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Random Fractional Differential Equations and Applications

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Publications

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Dedication

To my parents

To my sisters Fatma and Nabila

To my brothers Mohammed and Ghoulam-Allah

To my teacher Dahdoubi Fatima

To my friend Bouras Bouziane

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Mr. Abdelkader SENOUCI, Professor at the University of Tiaret
Mr. Berrabah BENDOUKHA, Professor at the University of Mostaganem
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Abstract

In this project, we are interested to study random fractional differential equations. They are defined as fractional differential equations involving second order random/stochastic elements for that we use the ideas of the fractional calculus to a mean-square setting.
We introduce a new class of problems for random fractional differential equations. The existence and uniqueness of random solutions is first obtained by means of appropriate fixed point theorems. New concepts on the sequential continuous and fractional derivative dependence are introduced. Some examples are discussed to illustrate our main results.
Then, a class of random differential equations of Airy type is introduced. The existence and uniqueness criterion for stochastic process solutions for the introduced class is discussed. New concepts and notions on $\beta-$differential dependence are also introduced and some related results are proved. Several illustrative examples are given at the end.

Keywords: Fractional differential equations, random differential equations, stochastic processes, existence of solution, mean square Caputo derivative, mean square calculus.

Résumé

Dans ce projet, nous sommes intéressés à étudier les équations différentielles fractionnaires aléatoires. Elles sont définies ici comme des équations différentielles fractionnaires impliquant des éléments aléatoires et / ou stochastiques de second ordre pour lesquels nous utilisons des idées du calcul fractionnaire dans un cadre de "mean square".
Nous introduisons une classe de problèmes liés aux équations différentielles fractionnaires aléatoires. L'existence et l'unicité des solutions est obtenue au moyen d'un théorème de point fixe. Nouveaux concepts sur la dépendance séquentielle continue et la dépendance aux dérivées fractionnaires sont introduits. Quelques exemples sont discutés pour illustrer nos principaux résultats.
Ensuite, une classe d'équations différentielles aléatoires de type Airy est introduite. Le critère d'existence et d'unicité des solutions de processus stochastiques pour la classe introduite est discuté. De nouveaux concepts et des notions sur la dépendance $\beta-$differential sont également introduits et certains résultats sont prouvés. Des exemples illustratifs sont aussi discutés; ils nous permettent d'assurer que la classes de fonctions avec lesquelles nous travaillons n'est pas vide.

Mots-clés: Équations différentielles fractionnaires, équations différentielles aléatoires, processus stochastiques, existence de solution, dérivée de Caputo au sense "mean square", calcul au sens "mean square".

الملخص

في هذا المشروع، نحن مهتمون بدراسة المعادلات التفاضلية الكسرية العشوائية. يتم تعريفها هنا على أنها معادلات تفاضلية كسرية تتضمن عناصر عشوائية من الدرجة الثانية لذلك نستخدم فكرة حساب التفاضل والتكامل الكسري إلى إعداد متوسط مربع .

نقدم مسائل جديدة للمعادلات التفاضلية الكسرية العشوائية. يتم الحصول على وجود وتفرد الحلول العشوائية عن طريق نظريات النقطة الثابتة المناسبة. يتم إدخال مفاهيم جديدة حول الاعتماد المستمر المتسلسل ومشتق بيتا الجزئي. تتم مناقشة بعض الأمثلة لتوضيح نتائجنا الرئيسية.

بعد ذلك ، يتم تقديم فئة من المعادلات التفاضلية العشوائية من النوع إيري. تتم مناقشة معايير الوجود والتفرد لحلول العمليات العشوائية للفئة المقدمة. كما يتم إدخال مفاهيم ومفاهيم جديدة حول الاعتماد على بيتا التفاضلي ويتم إثبات بعض النتائج ذات الصلة ، ويتم تقديم العديد من الأمثلة التوضيحية.

الكلمات المفتاحية: المعادلات التفاضلية الكسرية، المعادلات التفاضلية العشوائية، العمليات العشوائية، وجود الحل، تفاضل متوسط مربع كابوتو، الحساب بمعنى متوسط المربع.

Contents

Abbreviations

FDEs : Fractional Differential Equations
RDEs : Random Differential Equations
RFDEs : Random Fractional Differential Equations
cdf : cumulative distribution function
pmf : probability mass function
pdf : Probability density function
R-L : Riemann-Liouville

Symbols

$\mathbb{N}$: Natural number set
$\mathbb{N}^*$: Natural number set without zero
$\mathbb{R}$: Real number set
$E[\cdot]$: Expectation
$\sigma[\cdot]^2$: Variance
$\mathbb{R}_+$: Real positive number set
exp : Exponential
$n!$: The factorial of n
D : Mean square Caputo derivative
I : Mean square R-L integral

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Introduction

The concept of fractional calculus is originated from a question that was asked by Marquis de L'Hôpital $(1661-1704)$ to Gottfreid Wilhelm Leibniz $(1646-1716)$, in the years 1695, which was looking for the significance of Leibniz's notation $d^n y/dx^n$ for the derivative of order $n \in \mathbb{N}^* := \{0, 1, \ldots\}$ when $n = 1/2$ (What if $n = 1/2$?). In his reply, date 30 September 1695, Leibniz wrote to L'Hôpital as follows: "... *This is an apparent paradox from which, one day, useful consequences will be drawn...*"

The fractional calculus has progressively become a useful technique in mathematical modeling. It has developed a strong interest due to its appearance in diverse applications such as economics, optimal control, physics, engineering . [9, 15, 42, 46].

In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function called fractional differential equations (FDEs). FDEs have attracted a lot of interest in the fields of mathematics and applications: engineering, environmental science, physics, and economics. A recent and notable research on FDEs is found in [11, 12, 23, 26, 32].

The nature is the database of the knowledge we can have about the parameters that characterize a dynamic system in engineering or natural sciences. A deterministic dynamical system is specific. Unfortunately, most of the data available for describing and evaluating dynamic system attributes is not precise, unclear, or ambiguous. To put it another

way, evaluating the parameters of a dynamical system has an uncertainty. When our knowledge of the parameters on a dynamic system is probabilistic, the use of random or stochastic differential equations is a frequent strategy in mathematical modeling of such system.
Just a small amount of graduate-level literature exists on random differential equations (RDEs), and there is only one book about them despites their importance in scientific and engineering applications; it is a book by the author Helga Bunke [5] from 1972 and another one by Dobies Bobrowski [4] from 1987 completely committed to the theory of RDEs. They were presented as excellent models in several disciplines of science and engineering since random coefficients and uncertainties were taken into account, [18, 33, 44], as a result developing a fractional calculus that includes the uncertainty of real life is crucial.
The study of random fractional differential equations (RFDEs) become a popular research topic in recent years,[6, 43]. RFDEs are defined here as FDEs involving second order random elements.
Our interest in this dissertation comes from the fact that many scientific phenomena have been represented using FDEs.
Numerous random phenomena in nature that directly interest us can be mathematically represented in terms of limiting sums, derivatives, integrals, and differential/ integral equations. Thus, the development of a calculus related to stochastic processes is the next stage in our discussion.
We shall develop a calculus relevant to this approach, namely, the calculus in mean square or mean square calculus. The mean square calculus is important for several practical reasons. First of all, its significance comes from the use of uncomplicated and well-developed techniques . Secondly, the method for building mean square calculus and applying it to physical issues is largely the same as the calculus of ordinary (deterministic) functions. Engineers and scientists with a strong foundation in the analysis of ordinary functions will thus find

it simpler to understand. Furthermore, the mean square calculus is interesting since it is characterized in terms of moments, which are our main concerns.

Presently, there is little work on mean square fractional calculus, see [14, 16]. This amount of work will serve as the basis for our investigation into the mean square fractional calculus.

Organization of the Thesis

In this thesis, we study two new types of RFDEs using the theory of mean square fractional calculus. We are interested to the question of existence and uniqueness of random solution in a suitable Banach space. Additionally, we present new concepts on continuous and fractional derivative dependence.

This thesis is structured as follows :

Chapter 1 : Primary knowledge

The first chapter is dedicated to the basic concepts on random processes and mean square fractional calculus that the reader needs in the upcoming sections. For a clear understanding, we introduce some notions on probability theory. We cite also part of essential analysis. We end this chapter with a presentation of fundamental definitions and theorems of functional analysis.

Chapter 2 : Sequential Random Fractional Differential Equations: New Existence and New Data Dependence Results

This chapter is devoted to study a new class of RDEs with sequential fractional derivative in the sense of mean square. We present our results [49] which consist in studying the existence and the uniqueness of second order stochastic solution of the considered problem in a Banach space using fixed point theory. In the same context, new results on the continuous dependence and fractional derivative dependence are introduced. Some examples are also constructed to illustrate our results.

Chapter 3 : A Sequential Random Airy Type Problem of Fractional Order: Existence, Uniqueness and $\beta-$ Differential Dependence
This chapter is the subject of the main results of our contribution [50] which relates to studying a new class of random Airy type problem with n sequential mean square fractional derivative. We introduce the existence and uniqueness criteria by applying the Banach mapping principle. Then, we treat the questions of the continuous/differential dependence of the solutions with respect to the random data of the considered problem. Thus, we give some demonstrative examples to illustrate the obtained results.
Conclusion and Perspectives : This last part serves as a reminder of the numerous contributions provided to this thesis as well as the perspectives taken into consideration.
For the convenience of readers interested in additional research on these and other closely connected issues, a substantial Bibliography is provided at the end of this thesis.

Chapter 1

Preliminaries

In this chapter, we present the basic knowledge of probability theory. Then, we introduce some notions of random processes. After that, we move on to presenting the mean square fractional calculus. At the end, we give some important definitions and theorems of functional analysis.

1.1 Notions of Probability

The probability theory is a mathematical subfield. It has been created in 17^{th} century. It was focused on analysis of random phenomena. For example, Communication systems [20]. Let's define the fundamental terms for probability that will required.

Definition 1.1.1. *[36] Before rolling a die you do not know the result. This is an example of a random experiment. In particular, a random experiment is a process by which we observe something uncertain.*
There are certain terms associated with random experiments that are given as follows:
***Outcome**: A result of a random experiment.*
***Sample space**: The set of all possible outcomes.*
***Event**: A subset of the sample space.*

Definition 1.1.2. *[19] An ordered triple $(\Omega, \mathcal{F}, P)$ is called a probability space if*

- Ω *is a sample space;*
- $\mathcal{F}$ *is a* $\sigma-$*algebra of measurable subsets (events) of* Ω *;*
- P *is a probability measure on* $\mathcal{F}$*, that is,* P *satisfies the following Kolmogorov axioms :*

– *for any* $A \in \mathcal{F}$ *there exists a number* $P(A) \geq 0$ *called the probability of the event* A;

– *the probability measure is normalized, i.e.,* $P(\Omega) = 1$;

– *if events* $A_1, A_2, \ldots$ *satisfy* $A_i \cap A_j = \varnothing$ *for all* $i \neq j$,

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

Random Variables

In this paragraph we define random variables and some related concepts such as probability mass function, expected value and variance.

Definition 1.1.3. *[19] A random variable* X *is a deterministic function that assigns a real number to each outcome in the sample space, i.e., a mapping from the sample space* Ω *to the set of real numbers*

$$X : \Omega \to \mathbb{R}.$$

The sample space Ω *is the domain of the random variable and the set of all values taken on by the random variable is the range of the random variable.*
There are fundamentally two distinct types of random variables, discrete random variables and continuous random variables. If the range of random variable assumes values from a countable set, it is then a

discrete random variable. If the range can take infinitely many reals values, it is then a continuous random variable.

Cumulative Distribution Function

Definition 1.1.4. *[19] The cumulative distribution function (cdf) of a random variable X expresses the complete probability model of a random experiment as the following mathematical function:*

$$F_X(x) = P(X \leq x).$$

Probability Mass Function

Definition 1.1.5. *[19] The probability mass function (pmf) of a discrete random variable X expresses the complete probability model of a random experiment as the following mathematical function :*

$$f(x) = P(X = x).$$

The pmf must satisfy the following important properties :

- $f(x) \geq 0.$
- $\sum_x f(x) = 1.$

Probability Density Function

Definition 1.1.6. *[19] The probability density function (pdf) of a continuous random variable X, expresses the complete probability model of a random experiment as the following mathematical function:*

- $f(x) \geq 0$
- $\int_{-\infty}^{+\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = \int_a^b f(x)dx$

Expected Value

Definition 1.1.7. *[19] The expected value of a random variable X, denoted by $E[X]$ is defined as follows:*

$$E[X] = \begin{cases} \int_{-\infty}^{+\infty} x f(x) dx \text{ if } X \text{ continuous} \\ \sum_x x f(x) \text{ if } X \text{ discrete} \end{cases}$$

Variance

Definition 1.1.8. *[19] The variance of a random variable X can be expressed as follows :*

$$\sigma[X]^2 = E[(X - E[X])^2] = E[X^2] - [E[X]]^2.$$

Moments

Definition 1.1.9. *[19] The expected values of all the powers of a random variable, known as the moment of the random variable, can completely specify the distribution of the random variable. For the random variable X, assuming n is a positive integer, the nth moment , denoted by $E[X^n]$, is defined as follows:*

$$E[X^n] = \begin{cases} \int_{-\infty}^{+\infty} x^n f(x) dx \text{ if } X \text{ continuous} \\ \sum_x x^n f(x) \text{ if } X \text{ discrete} \end{cases}$$

1.2 Random Processes

Suppose to every outcome (sample point) ω in the sample space Ω of a random experiment, according to some rule, a function of time t is

assigned. The set or collection of all such functions that result from a random experiment, denoted by $X(t;w)$, is a random process or a stochastic process. [19]
The function $X(t;\omega)$ versus t, for w fixed, is a deterministic function and is called a realization, sample path, or sample function of the random process. For a given $\omega = \omega_i$, a specific function of time t, i.e. $X(t,\omega_i)$, is thus produced, and denoted by $x(t)$. For a specific time $t = t_k$, $X(t_k;\omega_i)$ is a random variable, and is called a time sample. For a specific ω_i and a specific $t_k, X(t_k;\omega_i)$ is simply a nonrandom constant. It is common to suppress ω, and simply let $X(t)$ denote a random process.[19]
The notion of a random process is an extension of the random variable concept. The difference is that in random processes, the mapping is done onto functions of time rather than onto constants (real numbers). The basic connection between the concept of a random process and the concept of a random variable is that at any time instant (for each choice of observation instant) the random process defines a random variable. However, the distributions at various times could be potentially different.[19]
Generally speaking, a stochastic or random process (in this thesis both terms will be used in an equivalent sense) is a family of random variables defined on a common probability space, indexed by the elements of an ordered set J, which is called the parameter set. Most often, J is taken to be an interval of time and the random variable indexed by an element $t \in J$ is said to describe the state of the process at time t.[45] Random processes considered here are specified by the following definition:

Definition 1.2.1. *[45] A random process is a family of random variables* $\{X(t), t \in J\}$*, defined on a common probability space* $\{\Omega, \mathcal{F}, P\}$*, where the parameter set* J *is a subset of the real line* $\mathbb{R}$.

Random Processes as Random Functions

Consider a random process $\{X(t), t \in J\}$. This random process is resulted from a random experiment, e.g., Observing the stock prices of a company over a period of time. Remember that any random experiment is defined on a sample space Ω. After observing the values of $X(t)$, we obtain a function of time such as the one showed in Figure 2.1[36].

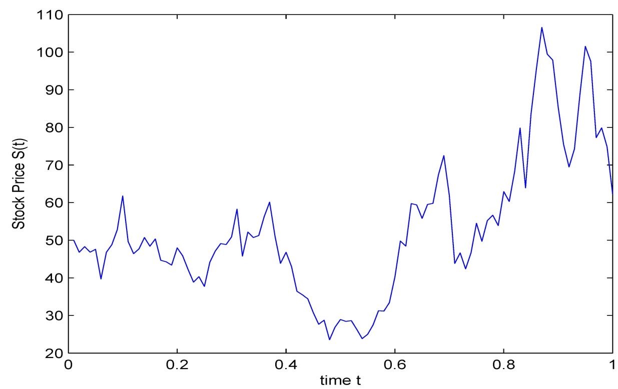

Figure 1.1: A possible realization of values of a stock observed as function of time. Here, S(t) is an example of a random process.

The function shown in this figure is just one of the many possible outcomes of this random experiment. We call each of these possible functions of $X(t)$ a sample function or sample path. It is also called a realization of $X(t)$.
From this point of view, a random process can be thought of as a random function of time. You are familiar with the concept of functions. The difference here is that $\{X(t), t \in J\}$ will be equal to one of many possible sample functions after we are done with our random experiment. In engineering applications, random processes are often referred to as random signals.[36]

Definition 1.2.2. *[36] A random process is a random function of time.*

Tools for Random Processes

As with random variables, we can mathematically describe a random process in terms of a cumulative distribution function, probability density function, or probability mass function.

Probability Function/Cumulative Function

Definition 1.2.3. *[36] Consider the random process $\{X(t), t \in J\}$. For any $t_0 \in J, X(t_0)$ is a random variable, so we can write its cdf*

$$F_{X(t_0)}(x) = P(X(t_0) \leq x).$$

If $t_1, t_2 \in J$, then we can find the joint cdf of $X(t_1)$ and $X(t_2)$ by

$$F_{X(t_1),X(t_2)}(x_1, x_2) = P(X(t_1) \leq x_1, X(t_2) \leq x_2).$$

More generally for $t_1, t_2, \ldots, t_n \in J$ we can write

$$F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n) = P(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n).$$

Similarly, we can write joint pdf or pmf depending on whether $X(t)$ is continuous-valued (the $X(t_i)$'s are continuous random variables) or discrete-valued (the $X(t_i)$'s are discrete random variables).

Mean and Correlation Functions

Definition 1.2.4. *(Mean Function of a Random Process).[19, 36] For a random process $\{X(t), t \in J\}$, the mean function $\mu_X(t) : J \to \mathbb{R}$, is defined as*

$$\mu_X(t) = E[X(t)] = \int_J x f_{X(t)}(x) dx.$$

In the next paragraph, we will introduce the essential of the theory of L_2 spaces. To do this, we refer to [41].

1.3 L_2-Spaces

Let us consider the properties of a class of real random variables $X_1, X_2, \ldots,$ whose second moments, $E[X_1^2], E[X_2^2], \ldots,$ are finite. They are called second order random variables. We have

$$E[XY]^2 \leq E[X^2]E[Y^2].$$

$$E[(X+Y)^2] < \infty.$$

$$E[(cX)^2] = c^2E[X^2], \forall c \in \mathbb{R}.$$

Hence, the class of all second order random variables on a probability space constitues a linear vector space if all equivalent random variables are identified. Two random variables X and Y are called equivalent if $P(X \neq Y) = 0$. Let us use the notation

$$< X, Y >= E[XY].$$

The relation $< X, Y >$ satisfies the properties of the inner product :

- $< X, Y >= E[XY] = E[YX] =< Y, X >$.
- $< cX, Y >= E[cXY] = cE[XY] = c < X, Y >$.
- $< X + Y, W >= E[(X + Y)W] = E[XW] + E[YW] =< X, W > + < Y, W >$.
- $< X, X >= E[XX] = E[X^2] \geq 0.$
- $< X, X >= E[X^2] = 0$ if and only if $X = 0$.

Define

$$\|X\|_2 = \sqrt{E[X^2]}.$$

It follows directly that $\|X\|_2$ possesses the norm properties :

- $\|X\|_2 \geq 0; \|X\|_2 = 0$ if and only if $X = 0$.

- $\|\alpha X\|_2 = |\alpha|\|X\|_2, \alpha \in \mathbb{R}$.
- Cauchy inequality: $\|X + Y\|_2 \leq \|X\|_2 + \|Y\|_2$.

Define the distance between X, Y by

$$d(X, Y) = \|X - Y\|_2.$$

The distance $d(X, Y)$ possesses the usual distance properties :

- $d(X, Y) \geq 0; d(X, Y) = 0$ if and only if $X = Y$with probability one.
- $d(X, Y) = d(Y, X)$
- $d(X, Y) \leq d(X, W) + d(W, Y)$

The linear vector space of second order random variables with the inner product, the norm , and the distance defined above is called a L_2 space.

Theorem 1.3.1. *L_2 spaces are complete in the sense that any Cauchy sequence in L_2 has a unique limit in L_2. That is , let the sequence $\{X_n\}$ be defined on the set of natural numbers. There is a unique element $X \in L_2$ such that*

$$\|X_n - X\|_2 \to 0 \text{ as } n \to \infty$$

if and only if

$$d(X_n, X_m) = \|X_n - X_m\|_2 \to 0 \text{ as } n, m \to \infty \text{ in any manner whatever.}$$

Proof. For proof, see Loève[1,p.161] [30] □

We thus see that L_2 spaces are complete normed linear spaces (Banach spaces) and complete inner product spaces (Hilbert spaces).

Second Order Stochastic Processes

Definition 1.3.2. *[41] Consider a stochastic process with index set* $J \subset \mathbb{R}$ *for which* $X(t_1), \ldots, X(t_m)$ *are elements of* L_2 *space for every set* $t_1, \ldots, t_m$. *Such a stochastic process is called a second order stochastic process and is characterized by*

$$\|X(t)\|_2^2 = E[X(t)X(t)], t \in J.$$

Mean Square Convergence

Definition 1.3.3. *[41] A sequence of random variables* $\{X_n\}$ *converges in mean square to a random variable* X *as* $n \to \infty$ *if*

$$\lim_{n\to\infty} \|X_n - X\|_2 = 0.$$

Other commonly used names are convergence in quadratic mean and second order convergence.

Theorem 1.3.4. *[41] Let* $\{X_n\}$ *be a sequence of second order random variables. If*

$$X_n \underset{\textit{mean square}}{\longrightarrow} X$$

then

$$\lim_{n\to\infty} E[X_n] \to E[X].$$

*In words, "*lim*" and "expectation" commute.*

Remark 1. *[45] For this kind of convergence, it is not necessary that the values* $X_n(\omega)$ *converge to* $X(\omega)$. *We shall call this kind of convergence, "convergence in quadratic mean " or "convergence in the mean square" .*

Proposition 1.3.1. *[45] If* X_n *such that* $E[X_n^2] < \infty$ *converges in the mean square, it may have at most one limit.*

Proof. [45] Suppose that

$$X_n \to U \text{ and } X_n \to V,$$

in mean square, where $U, V \in L_2$; then for all $n = 1, 2, \ldots,$

$$\|U - V\|_2 \leq \|X_n - U\|_2 + \|X_n - V\|_2 \to 0 \text{ as } n \to \infty.$$

Hence, $\|U - V\|_2 = 0$, which implies that $U = V$. □

Quadratic Mean Continuity

[45] Let $\{X(t), t \in J\}$ be an L_2 stochastic process with $J \subset \mathbb{R}$ an interval.

Definition 1.3.5. *A second order process $\{X(t), t \in J\}$ is said to be L_2 continuous (or continuous in quadratic mean) at a point $t_0 \in J$ if and only if*

$$X(t) \underset{\text{mean square}}{\to} X(t_0) \text{ as } t \to t_0,$$

that is

$$\|X(t) - X(t_0)\|_2^2 = E[(X(t) - X(t_0))^2] \to 0 \text{ as } t \to t_0.$$

1.4 Mean Square Calculus

In this section, we will present the definitions of the Riemann-Liouville (R-L) fractional integral and the Caputo fractional derivatives in the sens of mean square. Also, some of their properties are introduced.
We start this section by the introducing the definition of the well-known Gamma function of Euler..

Definition 1.4.1. *[23, 35] We call Gamma function the following integration*

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} \exp(-t) dt$$

where $s \in \mathbb{R}_+$.

Some of the properties of this function [23, 35]:

- $\Gamma(s+1) = s\Gamma(s), (s > 0)$.
- $\Gamma(n+1) = n!$ and $\Gamma(1) = 1, (n \in \mathbb{N})$.

The importance of this function is that it is effective for modeling phenomena containing continuous change.
There are different definitions and notations of fractional derivative and fractional integral in the deterministic theory of fractional calculus. And the choice of any of these definitions will depend on many factors, such as the function concerned. In our situation, the function in question is a random function of order two. Therefore, we need to define another type of fractional integral and derivative that fits this type of functions which is called mean square R-L fractional integral and mean square Caputo fractional derivative.

Mean square fractional integral and derivative

We present in the following definition the mean square R-L fractional integral.

Definition 1.4.2. *[37] Let $X(t), t \in J$, be a second order stochastic process and let $\beta > 0$. The mean square Riemann-Liouville (R-L) fractional integral to order β of $X(t)$ is given by*

$$I_a^\beta X(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)ds, t \in [a,b] \subset J. \tag{1.1}$$

Below, we give the definition of the mean square fractional derivative in the sense of Caputo.

Definition 1.4.3. *[37] Let $X(t), t \in J$, be a second order stochastic process and let $\beta > 0$ be such that $\beta \in]m-1, m], m \in \mathbb{N}^*$. The mean* *square Caputo fractional derivative of $X(t)$ at $t \in [a,b] \subset J$, is given by*

$$D_a^{\beta} X(t) = \begin{cases} I_a^{m-\beta} X^{(m)}(t) \quad , \quad \beta \in]m-1, m[\\ \\ \frac{d^m}{dt^m} X(t) \quad , \quad \beta = m. \end{cases}$$

Properties

Theorem 1.4.4. *[37] Let $X(t)$ and $Y(t)$ be second order stochastic processes for which $I_a^{\beta} X(t)$ and $I_a^{\beta} Y(t), \beta > 0$, exists for $t \in [a,b] \subset T$. Then*

- *(a)(Linearity)*

$$I_a^{\beta}[X(t) + Y(t)] = I_a^{\beta} X(t) + I_a^{\beta} Y(t).$$

- *(b)(Homogeneity)*

$$I_a^{\beta}[cX(t)] = cI_a^{\beta}[X(t)]$$

where c is a constant.

Proof. Letting

- $a = s_0 < \cdots < s_n = t \leq b$,
- $s_k^* \in [s_{k-1}, s_k]$ for $k = 1, \ldots, n$ and
- $\Delta_n = \max_k (s_k - s_{k-1})$

we have

- (a)

$$\begin{aligned}
I_a^\beta[X(t)+Y(t)] &= \int_a^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}[X(s)+Y(s)]ds \\
&= \lim_{\substack{n \longrightarrow \infty \\ \Delta n \to 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}[X(s_k^*)+Y((s_k^*))](s_k-s_{k-1}) \\
&= \lim_{\substack{n \longrightarrow \infty \\ \Delta n \to 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}X(s_k^*)(s_k-s_{k-1}) \\
&\quad + \lim_{\substack{n \longrightarrow \infty \\ \Delta n \to 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}Y((s_k^*))(s_k-s_{k-1}) \\
&= \int_a^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}X(s)ds + \int_a^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}Y(s)ds \\
&= I_a^\beta X(t) + I_a^\beta Y(t).
\end{aligned}$$

- (b) Similarly

$$\begin{aligned}
I_a^\beta[cX(t)] &= \lim_{\substack{n \longrightarrow \infty \\ \Delta n \to 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}cX(s_k^*)(s_k-s_{k-1}) \\
&= c \lim_{\substack{n \longrightarrow \infty \\ \Delta n \longrightarrow 0}} \sum_{k=1}^{n} \frac{(t-s_k^*)^{\beta-1}}{\Gamma(\beta)}X(s_k^*)(s_k-s_{k-1}) \\
&= cI_a^\beta X(t).
\end{aligned}$$

□

Theorem 1.4.5. *[37] Let $X(t)$ be a second order stochastic process such that $\dot{X}(t)$ exists and is mean square continuous on $[a,b] \subset J$. Then*

$$\lim_{\beta \to 0} I_a^\beta X(t) = X(t)$$

Theorem 1.4.6. *[37] Let $\beta \in (m-1, m)$ where $m \in \mathbb{N}^*$. Let $X(t)$ be a second order stochastic process such that $X^{(m)}(t)$ exists and is mean square continuous on $[a,b] \subset J$. Then*

$$\lim_{\beta \to m} D_a^\beta X(t) = X^{(m)}(t).$$

Proof.

$$\begin{aligned}\lim_{\beta\to m} D_a^\beta X(t) &= \lim_{\beta\to m} I_a^{m-\beta} X^{(m)}(t)\\ &= \lim_{\gamma\to 0} I_a^\gamma X^{(m)}(t)\\ &= X^{(m)}(t),\end{aligned}$$

where Theorem (1.4.5) has been used in the last step. □

1.5 Composition Rules

Theorem 1.5.1. *[37] Let $\beta > 0$ and $\alpha > 0$ and let $X(t)$ be a second order stochastic process such that $I_a^\alpha I_a^\beta X(t)$ exists for $t \in [a,b] \subset J$. Then*

$$I_a^\alpha I_a^\beta X(t) = I_a^{\alpha+\beta} X(t).$$

Theorem 1.5.2. *[37] Let $m \in \mathbb{N}^*$ and let $X(t)$ be a second order stochastic process. Then for $\alpha > 0$*

$$D_a^\alpha X^{(m)}(t) = D_a^{\alpha+m} X(t).$$

Theorem 1.5.3. *[37] Let $\alpha \in]n-1,n], n \in \mathbb{N}^*$, and let $X(t)$ be a second order stochastic process, then*

$$D_a^\alpha I_a^\alpha X(t) = X(t).$$

Theorem 1.5.4. *[37] Let $\beta \in]m-1,m], m \in \mathbb{N}^*$. Let $X(t)$ be a second order stochastic process. For $t \in [a,b] \subset J$ we have*

$$\mathbf{I}_a^\beta \mathbf{D}_a^\beta X(t) = X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a).$$

Proof.

$$I_a^\beta D_a^\beta X(t) \;=\; I_a^\beta I_a^{m-\beta} X^{(m)}(t) = I_a^m X^{(m)}(t)$$

we have using integration by parts: For $m = 1$

$$I_a^1 X^{(1)}(t) = \int_a^t X^{(1)}(s)ds = [X(s)]_a^t + 0 = X(t) - X(a);$$

For $m = 2$

$$\begin{aligned} I_a^2 X^{(2)}(t) &= \int_a^t \tfrac{(t-s)}{\Gamma(2)} X^{(2)}(s)ds \\ &= [\tfrac{(t-s)}{\Gamma(2)} X^{(1)}(s)]_a^t + \int_a^t X^{(1)}(s)ds \\ &= -\tfrac{(t-a)^1}{\Gamma(2)} X^{(1)}(a) - X(a) + X(t); \end{aligned}$$

Continuing in the same manner, we get

$$I_a^m X^{(m)}(t) = X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a).$$

Finally,

$$I_a^\beta D_a^\beta X(t) = X(t) - \sum_{j=0}^{m-1} \frac{(t-a)^j}{\Gamma(j+1)} X^{(j)}(a).$$

□

1.6 Fixed Points

In this last paragraph, we will introduce the Banach fixed point theorem [23, 39]. This theorem is essentially based on the following definitions:

Definition 1.6.1. *Let B a Banach space endowed with the norm $\|\cdot\|_B$ and H a map of B in B. We call a fixed point u of H any point $u \in B$ such that*

$$Hu = u.$$

Definition 1.6.2. *Let S be a normed vector space, of norm $\|\cdot\|_S$. A map H of S in S is said to be Lipschitz with constant $L \geq 0$ if it satisfies:*

$$\forall u, v \in S : \|f(u) - f(v)\|_S \leq L\|u - v\|_S.$$

Definition 1.6.3. *The Lipschitz map f is said to be a contraction if :*

$$L \in [0, 1[.$$

In the following theorem, we present the Banach contraction principle.

Theorem 1.6.4. *Let (E, d) be a Banach space and let $f : E \to E$ a contraction. Then f admits a unique fixed point.*

Chapter 2

Sequential Random Fractional Differential Equations: New Existence and New Data Dependence Results

In this chapter, we are concerned with sequential random fractional differential equations with two different orders and nonlocal conditions [49]. The existence and uniqueness of solutions for the problem is obtained using an appropriate fixed point theorem. Then, new concepts on the sequential continuous and fractional derivative dependence are introduced. At the end of this chapter, some results of stability on the random data as well on the deterministic case are discussed.

2.1 Position of the Problem

Several of the real problems we face in daily life involve some degree of uncertainty. For example, in communication networks, arrivals of data packets, occurrences of calls or connections of flows are mathematically modeled as random/stochastic processes. Accordingly, it has become better to model natural phenomena with differential equations so that the unknown function is a random process and the conditions are also random variables.

In this context, we find researcher's attention focused on the study of RFDEs. These studies can be found in the following references [7, 28, 29, 48].

In 2017, El- Sayed et al. [13] studied the following very interesting RFDEs with a nonlocal conditions:

$$\begin{cases} \mathbf{D}^{\alpha}X(t) &= c(t)f(X(t)) + b(t),\ t \in [0,T] \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k),\ a_k > 0, \end{cases}$$

where the authors used the concept of the mean square fractional derivative in the sense of Caputo of order $\alpha \in]0,1]$ and mean square fractional integral in the sense of R-L for mean continuous second order stochastic process. They have been interested to the existence and the uniqueness of the random solution. Also, by introducing the concepts of continuous and fractional derivative dependence, they proved the stability of some random/deterministic data dependence.

Then, based on the above paper, the authors of [40] have been concerned with the following random problem

$$\begin{cases} \mathbf{D}^{\alpha}X(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\alpha-1}X(t)\big) \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k) \\ X_1 &= X'(0), \end{cases}$$

where the authors investigated the Caputo mean square fractional derivative of order $\alpha \in]1,2]$. They investigated the stochastic solution uniqueness in an appropriate Banach space under various hypotheses. We note that the above-mentioned research did not address the topic of sequential derivatives and as we see in chapter 1, the mean square Caputo fractional derivatives possess neither semi-group nor commutative property i.e. $D^{\alpha+\beta}f \neq D^{\alpha}D^{\beta}f, D^{\alpha}D^{\beta}f \neq D^{\beta}D^{\alpha}f$, where $D^{\cdot}$

stands for mean square Caputo fractional differential operator. This renders the reduction of the order of RFDEs impossible. The advantage of considering the sequential derivative is that we can reduce the order of RFDEs.

Motivated by the above works, we are concerned with a new class of random differential equations with nonlocal conditions and two sequential fractional derivatives. So, we consider the problem:

$$\begin{cases} \mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\beta}X(t)\big), \\ X_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k),\ a_k > 0,\ \tau_k \in]0,T[\\ X_1 &= X^{(\beta)}(0), \end{cases} \tag{2.1}$$

where, $\mathbf{D}^{\alpha}$ and $\mathbf{D}^{\beta}$ represent the mean square Caputo fractional derivatives, with, α and β are in $]0,1]$, $X(\cdot)$ is a second order random function, X_0, X_1 are a second order random variable and a_k are positive real numbers, $f : \mathbb{L}_2(\Omega) \to \mathbb{R}$, $g : \mathbb{L}_2(\Omega) \to \mathbb{R}$, c and $b : J \to \mathbb{R}$, with, $J = [0,T]$.

2.2 Methodology

In this chapter and in the following one, we consider a complete probability space $\big(\Omega, \mathcal{F}, P\big)$. Over this space, we consider the stochastic process of order two $X(t)$, where $t \in J = [0,T]$.

Then, we consider the following space denoted by

$$\mathcal{C} = \mathcal{C}(J, \mathbb{L}_2(\Omega)) := \{X : J \to \mathbb{L}_2(\Omega) \text{ such that } E[X(t)^2] < \infty\}$$

which represents the set of all second order stochastic processes which are mean square continuous over J. This set is a Banach space equipped with the following norm

$$\|X\|_{\mathcal{C}} = \sup_{J} \|X(t)\|_2 \text{ where } \|X(t)\|_2 = \sqrt{E[X^2(t)]}.$$

2.3 A Sequential Random Integral Solution

In the following lemma, we present the random integral representation for (2.1).

Lemma 2.3.1. *The integral solution of* (2.1) *is given by:*

$$\begin{aligned} X(t) =& a^{-1}\Bigg[X_0 - X_1 \sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \\ & - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big] ds\Bigg] \\ & + X_1 \frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big] ds, \end{aligned} \tag{2.2}$$

where, $a = 1 + \sum_{k=1}^{n} a_k$.

Proof. We note

$$Y(t) := c(t)f\big(X(t)\big) + b(t)g\big(\mathbf{D}^{\beta}X(t)\big)$$

and we consider

$$\mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) = Y(t), \tag{2.3}$$

for which we apply the mean square R-L fractional integral of order α to (2.3) to obtain

$$\mathbf{D}^{\beta}X(t) = \gamma_0 + \mathbf{I}^{\alpha}Y(t), \tag{2.4}$$

where, $\gamma_0 \in \mathbb{R}$.

Again, we apply the mean square Riemann-Liouville fractional integral of order β to (2.4). We can write

$$X(t) = \gamma_1 + \gamma_0 \frac{t^{\beta}}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta}Y(t), \tag{2.5}$$

where, $\gamma_1 \in \mathbb{R}$.
We take $t = 0$ in (2.5), we get $X(0) = \gamma_1$, and we take $t = \tau_k$ in (2.5), we get,

$$X(\tau_k) = \gamma_1 + \gamma_0 \frac{\tau_k^\beta}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k},$$

so,

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \gamma_1 + \sum_{k=1}^{n} a_k \left[\gamma_1 + \gamma_0 \frac{\tau_k^\beta}{\Gamma(\beta+1)} + \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k} \right]. \tag{2.6}$$

The fractional derivative $\mathbf{D}^\beta$ of (2.5) is

$$X^{(\beta)}(t) = \gamma_0 + \mathbf{I}^\alpha Y(t),$$

and we take $t = 0$, we have

$$X^{(\beta)}(0) = \gamma_0 = X_1.$$

Substituting the value of γ_0 in (2.6), we get the value of γ_1

$$\gamma_1 = \frac{1}{1 + \sum_{k=1}^{n} a_k} \left[X_0 - X_1 \sum_{k=1}^{n} a_k \frac{\tau_k^\beta}{\Gamma(\beta+1)} - \sum_{k=1}^{n} a_k \, \mathbf{I}^{\alpha+\beta} Y(t)\big|_{t=\tau_k} \right].$$

The proof is thus achieved. □

2.4 A Unique Sequential Solution in $\|.\|_F$ Sense

For the purpose of studying the existence and the uniqueness of random solutions for (2.1), we define the following Banach space

$$F := \{X \in \mathcal{C}, \mathbf{D}^\beta X \in \mathcal{C}\},$$

equipped with the norm

$$\|X\|_F = \|X\|_{\mathcal{C}} + \|\mathbf{D}^\beta X\|_{\mathcal{C}}.$$

It is note that F is a Banach space; to prove this we use the fact that both $L_2(\Omega)$ and $\mathcal{C}$ are Banach spaces [27, 38].
Now, over F, we define the operator Φ as follows:

$$\begin{aligned} \Phi \ : F \ &\rightarrow \ F \\ X \ &\rightarrow \ \Phi X, \end{aligned}$$

$$\begin{aligned} \Phi X(t) := a^{-1}\Bigg[X_0 - X_1\sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \\ - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds\Bigg] \\ + X_1\frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)f(X(s)) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds, \end{aligned}$$

where, $a = 1 + m_{k=1}^{n} a_k$.
We will present the following hypotheses that are needed in the rest of this work :
Suppose that $f, g : \mathbb{L}_2(\Omega) \rightarrow \mathbb{R}$ and, $c, b : J \rightarrow \mathbb{R}$ are continuous functions. In addition, we assume that

(H1): $\exists K_1, K_2 > 0$, $\forall x, y \in \mathbb{L}_2(\Omega)$

$$||f(x) - f(y)||_2 \leq K_1||x - y||_2,$$

,

$$||g(x) - g(y)||_2 \leq K_2||x - y||_2$$

and, $\exists m_1, m_2 > 0$

$$f(0) \leq m_1 < \infty, g(0) \leq m_2 < \infty.$$

(H2): $\sup_{t\in J}|c(t)| = u < \infty$, and, $\sup_{t\in J}|b(t)| = v < \infty$.

Lemma 2.4.1. *Suppose that (H1) and (H2) hold. Then,*

$$X \in F \Rightarrow \Phi(X) \in \mathcal{C}.$$

Proof. We have to show that for all $t \in J$, the random variable $\Phi(X)(t)$ is an element of $L_2(\Omega)$ and that $\Phi(X)$ is a continuous function on J. Let us first notice that according to the hypotheses (H1) and (H2):

$$\begin{aligned} X \in F \ \Rightarrow \ & \|f(X(s))\|_2 \leq \|f(X(s)) - f(0)\|_2 + \|f(0)\|_2 \\ \leq \ & K_1 \|X(s)\|_2 + m_1 \leq K_1 \|X(s)\|_{\mathcal{C}} + m_1. \end{aligned} \tag{2.7}$$

In the same manner,

$$X \in F \ \Rightarrow \ \|g(\mathbb{D}^\beta X(s))\|_2 \leq K_2 \|\mathbb{D}^\beta X(s)\|_{\mathcal{C}} + m_2. \tag{2.8}$$

On the other hand, if we put

$$Y(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f(X(s)) + b(s) g\big(\mathbf{D}^\beta X(s)\big) \Big] ds$$

$$\Phi X(t) := a^{-1} X_0 + \left[\frac{t^\beta}{\Gamma(\beta+1)} - a^{-1} \sum_{k=1}^{n} a_k \frac{\tau_k^\beta}{\Gamma(\beta+1)} \right] X_1 - a^{-1} \sum_{k=1}^{n} a_k Y(\tau_k) + Y(t),$$

As, $L_2(\Omega)$ is a vector space and X_0, x_1 are two elements of $L_2(\Omega)$, it suffices to show that $\Phi(X)(t)$ is an element of $L_2(\Omega)$ to prove this property for $Y(t)$, for all $t \in J$. Using the properties of the norm, as well as estimates (2.7) and (2.8), we get:

$$\begin{aligned} \|Y(t)\|_2 \ \leq \ & (u(K_1 \|X(s)\|_{\mathcal{C}} + m_1) + v(K_2 \|\mathbb{D}^\beta X(s)\|_{\mathcal{C}} + m_2)) \int_0^t \tfrac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} ds \\ \leq \ & \tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} (u(K_1 \|X(s)\|_{\mathcal{C}} + m_1) + v(K_2 \|\mathbb{D}^\beta X(s)\|_{\mathcal{C}} + m_2)) < \infty. \end{aligned}$$

□

Let us now show the continuity of $\Phi(X)$. For all $t_0, t \in J, t_0 < t$, we have:

$$
\begin{aligned}
&\Phi X(t) - \Phi X(t_0) = \\
&X_1 \frac{t^\beta}{\Gamma(\beta+1)} + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds \\
&- X_1 \frac{t_0^\beta}{\Gamma(\beta+1)} - \int_0^{t_0} \frac{(t_0-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(t)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds, \\
=&\frac{t^\beta - t_0^\beta}{\Gamma(\beta+1)} X_1 + \int_0^{t_0} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds \\
&+ \int_{t_0}^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds \\
&- \int_0^{t_0} \frac{(t_0-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(t)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds, \\
&\Phi X(t) - \Phi X(t_0) \\
=&\frac{t^\beta - t_0^\beta}{\Gamma(\beta+1)} X_1 + \int_{t_0}^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds \\
&+ \int_0^{t_0} \Big[\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_0-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big] \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds.
\end{aligned}
$$

Using the above norm, we get

$$
\begin{aligned}
&||\Phi X(t) - \Phi X(t_0)||_2 \leq \\
&\frac{t^\beta - t_0^\beta}{\Gamma(\beta+1)} X_1 + \int_{t_0}^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[|c(s)| ||f\big(X(s)\big)||_2 + |b(s)| ||g\big(\mathbf{D}^\beta X(s)\big)||_2\Big] ds \\
&+ \int_0^{t_0} \Big[\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} - \frac{(t_0-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big] \Big[|c(s)| ||f\big(X(s)\big)||_2 + |b(s)| ||g\big(\mathbf{D}^\beta X(s)\big)||_2\Big] ds.
\end{aligned}
\tag{2.9}
$$

Using (2.7) and (2.8), we get

$$
\begin{aligned}
&||\Phi X(t)-\Phi X(t_0)||_{\mathcal{C}}\leq\\
&\frac{t^{\beta}-t_0^{\beta}}{\Gamma(\beta+1)}X_1+\frac{(t-t_0)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[u\big(K_1||X||_{\mathcal{C}}+m_1\big)+v\big(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2\big)\Big]\\
&+\left[\frac{-(t-t_0)^{\alpha+\beta}+t^{\alpha+\beta}-t_0^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\Big[u\big(K_1||X||_{\mathcal{C}}+m_1\big)+v\big(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2\big)\Big],\\
&||\Phi X(t)-\Phi X(t_0)||_{\mathcal{C}}\leq\\
&\frac{t^{\beta}-t_0^{\beta}}{\Gamma(\beta+1)}X_1+\left[\frac{t^{\alpha+\beta}-t_0^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right]\Big[u\big(K_1||X||_{\mathcal{C}}+m_1\big)+v\big(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2\big)\Big].
\end{aligned}
\tag{2.10}
$$

At the end we get:

$$\lim_{t\to t_0}\|\Phi(X)(t)-\Phi(X)(t_0)\|_{\mathcal{C}}=0.$$

That is, $\Phi(X)$ is right continuous from any point $t_0\in J$. On the same way, we show the continuity on the left.
This Completes the proof of Lemma 2.4.1.

Lemma 2.4.2. *Assume that (H1) and (H2) hold. Then,*

$$X\in F\Rightarrow \mathbf{D}^{\beta}\Phi(X)\in\mathcal{C}.$$

Proof. For all $t_0, t\in J, t_0<t$ we have

$$
\begin{aligned}
&\mathbf{D}^{\beta}\Phi X(t)-\mathbf{D}^{\beta}\Phi X(t_0)=\\
&X_1+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds\\
&-X_1-\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds.
\end{aligned}
\tag{2.11}
$$

Hence,

$$\begin{aligned}&||\mathbf{D}^{\beta}\Phi X(t)-\mathbf{D}^{\beta}\Phi X(t_0)||_2 \leq \\ &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)||_2+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)||_2\Big]ds \\ &-\int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|c(s)|||f\big(X(s)\big)||_2+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)||_2\Big]ds.\end{aligned} \tag{2.12}$$

Consequently,

$$\begin{aligned}&||\mathbf{D}^{\beta}\Phi X(t)-\mathbf{D}^{\beta}\Phi X(t_0)||_{\mathcal{C}} \leq \\ &\left[\frac{t^{\alpha}-t_0^{\alpha}}{\Gamma(\alpha+1)}\right]\Big[u\big(K_1||X||_{\mathcal{C}}+m_1\big)+v\big(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2\big)\Big].\end{aligned} \tag{2.13}$$

Finally,

$$\lim_{t\to t_0}||\mathbf{D}^{\beta}\Phi X(t)-\mathbf{D}^{\beta}\Phi X(t_0)||_{\mathcal{C}}=0.$$

So, $\mathbf{D}^{\beta}\Phi(X)$ is right continuous from any point $t_0 \in J$. On the same manner, we show the continuity on the left.
The proof of Lemma 2.4.2 is achieved. □

From Lemmas 2.4.1 and 2.4.2 immediately follows the following result.

Proposition 2.4.1. *The subspace F is stable for Φ.*

Let us pass to prove the existence and uniqueness of a stochastic solution for problem (2.1).

Theorem 2.4.3. *Assume that $(H1)$ and $(H2)$ hold. The random problem (2.1) has a unique solution provided that $A<1$, where,*

$$A := \left(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)\Big(uK_1+vK_2\Big).$$

Proof. Now, we consider $X, Y \in F$. Hence, we have

$$\begin{aligned}
&\Phi X(t) - \Phi Y(t) = \\
&- a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^{\beta} X(s)\big) \Big] ds \\
&+ \int_0^{t} \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^{\beta} X(s)\big) \Big] ds \\
&+ a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(Y(s)\big) + b(s) g\big(\mathbf{D}^{\beta} Y(s)\big) \Big] ds \\
&- \int_0^{t} \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(Y(s)\big) + b(s) g\big(\mathbf{D}^{\beta} Y(s)\big) \Big] ds.
\end{aligned} \tag{2.14}$$

This implies that

$$\begin{aligned}
&\Phi X(t) - \Phi Y(t) = -a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)\big(f\big(X(s)\big) - f\big(Y(s)\big)\big) \\
&+ b(s)\big(g\big(\mathbf{D}^{\beta} X(s)\big) - g\big(\mathbf{D}^{\beta} Y(s)\big)\big) \Big] ds \\
&+ \int_0^{t} \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)\big(f\big(X(s)\big) - f\big(Y(s)\big)\big) + b(s)\big(g\big(\mathbf{D}^{\beta} X(s)\big) - g\big(\mathbf{D}^{\beta} Y(s)\big)\big) \Big] ds.
\end{aligned} \tag{2.15}$$

Consequently,

$$
\begin{aligned}
&||\Phi X(t)-\Phi Y(t)||_2 \leq \\
&a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(Y(s)\big)||_2 \\
&+|b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big]ds \qquad (2.16)\\
&+\int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(Y(s)\big)||_2 \\
&+|b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big]]ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||\Phi X-\Phi Y||_{\mathcal{C}} \leq & a^{-1}\sum_{k=1}^{n} a_k \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||X-Y||_{\mathcal{C}}\Big] \\
&+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||\mathbf{D}^{\beta}(X-Y)||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-Y)||_{\mathcal{C}}\Big],
\end{aligned} \tag{2.17}
$$

thus,

$$
||\Phi X(t)-\Phi Y(t)||_{\mathcal{C}} \leq 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-Y||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-Y)||_{\mathcal{C}}\Big]. \tag{2.18}
$$

Moreover, we have

$$
\begin{aligned}
\mathbf{D}^{\beta}\Psi X(t)-\mathbf{D}^{\beta}\Psi Y(t) =& \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds \\
&-\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(Y(s)\big)+b(s)g\big(\mathbf{D}^{\beta}Y(s)\big)\Big]ds,
\end{aligned}
$$

$$
=\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)\big(f\big(X(s)\big)-f\big(Y(s)\big)\big)+b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}Y(s)\big)\big)\Big]ds. \tag{2.19}
$$

Hence, it yields that

$$||\mathbf{D}^{\beta}\Psi X(t) - \mathbf{D}^{\beta}\Psi Y(t)||_2 \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big[|c(s)|||f\big(X(s)\big) - f\big(Y(s)\big)||_2 + |b(s)|||(g\big(\mathbf{D}^{\beta}X(s)\big) - g\big(\mathbf{D}^{\beta}Y(s)\big)||_2\Big] ds, \tag{2.20}$$

so,

$$||\mathbf{D}^{\beta}\Psi X - \mathbf{D}^{\beta}\Psi Y||_{\mathcal{C}} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \Big[uK_1||X-Y||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-Y)||_{\mathcal{C}}\Big]. \tag{2.21}$$

By the inequalities (2.18) and (2.21), we get

$$||\Phi X - \Phi Y||_F \leq \Big[2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big] \Big[uK_1 + vK_2\Big] ||X-Y||_F. \tag{2.22}$$

At the end, we conclude that

$$||\Phi X - \Phi Y||_F \leq A||X-Y||_F. \tag{2.23}$$

Finally, Φ is contractive as $A < 1$.
This ends the proof. □

2.5 Dependence on Random Data

In this section, we establish new concepts for the above sequential RFDE with its nonlocal condition, in addition, we prove the results for the continuous and differentially dependence on random/deterministic data.

So let us consider the following sequential RFDE with the nonlocal conditions:

$$\begin{cases} \mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t) &= c(t)f(X(t)) + b(t)g\big(\mathbf{D}^{\beta}X(t)\big), \\ \tilde{X}_0 &= X(0) + \sum_{k=1}^{n} a_k X(\tau_k), \\ \tilde{X}_1 &= X^{(\beta)}(0), \end{cases} , \quad (2.24)$$

and, we study the dependence on the random data X_0 and X_1 of the solution of the random problem (2.1).

Definition 2.5.1. *The solution $X \in F$ of the random problem* (2.1) *is continuously and β-differentially dependent on the random data X_0 and X_1 if for all $\epsilon > 0$, $\exists \delta_0 > 0, \delta_1 > 0$ such that $||X_0 - \tilde{X}_0||_2 \leq \delta_0$, and, $||X_1 - \tilde{X}_1||_2 \leq \delta_1$, $\Rightarrow ||X - \tilde{X}||_F \leq \epsilon$.*

Theorem 2.5.2. *Assume that $(H1)$ and $(H2)$ hold. Then, the solution of the above problem is continuously and β-differentially dependent on X_0 and X_1.*

Proof. Let $X(t)$ as defined in (2.2) be the solution of the problem (2.1) and

$$\begin{aligned} \tilde{X}(t) =& a^{-1}\Bigg[\tilde{X}_0 - \tilde{X}_1 \sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \\ & - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds\Bigg] \\ & + \tilde{X}_1 \frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds, \end{aligned} \quad (2.25)$$

be the solution of the problem (2.24).

Then,

$$\begin{aligned}
&X(t)-\tilde{X}(t)=\\
&a^{-1}\big(X_0-\tilde{X}_0\big)-a^{-1}\sum_{k=1}^{n}a_k\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}\big(X_1-\tilde{X}_1\big)-a^{-1}\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\\
&\times\Big[c(s)\big(f\big(X(s)\big)-f\big(\tilde{X}(s)\big)\big)+b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\big)\Big]ds\\
&+\frac{t^{\beta}}{\Gamma(\beta+1)}\big(X_1-\tilde{X}_1\big)\\
&+\int_0^{t}\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[c(s)\big(f\big(X(s)\big)-f\big(\tilde{X}(s)\big)\big)+b(s)\big(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\big)\Big]ds,
\end{aligned} \tag{2.26}$$

we use $||\cdot||_2$ on J, we get

$$\begin{aligned}
&||X(t)-\tilde{X}(t)||_2\leq\\
&a^{-1}||X_0-\tilde{X}_0||_2+a^{-1}\sum_{k=1}^{n}a_k\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}||X_1-\tilde{X}_1||_2+a^{-1}\sum_{k=1}^{n}a_k\int_0^{\tau_k}\frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\\
&\times\Big[|c(s)|||f\big(X(s)\big)-f\big(\tilde{X}(s)\big)||_2+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)||_2\Big]ds\\
&+\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}||X_1-\tilde{X}_1||_2+\int_0^{t}\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|c(s)|||f\big(X(s)\big)-f\big(\tilde{X}(s)\big)||_2\\
&+|b(s)|||g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)||_2\Big]ds,
\end{aligned} \tag{2.27}$$

hence,

$$||X-\tilde{X}||_{\mathcal{C}} \leq a^{-1}\delta_0 + a^{-1}\sum_{k=1}^{n} a_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)}\delta_1 + \frac{t^{\beta}}{\Gamma(\beta+1)}\delta_1$$
$$+a^{-1}\sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big]ds$$
$$+\int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big]ds,$$

so

$$||X-\tilde{X}||_{\mathcal{C}} \leq a^{-1}\delta_0 + a^{-1}\sum_{k=1}^{n} a_k \frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1 + \frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1$$
$$+a^{-1}\sum_{k=1}^{n} a_k \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big]$$
$$+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big]. \tag{2.28}$$

So,

$$\begin{aligned}||X-\tilde{X}||_{\mathcal{C}} \leq & a^{-1}\delta_0 + \big(a^{-1}\sum_{k=1}^{n} a_k + 1\big)\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1 \\ & + \Big(a^{-1}\sum_{k=1}^{n} a_k\tau_k + 1\Big)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times \Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big],\end{aligned}$$

$$\begin{aligned}||X-\tilde{X}||_{\mathcal{C}} \leq & \delta_0 + 2\frac{T^{\beta}}{\Gamma(\beta+1)}\delta_1 + 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times \Big[uK_1||X-\tilde{X}||_{\mathcal{C}} + vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big].\end{aligned} \tag{2.29}$$

Furthermore,

$$\begin{aligned}\mathbf{D}^{\beta}X(t)-\mathbf{D}^{\beta}\tilde{X}(t)=&X_1+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(X(s)\big)+b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big]ds\\&-\tilde{X}_1-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)f\big(\tilde{X}(s)\big)+b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big]ds.\end{aligned}\tag{2.30}$$

Then, we have

$$||\mathbf{D}^{\beta}X-\mathbf{D}^{\beta}\tilde{X}||_{\mathcal{C}}\leq\delta_1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big].\tag{2.31}$$

Combining the inequalities (2.29) and (2.31), we obtain

$$\begin{aligned}||X-\tilde{X}||_F\leq&\delta_0+\Big(2\frac{T^{\beta}}{\Gamma(\beta+1)}+1\Big)\delta_1\\&+\Big(2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(uK_1+vK_2\Big)||X-\tilde{X}||_F\end{aligned}\tag{2.32}$$

witch implies that

$$||X-\tilde{X}||_F\leq\frac{\delta_0+\Big(2\frac{T^{\beta}}{\Gamma(\beta+1)}+1\Big)\delta_1}{1-A}=\epsilon.\tag{2.33}$$

This ends the proof. □

We pass to study the dependence on the deterministic data $a_k>0$ for the solution of (2.1).
We consider the sequential random FDE with the nonlocal conditions

$$\left\{\begin{array}{rcl}\mathbf{D}^{\alpha}\big(\mathbf{D}^{\beta}X\big)(t)&=&c(t)f(X(t))+b(t)g\big(\mathbf{D}^{\beta}X(t)\big),\\X_0&=&X(0)+\sum_{k=1}^{n}\tilde{a}_kX(\tau_k),\\X_1&=&X^{(\beta)}(0),\end{array}\right.,\tag{2.34}$$

and we introduce the following definition.

Definition 2.5.3. *The solution $X \in F$ of* (2.1) *is continuously and β-differentially dependent on the deterministic data a_k if for all $\epsilon > 0, \exists \delta > 0$ such that*
$|a_k - \tilde{a}_k| < \delta \Rightarrow ||X - \tilde{X}||_F \leq \epsilon.$

No, we present to the reader the following result.

Theorem 2.5.4. *Assume that $(H1)$ and $(H2)$ hold. Then, the solution of the sequential RFDE is continuously and β-differentially dependent on a_k.*

Proof. Before starting the proof, we introduce the following notations:

$$\begin{aligned}
\mathcal{K}_1 =& a^{-1} - \tilde{a}^{-1},\\
\mathcal{K}_2 =& \sum_{k=1}^{n} \left((\tilde{a}^{-1}\tilde{a}_k - a^{-1}a_k) \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} \right),\\
\mathcal{K}_3 =& \tilde{a}^{-1} \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(\tilde{X}(s)\big) + b(s) g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big) \Big] ds\\
& - a^{-1} \sum_{k=1}^{n} a_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^{\beta}X(s)\big) \Big] ds,\\
\mathcal{K}_4 =& \int_0^{t} \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^{\beta}X(s)\big) \Big] ds\\
& - \int_0^{t} \frac{(t - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(\tilde{X}(s)\big) + b(s) g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big) \Big] ds.
\end{aligned}$$

Let $X(t)$ (as defined in (2.2)) be the solution of (2.1) and

$$\tilde{X}(t) = \tilde{a}^{-1}\Bigg[X_0 - X_1 \sum_{k=1}^{n} \tilde{a}_k \frac{\tau_k^{\beta}}{\Gamma(\beta+1)} - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds\Bigg] + X_1 \frac{t^{\beta}}{\Gamma(\beta+1)} + \int_0^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds, \tag{2.35}$$

be the solution of (2.34).
Then,

$$X(t) - \tilde{X}(t) = \mathcal{K}_1 X_0 + \mathcal{K}_2 X_1 + \mathcal{K}_3 + \mathcal{K}_4. \tag{2.36}$$

Hence, we get

$$\begin{aligned} |\mathcal{K}_1| \leq & \sum_{k=1}^{n} |\tilde{a}_k - a_k| \\ \leq & n\delta, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \mathcal{K}_3 =& \tilde{a}^{-1}\Big(1 + \sum_{k=1}^{n} \tilde{a}_k\Big) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds \\ & - a^{-1}\Big(1 + \sum_{k=1}^{n} a_k\Big) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big] ds \\ & - \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(\tilde{X}(s)\big) + b(s)g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big)\Big] ds \\ & + a^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)f\big(X(s)\big) + b(s)g\big(\mathbf{D}^{\beta}X(s)\big)\Big] ds, \end{aligned} \tag{2.38}$$

so,

$$\begin{aligned}\mathcal{K}_3 =& -\int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)) \\ &+ b(s)(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big))\Big] ds \\ &+ (a^{-1} - \tilde{a}^{-1}) \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s) f\big(X(s)\big) + b(s) g\big(\mathbf{D}^\beta X(s)\big)\Big] ds \\ &+ \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[c(s)(f\big(X(s)\big) - f\big(\tilde{X}(s)\big)) \\ &+ b(s)(g\big(\mathbf{D}^\beta X(s)\big) - g\big(\mathbf{D}^\beta \tilde{X}(s)\big))\Big] ds,\end{aligned} \tag{2.39}$$

we know that

$$\sup_{t\in J} ||f\big(X(t)\big)||_2 \leq K_1 ||X||_{\mathcal{C}} + m_1,$$

and,

$$\sup_{t\in J} ||g\big(\mathbf{D}^\beta X(t)\big)||_2 \leq K_2 ||\mathbf{D}^\beta X||_{\mathcal{C}} + m_2.$$

Thanks to (H1) and (H2), we get

$$\begin{aligned}||\mathcal{K}_3||_2 \leq& \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[u K_1 ||X(s) - \tilde{X}(s)||_2 + v K_2 ||\mathbf{D}^\beta (X(s) - \tilde{X}(s))||_2\Big] ds \\ &+ n\delta \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[u(K_1 ||X(s)||_2 + m_1) + v(K_2 ||\mathbf{D}^\beta X(s)||_2 + m_2)\Big] ds \\ &+ \tilde{a}^{-1} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[u K_1 ||X(s) - \tilde{X}(s)||_2 + v K_2 ||\mathbf{D}^\beta (X(s) - \tilde{X}(s))||_2\Big] d\end{aligned} \tag{2.40}$$

hence,

$$\begin{aligned}||\mathcal{K}_3||_2 \leq & \frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2+vK_2||\mathbf{D}^{\beta}(X(s)-\tilde{X}(s))||_2\Big] \\ & + n\delta\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[u(K_1||X(s)||_2+m_1)+v(K_2||\mathbf{D}^{\beta}X(s)||_2+m_2)\Big] \\ & + \tilde{a}^{-1}\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2+vK_2||\mathbf{D}^{\beta}(X(s)-\tilde{X}(s))||_2\Big].\end{aligned} \tag{2.41}$$

Consequently,

$$||\mathcal{K}_4||_2 \leq \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2+vK_2||\mathbf{D}^{\beta}(X(s)-\tilde{X}(s))||_2\Big]. \tag{2.42}$$

Then,

$$\begin{aligned}||X(t)-\tilde{X}(t)||_2 \leq & |\mathcal{K}_1|||X_0||_2+||\mathcal{K}_2||_2||X_1||_2+||\mathcal{K}_3||_2+||\mathcal{K}_4||_2, \\ \leq & n\delta||X_0||_2+\sum_{k=1}^{n}\left(|\tilde{a}^{-1}\tilde{a}_k-a^{-1}a_k|\frac{\tau_k^{\beta}}{\Gamma(\beta+1)}\right)||X_1||_2 \\ & + \big(1+\tilde{a}^{-1}\big)\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times\Big[uK_1||X(s)-\tilde{X}(s)||_2+vK_2||\mathbf{D}^{\beta}(X(s)-\tilde{X}(s))||_2\Big] \\ & + n\delta\frac{\tau_k^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times\Big[u(K_1||X(s)||_2+m_1)+v(K_2||\mathbf{D}^{\beta}X(s)||_2+m_2)\Big] \\ & + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X(s)-\tilde{X}(s)||_2+vK_2||\mathbf{D}^{\beta}(X(s)-\tilde{X}(s))||_2\Big]\end{aligned} \tag{2.43}$$

We pass now to the sup over J, it yields that

$$\begin{aligned}||X(t)-\tilde{X}(t)||_{\mathcal{C}} \leq & n\delta||X_0||_2+\sum_{k=1}^{n}|\tilde{a}^{-1}\tilde{a}_k-a^{-1}a_k|\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2 \\ & +\big(1+\tilde{a}^{-1}\big)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times\Big[uK_1||X-\tilde{X}||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big] \\ & +n\delta\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ & \times\Big[u(K_1||X||_2+m_1)+v(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2)\Big] \\ & +\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big],\end{aligned} \tag{2.44}$$

$$\begin{aligned}||X-\tilde{X}||_{\mathcal{C}} \leq & n\delta\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2 \\ & +\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2)\big)\Big] \\ & +3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big].\end{aligned} \tag{2.45}$$

Also, we have

$$\begin{aligned}&\mathbf{D}^{\beta}X(t)-\mathbf{D}^{\beta}\tilde{X}(t)= \\ &\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[c(s)(f\big(X(s)\big)-f\big(\tilde{X}(s)\big))+b(s)(g\big(\mathbf{D}^{\beta}X(s)\big)-g\big(\mathbf{D}^{\beta}\tilde{X}(s)\big))\Big]ds\end{aligned} \tag{2.46}$$

so,

$$||\mathbf{D}^{\beta}X-\mathbf{D}^{\beta}\tilde{X}||_{\mathcal{C}}\leq\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big[uK_1||X-\tilde{X}||_{\mathcal{C}}+vK_2||\mathbf{D}^{\beta}(X-\tilde{X})||_{\mathcal{C}}\Big]. \tag{2.47}$$

By the inequalities (2.45) and (2.47), we observe that

$$\begin{aligned}||X-\tilde{X}||_F\leq & n\delta\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2\\ &+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m_2)\big)\Big]\\ &+\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\big(u(K_1||X||_{\mathcal{C}}+m_1)+v(K_2||\mathbf{D}^{\beta}X||_{\mathcal{C}}+m\\ &\leq\frac{n\delta}{1-L}\Big[||X_0||_2+\frac{T^{\beta}}{\Gamma(\beta+1)}||X_1||_2\Big]\\ &+\frac{n\delta}{1-L}\Big[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\big[uK_1+vK_2\big]||X-\tilde{X}||_F,\end{aligned} \tag{2.48}$$

where,

$$L=\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+3\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\big[uK_1+vK_2\big]. \tag{2.49}$$

□

2.6 Examples

In this paragraph, two examples will be presented to illustrate our theoretical results. In the first example, the functions f and g verify the lipschitz conditions and in the second example they do not satisfy the conditions.

Example 2.6.1. *We consider the following sequential RFDE*

$$\mathbf{D}^{1/4}\big(\mathbf{D}^{3/4}X\big)(t) = \frac{1}{17+t}f(X(t)) + \frac{1}{exp(t+1)}g\big(\mathbf{D}^{3/4}X(t)\big) \tag{2.50}$$

where $c(t)$ and $b(t)$ are defined on the interval $J=[0,2]$ by

$$c(t) = \frac{1}{17+t}, b(t) = \frac{1}{exp(t+1)},$$

and $f(X(t)), g\big(\mathbf{D}^{3/4}X(t)\big)$ are given by

$$f(X(t)) = \frac{cos(X(t)) + X(t)}{2}$$
$$and g\big(\mathbf{D}^{3/4}X(t)\big) = \sqrt{1/3}(sin(\mathbf{D}^{3/4}X(t)) + 1),$$

with the following conditions $E[X_0^2] = 1, E[X_1^2] = 2$.
The problem given in (2.50) satisfies the hypothesis (H1) and (H2) where $K_1 = \frac{1}{2}; K_2 = \sqrt{1/3}; u = \frac{1}{17}; v = \frac{1}{25};$ and we have

$$A = \frac{2\frac{2^1}{\Gamma(2)} + \frac{2^{1/4}}{\Gamma(5/4)}}{uK_1 + vK_2} = 0.2789,$$

then $A < 1$. Applying Theorem 3.4.2 we conclude that the problem (2.50) has a unique solution on J.
Under the same conditions ((H1), (H2)), the solution of the RFDE (2.50) is continuously and $3/4-$differentially dependent on X_0, X_1, which is illustrate by Theorem 3.5.2.
Also, by Theorem 3.5.4, we state that the unique solution of (2.50) is continuously and $3/4-$differentially dependent on a_k where the value of L given by (2.49) is equal to 0.3839.

Example 2.6.2. *The second example is given by*

$$\mathbf{D}^{1/4}\big(\mathbf{D}^{3/4}X\big)(t) = \frac{1}{17+t}f(X(t)) + \frac{1}{exp(t+1)}g\big(\mathbf{D}^{3/4}X(t)\big) \tag{2.51}$$

$$f(X(t)) = \frac{4}{3}cos(X(t)) + \sqrt{3}X(t),$$
$$g\big(\mathbf{D}^{3/4}X(t)\big) = \sqrt{7/2}cos(\mathbf{D}^{3/4}X(t)) + \mathbf{D}^{3/4}X(t).$$

The functions f and g do not verified the lipschitz conditions. So, we have

$$A = \frac{2\frac{2^1}{\gamma(2)} + \frac{2^{1/4}}{\gamma(5/4)}}{uK_1 + vK_2} = 1.1603.$$

Then applying Theorem 3.4.2 we conclude that the problem (2.50) has not a unique solution on J.

Chapter 3

A Sequential Random Airy Type Problem of Fractional Order: Existence, Uniqueness, and $\beta-$ Differential Dependence

In this chapter, a new class of sequential RFDEs of Airy type is introduced [50]. The existence and uniqueness criteria for stochastic process solutions for the introduced class are discussed. Some notions on $\beta-$differential dependence are also introduced. Then, new results on the $\beta-$dependence are discussed. In the end, some illustrative examples are proposed.

3.1 The studied problem

Many of random problems have been constructed by the Airy equation which is given by

$$Z'' - tZ = 0, \quad t \in \mathbb{R}.$$

In [10], the authors have been associated with the initial value problems for space-time-fractional Airy problem given by:

$$\frac{\partial^{\alpha} u(x_1,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x_1,t)}{\partial x_1^{\beta}}, 0 < \alpha \leq 1, 2 < \beta \leq 3, x_1 \in \mathbb{R},$$

with $u(x_1, 0) = \frac{1}{6}x_1^{\beta}$.
In [34], M.D. Ovidio and E. Orsingher have represented the law of the stable process $H^{v}(t), t > 0$ in terms of Airy functions.
In [31], the authors have been concerned with the M-Wright function in the time-fractional diffusion process and they have demonstrated that the auxiliary functions can be expressed in terms of the Airy functions.
An example from quantum mechanics is provided in the paper [24] where the exact solution of the Schrodinger equation, for the motion of a particle in a homogeneous external field can be described in terms of the Airy functions. Solutions of the Schrodinger equation involving the Airy functions are also given in [47].
For some other applications of Airy equations (and Airy functions) in elasticity theory, fluid mechanics, and quantum physics, the reader is invited to see the research works [1, 2, 22, 25].
We cite also the paper [3], where the authors have studied the following random fractional initial value problem of Airy type:

$$\begin{cases} ({}^{c}\mathbf{D}_{0_+}^{\alpha} Y)(t) - Bt^{\beta} Y(t) &= 0, t > 0, n-1 < \alpha \leq n, \beta > 0, \\ X^{(j)}(0) &= A_j, j = 0, 1, \ldots, n-1. \end{cases} \tag{3.1}$$

In this chapter, we examine a class of random fractional problems that generalize the traditional Airy-type differential equations in both the random and the fractional meanings. We shall specifically address the sequential random fractional generalized Airy-type problem shown below:

$$\begin{cases} \mathbf{D}^{\alpha_1}\cdots\mathbf{D}^{\alpha_n}Y(t) &= a_1A_1f_1(t,Y(t)) + a_2A_2f_2\big(t,\mathbf{D}^{\beta}Y(t)) + a_3A_3f_3\big(t,\mathbf{I}^{\rho}Y(t)\big), \\ X_0 &= Y(0), \\ X_i &= Y^{(\alpha_i+1)}(0), i=1,\dots,n-1, n\in\mathbb{N}^*, \\ t\in J=[0,T] &, \quad \alpha_i\in]0,1], 0<\beta<1, 0<\rho, \end{cases} \tag{3.2}$$

where: $\mathbf{D}^{\cdot}$ represents the mean square derivative in the sense of Caputo, $Y(\cdot)$ is a second order random function, $f_i : J\times\mathbb{L}_2(\Omega)\to\mathbb{L}_2(\Omega), i = 1,2,3,$ X_i are second order random variables $i = 0,\dots,n-1$, and A_1, A_2, A_3 are also second order random variables. a_1, a_2, a_k are real positive numbers.

It is essential to mention that if $n = 2, \alpha_1 = \alpha_2 = 1, a_2 = a_3 = 0$, then we get the common Airy equation.

Under some other considerations on the input data of (3.2), we can obtain the generalized Airy problem of [3].

These are two aspects that have motivated the study of the above problem.

3.2 Criteria for Existence and Uniqueness

We start our main result by proving the following random integral lemma.

Lemma 3.2.1. *The given problem (3.2) has the following random integral representation*

$$\begin{aligned} Y(t) &= \sum_{i=1}^{n-1}\frac{t^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}X_i + X_0 \\ &+ \int_0^t \frac{(t-s)^{\sum_{i=1}^{n}\alpha_i-1}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i\big)}\Big(a_1A_1f_1(s,Y(s)) + a_2A_2f_2\big(s,\mathbf{D}^{\beta}Y(s)) + a_3A_3f_3\big(s,\mathbf{I}^{\rho}Y(s)\big)\Big)ds. \end{aligned} \tag{3.3}$$

Proof. To prove the result, we first consider the following homogeneous linear differential problem:

$$\mathbf{D}^{\alpha_1}\cdots\mathbf{D}^{\alpha_n}Y(t) = W(t), \tag{3.4}$$

where, $W(t) := a_1A_1f_1(t, Y(t)) + a_2A_2f_2\big(t, \mathbf{D}^{\beta}Y(t)\big) + a_3A_3f_3\big(t, \mathbf{I}^{\rho}Y(t)\big)$. Utilizing the mean square Riemann-Liouville integral of order α_1, to (3.4), we can write

$$\mathbf{D}^{\alpha_2}\cdots\mathbf{D}^{\alpha_n}Y(t) = \gamma_1 + \mathbf{I}^{\alpha_1}W(t). \tag{3.5}$$

Once more, thanks to the square Riemann-Liouville integral of order α_2, we can state that

$$\mathbf{D}^{\alpha_3}\cdots\mathbf{D}^{\alpha_n}Y(t) = \gamma_2 + \mathbf{I}^{\alpha_2}\gamma_1 + \mathbf{I}^{\alpha_1+\alpha_2}W(t). \tag{3.6}$$

Consequently,

$$\mathbf{D}^{\alpha_3}\cdots\mathbf{D}^{\alpha_n}Y(t) = \gamma_2 + \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\gamma_1 + \mathbf{I}^{\alpha_1+\alpha_2}W(t). \tag{3.7}$$

Using the same arguments as before, we get the following statement

$$Y(t) = \sum_{i=1}^{n-1}\frac{t^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\gamma_i + \gamma_n + \mathbf{I}^{\sum_{i=1}^{n}\alpha_i}W(t), \tag{3.8}$$

where, $\gamma_i \in \mathbb{R}, i = 1\ldots, n$.
For $t = 0$, in (3.8) we have

$$Y(0) = \gamma_n$$

and by the first initial condition in (3.2), we get $\gamma_n = X_0$.
By differentiating of (3.8) α_{i+1}-times for $i = 1,\ldots,n-1$, and by taking $t = 0$, we obtain

$$\begin{aligned} Y^{(\alpha_n)}(0) =& \gamma_{n-1}, \\ &\vdots \\ Y^{(\alpha_2)}(0) =& \gamma_1. \end{aligned}$$

Also, we can see that

$$\gamma_1 = X_1,$$
$$\vdots$$
$$\gamma_{n-1} = X_{n-1}.$$

Substituting γ_i, $i = 0, \dots, n-1$, in (3.8), we get the desired representation.
The proof is thus achieved. □

Let now consider the Banach space defined by:

$$F := \{Y \in \mathcal{C}, \mathbf{D}^{\beta} Y \in \mathcal{C}\},$$

which is equipped with the norm

$$||Y||_F = \max(||Y||_{\mathcal{C}}, ||\mathbf{D}^{\beta} Y||_{\mathcal{C}}).$$

Let also introduce the random integral operator $H : F \to F$:

$$HY(t) = \sum_{i=1}^{n-1} \frac{t^{\sum_{j=i+1}^{n} \alpha_j}}{\Gamma(\sum_{j=i+1}^{n} \alpha_j + 1)} X_i + X_0$$
$$+ \int_0^t \frac{(t-s)^{\sum_{i=1}^{n} \alpha_i - 1}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i\big)} \Big(a_1 A_1 f_1(s, Y(s)) + a_2 A_2 f_2\big(s, \mathbf{D}^{\beta} Y(s)) + a_3 A_3 f_3\big(s, \mathbf{I}^{\rho} Y(s)\big) \Big) ds.$$

To facilitate the fastidious calculation, we consider the following notations and assumptions:

$(H1)$: There are three real positive numbers $K_1, K_2, K_3 > 0$, such that for all $Y_1, Y_2 \in \mathbb{L}_2(\Omega), t \in J$, the following inequalities are valid:

$$\|f_1(t, Y_1) - f_1(t, Y_2)\|_2 \le K_1 \|Y_1 - Y_2\|_2,$$
$$\|f_2(t, Y_1) - f_2(t, Y_2)\|_2 \le K_2 \|Y_1 - Y_2\|_2,$$
$$\|f_3(t, Y_1) - f_3(t, Y_2)\|_2 \le K_3 \|Y_1 - Y_2\|_2.$$

$(H2):$ There exist three positive real numbers $0 \leq r_1, r_2, r_3$, such that

$$\|f_1(t,0)\|_2 \leq r_1,$$
$$\|f_2(t,0)\|_2 \leq r_2,$$
$$\|f_3(t,0)\|_2 \leq r_3.$$

$$\rho = \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n} \alpha_j}}{\Gamma(\sum_{j=i+1}^{n} \alpha_j + 1)} \|X_i\|_2 + \|X_0\|_2,$$
$$\rho_1 = \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n} \alpha_j - \beta}}{\Gamma(\sum_{j=i+1}^{n} \alpha_j - \beta + 1)} \|X_i\|_2,$$
$$\phi = \frac{T^{\sum_{i=1}^{n} \alpha_i}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i + 1\big)} \Big(a_1\|A_1\|_2 r_1 + a_2\|A_2\|_2 r_2 + a_3\|A_3\|_2 r_3 \Big),$$
$$\phi_1 = \frac{T^{\sum_{i=1}^{n} \alpha_i}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i + 1\big)} \Big(a_1\|A_1\|_2 K_1 + a_2\|A_2\|_2 K_2 + a_3\|A_3\|_2 K_3 \Big),$$
$$\sigma = \frac{T^{\sum_{i=1}^{n} \alpha_i - \beta}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i - \beta + 1\big)} \Big(a_1\|A_1\|_2 r_1 + a_2\|A_2\|_2 r_2 + a_3\|A_3\|_2 r_3 \Big),$$
$$\sigma_1 = \frac{T^{\sum_{i=1}^{n} \alpha_i - \beta}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i - \beta + 1\big)} \Big(a_1\|A_1\|_2 K_1 + a_2\|A_2\|_2 K_2 + a_3\|A_3\|_2 K_3 \Big). \tag{3.9}$$

Now, we prove the existence of a unique stochastic process solution for our above Airy type problem.

Theorem 3.2.2. *Suppose satisfied the hypotheses (H1) and (H2). Then* (3.2) *has a unique stochastic process solution, under the condition that* $R < 1$, *where*

$$R := \max(\phi_1, \sigma_1).$$

Proof. To prove this theorem, we shall consider an arbitrary real positive number r, such that

$$r > \max(\frac{\rho+\phi}{1-\phi_1}, \frac{\rho_1+\sigma}{1-\sigma_1}).$$

We begin first by showing that $HB_r \subset B_r$, where

$$B_r = \{Y \in F : \|Y\|_F \leq r\}.$$

So, let $t \in J, Y \in B_r$. It is clear that by definition, we have

$$\|HY(t)\|_2 \leq \sum_{i=1}^{n-1} \frac{t^{\sum_{j=i+1}^{n} \alpha_j}}{\Gamma(\sum_{j=i+1}^{n} \alpha_j + 1)} \|X_i\|_2 + \|X_0\|_2$$
$$+ \int_0^t \frac{(t-s)^{\sum_{i=1}^{n} \alpha_i - 1}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i\big)} \Big(a_1 \|A_1\|_2 \|f_1(s, Y(s))\|_2 + a_2 \|A_2\|_2 \|f_2\big(s, \mathbf{D}^\beta Y(s)\big)\|_2$$
$$+ a_3 \|A_3\|_2 \|f_3\big(s, \mathbf{I}^\rho Y(s)\big)\|_2 \Big) ds.$$

Using both (H1) and (H2), we can state that

$$\begin{aligned} \|f_1(t, Y(t)) - f_1(t,0) + f_1(t,0)\|_2 &\leq \|f_1(t, Y(t)) - f_1(t,0)\|_2 + \|f_1(t,0)\|_2 \\ &\leq K_1 \|Y\|_2 + r_1. \end{aligned}$$

With the same arguments, we get

$$\begin{aligned} \|f_2(t, \mathbf{D}^\beta Y(t)) - f_2(t,0) + f_2(t,0)\|_2 &\leq \|f_2(t, \mathbf{D}^\beta Y(t)) - f_2(t,0)\|_2 + \|f_2(t,0)\|_2 \\ &\leq K_2 \|\mathbf{D}^\beta Y\|_2 + r_2, \end{aligned}$$

$$\begin{aligned} \|f_3(t, \mathbf{I}^\rho Y(t)) - f_3(t,0) + f_3(t,0)\|_2 &\leq \|f_3(t, \mathbf{I}^\rho Y(t)) - f_3(t,0)\|_2 + \|f_3(t,0)\|_2 \\ &\leq K_3 \|\mathbf{I}^\rho Y\|_2 + r_3. \end{aligned}$$

Therefore, it yields that

$$\|HY(t)\|_2 \leq \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n} \alpha_j}}{\Gamma(\sum_{j=i+1}^{n} \alpha_j + 1)} \|X_i\|_2 + \|X_0\|_2$$
$$+ \frac{T^{\sum_{i=1}^{n} \alpha_i}}{\Gamma\big(\sum_{i=1}^{n} \alpha_i + 1\big)} \Big(a_1 \|A_1\|_2 (K_1 \|Y\|_{\mathcal{C}} + r_1) + a_2 \|A_2\|_2 (K_2 \|\mathbf{D}^\beta Y\|_{\mathcal{C}} + r_2)$$
$$+ a_3 \|A_3\|_2 (K_3 \|\mathbf{I}^\rho Y\|_{\mathcal{C}} + r_3) \Big).$$

So, we obtain

$$\|HY\|_{\mathcal{C}} \leq \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\|X_i\|_2 + \|X_0\|_2$$
$$+ \frac{T^{\sum_{i=1}^{n}\alpha_i}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i+1\big)}\Big(a_1\|A_1\|_2(K_1\|Y\|_F + r_1) + a_2\|A_2\|_2(K_2\|Y\|_F + r_2)$$
$$+ a_3\|A_3\|_2(K_3\|Y\|_F + r_3)\Big)$$
$$\leq \rho + \phi + \phi_1 r < r. \tag{3.10}$$

On the other side, we can write

$$\|\mathbf{D}^{\beta}HY\|_{\mathcal{C}} \leq \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j-\beta}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j-\beta+1)}\|X_i\|_2$$
$$+ \frac{T^{\sum_{i=1}^{n}\alpha_i-\beta}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i-\beta+1\big)}\Big(a_1\|A_1\|_2(K_1\|Y\|_F + r_1) + a_2\|A_2\|_2(K_2\|Y\|_F + r_2)$$
$$+ a_3\|A_3\|_2(K_3\|Y\|_F + r_3)\Big)$$
$$\leq \rho_1 + \sigma + \sigma_1 r < r. \tag{3.11}$$

Thanks to (3.10) and (3.11), we can deduce that

$$\|HY\|_F \leq r.$$

We have thus proved that $HB_r \in B_r$.
Now, we prove that H is contractive.
Let $Y_1, Y_2 \in F, t \in J$. We have

$$HY_1(t) - HY_2(t) = \int_0^t \frac{(t-s)^{\sum_{i=1}^{n}\alpha_i-1}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i\big)}\Big(a_1A_1(f_1(s,Y_1(s)) - f_1(s,Y_2(s)))$$
$$+ a_2A_2(f_2\big(s,\mathbf{D}^{\beta}Y_1(s)) - f_2\big(s,\mathbf{D}^{\beta}Y_2(s))) + a_3A_3(f_3\big(s,\mathbf{I}^{\rho}Y_1(s)\big) - f_3\big(s,\mathbf{I}^{\rho}Y_2(s)\big))\Big)ds,$$

which leads to

$$\|HY_1(t) - HY_2(t)\|_2 \leq \int_0^t \frac{(t-s)^{\sum_{i=1}^n \alpha_i - 1}}{\Gamma\big(\sum_{i=1}^n \alpha_i\big)} \Big(a_1\|A_1\|_2 \|f_1(s, Y_1(s)) - f_1(s, Y_2(s))\|_2$$

$$+ a_2\|A_2\|_2 \|f_2\big(s, \mathbf{D}^\beta Y_1(s)\big) - f_2\big(s, \mathbf{D}^\beta Y_2(s)\big)\|_2$$

$$+ a_3\|A_3\|_2 \|f_3\big(s, \mathbf{I}^\rho Y_1(s)\big) - f_3\big(s, \mathbf{I}^\rho Y_2(s)\big)\|_2 \Big) ds.$$

Thanks to (H1), we have the following estimate

$$\begin{aligned} \|HY_1 - HY_2\|_{\mathcal{C}} &\leq \frac{T^{\sum_{i=1}^n \alpha_i}}{\Gamma\big(\sum_{i=1}^n \alpha_i + 1\big)} \\ &\times \Big(a_1\|A_1\|_2 (K_1\|Y_1 - Y_2\|_{\mathcal{C}}) + a_2\|A_2\|_2 (K_2\|\mathbf{D}^\beta Y_1 - \mathbf{D}^\beta Y_2\|_{\mathcal{C}}) \\ &+ a_3\|A_3\|_2 (K_3\|\mathbf{I}^\rho Y_1 - \mathbf{I}^\rho Y_2\|_{\mathcal{C}}) \Big) \\ &\leq \phi_1 \|Y_1 - Y_2\|_F. \end{aligned} \tag{3.12}$$

Some easy calculation will allow us to state that

$$\begin{aligned} \|\mathbf{D}^\beta (HY_1 - HY_2)\|_{\mathcal{C}} &\leq \frac{T^{\sum_{i=1}^n \alpha_i - \beta}}{\Gamma\big(\sum_{i=1}^n \alpha_i - \beta + 1\big)} \\ &\times \Big(a_1\|A_1\|_2 (K_1\|Y_1 - Y_2\|_{\mathcal{C}}) + a_2\|A_2\|_2 (K_2\|\mathbf{D}^\beta Y_1 - \mathbf{D}^\beta Y_2\|_{\mathcal{C}}) \\ &+ a_3\|A_3\|_2 (K_3\|\mathbf{I}^\rho Y_1 - \mathbf{I}^\rho Y_2\|_{\mathcal{C}}) \Big) \\ &\leq \sigma_1 \|Y_1 - Y_2\|_F. \end{aligned} \tag{3.13}$$

The inequalities (3.12) and (3.13) allow us to say that

$$\|HY_1 - HY_2\|_F \leq \max(\phi_1, \sigma_1) \|Y_1 - Y_2\|_F.$$

At the end of this proof, we can conclude that problem (3.2) has a unique stochastic process solution on J. □

Remark 2. *Theorem 3.2.2 was established with the norm of the* max *in F. However and as the proof given above shows, it remains true for the initial norm of F provided that replace the condition $R := \max(\phi_1, \sigma_1) < 1$ by the condition $\phi_1 + \sigma_1 < 1$. Indeed, this follows some inequalities*

$$\|HY_1 - HY_2\|_{\mathcal{C}} \leq \phi_1 \|HY_1 - HY_2\|_F$$

and

$$\|D^{\beta}(HY_1 - HY_2)\|_{\mathcal{C}} \leq \sigma_1 \|HY_1 - HY_2\|_F.$$

Note for example that the condition $\phi_1 + \sigma_1 < 1$ is automatically fulfilled if $2R < 1$.

3.3 Random Data and $\beta-$Dependence

Using the introduced norm of the above Banach space, we shall be concerned with introducing some random dependence definitions for the above fractional Airy type problem. Then, we prove some random variables data dependence results for the same problem.
To do this, we shall first consider the following auxiliary problem:

$$\begin{cases} \mathbf{D}^{\alpha_1} \cdots \mathbf{D}^{\alpha_n} Y(t) &= a_1 A_1 f_1(t, Y(t)) + a_2 A_2 f_2\big(t, \mathbf{D}^{\beta} Y(t)\big) \\ & \quad + a_3 A_3 f_3\big(t, \mathbf{I}^{\rho} Y(t)\big), 0 \leq a_i, \\ \tilde{X}_0 &= Y(0), \\ \tilde{X}_i &= Y^{(\alpha_i+1)}(0), i = 1, \dots, n-1, n \in \mathbb{N}^*, \\ t \in J = [0, T] &, \quad \alpha_i \in]0, 1], 0 < \beta < 1, 0 < \rho. \end{cases} \tag{3.14}$$

We introduce the following first definition.

Definition 3.3.1. *The solution Y of (3.2) is continuously and β-differentially dependent on the random data $X_i, i = 0, \dots, n-1, n \in \mathbb{N}^*$, if*

$$\forall \epsilon > 0, \exists \delta_i > 0, i = 0, \dots, n-1, \text{ such that } \|X_i - \tilde{X}_i\|_2 \leq \delta_i$$

the inequality : $\|Y-\tilde{Y}\|_F \leq \epsilon$ *holds.*

Now, we are ready to present to the reader the following main result.

Theorem 3.3.2. *Suppose that the conditions of Theorem 3.2.2 are valid. Then the solution of* (3.2) *is continuously and* β*-differentially dependent on* $X_i, i=0,\ldots,n-1, n\in\mathbb{N}^*$.

Proof. Let Y and $\tilde{Y}$ be the unique random solution of (3.2) and (3.14), where:

$$\tilde{Y}(t)=\sum_{i=1}^{n-1}\frac{t^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\tilde{X}_i+\tilde{X}_0$$
$$+\int_0^t\frac{(t-s)^{\sum_{i=1}^{n}\alpha_i-1}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i\big)}\Big(a_1A_1f_1(s,\tilde{Y}(s))+a_2A_2f_2\big(s,\mathbf{D}^\beta\tilde{Y}(s))+a_3A_3f_3\big(s,\mathbf{I}^\rho\tilde{Y}(s))\Big)ds. \tag{3.15}$$

We have

$$\begin{aligned}Y(t)-\tilde{Y}(t)&=\sum_{i=1}^{n-1}\frac{t^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}(X_i-\tilde{X}_i)+(X_0-\tilde{X}_0)\\&+\int_0^t\frac{(t-s)^{\sum_{i=1}^{n}\alpha_i-1}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i\big)}\Big(a_1A_1(f_1(s,Y(s))-f_1(s,\tilde{Y}(s)))\\&+a_2A_2(f_2\big(s,\mathbf{D}^\beta Y(s))-f_2\big(s,\mathbf{D}^\beta\tilde{Y}(s)))\\&+a_3A_3(f_3\big(s,\mathbf{I}^\rho Y(s)\big)-f_3\big(s,\mathbf{I}^\rho\tilde{Y}(s)))\Big)ds.\end{aligned} \tag{3.16}$$

So, we get

$$\|Y(t)-\tilde{Y}(t)\|_2 \leq \sum_{i=1}^{n-1} \frac{t^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\|X_i-\tilde{X}_i\|_2+\|X_0-\tilde{X}_0\|_2$$
$$+\int_0^t \frac{(t-s)^{\sum_{i=1}^{n}\alpha_i-1}}{\Gamma\big(\sum_{i=1}^{n}\alpha_i\big)}\Big(a_1\|A_1\|_2\|f_1(s,Y(s))-f_1(s,\tilde{Y}(s))\|_2$$
$$+a_2\|A_2\|_2\|f_2\big(s,\mathbf{D}^{\beta}Y(s)\big)-f_2\big(s,\mathbf{D}^{\beta}\tilde{Y}(s)\big)\|_2$$
$$+a_3\|A_3\|_2\|f_3\big(s,\mathbf{I}^{\rho}Y(s)\big)-f_3\big(s,\mathbf{I}^{\rho}\tilde{Y}(s)\big)\|_2\Big)ds. \tag{3.17}$$

Consequently, we obtain

$$\|Y-\tilde{Y}\|_{\mathcal{C}} \leq \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\delta_i+\delta_n+\phi_1\|Y-\tilde{Y}\|_F. \tag{3.18}$$

With the same arguments as before, we have

$$\|\mathbf{D}^{\beta}(Y-\tilde{Y})\|_{\mathcal{C}} \leq \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j-\beta}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j-\beta+1)}\delta_i+\sigma_1\|Y-\tilde{Y}\|_F. \tag{3.19}$$

By the inequalities (3.22) and (3.23), we get

$$\|Y-\tilde{Y}\|_F \leq \max\Big(\sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\delta_i+\delta_n, \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j-\beta}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j-\beta+1)}\delta_i\Big)$$
$$+\max(\phi_1,\sigma_1)\|Y-\tilde{Y}\|_F. \tag{3.20}$$

This leads to

$$\|Y-\tilde{Y}\|_F \leq \frac{\max\big(\sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j+1)}\delta_i+\delta_n, \sum_{i=1}^{n-1} \frac{T^{\sum_{j=i+1}^{n}\alpha_j-\beta}}{\Gamma(\sum_{j=i+1}^{n}\alpha_j-\beta+1)}\delta_i\big)}{1-R}, \tag{3.21}$$

where $R = \max(\phi_1, \sigma_1)$.
The proof is thus complete. □

3.4 Applications

This section deals with two examples to review the main results by a numerical point of view.

Example 3.4.1. *We consider the following problem*

$$\begin{aligned} \mathbf{D}^{0.7}\mathbf{D}^{0.4}Y(t) &= 1.5A_1\tfrac{\cos Y(t)+Y(t)}{33(t^2+2)} + \sqrt{3}A_2\tfrac{\mathbf{D}^{\frac{1}{25}}Y(t)+\sin \mathbf{D}^{\frac{1}{25}}Y(t)}{t+31} \\ &\quad +\tfrac{1}{2}A_3\tfrac{2\cos \mathbf{I}^{\frac{3}{2}}Y(t)+2\sin \mathbf{I}^{\frac{3}{2}}Y(t)}{\exp(\sqrt{t}+23)}, \end{aligned} \tag{3.22}$$

such that $E(X_0^2) = 1, E(X_1^2) = 3, E(A_1^2) = 4, E(A_2^2) = 1, E(A_3^2) = 16$, *where* $t \in J = [0, 7]$.
We have $K_1 = \frac{1}{66}, K_2 = \frac{1}{31}, K_3 = \frac{2}{\exp 23}$, $r_1 = \frac{1}{33(t^2+2)}, r_2 = 0, r_3 = \frac{2}{\exp(\sqrt{t}+23)}$.
We get $\rho = 5.2515, \rho_1 = 3.9203, \phi = 0.3694, \phi_1 = 0.8234, \sigma = 0.3482, \sigma_1 = 0.7763$, *and* $R = max(\phi_1, \sigma_1) = 0.8234 < 1$.
Thanks to Theorem 3.2.2, the problem (3.22) *has a unique stochastic process solution on* $J = [0, 7]$.

Example 3.4.2. *Consider the following problem*

$$\mathbf{D}^{0.6}\mathbf{D}^{0.6}\mathbf{D}^{0.9}Y(t) = 0.75A_1f_1(t, Y(t)) + 2A_2f_2\big(t, \mathbf{D}^{\frac{2}{33}}Y(t)\big) + A_3f_3\big(t, \mathbf{I}^3Y(t)\big), \tag{3.23}$$

such that $E(X_0^2) = 2, E(X_1^2) = 1, E(X_2^2) = 5, E(A_1^2) = 9, E(A_2^2) =$

$16, E(A_3^2) = 1,$ *where* $t \in J = [0, 5],$ *and*

$$f_1(t, Y(t)) = \frac{1}{2t + 43}(\sin Y(t) + \cos Y(t)),$$
$$f_2(t, \mathbf{D}^{\frac{2}{23}} Y(t)) = \frac{1}{\sqrt{t} + \exp(27)}(\mathbf{D}^{\frac{2}{23}} Y(t) + \cos \mathbf{D}^{\frac{2}{23}} Y(t)),$$
$$f_3(t, \mathbf{I}^3 Y(t)) = \frac{\mathbf{I}^3 Y(t)}{t^3 + 47}.$$

We have $K_1 = \frac{1}{43}, K_2 = \frac{1}{\exp(27)}, K_3 = \frac{1}{47}, r_1 = \frac{1}{2t+43}, r_2 = \frac{1}{\sqrt{t}+\exp(47)}, r_3 = 0.$

Using our data, we find $\rho = 19.7213, \rho_1 = 17.1148, \phi = 0.6992, \phi_1 = 0.9835, \sigma = 0.6718, \sigma_1 = 0.9450,$ *and* $R = max(\phi_1, \sigma_1) = 0.9835 < 1.$

Then, by Theorem 3.3.2, the problem (3.23) *is continuously and* $\frac{2}{33}-$*differentially dependent on* $X_i, i = 0, 1, 2.$

Conclusion

We have been concerned with study random fractional differential equations using the mean square fractional calculus. The first chapter allowed us to be familiar with the concepts of stochastic processes. It included some basic results on mean square fractional calculus and some fixed point theory notions. In the second chapter, we have been concerned with a new class of sequential random differential equations with mean square Caputo fractional derivatives. We have proved the existence and uniqueness of random solutions under some conditions on the data of the problem. Additionally, we have introduced and then presented new concepts and new results for continuous and fractional derivative dependence for solutions for the studied problem.

In the third chapter, we have studied a class of random fractional differential problems. The considered class generalizes the classical Airy differential problem both in the random and in the fractional senses. We have established new conditions to prove the existence of a unique stochastic process solution. Then, new notions on $\beta-$differential dependence have been introduced, and new results on the dependence have been established.

In the future, the Ulam-Hyers stability will be analyzed. Then, it will be compared with the continuous dependence results of the present work. Also, we intend to study nth order stochastic process in L_p spaces. These are some open questions that need to be studied...

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