpha < \frac{\log_{p+1}|A_0(z)|}{\log_q\left(\frac{1}{1-|z|}\right)},$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have

$$|A_0(z)| > \exp_{p+1}\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$> \exp_{p+1}\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant |A_i(z)|, \qquad (i = 1, 2, ..., k-1),$$

as $|z| \to 1^-$ for $z \in H$. Then, by applying Theorem A, we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \alpha,$$

for $0 \leqslant \beta < \alpha$. By taking $\beta = \mu - 2\varepsilon$ and $\alpha = \mu - \varepsilon$ for any given ε $(0 < 2\varepsilon < \mu)$, we get

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu,$$

and by Lemma 2.2.4 we get (2.4.4). Hence, we can easily obtain the conclusion of Theorem 2.1.4.

Proof of Theorem 2.1.5 Suppose that every solution f of equation (0.0.1) not being identically equal to 0. By (2.1.10), we know that

$$\exists (\lambda - \varepsilon) \in \mathbb{R} : \qquad \liminf_{|z|\to 1^-, z\in H} \frac{\log_p T(|z|, A_0)}{\log_q\left(\frac{1}{1-|z|}\right)} > \lambda - \varepsilon > \mu - \varepsilon,$$

Obviously

$$\frac{\log_p T(|z|, A_0)}{\log_q\left(\frac{1}{1-|z|}\right)} > \mu - \varepsilon, \tag{2.4.5}$$

az $|z| \to 1^-$ for $z \in H$. By (2.1.11) and (2.4.5), we obtain

$$T(|z|, A_0) > \exp_p\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$> \exp_p\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant T(|z|, A_i), \qquad (i = 1, 2, ..., k-1), \tag{2.4.6}$$

az $|z| \to 1^-$ for $z \in H$. By applying Theorem 2.1.2 $(A_k(z) \equiv 1)$, we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu. \tag{2.4.7}$$

By Lemma 2.2.4, we get

$$\begin{aligned} \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) &\leqslant \max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, ..., k-1\right\} \\ &= \sigma_{M,[p,q]}(A_0) = \mu. \end{aligned} \tag{2.4.8}$$

Therefore, by (2.4.7) and (2.4.8), we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Proof of Theorem 2.1.6 Set

$$\begin{aligned} \alpha_0 &= \liminf_{|z|\to 1^-, z\in H} \frac{\log_p T(|z|, A_0)}{\log_q\left(\frac{1}{1-|z|}\right)}, \\ \alpha_i &= \limsup_{|z|\to 1^-, z\in H} \frac{\log_p T(|z|, A_i)}{\log_q\left(\frac{1}{1-|z|}\right)}, \qquad (i = 1, 2, ..., k-1). \end{aligned}$$

By (1.12), there exist real numbers α, β such that $\alpha_i < \beta < \alpha < \alpha_0$, $i = 1, 2, ..., k-1$. It yields

$$\frac{\log_p T(|z|, A_i)}{\log_q\left(\frac{1}{1-|z|}\right)} < \beta < \alpha < \frac{\log_p T(|z|, A_0)}{\log_q\left(\frac{1}{1-|z|}\right)},$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have

$$T(|z|, A_0) > \exp_{p+1}\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$> \exp_{p+1}\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant T(|z|, A_i), \qquad (i = 1, 2, ..., k-1),$$

as $|z| \to 1^-$ for $z \in H$. Then, by applying Theorem B, we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \alpha,$$

for $0 \leqslant \beta < \alpha$. By taking $\beta = \mu - 2\varepsilon$ and $\alpha = \mu - \varepsilon$ for any given $\varepsilon \ (0 < 2\varepsilon < \mu)$, we get

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geqslant \mu,$$

and by Lemma 2.2.4 we get $(2.4.4)$. Hence, we can easily obtain the conclusion of Theorem 2.1.6.

Proof of Theorems 2.1.7 and 2.1.8 By using a similar proof as in Theorem 2.1.3 or Theorem 2.1.4, then for $\mu > 0$ *and for all* ε $(0 < 2\varepsilon < \mu)$ sufficiently small, we have

$$|A_0(z)| > \exp_{p+1}\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$> \exp_{p+1}\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant |A_i(z)|, \qquad (i = 1, 2, ..., k),$$

as $|z| \to 1^-$ for $z \in H$. Hence, by applying Theorem 2.1.1, we get the result.

Proof of Theorems 2.1.9 and 2.1.10 By using a similar proof as in Theorem 2.1.5 or Theorem 2.1.6, then for $\mu > 0$ and for all ε $(0 < 2\varepsilon < \mu)$ sufficiently small, we have

$$T(|z|, A_0) > \exp_p\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

$$> \exp_p\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \geqslant T(|z|, A_i), \qquad (i = 1, 2, ..., k),$$

as $|z| \to 1^-$ for $z \in H$. Hence, by applying Theorem 2.1.2, we get the result.

Chapter 3

The Fixed Points of Solutions and Their Arbitrary-order Derivatives of Linear Differential Equations in The Unit Disc

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades, see [13]. However, there are few studies on the fixed points of solutions of differential equations, specially in the unit disc.

In 2000, Chen[12] studied firstly the problem on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients. Later, Zhang and Chen [22] consider the question of the existence of fixed points of the derivatives of solutions of complex linear differential equations in D. Since a few months, Chen *et al.* [10] investigated the fixed points of solutions and their arbitrary order derivatives of equations (0.0.1) and (0.0.2), and they obtained the following results.

Theorem E (see [10]) *Assume that the assumptions of Theorem C hold. Then every solution* $f \not\equiv 0$ *of* (0.0.1) *satisfies*

$$\bar{\lambda}_n\left(f^{(j)}-z\right)=\bar{\lambda}_n\left(f-z\right)=\sigma_n\left(f\right)=\infty,$$

$$\bar{\lambda}_{n+1}\left(f^{(j)}-z\right)=\bar{\lambda}_{n+1}\left(f-z\right)=\sigma_{n+1}\left(f\right)=\mu. \qquad (j=1,2,...).$$

In this chapter, we will represent the results of Chen *et al.* [10] concerning the fixed points of solutions and thier arbitrary-order of equations (0.0.1) and (0.0.2) by replacing the iterated order by $[p,q]-$order, and we get our theorems as follows.

Theorem 3.0.1 *Assume that the assumptions of Theorem 2.1.3 or Theorem 2.1.4 hold. Then every solution* $f \not\equiv 0$ *of equation* (0.0.1) *satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)}-z\right)=\bar{\lambda}_{[p,q]}\left(f-z\right)=\sigma_{[p,q]}\left(f\right)=\infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)}-z\right)=\bar{\lambda}_{[p+1,q]}\left(f-z\right)=\sigma_{[p+1,q]}\left(f\right)=\mu. \quad (j=1,2,...).$$

Proof. Suppose that every solution f of equation (0.0.1) not being identically equal to 0.
First step. We consider the fixed points of $f(z)$. Define the function g by setting

$$g(z):=f(z)-z, \qquad z\in D.$$

Then, It follows from (0.0.1) that

$$g^{(k)}+A_{k-1}g^{(k-1)}+\cdots+A_1g'+A_0g=-A_1-zA_0, \tag{3.0.1}$$

and by Theorem 2.1.3 or Theorem 2.1.4, we have

$$\begin{gathered}\sigma_{[p,q]}(g)=\sigma_{[p,q]}(f)=\infty, \qquad \sigma_{[p+1,q]}(g)=\sigma_{[p+1,q]}(f)=\mu,\\ \bar{\lambda}_{[p+1,q]}(g)=\bar{\lambda}_{[p+1,q]}(f-z).\end{gathered} \tag{3.0.2}$$

Now, we prove that $-A_1-zA_0\not\equiv 0$. Assume that $-A_1-zA_0\equiv 0$. Clearly $A_0\not\equiv 0$. Then $\lim\limits_{|z|\to 1^-, z\in H}\left|\frac{A_1}{A_0}\right|=1$ and by (2.4.2), we have

$$\left|\frac{A_1}{A_0}\right|<\frac{\exp_{p+1}\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}}{\exp_{p+1}\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}}\to 0,$$

az $|z|\to 1^-$ for $z\in H$. Then $\lim\limits_{|z|\to 1^-, z\in H}\left|\frac{A_1}{A_0}\right|=0$. It is easy to see the contradiction. Hence, $-A_1-zA_0\not\equiv 0$. Next by Lemma 2.2.5, we get

$$\max\left\{\sigma_{[p,q]}(A_i)\,(i=0,1,...,k-1)\,,\sigma_{[p,q]}(-A_1-zA_0)\right\}<\infty.$$

We deduce by using $(3.0.1),(3.0.2)$ and Lemma 2.2.6 that

$$\bar{\lambda}_{[p,q]}(g)=\sigma_{[p,q]}(g)=\infty, \qquad \bar{\lambda}_{[p+1,q]}(g)=\sigma_{[p+1,q]}(g)=\mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f-z)=\bar{\lambda}_{[p,q]}(g)=\sigma_{[p,q]}(g)=\sigma_{[p,q]}(f)=\infty,$$

$$\bar{\lambda}_{[p+1,q]}(f-z)=\bar{\lambda}_{[p+1,q]}(g)=\sigma_{[p+1,q]}(g)=\sigma_{[p+1,q]}(f)=\mu.$$

Second step. For the following proof, we use the principle of mathematical induction. Set $A_k(z)\equiv 1$, then $|A_k(z)|\leqslant \exp_{p+1}\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$. We consider the fixed points of $f^{(j)}(z)\,(j=1,2,...)$.

• Define the function g_1 by setting

$$g_1(z):=f'(z)-z,\ z\in D.$$

Then, by $(3.0.2)$, we obtain

$$\sigma_{[p,q]}(g_1)=\sigma_{[p,q]}(f')=\infty,\qquad \sigma_{[p+1,q]}(g_1)=\sigma_{[p+1,q]}(f')=\mu,$$
$$\bar{\lambda}_{[p+1,q]}(g_1)=\bar{\lambda}_{[p+1,q]}(f'-z).\tag{3.0.3}$$

It follows by $(0.0.2)$, that

$$\frac{A_k}{A_0}f^{(k+1)}+\left(\left(\frac{A_k}{A_0}\right)'+\frac{A_{k-1}}{A_0}\right)f^{(k)}+\cdots+\left(\left(\frac{A_2}{A_0}\right)'+\frac{A_1}{A_0}\right)f''+\left(\left(\frac{A_1}{A_0}\right)'+1\right)f'=0.\tag{3.0.4}$$

Multiplying (3.0.4) by A_0, we obtain

$$A_{k,1}f^{(k+1)}+A_{k-1,1}f^{(k)}+\cdots+A_{1,1}f''+A_{0,1}f'=0.\tag{3.0.5}$$

Substituing $f'=g_1+z$ into $(3.0.5)$, we obtain

$$A_{k,1}g_1^{(k)}+A_{k-1,1}g_1^{(k-1)}+\cdots+A_{1,1}g_1'+A_{0,1}g_1=F_1,\tag{3.0.6}$$

where

$$A_{k,1}=1,\ A_{i,1}=A_0\left(\left(\frac{A_{i+1}}{A_0}\right)'+\frac{A_i}{A_0}\right)\quad(i=1,2,...,k-1),\tag{3.0.7}$$

$$A_{0,1}=A_0\left(\left(\frac{A_1}{A_0}\right)'+1\right),\tag{3.0.8}$$

$$F_1=-(A_{1,1}+zA_{0,1}).\tag{3.0.9}$$

Next we prove that $A_{0,1}\not\equiv 0$ and $F_1\not\equiv 0$. Assume that $A_{0,1}\equiv 0$, then $\frac{A_1}{A_0}=-z+C_0$ where C_0 is an arbitrary constant. Hence, we have $A_1+(z-C_0)A_0=0$. Then, $f_0=z-C_0$ is a solution of (0.0.1) and $\sigma_{[p,q]}(f_0) < \infty$. This contradicts $(3.0.2)$. Now, assume that $F_1 \equiv 0$. By $(3.0.5)$ and $(3.0.9)$, we know that the function f_1 such that $f_1' = z$ is a solution of equation $(0.0.1)$ and $\sigma_{[p,q]}(f_1) < \infty$. This contradicts $(3.0.2)$. Therefore, $A_{0,1} \not\equiv 0$ and $F_1 \not\equiv 0$. It follows by $(3.0.7) - (3.0.9)$ and Lemma 2.2.5 that

$$\max\left\{\sigma_{[p,q]}(A_{i,1})\,(i = 0, 1, ..., k-1),\, \sigma_{[p,q]}(F_1)\right\} < \infty.$$

We deduce by using $(3.0.3)$, $(3.0.6)$ and Lemma 2.2.6 that

$$\bar{\lambda}_{[p,q]}(g_1) = \sigma_{[p,q]}(g_1) = \infty, \qquad \bar{\lambda}_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(g_1) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f' - z) = \bar{\lambda}_{[p,q]}(g_1) = \sigma_{[p,q]}(g_1) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f' - z) = \bar{\lambda}_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(f) = \mu.$$

• Set $g_2(z) = f''(z) - z,\ z \in D$. Then, by using a similar discussion as in the case of the function g_1, we can get

$$A_{k,2}f^{(k+2)} + A_{k-1,2}f^{(k+1)} + \cdots + A_{1,2}f^{(3)} + A_{0,2}f'' = 0$$

and

$$A_{k,2}g_2^{(k)} + A_{k-1,2}g_2^{(k-1)} + \cdots + A_{1,2}g_2' + A_{0,2}g_2 = F_2,$$

where

$$\begin{aligned} A_{k,2} &= 1,\ A_{i,2} = A_{0,1}\left(\left(\frac{A_{i+1,1}}{A_{0,1}}\right)' + \frac{A_{i,1}}{A_{0,1}}\right)\ (i = 1, 2, ..., k-1), \\ A_{0,2} &= A_{0,1}\left(\left(\frac{A_{1,1}}{A_{0,1}}\right)' + 1\right), \\ F_2 &= -\left(A_{1,2} + zA_{0,2}\right). \end{aligned}$$

Therefore, by the same procedure as for g_1, we obtain

$$\bar{\lambda}_{[p,q]}(f'' - z) = \bar{\lambda}_{[p,q]}(g_2) = \sigma_{[p,q]}(g_2) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f'' - z) = \bar{\lambda}_{[p+1,q]}(g_2) = \sigma_{[p+1,q]}(g_2) = \sigma_{[p+1,q]}(f) = \mu.$$

• Now, assume that

$$\begin{aligned} &A_{0,s} \not\equiv 0, \\ &\bar{\lambda}_{[p,q]}\left(f^{(s)} - z\right) = \sigma_{[p,q]}(f) = \infty, \\ &\bar{\lambda}_{[p+1,q]}\left(f^{(s)} - z\right) = \sigma_{[p+1,q]}(f) = \mu \end{aligned} \tag{3.0.10}$$

for all $s = 0, 1, ..., j-1$, and we prove that for $s = j$ we have (3.0.10) hold. Set $g_j(z) = f^{(j)}(z) - z$, $z \in D$. Then, by using (3.0.2), we obtain

$$\sigma_{[p,q]}(g_j) = \sigma_{[p,q]}\left(f^{(j)}\right) = \infty, \qquad \sigma_{[p+1,q]}(g_j) = \sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu,$$
$$\bar{\lambda}_{[p+1,q]}(g_j) = \bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right), \tag{3.0.11}$$

by following the same procedure as before, we have

$$A_{k,j}f^{(k+j)} + A_{k-1,j}f^{(k+j-1)} + \cdots + A_{1,j}f^{(j+1)} + A_{0,j}f^{(j)} = 0,$$

then, we get

$$A_{k,j}g_j^{(k)} + A_{k-1,j}g_j^{(k-1)} + \cdots + A_{1,j}g_j' + A_{0,j}g_j = F_j, \tag{3.0.12}$$

where

$$\begin{aligned} A_{k,j} &= 1,\ A_{i,j} = A_{0,j-1}\left(\left(\frac{A_{i+1,j-1}}{A_{0,j-1}}\right)' + \frac{A_{i,j-1}}{A_{0,j-1}}\right)\ (i = 1, 2, ..., k-1), \\ A_{0,j} &= A_{0,j-1}\left(\left(\frac{A_{1,j-1}}{A_{0,j-1}}\right)' + 1\right) \not\equiv 0\ (A_{0,0} = A_0, \quad A_{1,0} = A_1), \\ F_j &= -(A_{1,j} + zA_{0,j}) \not\equiv 0. \end{aligned}$$

We deduce by using (3.0.11), (3.0.12) and Lemma 2.2.6 that

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(g_j) = \sigma_{[p,q]}(g_j) = \sigma_{[p,q]}\left(f^{(j)}\right) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(g_j) = \sigma_{[p+1,q]}(g_j) = \sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu, \quad j = 1, 2, ...$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) = \mu, \qquad (j = 1, 2, ...).$$

□

Theorem 3.0.2 *Assume that the assumptions of Theorem 2.1.5 or Theorem 2.1.6 hold. Then every solution $f \not\equiv 0$ of equation (0.0.1) satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) = \mu, \quad (j = 1, 2, ...).$$

Proof. Suppose that every solution f of equation (0.0.1) not being identically equal to 0. By applying Theorem 2.1.5 or Theorem 2.1.6, we get

$$\sigma_{[p,q]}(f) = \infty, \qquad \sigma_{[p+1,q]}(f) = \mu.$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$, then we can easily obtain

$$\begin{aligned} T(r, A_1) &= T(r, -zA_0) \leqslant T(r, A_0) + T(r, z), \\ T(r, A_0) &= T\left(r, \tfrac{A_1}{-z}\right) \leqslant T(r, A_1) + T(r, z) + O(1). \end{aligned} \tag{3.0.13}$$

It follows from (3.0.13) that

$$1 - \frac{T(r,z) + O(1)}{T(r,A_0)} \leqslant \frac{T(r,A_1)}{T(r,A_0)} \leqslant 1 + \frac{T(r,z)}{T(r,A_0)}. \tag{3.0.14}$$

By following the same reasoning as in the proof of Theorem 2.1.5, we get

$$\begin{aligned} T(r, A_1) \leq \exp_p\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} &< \exp_p\left\{(\mu - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\} \\ &< T(r, A_0), \end{aligned} \tag{3.0.15}$$

as $r = |z| \to 1^-$ for $z \in H$. By using (3.0.15), we have

$$\frac{T(r,z)}{T(r,A_0)} \leqslant \frac{T(r,z)}{\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}} \to 0, \tag{3.0.16}$$

as $|z| \to 1^-$ for $z \in H$. Then, by (3.0.14) and (3.0.16), we get

$$\lim_{|z|\to 1^-, z\in H} \frac{T(r,A_1)}{T(r,A_0)} = 1. \tag{3.0.17}$$

On the other hand, we have

$$\begin{aligned} &\text{for } p = 1, \qquad \frac{T(r,A_1)}{T(r,A_0)} < \frac{1}{\exp\left\{\varepsilon\log_q\left(\frac{1}{1-|z|}\right)\right\}} < 1, \\ &\text{for } p \geqslant 2, \qquad \frac{T(r,A_1)}{T(r,A_0)} < \frac{\exp_p\left\{(\mu-2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}}{\exp_p\left\{(\mu-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}} \to 0. \end{aligned} \tag{3.0.18}$$

It follows by (3.0.18) that

$$\lim_{|z|\to 1^-, z\in H} \frac{T(r,A_1)}{T(r,A_0)} \neq 1. \tag{3.0.19}$$

Obviously, (3.0.17) contradicts with (3.0.19). Hence, $-A_1 - zA_0 \not\equiv 0$. Set $A_k(z) \equiv 1$, then $T(r, A_k) \leqslant \exp_p\left\{(\mu - 2\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$. Clearly, $A_0 \not\equiv 0$. We can get the conclusion of Theorem 3.0.2, by reasoning in the same way as we did in the proof of Theorem 3.0.1.

□

Theorem 3.0.3 *Assume that the assumptions of one of Theorem 2.1.7 to Theorem 2.1.10 hold. Then every meromorphic (or analytic) solution $f \not\equiv 0$ of equation (0.0.2) satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(f - z) = \sigma_{[p,q]}(f) \geqslant \mu \qquad (j = 1, 2, ...).$$

Proof. Suppose that every solution f of equation (0.0.2) not being identically equal to 0. By applying one of Theorem 2.1.7 to Theorem 2.1.10, we get

$$\sigma_{[p,q]}(f) = \infty, \qquad \sigma_{[p+1,q]}(f) \geqslant \mu.$$

Then, we can get the conclusion of Theorem 3.0.3, by reasoning in the same way as we did in the proof of theorem 3.0.1 and Theorem 3.0.2 by using $\sigma_{[p+1,q]}(f) \geqslant \mu$ instead of $\sigma_{[p+1,q]}(f) = \mu$, and $\sigma_{[p+1,q]}\left(f^{(j)}\right) \geqslant \mu$ instead of $\sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu$ $(j = 1, 2, ...)$. □

Example 3.0.1 *Consider the following equation*

$$K_2(z)\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}f'' + K_1(z)\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}f'$$
$$+ K_0(z)\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}f = 0, \qquad (3.0.20)$$

where K_0, K_1 and K_2 are analytic functions in the unit disc D such that

$$\begin{cases} \sigma_{M,[3,2]}(K_0) > 2 \quad and \quad |K_0| > 1. \\ \sigma_{M,[3,2]}(K_1) < 1 \quad and \quad |K_1| < 1. \\ \sigma_{M,[3,2]}(K_2) < 1 \quad and \quad |K_2| < 1. \end{cases}$$

Let $H = \{z \in \mathbb{C} : |z| = r < 1$ and $\arg z = 0\} \subset D$ a set of complex numbers satisfying $\overline{dens_D}\{|z| : z \in H\} = 1 > 0$.

In the equation (3.0.20) we have for all ε $(0 < \varepsilon < 1)$ sufficiently small :

$A_0(z) = K_0(z)\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}$, $A_i(z) = K_i(z)\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}$, $i = 1, 2$, *then we get*

$$max\left\{\sigma_{M,[3,2]}(A_1), \sigma_{M,[3,2]}(A_2)\right\} < \sigma_{M,[3,2]}(A_0).$$

In the other hand

$$\begin{aligned}|A_0(z)| &= |K_0(z)|\left|\exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-z}\right)\right\}\right| \\ &> \exp_4\left\{\left(2-\frac{\varepsilon}{2}\right)\log_2\left(\frac{1}{1-r}\right)\right\},\end{aligned}$$

$$\Longrightarrow \frac{\log_4|A_0(z)|}{\log_2\left(\frac{1}{1-r}\right)} > 2-\frac{\varepsilon}{2} \Longrightarrow \liminf_{r\to 1^-, z\in H}\frac{\log_4|A_0(z)|}{\log_2\left(\frac{1}{1-r}\right)} \geqslant 2-\frac{\varepsilon}{2} > 2-\varepsilon,$$

and

$$\begin{aligned}|A_i(z)| &= |K_i(z)|\left|\exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-z}\right)\right\}\right| \\ &< \exp_4\left\{(2-2\varepsilon)\log_2\left(\frac{1}{1-r}\right)\right\}, \quad i = 1, 2,\end{aligned}$$

az $r \to 1^-$ *for* $z \in H$.

It is clear that the conditions of Theorem 2.1.7 hold with $\mu = 2$, $p = 3$ *and* $q = 2$ *on the set* H *such that* $\overline{dens_D}\{|z| : z \in H\} > 0$.

By Theorem 3.0.3, every meromorphic (or analytic) solution $f \not\equiv 0$ *of equation* (3.0.20) *satisfies*

$$\bar{\lambda}_{[3,2]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[3,2]}(f - z) = \sigma_{[3,2]}(f) = \infty,$$

and

$$\bar{\lambda}_{[4,2]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[4,2]}(f - z) = \sigma_{[4,2]}(f) \geqslant 2, \quad (j = 1, 2, ...).$$

CONCLUSION

Overall, the subject of this thesis was devoted to the growth and fixed points of solutions of linear differential equations of the form

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,$$

in the case where $A_i(z) \not\equiv 0$ $(i = 0, 1, ..., k$ and $k \geqslant 2)$ are analytic functions in the unit disc by using the concept of $[p, q]$-order.

During this work, we mentioned some results, in which we studied the $[p, q]$-order and the $[p, q]$-exponent of convergence of the sequence of distinct fixed points of solutions and their arbitrary order derivatives of general high-order linear differential equations cited above, and this leads us to ask the following questions:

Is it possible to obtain similar results for a sector of the unit disc?
And can we generalize the results when the coefficients are meromorphic functions and for non-homogeneous linear differential equations? And under what conditions would this generalization be possible?

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