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Option : Functional Analysis

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Growth of Entire solutions of Certain Classes of Non-Linear Differential Equations

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Submitted

by:

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Contents

Introduction

Nevanlinna theory was created to provide a quantitative measure of the value distribution of meromorphic functions. This theory originated over ninety years ago and still plays a very important role in the study of solutions of linar/non-linear differential equations in the complex domain.

This thesis is divided into introduction and two chapters. In the first chapter, we shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions. For example, the characteristic function $T(r,f)$, the counting function of the poles $N(r,f)$, and the proximity function $m(r,f)$ (see, [4],[3]). We use $\sigma(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros of f. The first and the second fundamental theorems are main parts of the theory. The first main theorem gives an upper bound for the counting function $N\left(r, \frac{1}{f-a}\right)$ for any $a \in \mathbb{C}$ and for large r, while the second main theorem provides a lower bound on the sum of any finite collection of counting functions $N\left(r, \frac{1}{f-a_j}\right)$ where $a_j \in \mathbb{C}$ and large r. In addition the identity

$$m\left(r, \frac{f'}{f}\right) = o\left(T(r,f)\right), \ r \to \infty \tag{1}$$

is very important in Nevanlinna theory. Relation (1) is called the lemma on the logarithmic derivative and it is an essential part of the proof of the second main theorem. Finally, we recall an identity originally due to Valiron [8] and later generalized by Mohon'ko [6], has proved to be an extremely useful tool in the study of meromorphic solutions of differential, difference and functional equations.

In the second chapter, we study the the non-linear differential equation

$$f\left(z\right)^{n_0}\left(f'\left(z\right)\right)^{n_1}\left(f''\left(z\right)\right)^{n_2}...\left(f^{(k)}\left(z\right)\right)^{n_k}=H(z),$$

where $H(z)$ is a non-vanishing entire function, n_0, $n_1, ..., n_k$ are non-negative integers such that $n_0 n_k \geq 1$. In fact, we prove that any non-constant entire solution of the above equation has the same growth rare as $H(z)$.

Chapter 1

Nevanlinna theory of meromorphic functions

This chapter is devoted to reviewing the basic facts and notations in the Nevanlinna theory needed for the material included in this thesis; for more details, we refer the reader to ([4], [3]) .

1.1 Poisson-Jensen and Jensen formulas

The starting point for Nevanlinna's theory is the following Poisson-Jensen formula.

Theorem 1.1 (Poisson-Jensen formula) [4] *Let* $f(z)$ *be a meromorphic function such that* $f(0) \neq 0, \infty$ *and let* $a_1, a_2, \ldots$ (*resp.* $b_1, b_2, \ldots$) *denote its zeros* (*resp. poles*), *each taken into account according to its multiplicity. If* $z = re^{i\theta}$ *and* $0 \leq r < R < \infty$, *then*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\varphi})\right| \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} d\varphi + \sum_{|a_j|<R} \log\left|\frac{R(z-a_j)}{R^2 - \overline{a_j}z}\right| - \sum_{|b_k|<R} \log\left|\frac{R(z-b_k)}{R^2 - \overline{b_k}z}\right|. \tag{1.1}$$

In partiluclar, by taking $z = 0$, we may derive the so-called Jensen formula.

Theorem 1.2 (Jensen formula) *[4] Let $f(z)$ be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) denote its zeros (resp. poles), each taken into account according to its multiplicity. Then, we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) - \sum_{|a_i|<r} \log\left(\frac{r}{|a_i|}\right). \quad (1.2)$$

Proof. *We give the proof for the case that $f(z)$ has no zeros or poles on $|z| = r$. Denote*

$$g(z) := f(z)\,\frac{\prod\limits_{|a_j|<r} \frac{r^2 - \bar{a}_j z}{r(z-a_j)}}{\prod\limits_{|b_j|<r} \frac{r^2 - \bar{b}_j z}{r(z-b_j)}}.$$

Then we have $g(z) \neq 0, \infty$ in $|z| \leq r$, hence $\log|g(z)|$ is a harmonic function. By the mean value property of classical harmonic functions,

$$\ln|g(0)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln\left|g\left(re^{i\varphi}\right)\right| d\varphi. \quad (1.3)$$

But since

$$|g(0)| = |f(0)|\,\frac{\prod\limits_{|a_j|<r} \frac{r}{|a_j|}}{\prod\limits_{|b_j|<r} \frac{r}{|b_j|}},$$

we get

$$\ln|g(0)| = \ln|f(0)| + \sum_{|a_j|<r} \ln\frac{r}{|a_j|} - \sum_{|b_j|<r} \ln\frac{r}{|b_j|}. \quad (1.4)$$

Finally, for any $z = re^{i\varphi}$, we have

$$\left|\frac{r^2 - \bar{a}_j z}{r(z-a_j)}\right| = \left|\frac{r^2 - \bar{b}_j z}{r(z-b_j)}\right| = \left|\frac{r^2 - \bar{a}_j re^{i\varphi}}{r(re^{i\varphi}-a_j)}\right| = \left|\frac{r^2 - \bar{b}_j re^{i\varphi}}{r(re^{i\varphi}-b_j)}\right| = 1,$$

for all a_j, b_j, *hence* $|g(re^{i\varphi})| = |f(re^{i\varphi})|$. *Combining this fact with* (1.3) *and* (1.4), *we obtain the assertion.* ■

1.2 Nevanlinna characteristic functions

Before we define the so-called **Nevanlinna characteristic function**, we need first to introduce the concept of the truncated logarithm denoted by $\log^+ x$. Indeed, for a positive real number x, the truncated logarithm $\log^+ x$ is defined by

$$\log^+ x \quad := \quad \max\{\log x \; ; 0\} \quad = \quad \begin{cases} \log x & if \quad x > 1. \\ 0 & if \quad 0 \leq x \leq 1. \end{cases}$$

Basic properties of this truncated logarithm are contained in the following lemmas.

Lemma 1.1 [4] *We have the following properties :*

a) $\log x \leq \log^+ x \quad (x \geq 0)$.

b) $\log^+ x \leq \log^+ y \quad (0 \leq x \leq y)$.

c) $\log x = \log^+ x - \log^+ \frac{1}{x} \quad (x > 0)$.

d) $|\log x| = \log^+ x + \log^+ \frac{1}{x} \quad (x > 0)$.

e) $\log^+ \left(\prod_{j=1}^{n} x_j \right) \leq \sum_{j=1}^{n} \log^+ x_j \quad (x_j \geq 0, j = 1, \ldots, n)$.

f) $\log^+ \left(\sum_{j=1}^{n} x_j \right) \leq \sum_{j=1}^{n} \log^+ x_j + \log n \quad (x_j \geq 0, j = 1, \ldots, n)$.

Proof. (a) and (b) are immediate consequences of the definition of truncated logarithme.

c) We have

$$\begin{aligned}\log^+ x - \log^+ \frac{1}{x} &= \max(\log x, 0) - \max\left(\log \frac{1}{x}, 0\right)\\ &= \max(\log x, 0) - \max(-\log x, 0)\\ &= \max(\log x, 0) + \min(\log x, 0)\\ &= \log x.\end{aligned}$$

d) For any real number $x \geq 0$, we have

$$\begin{aligned}\log^+ x + \log^+ \frac{1}{x} &= \max(\log x, 0) + \max\left(\log \frac{1}{x}, 0\right)\\ &= \max(\log x, 0) + \max(-\log x, 0)\\ &= \max(\log x, 0) - \min(\log x, 0)\\ &= |\log x| .\end{aligned}$$

e) Assume that $\prod_{j=1}^{m} x_j > 1$, since the case $\prod_{j=1}^{m} x_j \leq 1$ is trivial. Now, by making use of (a) we have

$$\begin{aligned}\log^+\left(\prod_{j=1}^{m} x_j\right) &= \log\left(\prod_{j=1}^{m} x_j\right)\\ &= \sum_{j=1}^{m} \log x_j\\ &\leq \sum_{j=1}^{m} \log^+ x_j \ .\end{aligned}$$

f) From (b) and (e) above, we obtain

$$\begin{aligned}
\log^+\left(\sum_{j=1}^{m} x_j\right) &\leq \log^+\left(m \max_{1\leq j\leq m} x_j\right) \\
&\leq \log m + \log^+\left(\max_{1\leq j\leq m} x_j\right) \\
&\leq \log m + \sum_{j=1}^{m} \log^+ x_j.
\end{aligned}$$

■

Lemma 1.2 [3] *For all $a \in \mathbb{C}$, we have*

$$\log^+ |a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta. \tag{1.5}$$

Proof. Denote $f(z) = a - z$, and suppose that $|a| < 1$. By using Jensen formula (1.2) with $r = 1$, we obtain

$$\begin{aligned}
\log |a| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(e^{i\theta})\right| d\theta - \log\frac{1}{|a|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta + \log |a|,
\end{aligned}$$

hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta = 0 = \log^+ |a|.$$

If $|a| \geq 1$, then f has no zeros in the disc $|z| < 1$. Therefore,

$$\log^+ |a| = \log |a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta.$$

■

In order to define the characteristic function of Nevanlinna,we need to define the Nevanlinna counting function $N(r, f)$ and the Nevanlinna proximity function $m(r, f)$.

Definition 1.1 (Unintegrated counting function) [4] *Let $a \in \mathbb{C}$ and let f be a meromorphic function such that $f \not\equiv a$. Then,*

- *$n(r, a, f)$ denotes the number of roots of the equation $f(z) = a$ in the disc $|z| \le r$, each root is counted according to its multiplicity.*
- *In addition, $n(r, \infty, f)$ denotes the number of poles of f in the disc $|z| \le r$, each pole is counted according to its multiplicity.*

Definition 1.2 (Counting function) [4] *Let $f(z)$ be a meromorphic function. For $a \in \mathbb{C}$, We define counting function by*

$$N(r, a, f) = N\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r,$$

provided that $f(z) \not\equiv a$, and

$$N(r, \infty, f) = N(r, f) := \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r.$$

Lemma 1.3 [4] *Let $f(z)$ be a meromorphic function with a-points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $|z| \le r$ such that*

$$0 < |\alpha_1| \le |\alpha_2| \le \cdots \le |\alpha_n| \le r,$$

each counted according to its multiplicity. Then

$$\int_0^r \frac{n(t, a, f)}{t} dt = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt = \sum_{0 < |\alpha_i| \le r} \log \frac{r}{|\alpha_i|}. \tag{1.6}$$

Proof. *Denoting* $r_i = |\alpha_i|$, *for* $i = 1, ..., n,$ *we obtain*

$$\sum_{0<|\alpha_i|\le r} \log \frac{r}{|\alpha_i|} = \sum_{i=1}^{n} \log \frac{r}{r_i} = \log \frac{r^n}{r_{1\ldots r_n}} = n \log r - \sum_{i=1}^{n} \log r_i$$
$$= \sum_{i=1}^{n-1} i(\log r_{i+1} - \log r_i) + n(\log r - \log r_n)$$
$$= \sum_{i=1}^{n-1} i \int_{r_i}^{r_{i+1}} \frac{dt}{t} + n \int_{r_n}^{r} \frac{dt}{t} = \int_{0}^{r} \frac{n(t,a)}{t} dt.$$

■

The following result plays a key role in the proof of the first main theorem of Nevanlinna

Proposition 1.1 [4] *Let* $f(z)$ *be a meromorphic function with the Laurent expansion*

$$f(z) = \sum_{j=m}^{+\infty} c_j z^j, \quad c_m \neq 0,\ m \in \mathbb{Z}.$$

Then

$$\log |c_m| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\varphi}) \right| d\varphi + N(r,f) - N\left(r, \frac{1}{f}\right).$$

Proof. *Define the meromorphic function* $h(z)$ *by setting*

$$h(z) := f(z) z^{-m}, \ z \in \mathbb{C}$$

Clearly, $m = n(0,0,f) - n(0,\infty,f)$ *and* $h(0) \neq 0, \infty.$ *The functions* $h(z)$ *and* $f(z)$ *have the same poles and zeros in* $0 < |z| \le r.$ *The Jensen formula, together with*

Lemma **1.3***, yields*

$$
\begin{aligned}
\log|c_m| &= \log|h(0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\varphi})\right| d\varphi + \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) - \sum_{|a_i|<r} \log\left(\frac{r}{|a_i|}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})r^{-m}\right| d\varphi \\
&+ \int_0^r \frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt - \int_0^r \frac{n(t,0,f)-n(0,0,f)}{t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi - m\log r \\
&+ \int_0^r \frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt - \int_0^r \frac{n(t,0,f)-n(0,0,f)}{t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + - \left[n(0,0,f)-n(0,\infty,f)\right]\log r \\
&+ \int_0^r \frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt - \int_0^r \frac{n(t,0,f)-n(0,0,f)}{t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right).
\end{aligned}
$$

■

Definition 1.3 (Proximity function) [4] *Let $f(z)$ be a meromorphic function. For $a \in \mathbb{C}$, we define the proximity function of $f(z)$ by*

$$m(r,a,f) = m\left(r,\frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\varphi})-a|}d\varphi,$$

provided that $f(z) \not\equiv a$, and

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi.$$

Now, we are ready to define the characteristic function of Nevanlinna $T(r, f)$.

Definition 1.4 (Characteristic function) [4] *For a meromorphic function $f(z)$, we define its characteristic function as*

$$T(r,f) := m(r,f) + N(r,f).$$

Remark 1.1 *Observe that if $f(z)$ is an entire function, which means that it has no poles within the disc $|z| \leq r$, and so $N(r, f) = 0$. Therefore,*

$$T(r,f) = m(r,f).$$

Example 1.1 *For the function $f(z) = e^z$, we have $N(r, f) = 0$. On the other hand,*

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f(re^{i\varphi}) \right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{re^{i\varphi}} \right| d\varphi \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\varphi d\varphi \\ &= \frac{r}{\pi}. \end{aligned}$$

Hence,

$$T(r, e^z) = m(r, e^z) + N(r, e^z) = \frac{r}{\pi} + 0 = \frac{r}{\pi}.$$

Next, we present some elementary properties of the characteristic functions.

Proposition 1.2 [4] *Let $f(z), f_1(z), \ldots, f_n(z)$ $(n \geq 1)$ be meromorphic functions and a, b, c and d be complex constants such that $ad - bc \neq 0$. Then*

1.

$$T\left(r, \prod_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} T(r, f_i).$$

2.

$$T\left(r,\sum_{i=1}^{n} f_i\right) \leq \sum_{i=1}^{n} T\left(r,f_i\right)+\log n.$$

3.

$$T\left(r,f^m\right)=mT\left(r,f\right),\ \forall m\in\mathbb{N}.$$

4.

$$T\left(r,\frac{af+b}{cf+d}\right)=T\left(r,f\right)+O\left(1\right)\ \ as\ r\to+\infty\ ,\ f\not\equiv-\frac{d}{c}.$$

Example 1.2 *Let $f(z)=\tan z$, we obtain*

$$f(z)=\tan z=\frac{\sin z}{\cos z}=\frac{e^{iz}-e^{-iz}}{2i}\frac{2}{e^{iz}+e^{-iz}}=-i\frac{e^{2iz}-1}{e^{2iz}+1}.$$

Hence, by making use of (4), we have

$$T(r,f)=T(r,-i\frac{e^{2iz}-1}{e^{2iz}+1})=T(r,e^{2iz})+O(1)=\frac{2r}{\pi}+O(1).$$

Proposition 1.3 *If $f\left(z\right)$ is a transcendental meromorphic function, then*

$$\liminf_{r\to\infty}\frac{T(r,f)}{\log r}=\infty.$$

In other words, $\log r=o\left(T(r,f)\right)$, *as* $r\to\infty$.

The following result due to Cartan tells us that on average the integrated counting function is larger than the proximity function, when we allow the target values vary over the boundary of a disc.

Theorem 1.3 (Cartan) ([3]) *Suppose that f is meromorphic in $|z|<R$. Then*

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi} N\left(r,e^{i\theta},f\right)d\theta+\log^+|f(0)|\,,\ \ (0<r<R). \tag{1.7}$$

Proof. By applying the Jensen formula (1.2) for the function $f(z) - e^{i\theta}$, we obtain

$$\log\left|f(0) - e^{i\theta}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\varphi + N(r,f) - N\left(r, e^{i\theta}, f\right). \qquad (1.8)$$

Intergarting both sides of (1.8) with respect to θ, yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta}\right| d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\varphi\right] d\theta \\ &+ N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\theta\right] d\varphi \\ &+ N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta.\end{aligned}$$

By using (1.5), we deduce

$$\begin{aligned}\log^+|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\varphi})\right| d\varphi + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta \\ &= m(r,f) + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta \\ &= T(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta.\end{aligned}$$

Hence, the formula (1.7) follows. ■

1.3 The first main theorem

We now move on to some essential theorems of Nevanlinna theory on which many of our subsequent results are based. We start with the so-called first main theorem of Nevanlinna.

Theorem 1.4 (First main theorem of Nevanlinna) *[4] Let $f(z)$ be a meromorphic function with the Laurent expansion*

$$f(z) - a = \sum_{j=m}^{+\infty} c_j z^j,\ c_m \neq 0,\ m \in \mathbb{Z},\ a \in \mathbb{C}.$$

Then, we have

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) - \log|c_m| + \varphi(r, a) \tag{1.9}$$

where $|\varphi(r, a)| \leq \log^+ |a| + \log 2$.

Proof. Assume first $a = 0$. By Lemma **1.1** and Proposition **1.1**, we obtain

$$\begin{aligned}
\log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta + N(r, f) - N\left(r, \frac{1}{f}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})|} d\theta + N(r, f) - N\left(r, \frac{1}{f}\right) \\
&= m(r, f) - m\left(r, \frac{1}{f}\right) + N(r, f) - N\left(r, \frac{1}{f}\right). \\
&= T(r, f) - T\left(r, \frac{1}{f}\right)
\end{aligned}$$

Hence

$$T(r, \frac{1}{f}) = T(r, f) - \log|c_m|, \tag{1.10}$$

which is the assertion with $\varphi(r,0)=0$. Let's deal now with the case $a\neq 0$. To satrt with, we define $h\left(z\right):=f\left(z\right)-a$. Clearly

$$N\left(r,\frac{1}{h}\right)=N\left(r,\frac{1}{f-a}\right),N\left(r,h\right)=N(r,f)andm\left(r,\frac{1}{h}\right)=m\left(r,\frac{1}{f-a}\right).$$

Recall that

$$\log^{+}\left|h\left(z\right)\right|=\log^{+}\left|f\left(z\right)-a\right|\leq\log^{+}\left|f\left(z\right)\right|+\log^{+}\left|a\right|+\log 2.$$

$$\log^{+}\left|f\left(z\right)\right|=\log^{+}\left|h+a\right|\leq\log^{+}\left|h\right|+\log^{+}\left|a\right|+\log 2.$$

Integrating the above two inequalities, we see that

$$m(r,h)\leq m(r,f)+\log^{+}|a|+\log 2$$

and

$$m(r,f)\leq m(r,h)+\log^{+}|a|+\log 2.$$

Hence $\varphi(r,a):=m(r,h)-m(r,f)$ satisfies

$$-(\log^{+}|a|+\log 2)\leq\varphi(r,a)\leq\log^{+}|a|+\log 2\Leftrightarrow|\varphi(r,a)|\leq\log^{+}|a|+\log 2.$$

Applying (1.10) for $h(z)$ we obtain

$$\begin{aligned} T(r,\frac{1}{f-a}) &= T(r,\frac{1}{h}) = m(r,\frac{1}{h}) + N(r,\frac{1}{h}) \\ &= m(r,h) + N(r,h) - \log|c_m| \\ &= m(r,f) + \varphi(r,a) + N(r,f) - \log|c_m| \\ &= T(r,f) + \varphi(r,a) - \log|c_m|\,. \end{aligned}$$

■

Remark 1.2 *The first maun theorem may be expressed as*

$$T(r,\frac{1}{f-a}) = T(r,f) + O(1), \quad r \to \infty \tag{1.11}$$

for all $a \in \mathbb{C}$.

When $f(z)$ is an entire function, the maximum modulus $M(r,f) := \max\limits_{|z|=r}|f(z)|$ and $T(r,f)$ may be related by the following proposition:

Proposition 1.4 [4] *Let* $g(z)$ *be an entire function and assume that* $0 < r < R < \infty$ *and that the maximum modulus* $M(r,g) = \max_{|z|=r}|g(z)|$, *satisfies* $M(r,g) \geq 1$. *Then*

$$T(r,g) \leq \log M(r,g) \leq \frac{R+r}{R-r}T(R,g).$$

Proof. *The first inequality is trivial:*

$$T(r,g) = m(r,g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|g\left(re^{i\varphi}\right)\right| d\varphi \leq \log^+ M(r,g) = \log M(r,g).$$

To prove the second inequality, take z_0 *such that* $z_0 = re^{i\theta}$ *and that* $|g(z_0)| = M(r,g)$.

Recall that

$$\left|\frac{R(z-g)}{R^2-\bar{a}_i z}\right| < 1$$

whenever $|z| < R$. *Therefore, the Poisson-Jensen formula results in*

$$\begin{aligned}
\log M(r,g) = \log|g(z_0)| &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}\log|g(Re^{i\varphi})|d\varphi \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{(R-r)(R+r)}{(R-r)^2+2Rr(1-\cos(\theta-\varphi))}\log^+|g(Re^{i\varphi})|d\varphi \\
&\leq \frac{R+r}{R-r}m(R,g) = \frac{R+r}{R-r}T(R,g).
\end{aligned}$$

■

1.4 Growth order and maximum modulus of meromorphic functions

The aim of this section is to establish a link between the maximum modulus function $M(r,f)$ and the Nevanlinna characteristic $T(r,f)$ in the case when f is an entire functions. In fact, we show that for entire functions in the complex plane both functions give a similar measure of growth.

Definition 1.5 (Order of growth) ([4], [3]) *Let* f *be a meromorphic function. The order of growth of* f *is defined by*

$$\sigma(f) := \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r}.$$

Example 1.3 *Let function* $f(z) = \dfrac{e^z+1}{e^z-1}$. *We have,*

$$\sigma(f(z)) = \overline{\lim_{r\to+\infty}} \frac{\log\left(\frac{r}{\pi}+O(1)\right)}{\log r} = 1.$$

The following properties can be proved easily by using the limit definition.

Theorem 1.5 *Let* f, g *be nonconstant meromorphic functions. Then*

1. $\sigma(f+g) \leq \max\{\sigma(f);\sigma(g)\}$.
2. $\sigma(fg) \leq \max\{\sigma(f);\sigma(g)\}$.
3. *If* $\sigma(g) < \sigma(f)$ *then* $\sigma(f+g) = \sigma(fg) = \sigma(f)$.

By using Proposition 1.4, we can show that $T(r,f)$ and $\log M(r,f)$ have the same gowth rate.

Corollary 1.1 *Let* f *be an entire function. Then*

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r}.$$

In order to measure the growth density of the zeros of f, we introduce the concept of exponent of convergence.

Definition 1.6 (The exponent of convergence) [3] *Let* f *be a meromorphic function. Then the exponent of convergence of the sequence of zeros of* $f(z)$ *is defined by*

$$\lambda(f) := \limsup_{r\to+\infty} \frac{\log^+ N\left(r,\frac{1}{f}\right)}{\log r},$$

Where $N(r,\frac{1}{f})$ *is the counting function of zeros of* $f(z)$ *in* $\{z : |z| < r\}$.

1.5 Second main theorem

One of the most important technical results of Nevanlinna theory is the lemma of the logarithmic derivative. This result asserts that the proximity function of the logarithmic derivative of a meromorphic function f should be small compared to the characteristic function of f. This result is essential in the classical proof of the second main theorem of Nevanlinna theory.

Theorem 1.6 (Lemma on logarithmic derivative) [3] *Let f be a transcendental meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = S\left(r, f\right),$$

where $S\left(r,f\right) := O\left(\log T\left(r,f\right) + \log r\right)$ outside of a possible exceptional set $E \subset [0, +\infty)$ with finite linear measure, i.e., $\int_E dt < \infty$. In addition, if f is of finite order of growth, then

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

The following is an immediate consequence of the lemma on the logarithmic derivative.

Corollary 1.2 [3] *Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S\left(r, f\right),$$

and

$$T\left(r, f^{(k)}\right) \leq (k+1)\, T\left(r, f\right) + S\left(r, f\right).$$

Proof. We prove this Corollary by induction. When $k = 1$, the first statement is just the lemma on the logarithmic derivative. Assume now that we have proved

$$m\left(r, \frac{f^{(l)}}{f}\right) = S(r, f),$$

for some $l \geq 1$. Then

$$m\left(r, f^{(l)}\right) \leq m\left(r, f\right) + m\left(r, \frac{f^{(l)}}{f}\right) = m\left(r, f\right) + S\left(r, f\right).$$

On the other hand, if f has a pole of order p at some point z_0, then $f^{(l)}$ has a pole of order $l + p \geq l + 1$ at z_0. It follows that

$$N\left(r, f^{(l)}\right) \leq (l+1)\, N(r, f).$$

So

$$T\left(r, f^{(l)}\right) \leq (l+1)\, T(r, f) + S(r, f).$$

This is the second claim in the statement of the lemma in the case $k = l$. Hence

$$m\left(r, \frac{f^{(l+1)}}{f^{(l)}}\right) = S(r, f^{(l)}) = S(r, f).$$

Finally

$$m\left(r, \frac{f^{(l+1)}}{f}\right) \leq m\left(r, \frac{f^{(l+1)}}{f^{(l)}}\right) + m\left(r, \frac{f^{(l)}}{f}\right) = S(r, f) + S(r, f) = S(r, f).$$

■

From now on, the notation $S(r, f)$ will stands for any quantity that satisfies

$$S(r, f) = o(T(r, f),$$

as $r \to \infty$, outiside a possible exceptional set of finite linear measure. Observe, that

$$O\left(\log T\left(r, f\right) + \log r\right) = o(T(r, f), \ r \to \infty.$$

We are ready now to state the second main theorem of Nevanlinna.

Theorem 1.7 (Second main theorem) *[4] Let $f\left(z\right)$ be a non-constant meromorphic function, let $q \geq 2$ and let $a_1, ..., a_q \in \mathbb{C}$ be distinct points. then*

$$m\left(r, f\right) + \sum_{n=1}^{q} m\left(r, \frac{1}{f - a_n}\right) \leq 2T(r, f) + S(r, f).$$

Corollary 1.3 *[3] Let $f\left(z\right)$ be meromorphic and non-constant, and let $a_1, a_2, ..., a_q \in \mathbb{C}$ be $q \geq 2$ distinct points. Then*

$$\left(q - 1\right) T\left(r, f\right) \leq N\left(r, f\right) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) + S\left(r, f\right).$$

Proof. Suppose first that $a_i \in \mathbb{C}$ for all $i \in \{1, ..., q\}$. We add $\sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right)$ to the left and right hand side of the inequality in Theorem 1.7. We have

$$T\left(r, f\right) + \sum_{i=1}^{q} T\left(r, \frac{1}{f - a_i}\right) \leq 2T\left(r, f\right) + N\left(r, f\right) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) + S\left(r, f\right),$$

by the theorem 1.4, we obtain

$$\left(q - 1\right) T\left(r, f\right) \leq N\left(r, f\right) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) + S\left(r, f\right).$$

■

1.6 Some applications in differential equations

We have already developed enough tools to be able to conclude some remarkable global results about solutions of differential equations.

Theorem 1.8 [2]*Every transcendental meromorphic solution of the first Painlevé equation,*

$$y'' - 6y^2 + z = 0 \tag{1.12}$$

has infinitely many poles.

Proof. Write equation (1.12) as

$$y^2 = 6^{-1}\left(y\frac{y''}{y} + z\right),$$

and recall that $m\left(r, y^2\right) = 2m\left(r, y\right)$. Hence

$$\begin{aligned} 2m\left(r, y\right) = m\left(r, y^2\right) &= m\left(r, 6^{-1}\left(y\frac{y''}{y} + z\right)\right) \\ &\leq m\left(r, 6^{-1}\right) + m\left(r, y\frac{y''}{y} + z\right) \\ &\leq m\left(r, y\frac{y''}{y}\right) + m\left(r, z\right) + \log 2 \\ &\leq m\left(r, y\right) + m\left(r, \frac{y''}{y}\right) + O\left(\log r\right) \\ &= m\left(r, y\right) + S\left(r, y\right) + O\left(\log r\right). \end{aligned}$$

So $m\left(r, y\right) = S\left(r, y\right) + O\left(\log r\right)$. if y has only finitely many poles, that is $N(r, y) = O\left(\log r\right)$,then

$$T(r, y) = m(r, y) + N(r, y) = S\left(r, y\right) + O(\log r).$$

From Proposition 1.3 that if y is transcendental then $\log r = o\left(T\left(r, y\right)\right)$. Therefore our solution y satisfies $T\left(r, y\right) = S(r, y)$, which means that

$$T(r, y) = o\left(T(r, y)\right)$$

which is clearly a contradiction. ■

The Valiron-Mohon'ko theorem has been proven to be an extremely useful tool in the study of meromorphic solutions of differential. It is stated as follows.

Theorem 1.9 (Valiron-Mohon'ko) ([3], [8]) *Let*

$$R\left(z, f(z)\right) := \frac{a_0(z) + a_1(z)f(z) + ... + a_p(z)f^p(z)}{b_0(z) + b_1(z)f(z) + ... + b_q(z)f^q(z)},$$

be a rational function of $f\left(z\right)$ *of degree* $d = \max\left(p, q\right)$ *with coefficients* $a_i\left(z\right)$ *and* $b_j\left(z\right)$ *satisfying*

$$T(r, a_i) = S(r, f) \text{ and } T(r, b_j) = S(r, f).$$

Then

$$T\left(r, R\left(z, f(z)\right)\right) = dT(r, f) + S(r, f).$$

The next application is originally due to Malmquist from 1913. An alternate proof and a generalization was given by Yosida using Nevanlinna theory in 1930's.

Theorem 1.10 (Malmquist's theorem) ([5], [3]) *Let* $f\left(z\right)$ *be a meromorphic solution of the equation*

$$f'\left(z\right) = R\left(z, f\right) := \frac{a_0(z) + a_1(z)f(z) + ... + a_p(z)f^p(z)}{b_0(z) + b_1(z)f(z) + ... + b_q(z)f^q(z)}, \tag{1.13}$$

where the coefficients $a_i\left(z\right)$ *and* $b_j\left(z\right)$ *satisfy*

$$T\left(r, a_i\right) = S(r, f) \text{ and } T(r, b_j) = S(r, f),$$

Then equation (1.13) *reduces to the Riccati equation*

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f^2(z). \tag{1.14}$$

Proof. Using the Valiron-Mohon'ko theorem and second part of Corollary 1.2 with $k = 1$, we have

$$dT(r,f) + S(r,f) = T\left(r, R\left(z, f(z)\right)\right) = T\left(r, f'\right) \leq 2T\left(r, f\right) + S(r,f).$$

and so $d \leq 2$. Hence (1.13) takes the form

$$f'\left(z\right) = \frac{a_0(z) + a_1(z)f(z) + a_2(z)f^2(z)}{b_0(z) + b_1(z)f(z) + b_2(z)f^2(z)}, \tag{1.15}$$

where the coefficients are rational. Choose $\alpha \in \mathbb{C}$ such that

$$a_0(z) + a_1(z)\alpha + a_2(z)\alpha^2 \not\equiv 0,$$

and

$$b_0(z) + b_1(z)\alpha + b_2(z)\alpha^2 \not\equiv 0.$$

Then, by substituting

$$\omega\left(z\right) = \frac{1}{f\left(z\right) - \alpha}$$

into (1.15), it follows that

$$\begin{aligned}\left(\frac{1}{\omega} + \alpha\right)' = -\frac{\omega'}{\omega^2} &= \frac{a_0(z) + a_1(z)\left(1/\omega + \alpha\right) + a_2(z)\left(1/\omega + \alpha\right)^2}{b_0(z) + b_1(z)\left(1/\omega + \alpha\right) + b_2(z)\left(1/\omega + \alpha\right)^2} \\ &= \frac{a_0(z)\omega^2 + a_1(z)\left(1 + \alpha\omega\right) + a_2(z)\left(1 + \alpha\omega\right)^2}{b_0(z)\omega^2 + b_1(z)\left(1 + \alpha\omega\right) + b_2(z)\left(1 + \alpha\omega\right)^2} \\ &= \frac{\left(a_0(z) + a_1(z)\alpha + a_2(z)\alpha^2\right)\omega^2 + \left(a_1(z) + 2a_2(z)\alpha\right)\omega + a_2(z)}{\left(b_0(z) + b_1(z)\alpha + b_2(z)\alpha^2\right)\omega^2 + \left(b_1(z) + 2b_2(z)\alpha\right)\omega + b_2(z)},\end{aligned}$$

where the right hand side is irreducible, since $R(z,f)$ is irreducible.

Therefore,

$$\omega' = Q(z,\omega), \tag{1.16}$$

where

$$Q(z,\omega) = -\frac{\omega^2\left[\left(a_0(z)+a_1(z)\alpha+a_2(z)\alpha^2\right)\omega^2+\left(a_1(z)+2a_2(z)\alpha\right)\omega+a_2(z)\right]}{\left(b_0(z)+b_1(z)\alpha+b_2(z)\alpha^2\right)\omega^2+\left(b_1(z)+2b_2(z)\alpha\right)\omega+b_2(z)}.$$

Now, by (1.16), it follows that

$$\begin{aligned} 2T(r,\omega)+S(r,\omega) \geq T(r,\omega') &= T(r,Q(z,\omega)) \\ &= \deg_\omega(Q(z,\omega))\,T(r,\omega)+O(\log r), \end{aligned}$$

and so $\deg_\omega(Q(z,\omega)) \leq 2$. But this is possible only if $(b_1(z)+2b_2(z)\alpha) \equiv 0 \equiv b_2(z)$, and so $b_1(z) \equiv b_2(z) \equiv 0$ yielding the assertion. ■

Chapter 2

Entire solutions of certain classes of non-linear differential equations

2.1 Introduction

Nevanlinna's value distribution theory has found many applications in the study of entire and meromorphic solutions of non-linear differential equations. In this chapter we take a look at a few examples of such applications, and introduce some rather general tools for studying the value distribution of meromorphic solutions of differential equations. In fact, we study the the non-linear differential equation

$$f(z)^{n_0}\left(f'(z)\right)^{n_1}\left(f''(z)\right)^{n_2}...\left(f^{(k)}(z)\right)^{n_k}=H(z), \tag{2.1}$$

where $H(z)$ is a non-vanishing entire function, $n_0, n_1, ..., n_k$ are non-negative integers such that $n_0 n_k \geq 1$.

2.2 Growth of entire solutions

It is natural to ask if there exists a relation betweenn the growth of solutions of (2.1) and that of $H(z)$. Our next result shows that they have the same growth rate.

Theorem 2.1 *[1] If f is an entire solution of a monomial differential equation* (2.1), *then we have*

$$\frac{1}{q}T(r,H)+S(r,f)\leq T(r,f)\leq\frac{1}{n_0}T(r,H)+S(r,f), \tag{2.2}$$

where $q=n_0+n_1+...+n_k$.

Proof. Let f be an entire solution of (2.1). It follows from the first statement of Corollary 1.2, that for any entire function g, we have

$$T(r,g')\leq T(r,g)+S(r,g)$$

From this and (2.1), we have

$$\begin{aligned}
T(r,H)&=T\left(r,f^{n_0}(f')^{n_1}(f'')^{n_2}...\left(f^{(k)}\right)^{n_k}\right)\\
&\leq T(r,f^{n_0})+T\left(r,(f')^{n_1}\right)+...+T\left(r,\left(f^{(k)}\right)^{n_k}\right)\\
&=n_0T(r,f)+n_1T(r,f')+...+n_kT\left(r,f^{(k)}\right)\\
&\leq(n_0+n_1+...+n_k)T(r,f)+S(r,f)\\
&=qT(r,f)+S(r,f),
\end{aligned}$$

where $q:=n_0+n_1+...+n_k$. This proves the left hand side of (2.2)

$$\frac{1}{q}T(r,H)+S(r,f)\leq T(r,f). \tag{2.3}$$

It remains now to prove the second estimate. First, we may write equation (2.1) as follows

$$f(z)^q = \frac{(f(z))^{q-n_0}}{(f'(z))^{n_1}(f''(z))^{n_2}\dots(f^{(k)}(z))^{n_k}}H(z),$$

wich implies that

$$\begin{aligned} qT(r,f) = T(r,f^q) &= T\left(r, \frac{f^{q-n_0}}{(f')^{n_1}(f'')^{n_2}\dots(f^{(k)})^{n_k}}H\right) \\ &\leq T(r,H) + T\left(r, \frac{f^{q-n_0}}{(f')^{n_1}(f'')^{n_2}\dots(f^{(k)})^{n_k}}\right) + O(1). \end{aligned}$$

Then form Theorem 1.4

$$qT(r,f) \leq T(r,H) + T(r,\omega) + O(1), \tag{2.4}$$

where

$$\omega := \frac{(f'(z))^{n_1}(f''(z))^{n_2}\dots\left(f^{(k)}(z)\right)^{n_k}}{(f(z))^{q-n_0}}.$$

Since $q - n_0 = n_1 + \dots + n_k$, we may rewrite ω as

$$\omega = \left(\frac{f'(z)}{f(z)}\right)^{n_1}\left(\frac{f''(z)}{f(z)}\right)^{n_2}\dots\left(\frac{f^{(k)}(z)}{f(z)}\right)^{n_k}. \tag{2.5}$$

Therefore, by using the lemma on the logarithmic derivatives, we obtain

$$m(r,\omega) = m\left(r, \left(\frac{f'}{f}\right)^{n_1}\left(\frac{f''}{f}\right)^{n_2}\dots\left(\frac{f^{(k)}}{f}\right)^{n_k}\right) \leq \sum_{i=1}^{k} n_i m\left(r, \frac{f^{(i)}}{f}\right) = S(r,f). \tag{2.6}$$

Hence, By (2.4) and (2.6), we conclude

$$qT(r,f) \leq T(r,H) + N(r,\omega) + S(r,f). \tag{2.7}$$

We now estimate $N(r,\omega)$ by means of $N(r,\frac{1}{H})$. Let z_0 be a pole of ω. Then form (2.5), z_0 must be a zero of f, and form (2.1), z_0 must be a zero of $H(z)$. Let α denote the multiplicity of the pole z_0 of ω, let s denote the multiplicity of the zero z_0 of f, and let β denote the multiplicity of the zero z_0 of $H(z)$. We distibguish two cases:

- If $s > k$, then by (2.5), we obtain

$$\begin{aligned}\alpha &= \sum_{j=1}^{k} j n_j = \sum_{j=1}^{k} (j+k-k) n_j = \sum_{j=1}^{k} (j-k) n_j + \sum_{j=1}^{k} k n_j \\ &= \sum_{j=1}^{k} (j-k) n_j + k(q-n_0).\end{aligned}$$

Hence,

$$\alpha = kq - \sum_{j=0}^{k} (k-j) n_j,$$

and using (2.1), we have

$$\beta = \sum_{j=0}^{k} (s-j) n_j \geq \sum_{j=0}^{k-1} (s-j) n_j \geq \sum_{j=0}^{k-1} (k-j) n_j,$$

as $s > k$. Therefore,

$$\frac{1}{\beta} \leq \frac{1}{\sum_{j=0}^{k-1} (k-j) n_j},$$

as $k > 0$. Multiplying the above inequality by α, yields that

$$\frac{\alpha}{\beta} \leq \frac{kq}{\sum_{j=0}^{k-1} (k-j) n_j} - 1.$$

Since $\sum_{j=0}^{k-1}(k-j)\,n_j \geq kn_0$, we rewrite

$$\frac{\alpha}{\beta} \leq \frac{q}{n_0} - 1. \tag{2.8}$$

- If $1 \leq s \leq k$, Then by (2.5), we have

$$\begin{aligned}\alpha &\leq \sum_{j=1}^{s-1} jn_j + s\sum_{j=s}^{k} n_j \\ &= \sum_{j=1}^{s-1} jn_j + s\left(q - \sum_{j=0}^{s-1} n_j\right) \\ &= \sum_{j=0}^{s-1} jn_j + sq - \sum_{j=0}^{s-1} sn_j.\end{aligned}$$

Hence,

$$\alpha \leq sq - \sum_{j=0}^{s-1}(s-j)\,n_j,$$

and by (2.1), we get

$$\beta \geq \sum_{j=0}^{s-1}(s-j)\,n_j.$$

Therefore,

$$\frac{\alpha}{\beta} \leq \frac{sq}{\sum_{j=0}^{s-1}(s-j)\,n_j} - 1.$$

Since $\sum_{j=0}^{s-1}(s-j)\,n_j \geq sn_0$, we rewrite

$$\frac{\alpha}{\beta} \leq \frac{q}{n_0} - 1. \tag{2.9}$$

Since (2.8) and (2.9) cover all the possible values of s, we obtain that at every pole z_0 of ω,

$$\frac{\alpha}{\beta} \leq \frac{q}{n_0} - 1.$$

Then, it follows that

$$N(r, \omega) \leq \left(\frac{q}{n_0} - 1\right) + N\left(r, \frac{1}{H}\right). \tag{2.10}$$

Now let us rewrite the following using (2.10) into (2.7), we get

$$qT(r, f) \leq \left(\frac{q}{n_0} - 1\right) + N\left(r, \frac{1}{H}\right) + T\left(r, H\right) \ + S(r, f).$$

By Theorem 1.4,we obtain

$$N\left(r, \frac{1}{H}\right) \leq T(r, \frac{1}{H}) = T\left(r, H\right) + O\left(1\right),$$

and

$$T(r, f) \leq \frac{1}{n_0} T\left(r, H\right) \ + S(r, f). \tag{2.11}$$

Using (2.3)and (2.11) we get the proof of the theorem. ■

The following results are immediate consequences of Theorem 2.1

Corollary 2.1 *Let f be an entire function, and let*

$$M[z, f] := f\left(z\right)^{n_0} \left(f'\left(z\right)\right)^{n_1} \left(f''\left(z\right)\right)^{n_2} ... \left(f^{(k)}\left(z\right)\right)^{n_k}.$$

Then,

$$T(r, M) \leq qT\left(r, f\right) \ + S(r, f),$$

where $q = n_0 + n_1 + ... + n_k$ *is the degree of* $M[z, f]$*. In addition, if* $n_0 \geq 1$*, we have*

$$T(r, f) \leq \frac{1}{n_0} T(r, M) + S(r, f).$$

Corollary 2.2 *If* f *is an entire solution of a monomial differential equation* (2.1)*, then*

$$\sigma(H) = \sigma(f).$$

We close this section by the following example which illustrates Theorem 2.1.

Example 2.1 *Consider the differential equation*

$$H(z) = \prod_{k=1}^{n} \left(e^z + (-1)^k e^{-z} \right).$$

We see that this equation satisfies all hypotheses of Theorem 2.1. By calculation we show that $f(z) = e^z + e^{-z}$ *is a solution of this equation and* $\sigma(f) = 1$*. Then by Corollary 2.2, we have*

$$\sigma(H) = \sigma(f) = 1.$$

There are examples of (2.1) that possess entire solutions which satisfy the equality

$$T(r, H) \leq qT(r, f) + S(r, f). \tag{2.12}$$

However, the next example shows that

2.3 Some remarks

Now, we give some examples to show that (2.12) does not always hold for equations of the form (2.1). To sart with, we recall that Toppila [7] created a transcendental entire function g that satisfies the following property:

Proposition 2.1 *For an absolute constant c satisfying* $0 < c < 1$*, we have*

$$T(r, g^{'}) \leq cT\left(r, g\right), \tag{2.13}$$

Theorem 2.2 *for all r.*

Consider the differential equation (2.1) with $H(z) = gg^{(k)}$. Obviously, $f = g$ is an entire solution of this equation. Since g is entire, we have

$$T(r, g^{^{(k)}}) \leq T\left(r, g^{'}\right) + S\left(r, g^{'}\right)$$

Then from (2.13), we obtain

$$T(r, H) \leq T(r, g) + T\left(r, g^{'}\right) + S\left(r, g^{'}\right) \leq (1 + c)\, T(r, g) + S\left(r, g\right),$$

which shows that (2.12)does not hold for this example.

We next use Toppila's example to show that the factor $\frac{1}{n_0}$ in the right inequality in (2.2) can not be replaced by $\frac{1}{n_j}$ for any j satisfying $1 \leq j \leq k$. Consider the equation

$$f^{n_0} \left(f'\right)^{n_1} \left(f''\right)^{n_2} ... \left(f^{(k)}\right)^{n_k} = g^{n_0} \left(g'\right)^{n_1} \left(g''\right)^{n_2} ... \left(g^{(k)}\right)^{n_k}, \tag{2.14}$$

where g is Toppila's transcendental entire function which satisfies (2.13), $k \geq 1$, $n_0 \geq 1$ and $n_k \geq 1$. Obviously, $f = g$ is a solution of (2.14). Let $j \in \{1, 2, ..., k\}$ where $n_j > 0$. Set

$$H = g^{n_0} \left(g'\right)^{n_1} \left(g'\right)^{n_2} ... \left(g^{(k)}\right)^{n_k}$$

and $q = n_0 + n_1 + ... + n_k$. From elementary Nevanlinna estimations, we obtain

$$\begin{aligned} T(r,H) &= T\left(r, g^{n_0}(g')^{n_1}(g'')^{n_2}...\left(g^{(k)}\right)^{n_k}\right) \\ &= T\left(r, g^{n_0}(g')^{n_1}(g'')^{n_2}...\left(g^{(k)}\right)^{n_k}\frac{\left(g^{(j)}\right)^{n_j}}{\left(g^{(j)}\right)^{n_j}}\right) \\ &\le (q-n_j)\,T(r,g) + n_j T\left(r, g^{(j)}\right) + S(r,g) \\ &\le (q-n_j)\,T(r,g) + n_j T(r,g') + S(r,g') + S(r,g) \qquad \text{for } j \in \{1,2,...,k\}. \end{aligned}$$

Then by (2.13), it follows that

$$T(r,H) \le (q-n_j)\,T(r,g) + cn_j T(r,g) + S(r,g) \tag{2.15}$$

$$\le (q-n_j+cn_j)\,T(r,g) + S(r,g), \tag{2.16}$$

where c is an constant that satisfies $0 < c < 1$. We now choose the integers $n_0, n_1, ..., n_k$ so that

$$\frac{q}{n_j} < 2 - c. \tag{2.17}$$

It is easy to see that integers $n_0, n_1, ..., n_k$ can be chosen so that (2.17) holds, because $2-c$ is an absolute constant larger than 1 and $\frac{q}{n_j}$ can be made as close to 1 as desired by letting n_j be arbitrarily large and keeping the other integers fixed. If we now assume that

$$T(r,g) \le \frac{1}{n_j}T(r,H) + S(r,g), \tag{2.18}$$

then by substituting (2.16) into (2.18), we obtain

$$T(r,g) \le \left(\frac{q}{n_j} - 1 + c\right)T(r,g) + S(r,g),$$

or

$$\left(2-c-\frac{q}{n_j}\right)T\left(r,g\right)\leq S(r,g),$$

which is a contradiction, since

$$2-c-\frac{q}{n_j}>0$$

from (2.17). This contradiction shows that (2.18) does not hold, which means that we cannot replace $\frac{1}{n_0}$ with $\frac{1}{n_j}$ for any j satisfying $1\leq j\leq k$ in the right inequality in (2.2).

Conclusion

In this thesis we considered the differential monomial

$$H_i(z) := f\left(z\right)^{n_{0i}} \left(f'\left(z\right)\right)^{n_{1i}} \left(f''\left(z\right)\right)^{n_{2i}} ... \left(f^{(k)}\left(z\right)\right)^{n_{ki}},$$

where f is an entire function, and showed that f and H have the same growth rate. Indeed, we proved that

$$\frac{1}{q}T(r, H_i) + S(r, f) \leq T(r, f) \leq \frac{1}{n_0}T(r, H_i) + S(r, f),$$

where $q = n_{0i} + n_{1i} + ... + n_{ki}$. This result could be of great use in the study of non-linear differential equations, in particular, the growth of solutions.

It is natural now to consider the growth problem of a general differential polynomials, i.e., a combination of differential monomials. In other words, let

$$P\left(z, f\right) = \sum_{i=1}^{n} \alpha_i H_i(z),$$

where α_i are some constants. The question is: under whet conditions f and $P\left(z, f\right)$ have the same growth rate? We hope to find an answer to this question in our future work.

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