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Some results on the growth and oscillation of solutions of differential equations with meromorphic function coefficients of $[p,q]-\varphi$ order

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MASTER

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Contents

Introduction

Nevanlinna theory is a branch of complex analysis that deals with the study of meromorphic functions. It was introduced by a Finnish mathematician Rolf Nevanlinna in the early 20th century [21].

This theory has applications in many areas of mathematics, including algebraic geometry, number theory, and differential equations.
The main focus of Nevanlinna theory is on the distribution of values of meromorphic functions.

It provides a way to measure their growth, using the Nevanlinna characteristic function. And that's the reason why, this theory has became an important tool in the study of differential equations.

Wittich was the first one who made a systematic study in the application of Nevanlinna theory into complex differential equations in [26]. Since that, many problems has been studied and solved by several mathematicians.

Several researchers have studied the properties of solutions of linear differential equations of the second order or of higher order with entire or meromorphic functions by giving information on the hyper order, iterated order and the $[p,q]$-order of the solutions of these equations (see [2], [15], [16], [18], [22]....etc).

In recent years, some scientists have been interested in determining the properties of the solution of equations whose coefficients are entire or meromorphic functions in the complex plane (see [5], [20], [24]...etc)

As far as we know, in [6] Chyzhykov, Heittokangas and Rättyä introduced the concept of order φ-order in order to study the growth of solutions of linear differential equations in the complex plane.

Inspired by these works, in this thesis, we have studied some properties of solutions to certain differential equations of complex variable function coefficients.

This work is composed of an introduction, and three chapters.

In the first chapter, we introduce the elementary definitions of the Nevanlinna theory, and some other notations that we will need in the next chapters.

In order to demonstrate the results mentioned in the third chapter, besides some proved lemmas, we needed to demonstrate some auxiliary lemmas, that are mentioned in the second chapter.

The last chapter is devoted to the results obtained in the article [23], where we have studied the growth and oscillation of solutions to the homogeneous differential equation

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1}$$

where $A_j(z)$ $(j = 0, 1, \cdots, k)$ with $A_k(z) \not\equiv 0$ are meromorphic functions with finite $[p,q]-\varphi$ order.

Under some conditions, we have proved that every non-transcendental meromorphic solution $f \not\equiv 0$ of (1) is a polynomial with $deg f \leq s-1$ and every transcendental meromorphic solution f of (1) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\left\{\sigma, \mu_{[p,q]}(f,\varphi)\right\}$ satisfies

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty, \sigma \leq \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).$$

As a result, under the same hypothesis we have obtained that if ψ is a transcendental meromorphic function that satisfies a certain condition then, every transcendental meromorphic solution f of equation (1) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ satisfies

$$\begin{aligned}\sigma \leq \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) &= \lambda_{[p+1,q]}(f-\psi,\varphi)\\ &= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).\end{aligned}$$

In the second part of this chapter, we have studied the growth of solutions to the non-homogeneous differential equation

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{2}$$

where $A_j(z)$ $(j = 0, 1, \cdots, k)$ with $F(z) \not\equiv 0$ are meromorphic functions with finite $[p,q]-\varphi$ order.

Under some conditions, we have obtained that every non-transcendental meromorphic solution $f \not\equiv 0$ of (2) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (2) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\left\{\sigma, \mu_{[p,q]}(f,\varphi)\right\}$ satisfies

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty$$

and

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).$$

As a result, under the same hypothesis we have obtained that if ψ is a transcendental meromorphic function that satisfies a certain condition then, every transcendental meromorphic solution f of equation (2) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ satisfies

$$\begin{aligned}\sigma \leq \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) &= \lambda_{[p+1,q]}(f-\psi,\varphi)\\ &= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).\end{aligned}$$

These results can be considered as a generalization of some previous results.

We conclude this work with a conclusion and some perspectives.

Chapter 1

Nevanlinna theory

In this chapter we introduce some necessary and elementary definitions, notations and results that we will need later in the next two chapters.

1 Poisson-Jensen and Jensen formula

Theorem 1.1 *(Poisson-Jensen formula [7], [9]) Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \dots (resp.\ b_1, b_2, \dots)$ denote its zeros (resp. poles), each taken into account according to its multiplicity. If $z = re^{i\theta}$ and $0 \leqslant r < R < \infty$, then*

$$\begin{aligned} \log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2r\cos(\theta - \varphi) + r^2} d\varphi \\ &+ \sum_{|a_j|<R} \log\left|\frac{R(z - a_j)}{R^2 - \overline{a_j}z}\right| - \sum_{|b_k|<R} \log\left|\frac{R(z - b_k)}{R^2 - \overline{b_k}z}\right|. \end{aligned} \tag{1.1}$$

Theorem 1.2 *(Jensen formula [17]) Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \dots (resp.\ b_1, b_2, \dots)$ denote its zeros (resp. poles), each taken into account acording to its multiplicity. Then, we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + \sum_{|b_j|<r} \log\left(\frac{r}{|b_j|}\right) - \sum_{|a_j|<r} \log\left(\frac{r}{|a_j|}\right) \tag{1.2}$$

Proof. Proving formula (1.2) when f has no zeros or poles on $|z| = r$. Let

$$g(z) := f(z) \prod_{|a_j|<r} \left(\frac{r^2 - \overline{a_j}z}{r(z - a_j)}\right) \prod_{|b_k|<r} \left(\frac{r^2 - \overline{b_k}z}{r(z - b_k)}\right)^{-1}, \tag{1.3}$$

then $g \neq 0, \infty$ in $|z| < r$ and $\log|g(z)|$ is a harmonic function. By the mean property of classical harmonic functions, we have

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g(re^{i\varphi})\right| d\varphi. \tag{1.4}$$

Since

$$|g(0)| = |f(0)| \prod_{|a_j|<r} \left(\frac{r}{|a_j|}\right) \prod_{|b_k|<r} \left(\frac{r}{|b_k|}\right)^{-1}, \tag{1.5}$$

we obtain

$$\log|g(0)| = \log|f(0)| + \sum_{|a_j|<r} \log\left(\frac{r}{|a_j|}\right) - \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right). \tag{1.6}$$

For $z = re^{i\varphi}$, we have

$$\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right| = \left|\frac{r^2 - \overline{b_k}z}{r(z-b_k)}\right| = 1$$

for all a_j, b_k. Then

$$\log\left|g(re^{i\varphi})\right| = \log\left|f(re^{i\varphi})\right|. \tag{1.7}$$

Substituting (1.6) and (1.7) in (1.4), we obtain

$$\log|f(0)| + \sum_{|a_j|<r} \log\left(\frac{r}{|a_j|}\right) - \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi,$$

hence, the formula of Jensen. ■

2 Nevanlinna characteristic function

Definition 1.1 *(Integrated counting function [17]) $a \in \mathbb{C}$ is given. Let f be a meromorphic function such that $f \not\equiv a$.*
Then $n(r,a,f)$ denotes the number of roots of the equation $f(z) - a = 0$ in the disc $\{z : |z| \leqslant r\}$, each root according to its multiplicity.
Similarly $\overline{n}(r,a,f)$ counts the number of distinct roots of $f(z) - a = 0$ in the disc $\{z : |z| \leqslant r\}$.
And $n(r,\infty,f)$ denotes the number of poles of f in the disc $\{z : |z| \leqslant r\}$, each pole according to its multiplicity.
Similarly $\overline{n}(r,\infty,f)$ counts the number of distinct poles of f in the disc $\{z : |z| \leqslant r\}$,.

Example 1.1 *Let $f(z) = \sin^2(z)$, we have*

$$n(r,0,f) = 2 + 4\left[\frac{r}{\pi}\right] \quad \textit{and} \quad \overline{n}(r,0,f) = 1 + 2\left[\frac{r}{\pi}\right].$$

Example 1.2 *Let $f(z) = \frac{\tan(z)}{z^4}$, we have*

$$n(r,\infty,f) = 4 \quad \textit{and} \quad \overline{n}(r,\infty,f) = 1.$$

Definition 1.2 *(Counting function [9]) Let f be a meromorphic function. For $a \in \mathbb{C}$, we define the a-point function of f by*

$$\mathrm{N}(r,a,f) = \mathrm{N}\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t} dt + n(0,a,f)\log r, \quad f \not\equiv a,$$

and

$$\mathrm{N}(r,\infty,f) = \mathrm{N}(r,f) := \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt + n(0,\infty,f)\log r.$$

Similarly, we define the a-point distinct function of f by

$$\overline{\mathrm{N}}(r,a,f) = \overline{\mathrm{N}}\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{\overline{n}(t,a,f) - \overline{n}(0,a,f)}{t} dt + \overline{n}(0,a,f)\log r, \quad f \not\equiv a,$$

and

$$\overline{\mathrm{N}}(r,\infty,f) = \overline{\mathrm{N}}(r,f) := \int_0^r \frac{\overline{n}(t,\infty,f) - \overline{n}(0,\infty,f)}{t} dt + \overline{n}(0,\infty,f)\log r.$$

Example 1.3 *Let* $f(z)=\frac{\exp(z)}{z^3}$*, we have*

$$n(t,\infty,f)=n(0,\infty,f)=3 \quad and \quad \overline{n}(t,\infty,f)=\overline{n}(0,\infty,f)=1,$$

hence

$$\mathrm{N}(r,f)=\int_0^r \frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt+n(0,\infty,f)\log=3\log r$$

and

$$\overline{\mathrm{N}}(r,f)=\int_0^r \frac{\overline{n}(t,\infty,f)-\overline{n}(0,\infty,f)}{t}dt+\overline{n}(0,\infty,f)\log r=\log r.$$

Lemma 1.1 *([9], [17]) Let* f *be a meromorphic function with a-points* $\alpha_1,\alpha_2,\ldots,\alpha_n$ *in* $\{z: |z|\le r\}$ *such that* $0<|\alpha_1|\le|\alpha_2|\le\ldots\le|\alpha_n|\le r$*, and* $f(0)\ne 0$*, each counted according to its multiplicity. Then*

$$\begin{aligned}\int_0^r \frac{n(t,a,f)}{t}dt &= \int_0^r \frac{n(t,a,f)-n(0,a,f)}{t}dt\\ &= \sum_{0<|\alpha_j|\le r}\log\left(\frac{r}{|\alpha_j|}\right).\end{aligned} \tag{1.8}$$

Proof. ([9], [17]) Denoting $|\alpha_j|=r_j$ for $j=1,\ldots,n$, we obtain

$$\begin{aligned}\sum_{0<|\alpha_j|\le r}\log\left(\frac{r}{|\alpha_j|}\right) &= \sum_{j=1}^{n}\log\left(\frac{r}{r_j}\right)=n\log r-\sum_{j=1}^{n}\log r_j\\ &= \sum_{j=1}^{n} j\left(\log r_{j+1}-\log r_j\right)+n\left(\log r-\log r_n\right)\\ &= \sum_{j=1}^{n}\int_{r_j}^{r_{j+1}}\frac{j}{t}dt+\int_{r_n}^{r}\frac{n}{t}dt\\ &= \int_0^r \frac{n(t,a,f)}{t}dt.\end{aligned}$$

■

Proposition 1.1 *([9], [17]) Let* f *be a meromorphic function represented by its Laurent expansion in original point*

$$f(z)=\sum_{j=m}^{+\infty}c_j z^j,\quad c_m\ne 0,\quad m\in\mathbb{Z}.$$

Then

$$\log|c_m|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\varphi})\right|d\varphi+\mathrm{N}(r,f)-\mathrm{N}\left(r,\frac{1}{f}\right).$$

Definition 1.3 [17] *For any real number* $x\ge 0$*, we define*

$$\log^+ x:=\max(0,\log x).$$

The following lemma contains some properties of $\log^+ x$.

Lemma 1.2 [17] *Let* $x, y, x_j, j=1,\ldots,n$ *strictly positive real numbers. Then we have*

1. $\log x\le\log^+ x$,

2. $\log^+ x \le \log^+ y \quad (0 \le x \le y)$,

3. $\log x = \log^+ x - \log^+ \frac{1}{x} \quad (x > 0)$,

4. $\left|\log x\right| = \log^+ x + \log^+ \frac{1}{x} \quad (x > 0)$.

5. $\log x \le \log^+ x \quad (x \ge 0)$,

6. $\log^+ \left(\prod_{j=1}^{n} x_j \right) \le \sum_{j=1}^{n} \log^+ x_j$,

7. $\log^+ \left(\sum_{j=1}^{n} x_j \right) \le \log n + \sum_{j=1}^{n} \log^+ x_j$.

Proof. The properties 1, 2 are immediate consequences of the Definition 1.3 and the monotonicity of the logarithmic function.

Proving 3, 4, 5 et 6. For 3, we have ([1])

$$\begin{aligned} \log x^+ - \log^+ \frac{1}{x} &= \max\left(\log x, 0\right) - \max\left(\log \frac{1}{x}, 0\right) \\ &= \max\left(\log x, 0\right) - \max\left(-\log x, 0\right) \\ &= \log x. \end{aligned}$$

The property 4. is obtained as follows ([1])

$$\begin{aligned} \log x^+ + \log^+ \frac{1}{x} &= \max\left(\log x, 0\right) + \max\left(\log \frac{1}{x}, 0\right) \\ &= \max\left(\log x, 0\right) + \max\left(-\log x, 0\right) \\ &= \left|\log x\right|. \end{aligned}$$

For the property 5., if $\prod_{j=1}^{n} x_j \leqslant 1$, then the inequality is obvious. Suppose that $\prod_{j=1}^{n} x_j > 1$. Hence,

$$\begin{aligned} \log^+ \left(\prod_{j=1}^{n} x_j \right) &= \log \left(\prod_{j=1}^{n} x_j \right) \\ &= \sum_{j=1}^{n} \log x_j \\ &\le \sum_{j=1}^{n} \log^+ x_j, \text{ according to 1.} \end{aligned}$$

Finally, we get the property 6. using properties 2 et 5. In fact

$$\begin{aligned} \log^+ \left(\sum_{j=1}^{n} x_j \right) &\leqslant \log^+ (n \max_{1 \le j \le n} x_j) \\ &\leqslant \log n + \log^+ (\max_{1 \le j \le n} x_j) \\ &\leqslant \log n + \sum_{j=1}^{n} \log^+ x_j. \end{aligned}$$

■

Lemma 1.3 [9] *For all $a \in \mathbb{C}$, we have*

$$\log^+ |a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta. \tag{1.9}$$

Definition 1.4 *(Proximity function)([9], [17]) Let f be a meromorphic function. For $a \in \mathbb{C}$, we define the proximity function of f by*

$$m(r,a,f) = m\left(r, \frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\left|f\left(re^{i\varphi}\right) - a\right|} d\varphi, \quad f \not\equiv a,$$

and

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi.$$

Example 1.4 *Let $f(z) = \frac{\exp(az)}{z^m}$, $a \in \mathbb{C}^*$, $m \in \mathbb{N}^*$. We have*

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{\exp(|a|e^{i\arg a} re^{i\varphi})}{\left(re^{i\varphi}\right)^m}\right| d\varphi \\ &= \frac{ar}{\pi} - \frac{m\log r}{2}. \end{aligned}$$

Definition 1.5 *(Characteristic function)([17]) For a meromorphic function f, we define its characteristic function as*

$$\mathrm{T}(r,f) := m(r,f) + \mathrm{N}(r,f).$$

Example 1.5 *Let $f(z) = \exp(az^n)$, $a \in \mathbb{C}^*$, $m \in \mathbb{N}$. We have*

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left(|a|e^{i\arg a}\left(re^{i\varphi}\right)^n\right)\right| d\varphi \quad (a = |a|e^{i\arg a}, \ -\pi < \arg a \le \pi) \\ &= \frac{|a|r^n}{\pi}. \end{aligned}$$

Since f is an entire function, then

$$\mathrm{T}(r,f) = m(r,f) = \frac{|a|r^n}{\pi}.$$

Example 1.6 *Let $f(z) = \frac{\exp(\pi z)}{z^m}$, $m \in \mathbb{N}^*$. We have*

$$n(t,\infty,f) = n(0,\infty,f) = m,$$

then

$$\mathrm{N}(r,f) = m\log r.$$

According to Example 1.4 we have

$$m(r,f) = r - \frac{m\log r}{2}.$$

Hence

$$\mathrm{T}(r,f) = \frac{m\log r}{2}.$$

3 The first main theorem

Theorem 1.3 ***(First main theorem of Nevanlinna)***([9], [17]) *Let f be a meromorphic function with the Laurent expansion*

$$f(z)=\sum_{j=m}^{+\infty} c_j z^j,\quad c_m\neq 0,\quad m\in\mathbb{Z},\qquad z\in\mathbb{C}.$$

Then, for all complex number a, we have

$$\mathrm{T}\left(r,\frac{1}{f-a}\right)=\mathrm{T}(r,f)-\log|c_m|+\varphi(r,a) \tag{1.10}$$

where

$$|\varphi(r,a)|\leq \log^+|a|+\log 2.$$

Proof. Assume first that $a=0$. By Proposition 1.1 and Lemma 1.2(3), we obtain

$$\begin{aligned}\log|c_m| &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi-\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{\left|f(re^{i\varphi})\right|}d\varphi+\mathrm{N}(r,f)-\mathrm{N}\left(r,\frac{1}{f}\right)\\ &= m(r,f)-m\left(r,\frac{1}{f}\right)+\mathrm{N}(r,f)-\mathrm{N}\left(r,\frac{1}{f}\right)\\ &= \mathrm{T}(r,f)-\mathrm{T}\left(r,\frac{1}{f}\right)\end{aligned}$$

hence

$$\mathrm{T}\left(r,\frac{1}{f-a}\right)=\mathrm{T}(r,f)-\log|c_m| \tag{1.11}$$

with $\varphi(r,0)\equiv 0$. Suppose now, that $a\neq 0$. We define $h(z)=f(z)-a$, then

$$\mathrm{N}\left(r,\frac{1}{h}\right)=\mathrm{N}\left(r,\frac{1}{f-a}\right),$$

$$m\left(r,\frac{1}{h}\right)=m\left(r,\frac{1}{f-a}\right),$$

$$\mathrm{N}(r,h)=\mathrm{N}(r,f).$$

Moreover,

$$\begin{aligned}\log^+|h| &= \log^+|f-a|\leq\log^+|a|+\log 2,\\ \log^+|f| &= \log^+|f-a+a|=\log^+|h+a|\leq\log^+|h|+\log^+|a|+\log 2,\end{aligned}$$

Integrating these inequalities we see that

$$\begin{aligned}m(r,h) &\leq m(r,f)+\log^+|a|+\log 2,\\ m(r,f) &\leq m(r,h)+\log^+|a|+\log 2.\end{aligned}$$

we put

$$\varphi(r,a):=m(r,h)-m(r,f)$$

satisfies $|\varphi(r,a)|\leq\log^+|a|+\log 2$. By applying the formula (1.11) for h, we obtain

$$\begin{aligned}\mathrm{T}\left(r,\frac{1}{f-a}\right) &= \mathrm{T}\left(r,\frac{1}{h}\right)=\mathrm{T}(r,h)-\log|c_m|\\ &= m(r,h)+\mathrm{N}(r,h)-\log|c_m|\\ &= \varphi(r,a)+m(r,f)+\mathrm{N}(r,f)-\log|c_m|\end{aligned}$$

hence, the result. ■

Theorem 1.4 ***(Nevanlinna)***([17]) *Let f be a meromorphic function not being identically equal to a constant. Then, for all $a \in \mathbb{C}$, we have*

$$\mathrm{T}\left(r, \frac{1}{f-a}\right) = \mathrm{T}(r,f) + \mathrm{O}(1) \quad r \to +\infty.$$

The following diagram summarises the properties of the Novanlinna characteristic function

Proposition 1.2 *([7], [9], [17]) Let $f, f_1, f_2, \ldots, f_n$ $(n \geq 1)$ be meromorphic functions and a, b, c and d be complex constants such that $ad - bc \neq 0$. Then*

1. $\mathrm{T}\left(r, \sum_{k=1}^{n} f_k\right) \leq \sum_{k=1}^{n} \mathrm{T}(r, f_k) + \log n,$
2. $\mathrm{T}\left(r, \prod_{k=1}^{n} f_k\right) \leq \sum_{k=1}^{n} \mathrm{T}(r, f_k),$
3. $\mathrm{T}\left(r, f^m\right) = m\mathrm{T}(r,f) \quad \forall \ m \in \mathbb{N}^*,$
4. $\mathrm{T}\left(\frac{af+b}{cf+d}\right) = \mathrm{T}(r,f) + \mathrm{O}(1) \quad as \ r \to +\infty \quad f \not\equiv -\frac{d}{c}.$

Proof.

1. We have

$$\mathrm{T}\left(r, \sum_{j=1}^{n} f_j\right) = m\left(r, \sum_{j=1}^{n} f_j\right) + \mathrm{N}\left(r, \sum_{j=1}^{n} f_j\right)$$

and

$$\begin{aligned} m\left(r, \sum_{j=1}^{n} f_j\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\sum_{j=1}^{n} f_j(re^{i\theta})\right| d\theta \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j=1}^{n} \log^+ \left|f_j(re^{i\theta})\right| + \log n\right) d\theta \\ &= \sum_{j=1}^{n} m(r, f_j) + \log n. \end{aligned}$$

On the other hand, since the the multiplicity of the pole of $\sum_{j=1}^{n} f_j$ in z_0 does not exceed the sum of the multiplicities of the poles of $f_j \, (j = 1, \ldots, n)$ in z_0, then

$$\mathrm{N}\left(r, \sum_{j=1}^{n} f_j\right) \leqslant \sum_{j=1}^{n} \mathrm{N}(r, f_j).$$

Hence,

$$\begin{aligned} \mathrm{T}\left(r, \sum_{j=1}^{n} f_j\right) &= m\left(r, \sum_{j=1}^{n} f_j\right) + \mathrm{N}\left(r, \sum_{j=1}^{n} f_j\right) \\ &\leqslant \sum_{j=1}^{n} \mathrm{T}(r, f_j) + \log n\,. \end{aligned}$$

2. We have

$$\mathrm{T}\left(r,\prod_{j=1}^{n}f_j\right)=m\left(r,\prod_{j=1}^{n}f_j\right)+\mathrm{N}\left(r,\prod_{j=1}^{n}f_j\right).$$

Since

$$\begin{aligned} m\left(r,\prod_{j=1}^{n}f_j\right) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\prod_{j=1}^{n}f_j(re^{i\theta})\right|d\theta \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{n}\log^+\left|f_j(re^{i\theta})\right|d\theta \\ &= \sum_{j=1}^{n}m(r,f_j), \end{aligned}$$

and the fact that the multiplicity of the pole of $\prod_{j=1}^{n}f_j$ in z_0 does not exceed the sum of the multiplicities of the poles of $f_j\,(j=1,...,n)$ in z_0, gives

$$\mathrm{N}\left(r,\sum_{j=1}^{n}f_j\right)\leqslant\sum_{j=1}^{n}\mathrm{N}(r,f_j).$$

As a consequence

$$\mathrm{T}\left(r,\prod_{j=1}^{n}f_j\right)\leqslant\sum_{j=1}^{n}\mathrm{T}(r,f_j).$$

3. We have $|f|\leqslant 1$ is equivalent to $|f|^n\leqslant 1$.
a) If $|f|\leqslant 1$, then

$$m(r,f^n)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f^n(re^{i\theta})\right|d\theta=0$$

and

$$\mathrm{N}(r,f^n)=n\mathrm{N}(r,f).$$

Hence

$$\mathrm{T}(r,f^n)=n\mathrm{T}(r,f).$$

b) If $|f|>1$, then

$$\begin{aligned} m(r,f^n) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f^n(re^{i\theta})\right|d\theta \\ &= nm(r,f), \end{aligned}$$

and

$$\mathrm{N}(r,f^n)=n\mathrm{N}(r,f).$$

Therefore,

$$\mathrm{T}(r,f^n)=n\mathrm{T}(r,f).$$

4. Let $g=\frac{af+b}{cf+d}$, with $ad-cb\neq 0$ then we have

$$gcf+gd=af+b\;\Leftrightarrow\;f=\frac{b-gd}{gc-a}.$$

Therefore, it is sufficient to show prove that $T(r,g) \leqslant T(r,f) + O(1)$. We distinguish two cases. Case 1. If $c = 0$, then

$$\begin{aligned} T(r,g) &= T\left(r, \frac{af+b}{d}\right) \\ &\leqslant T\left(r, \frac{a}{d}\right) + T(r,f) + T\left(r, \frac{b}{d}\right) + \log 2 \\ &= T(r,f) + O(1). \end{aligned}$$

Case 2. If $c \neq 0$, then

$$\begin{aligned} T(r,g) &= T\left(r, \frac{af+b}{cf+d}\right) \\ &= T\left(r, \frac{\frac{a}{c}(cf+d) - \frac{ad}{c} + b}{cf+d}\right) \\ &= T\left(r, \frac{a}{c} + \frac{cb-ad}{c^2} . \frac{1}{f+\frac{d}{c}}\right) \\ &\leqslant T\left(r, \frac{cb-ad}{c^2} . \frac{1}{f+\frac{d}{c}}\right) + O(1) \\ &\leqslant T\left(r, \frac{1}{f+\frac{d}{c}}\right) + O(1) \\ &\leqslant T\left(r, f + \frac{d}{c}\right) + O(1) \\ &\leqslant T(r,f) + O(1). \end{aligned}$$

■

Theorem 1.5 ([17]) *A meromorphic function f is rational if and only if $T(r,f) = O(\log r)$.*

4 The growth of an entire or meromorphic function

4.1 Order of growth

Definition 1.6 ([9], [17]) *Let f be a meromorphic function. The order of growth of f is defined by*

$$\rho(f) := \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r}$$

and if f is an entire function, then

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log M(r,f)}{\log r}.$$

Proposition 1.3 ([7], [9], [17]) *Let f, g be non-constant meromorphic functions. Then*

1. $\rho(f+g) \leq \max\{\rho(f), \rho(g)\}$.

2. $\rho(fg) \leq \max\{\rho(f), \rho(g)\}$.

3. *If* $\rho(g) \leq \rho(f)$ *then* $\rho(f+g) = \rho(fg) = \rho(f)$.

Example 1.7 *Let $f(z) = \cosh(z)$, we have*

$$\mathrm{M}(r, f) = \cosh r \sim \frac{e^r}{2}, \quad r \to \infty.$$

Hence

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\log \mathrm{M}(r, f)}{\log r} = 1.$$

4.2 Hyper-order of function

Definition 1.7 *([16], [17]) Let f be a meromorphic function. The hyper-order of f is defined by*

$$\rho_2(f) := \limsup_{r\to+\infty} \frac{\log\log \mathrm{T}(r, f)}{\log r},$$

and if f is entire, then

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log \mathrm{T}(r, f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log \mathrm{M}(r, f)}{\log r}.$$

Remark 1.1 *If $\rho(f) < +\infty$, then $\rho_2(f) = 0$.*

4.3 p-Iterated-order of function

In order to generalize some results on the properties of solutions of certain differential equations, we need to define the p-iterated order of a meromorphic or entire function, but first we need to define the following expressions on the exponential and its reciprocal function.
For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. And for all $r \in (0, +\infty)$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$.

Proposition 1.4 *([4]) Let $x_i \in \mathbb{R}$ such that $x_i > 1$ and $i = 1, \ldots, n$, we have*

1. $\log_p \left(\sum_{i=1}^{n} x_i \right) \le \sum_{i=1}^{n} \log_p x_i + \mathrm{O}(1),$

2. $\log_p \left(\prod_{i=1}^{n} x_i \right) \le \sum_{i=1}^{n} \log_p x_i + \mathrm{O}(1),$

Proof. In order to prove the proposition, we use the mathematical induction.

1. For $p = 1$, we have $\log \left(\sum_{i=1}^{n} x_i \right) \le \sum_{i=1}^{n} \log x_i + \mathrm{O}(1)$.

Suppose that $\log_p \left(\sum_{i=1}^{n} x_i \right) \le \sum_{i=1}^{n} \log_p x_i + \mathrm{O}(1)$ is verified, and proving it For the $p+1$ order. We have

$$\begin{aligned} \log_{p+1} \left(\sum_{i=1}^{n} x_i \right) &= \log \left(\log_p \left(\sum_{i=1}^{n} x_i \right) \right) \\ &\le \log \left(\sum_{i=1}^{n} \log_p x_i + \mathrm{O}(1) \right) \\ &\le \sum_{i=1}^{n} \log_{p+1} x_i + \mathrm{O}(1). \end{aligned}$$

2. For $p = 1$, we have $\log\left(\prod_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} \log x_i + O(1)$.

Suppose that $\log_p\left(\prod_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} \log_p x_i + O(1)$ is verified, and proving it For the $p+1$ order. We have

$$\begin{aligned}\log_{p+1}\left(\prod_{i=1}^{n} x_i\right) &= \log\left(\log_p\left(\prod_{i=1}^{n} x_i\right)\right) \\ &\leq \log\left(\sum_{i=1}^{n} \log_p x_i + O(1)\right) \\ &\leq \sum_{i=1}^{n} \log_{p+1} x_i + O(1).\end{aligned}$$

■

Definition 1.8 *([15],[17]) Let f be a meromorphic function. The p-iterated-order of f is defined by*

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r},$$

and if f is entire, then

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}.$$

5 The [p,q]-order, Lower [p,q]-order and [p,q]-convergence exponent of an entire or meromorphic function

5.1 The [p,q]-order

Definition 1.9 *([14], [18]) Let f be a meromorphic function. The [p,q]-order of f is defined by*

$$\rho_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r},$$

and if f is entire, then

$$\rho_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Example 1.8 *Let $f(z) = e^{e^{2z^3}}$, we have* $\rho_{[2,1]}(f) = 3$.

5.2 The Lower [p,q]-order

Definition 1.10 *[27] Let f be a meromorphic function. The lower [p,q]-order of f is defined by*

$$\mu_{[p,q]}(f) := \liminf_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r},$$

and if f is entire, then

$$\mu_{[p,q]}(f) = \liminf_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r} = \liminf_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Remark 1.2 *For $p = q = 1$ in Definition 1.10, we obtain the definition of lower order of f. For $p = 2, q = 1$ in Definition 1.10, we obtain the definition of lower hyper order of f.*

5.3 The [p,q]-convergence exponent

Definition 1.11 ([18], [19]) *Let f be a meromorphic function. The [p,q]-convergence exponent of the sequence of a-points of f is defined by*

$$\lambda_{[p,q]}(f-a) = \lambda_{[p,q]}(f,a) := \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f-a}\right)}{\log_q r}.$$

If $a = 0$, then the [p,q]-convergence exponent of the zero-sequence of f is defined by

$$\lambda_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.$$

If $a = \infty$, then the [p,q]-convergence exponent of the pole-sequence of f is defined by

$$\lambda_{[p,q]}\left(\frac{1}{f}\right) = \limsup_{r\to+\infty} \frac{\log_p N(r,f)}{\log_q r}.$$

Similarly, the [p,q]-convergence exponent of the distinct zero-sequence of f is defined by

$$\overline{\lambda}_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r},$$

and the [p,q]-convergence exponent of the distinct pole-sequence of f is defined by

$$\overline{\lambda}_{[p,q]}\left(\frac{1}{f}\right) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r,f)}{\log_q r}.$$

Remark 1.3 *For $p = q = 1$ in Definition 1.11, we obtain the definition of convergence exponent of f.*
For $p = 2, q = 1$ in Definition 1.11, we obtain the definition of hyper convergence exponent of f.
For $q = 1$ in Definition 1.11, we obtain the definition of p-iterative convergence exponent of f.

5.4 $[p,q]-\varphi$ order and $[p,q]-\varphi$ lower order of meromorphic functions and entire functions

Definition 1.12 *([24])Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function, and p, q be positive integers satisfying $p \geqslant q \geqslant 1$. Then, the $[p,q]-\varphi$ order and $[p,q]-\varphi$ lower order of meromorphic function f are respectively defined by*

$$\rho_{[p,q]}(f,\varphi) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f,\varphi) = \liminf_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q \varphi(r)}.$$

Definition 1.13 *([24]) Let f be a meromorphic function. Then, the $[p,q]-\varphi$ exponent of convergence of zero-sequence (distinct zero-sequence) of f is defined by*

$$\lambda_{[p,q]}(f,\varphi)=\limsup_{r\to+\infty}\frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},$$

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\limsup_{r\to+\infty}\frac{\log_p \overline{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Remark 1.4 *([24]) If $\varphi(r)=r$ in the Definitions 1.12 and 1.13 then we will get the standard definitions of the $[p,q]-$ order and $[p,q]-$ exponent of convergence.*

Remark 1.5 *([24]) Throughout this manuscript, we assume that $\varphi:[0,+\infty)\to(0,+\infty)$ is non-decreasing unbounded function and always satisfies the following two conditions:*

1. $\lim\limits_{r\to+\infty}\dfrac{\log_{p+1} r}{\log_q \varphi(r)}=0.$

2. $\lim\limits_{r\to+\infty}\dfrac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)}=1$ *for some* $\alpha_1>1.$

By using Remark 1.5, we are able to obtain the following proposition.

Proposition 1.5 *([5]) Assume that φ satisfies conditions $1-2$ of Remark 1.5.*
1. If f is a meromorphic function, then

$$\lambda_{[p,q]}(f,\varphi)=\limsup_{r\to+\infty}\frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}=\limsup_{r\to+\infty}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},$$

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\limsup_{r\to+\infty}\frac{\log_p \overline{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}=\limsup_{r\to+\infty}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

2. 1. If f is an entire function, then

$$\rho_{[p,q]}(f,\varphi)=\limsup_{r\to+\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)}=\limsup_{r\to+\infty}\frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f,\varphi)=\liminf_{r\to+\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)}=\liminf_{r\to+\infty}\frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)}.$$

Example 1.9 *Let $\varphi(r)=\log_2 r$, and $f(z)=e^{e^z}$. For $p=4$, and $q=1$ we have*

$$\begin{aligned}\rho_{[4,1]}(f,\varphi) &= \limsup_{r\to+\infty}\frac{\log_{4+1} M(r,f)}{\log \varphi(r)}\\ &= \limsup_{r\to+\infty}\frac{\log_5 e^{e^r}}{\log\log_2 r}\\ &= 1.\end{aligned}$$

6 Results from function theory

6.1 Hadamard factorization theorem

Definition 1.14 *(Canonical product)([7], [17]) Let f be a transcendental meromorphic function such that $z_1, z_2, \dots$ denotes its zeros with $0 < |z_1| \leq |z_2| \leq \dots$. Let p be the smallest integer such that the series $\sum_{n=1}^{+\infty} \frac{1}{|z_n|^{p+1}}$ converges. We call*

$$\begin{aligned} E(u,0) &= (1-u), \\ E(u,p) &= (1-u)\exp\left(u + \frac{u^2}{2} + \dots + \frac{u^p}{p}\right) \quad p = 1,2,\dots \end{aligned}$$

principal factors. the infinite product

$$P(z) = \prod_{n=1}^{+\infty} E\left(\frac{z}{z_n}, p\right)$$

converges uniformly in each bounded domain in $\mathbb{C}$ and $P(z)$ is called the canonical product of f formed by the zeros of f. The integer p is called the genus of the canonical product.

Theorem 1.6 *([7], [17]) Let f be a meromorphic function of a finite order $\rho(f)$ and let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ the zeros and the poles of f in $\mathbb{C} \setminus 0$, respectively. Suppose that f can be represented as*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \dots, (c_k \neq 0),$$

in the neighborhood of $z = 0$. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

such that $Q(z)$ is a polynomial with degree less or equal to $\rho(f)$ and $P_1(z)$ and $P_2(z)$ are the canonical products of f formed by its zeros and poles of f.

7 Some elements of Wiman-Valiron theory

Definition 1.15 ([10], [25], [26]) *Let $f(z) = \sum_{n \geq 0} a_n z^n$ be an entire function. We define the maximal term of f by*

$$\mu(r,f) = \max_{n \geq 0} |a_n| r^n$$

and we define the central index of f by

$$\nu_r(f) = \max\{m : \mu(r,f) = |a_m| r^m\}.$$

Example 1.10 *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$. For sufficiently large r, we have*

$$\mu(r,f) = \max_{n \geq 0} |a_n| r^n = |a_n| r^n$$

Hence

$$\nu_r(f) = n.$$

8 Measure and density of sets

Definition 1.16 ***(Linear measure)*** ([15], [16]) *The linear measure of a set* $G \subset [0, +\infty)$ *is given by*

$$m(G) = \int_G dt.$$

Example 1.11 *Let* $G = [0, \pi] \cup [5, 9]$*, we have*

$$m(G) = \int_G dt = \int_{[0,\pi]\cup[5,9]} dt = 4 + \pi.$$

Definition 1.17 ***(Logarithmic measure)*** ([15], [16]) *The logarithmic measure of a set* $G \subset [1, +\infty)$ *is given by*

$$m_l(G) = \int_G \frac{dt}{t}.$$

Example 1.12 *Let* $G = [1, e^5]$*, we have*

$$m_l(G) = \int_G \frac{dt}{t} = \int_{[1,e^5]} \frac{dt}{t} = 5.$$

Definition 1.18 ***(Upper density)***([15], [16]) *The upper density of a set* $G \subset [0, +\infty)$ *is given by*

$$\overline{dens}(G) = \limsup_{r\to+\infty} \frac{m\,(G \cap [0, r])}{r}.$$

Example 1.13 *Let* $G = [0, +\infty)$*, we have*

$$\begin{aligned} \overline{dens}(G) &= \limsup_{r\to+\infty} \frac{m\,(G \cap [0, r])}{r} \\ &= \limsup_{r\to+\infty} \frac{m([0, r])}{r} \\ &= 1. \end{aligned}$$

Definition 1.19 ***(Upper logarithmic density)*** ([15], [16]) *The upper logarithmic density of a set* $G \subset [1, +\infty)$ *is given by*

$$\overline{\log dens}(G) = \limsup_{r\to+\infty} \frac{m_l\,(G \cap [1, r])}{\log r}.$$

Example 1.14 *Let* $G = [1, +\infty)$*, we have*

$$\begin{aligned} \overline{\log dens}(G) &= \limsup_{r\to+\infty} \frac{m_l\,(G \cap [1, r])}{\log r} \\ &= \limsup_{r\to+\infty} \frac{m_l([1, r])}{\log r} \\ &= 1. \end{aligned}$$

Chapter 2

Auxiliary lemmas

Introduction In this chapter, we will demonstrate some auxiliary lemmas that we will need in the proof of our results.

Proposition 2.1 *([2]) For all* $G \subset [1, +\infty)$ *the following statements hold:*

1. *If* $m_l(G) = +\infty$, *then* $m(G) = +\infty$,
2. *If* $\overline{dens}(G) > 0$, *then* $m(G) = +\infty$,
3. *If* $\overline{\log dens}(G) > 0$, *then* $m_l(G) = +\infty$.

Lemma 2.1 *([8]) Let* f *be a transcendental meromorphic function in the plane, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_1 \subset (1, +\infty)$ *that has a finite logarithmic measure, and a constant* $B > 0$ *depending only on* α *and (i,j) ((i,j) positive integers with* $i > j$*) such that for all z with* $|z| = r \notin [0,1] \cup E_1$, *we have*

$$\left|\frac{f^{(i)}(z)}{f^{(j)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{i-j}.$$

Lemma 2.2 *(Wiman-Valiron, [10],[25]) Let f be a transcendental entire function, and let z be a point with* $|z| = r$ *at which* $|f(z)| = M(r, f)$*. Then the estimation*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) (j \geq 1 \textit{ is an integer})$$

holds for all $|z|$ *outside a set* E_2 *of* r *of finite logarithmic measure, where* $\nu_f(r)$ *is the central index of* f.

Let p, q be positive integers and satisfy $p \geq q \geq 1$.

Lemma 2.3 *([27]) Let* f *be an entire function of* $[p,q] - \varphi$ *order and let* $\nu_f(r)$ *be the central index of* f. *Then*

$$\limsup_{r\to+\infty} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \rho_{[p,q]}(f, \varphi), \quad \liminf_{r\to+\infty} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \mu_{[p,q]}(f, \varphi).$$

Lemma 2.4 *([24]) Let* f *and* g *be non-constant meromorphic functions of* $[p,q] - \varphi$ *order. Then we have*

$$\rho_{[p,q]}(f + g, \varphi) \leq \max\{\rho_{[p,q]}(f, \varphi), \rho_{[p,q]}(g, \varphi)\}$$

and

$$\rho_{[p,q]}(fg,\varphi) \le \max\{\rho_{[p,q]}(f,\varphi), \rho_{[p,q]}(g,\varphi)\}.$$

Furthermore, if $\rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(g,\varphi)$, *then we obtain*

$$\rho_{[p,q]}(f+g,\varphi) = \rho_{[p,q]}(fg,\varphi) = \rho_{[p,q]}(f,\varphi).$$

Lemma 2.5 *([5]) Let* $p \ge q \ge 1$ *be integers, and let* f *and* g *be non-constant meromorphic functions* $\rho_{[p,q]}(f,\varphi)$ *as* $[p,q]-\varphi$ *order and* $\mu_{[p,q]}(g,\varphi)$ *as lower* $[p,q]-\varphi$ *order.Then we have*

$$\mu_{[p,q]}(f+g,\varphi) \le \max\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\}$$

and

$$\mu_{[p,q]}(fg,\varphi) \le \max\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\}.$$

Furthermore, if $\mu_{[p,q]}(g,\varphi) > \rho_{[p,q]}(f,\varphi)$, *then we obtain*

$$\mu_{[p,q]}(f+g,\varphi) = \mu_{[p,q]}(fg,\varphi) = \mu_{[p,q]}(g,\varphi).$$

Lemma 2.6 *([18]) Let* f *be a meromorphic function of* $[p,q]-\varphi$ *order. Then* $\rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f',\varphi)$.

Lemma 2.7 *([17]) Let* $\varphi : [0,+\infty) \to \mathbb{R}$ *and* $\psi : [0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \le \psi(r)$ *for all* $r \notin (E_3 \cup [0,1])$, *where* E_3 *is a set of finite logarithmic measure. Let* $\alpha_2 > 1$ *be a given constant. Then, there exists an* $r_1 = r_1(\alpha_2) > 0$ *such that* $\varphi(r) \le \psi(\alpha_2 r)$ *for all* $r > r_1$.

Lemma 2.8 ([9]) *Let* f *be a meromorphic function and let* $k \in \mathbb{N}$. *Then,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(rT\left(r,f\right)\right)\right)$$

outside a set $E_4 \subset (0,+\infty)$ *with a finite linear measure, and if* f *is of finite order of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log r\right).$$

Lemma 2.9 *([5]) Let* f_1, f_2 *be a meromorphic functions of* $[p,q]-\varphi$ *order satisfying* $\rho_{[p,q]}(f_1,\varphi) > \rho_{[p,q]}(f_2,\varphi)$, *where* φ *only satisfies* $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ *for some* $\alpha_1 > 1$. *Then there exists a set* $E_5 \subset [1,+\infty)$ *having infinite logarithmic measure such that for all* $r \in E_5$ *we have*

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

Lemma 2.10 [23] *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function , where* $g(z)$, $d(z)$ *are entire functions satisfying* $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \le \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) \le +\infty$ *and* $\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu$. *Then there exists a set* $E_6 \subset (1,+\infty)$ *of finite logarithmic measure such that for all* $|z| = r \notin ([0,1] \cup E_6)$ *and* $|g(z)| = M(r,g)$, *we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)), n \in \mathbb{N},$$

where $\nu_g(r)$ *denote the central index of* g.

Proof. We use the mathematical induction in order to obtain the following expression

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1 \dots j_n)} C_{j j_1 \dots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \dots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.1}$$

where $C_{j j_1 \dots j_n}$ are constants and $j + j_1 + 2j_2 + \dots + n j_n = n$. Then

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1 \dots j_n)} C_{j j_1 \dots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \dots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.2}$$

By Lemma 2.2, we can find a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r, g)$, we get

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1))(j = 1, 2, \dots, n), \tag{2.3}$$

where $\nu_g(r)$ is the central index of g. By replacing (2.3) into (2.2) we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n \left[(1 + o(1)) + \sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z}\right)^{j-n} (1 + o(1)) \sum_{(j_1 \dots j_n)} C_{j j_1 \dots j_n} \left(\frac{d'}{d}\right)^{j_1} \times \dots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}\right] \tag{2.4}$$

From the fact that $\rho_{[p,q]}(d, \varphi) = \beta < \mu$, we obtain for all $\epsilon(0 < 2\epsilon < \mu - \beta)$ and sufficiently large r

$$T(r, d) \le \exp_p \left\{ \left(\beta + \frac{\epsilon}{2}\right) \log_q \varphi(r) \right\}.$$

From Lemma 2.1, for some $\alpha_1 (0 < \alpha_1 < \alpha)$ with α be a given constant, there exist a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ and a constant $B > 0$ so that for all z verifying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\begin{aligned} \left|\frac{d^{(m)}(z)}{d(z)}\right| &\le B\,[T(\alpha_1 r, d)]^{m+1} \\ &\le B\left[\exp_p\left\{\left(\beta + \frac{\epsilon}{2}\right)\log_q \varphi(\alpha_1 r)\right\}\right]^{m+1} \\ &= B\left[\exp_p\left\{\left(\beta + \frac{\epsilon}{2}\right)\frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} \log_q \varphi(r)\right\}\right]^{m+1}. \end{aligned} \tag{2.5}$$

By remark 1.5, we obtain

$$\left|\frac{d^{(m)}(z)}{d(z)}\right| \le \exp_p \left\{ (\beta + \epsilon) \log_q \varphi(r) \right\}^m, m = 1, 2, \dots, n. \tag{2.6}$$

By using Lemma 2.3 and $\mu_{[p,q]}(g, \varphi) = \mu_{[p,q]}(f, \varphi) = \mu$, as a result we have

$$\nu_g(r) > \exp_p \left\{ (\mu - \epsilon) \log_q \varphi(r) \right\}$$

for sufficiently large r. Since $j_1 + 2j_2 + \dots + n j_n = n - j$, we get

$$\begin{aligned} \left|\left(\frac{\nu_g(r)}{z}\right)^{j-n} \left(\frac{d'}{d}\right)^{j_1} \times \dots \times \left(\frac{d^{(n)}}{d}\right)^{j_n}\right| &\le \left[\frac{\exp_p\left\{(\mu - \epsilon)\log_q \varphi(r)\right\}}{r}\right]^{j-n} \\ &\times \left[\exp_p\left\{(\beta + \epsilon)\log_q \varphi(r)\right\}\right]^{n-j} \\ &\le \left[\frac{r \exp_p\{(\beta + \epsilon)\log_q \varphi(r)\}}{r \exp_p\{(\mu - \epsilon)\log_q \varphi(r)\}}\right]^{n-j} \to 0, \end{aligned} \tag{2.7}$$

as $r \to +\infty$, where $|z| = r \notin [0,1] \cup E_6$, $E_6 = E_1 \cup E_2$ and $|g(z)| = M(r,g)$. Using (2.4) and (2.7), we obtain our assertion. ∎

Lemma 2.11 [23] *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \le \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) \le +\infty$ and $\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu$. Then there exists a set $E_7 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_7)$ and $|g(z)| = M(r,g)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le r^{2s}, (s \in \mathbb{N}).$$

Proof. From Lemma 2.10 we can find a set E_6 of finite logarithmic measure such that the estimation

$$\frac{f^{(s)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^s (1+o(1)) \quad (s \ge 1 \text{is an integer}) \tag{2.8}$$

is verified for all $|z| = r \notin ([0,1] \cup E_6)$ and $|g(z)| = M(r,g)$, where $\nu_g(r)$ is the central index of g. Then again, from Lemma 2.3, for any given $\varepsilon (0 < \varepsilon < 1)$, we can find $R > 1$ such that for all $r > R$, we have

$$\nu_g(r) > \exp_p\{(\mu - \varepsilon)\log_q \varphi(r)\}. \tag{2.9}$$

If $\mu = +\infty$, then we can replace $\mu - \varepsilon$ by a large enough real number M. Let $E_7 = [1,R] \cup E_6$, then $m_l(E_7) < +\infty$. Finally, by (2.8) and (2.9) we get

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| = \left|\frac{z}{\nu_g(r)}\right|^s \frac{1}{|1+o(1)|} \le \frac{r^s}{\left(\exp_p\{(\mu-\varepsilon)\log \varphi(r)\}\right)^s} \le r^{2s} \tag{2.10}$$

where $|z| = r \notin [0,1] \cup E_7$, $r \to +\infty$ and $|g(z)| = M(r,g)$. ∎

Lemma 2.12 [23] *Let f be an entire function such that $\rho_{[p,q]}(f,\varphi) < +\infty$. Then there exists entire functions $\beta_2(z)$ and $D(z)$ such that*

$$f(z) = \beta_2(z)e^{D(z)}$$

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(\beta_2,\varphi), \rho_{[p,q]}\left(e^{D(z)},\varphi\right)\right\}$$

and

$$\rho_{[p,q]}(\beta_2,\varphi) = \limsup_{r\to+\infty} \frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$|\beta_2(z)| \ge \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(\beta_2,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\} \quad (r \notin E_8),$$

where $E_8 \subset (1,+\infty)$ is a set of r finite linear measure.

Proof. Using Theorem 12.4 in [13] and Theorem 2.2 in [11], we obtain that $f(z)$ can be written as

$$f(z) = \beta_2(z)e^{D(z)},$$

such that

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(\beta_2,\varphi), \rho_{[p,q]}\left(e^{D(z)},\varphi\right)\right\}.$$

On the other hand, by a similar proof in Proposition 6.1 in [12], for any given $\varepsilon > 0$, we obtain

$$|\beta_2(z)| \ge \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(\beta_2,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\} \quad (r \notin E_8),$$

where $E_8 \subset (1,+\infty)$ is a set of r of finite linear measure. ∎

Lemma 2.13 [23] *Suppose that f is a meromorphic function such that $\rho_{[p,q]}(f,\varphi) < +\infty$. then, there exists entire functions $h_1(z)$, $h_2(z)$ and $L(z)$ such that*

$$f(z) = \frac{h_1(z)e^{L(z)}}{h_2(z)} \tag{2.11}$$

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(h_1,\varphi), \rho_{[p,q]}(h_2,\varphi), \rho_{[p,q]}(e^{L(z)},\varphi)\right\}. \tag{2.12}$$

Moreover, for any given $\varepsilon > 0$, we have

$$\begin{aligned} &\exp\left\{-\exp_p\left\{(\rho_{[p,q]}(f,\varphi)+\varepsilon)\log\varphi(r)\right\}\right\} \leq |f(z)| \\ &\leq \exp_{p+1}\left\{(\rho_{[p,q]}(f,\varphi)+\varepsilon)\log\varphi(r)\right\} \quad (r \notin E_9) \end{aligned} \tag{2.13}$$

Where $E_9 \subset (1,+\infty)$ is a set of r of finite linear measure.

Proof. By Hadamard factorization theorem, f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leq \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) < +\infty$$

and

$$\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu.$$

Using Lemma 2.12, we can find entire functions $h(z)$ and $L(z)$ such that

$$g(z) = h(z)e^{L(z)}, \quad \rho_{[p,q]}(g,\varphi) = \max\left\{\rho_{[p,q]}(h,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}.$$

Then there exists functions $h(z)$, $L(z)$ and $d(z)$ such that

$$f(z) = \frac{h(z)e^{L(z)}}{d(z)}$$

and

$$\rho_{[p,q]}(f,\varphi) = \max\left\{\rho_{[p,q]}(h,\varphi), \rho_{[p,q]}(d,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\}.$$

Therefore (2.11) and (2.12) hold. By setting $f(z) = \frac{h_1(z)e^{L(z)}}{h_2(z)}$, where $h_1(z), h_2(z)$ are the canonical products formed with the zeros and poles of f respectively. By using the definition of $[p,q]-\varphi$ order, for sufficiently large r and any given $\varepsilon > 0$, we have

$$\begin{aligned} |h_1(z)| &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(h_1,\varphi)+\frac{\varepsilon}{3}\right)\log_q\varphi(r)\right\} \\ |h_2(z)| &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(h_2,\varphi)+\frac{\varepsilon}{3}\right)\log_q\varphi(r)\right\}. \end{aligned} \tag{2.14}$$

From $\max\left\{\rho_{[p,q]}(h_1,\varphi), \rho_{[p,q]}(h_2,\varphi), \rho_{[p,q]}\left(e^{L(z)},\varphi\right)\right\} = \rho_{[p,q]}(f,\varphi)$ we get

$$|h_1(z)| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q\varphi(r)\right\} \tag{2.15}$$

$$|h_2(z)| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q\varphi(r)\right\}, \tag{2.16}$$

$$\left|e^{L(z)}\right| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q\varphi(r)\right\}. \tag{2.17}$$

Through the use of Lemma 2.12, we can find a set $E_9 \subset (1,+\infty)$ of r with a finite linear measure such that for any given $\varepsilon > 0$ we have that

$$\begin{aligned}|h_1(z)| &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h_1,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}\\ &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}, \quad (r \notin E_9),\end{aligned} \tag{2.18}$$

$$\begin{aligned}|h_2(z)| &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(h_2,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}\\ &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}, \quad (r \notin E_9).\end{aligned} \tag{2.19}$$

By (2.15), (2.17) and (2.19), for sufficiently large $r \notin E_9$ and any given $\varepsilon > 0$, we have

$$\begin{aligned}|f(z)| &= \frac{|h_1(z)||e^{L(z)}|}{|h_2(z)|}\\ &\leq \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}}{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}}\\ &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}.\end{aligned}$$

On the other side, we know that $\rho_{[p-1,q]}(L,\varphi) = \rho_{[p,q]}(e^L,\varphi) \leq \rho_{[p,q]}(f,\varphi)$, and $e^{|L(z)|} \geq e^{-|L(z)|}$. From the definition of $[p,q]-\varphi$ order, we get

$$\begin{aligned}|L(z)| &\leq M(r,L)\\ &\leq \exp_p\left\{\left(\rho_{[p-1,q]}(L,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\\ &\leq \exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}.\end{aligned}$$

Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$|e^{L(z)}| \geq e^{-|L(z)|} \geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}. \tag{2.20}$$

By making use of (2.16), (2.18) and (2.20), we can get

$$\begin{aligned}|f(z)| &= \frac{|h_1(z)||e^{L(z)}|}{|h_2(z)|}\\ &\geq \frac{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}}{\exp_{p+1}\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}}\\ &= \exp\left\{-3\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\frac{\varepsilon}{3}\right)\log_q \varphi(r)\right\}\right\}\\ &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\}.\end{aligned}$$

Finally Lemma 2.13 is proved. ■

Lemma 2.14 [23] *Let* $G \subset (1,+\infty)$ *be a set with a positive upper logarithmic density, and let* $A_j(z)(j=0,1,\ldots,k)$ *with* $A_k(z) \neq 0$ *and* $F(z)$ $(F(z)=0$ *or* $F(z)\neq 0)$ *be meromorphic functions with finite* $[p,q]-\varphi$ *order and* f *is a solution of equation*

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z). \tag{2.21}$$

If there exist a positive constant $\sigma > 0$ *and an integer* s, $0 \le s \le k$, *such that for sufficiently small* $\varepsilon > 0$, *we have*

$$|A_s(z)| \ge \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\}$$

as $|z| = r \in G, r \to \infty$ *and* $\max\{\rho_{[p,q]}(A_j,\varphi)(j \ne s), \rho_{[p,q]}(F,\varphi)\} < \sigma$, *then we have*

$$\rho_{[p,q]}(A_s,\varphi) = \delta \ge \sigma.$$

Proof. By using the proof by contradiction, we assume that $\rho_{[p,q]}(A_s,\varphi) = \delta < \sigma$. From the hypotheses of Lemma 2.14, we can find a positive constant $\sigma > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$|A_s(z)| \ge \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\} \tag{2.22}$$

for $|z| = r \in G, r \to +\infty$, where $G \subset (1,+\infty)$ is a set that has a positive upper logarithmic density (by Proposition 2.1, we have $m_l(G) = +\infty$). By Lemma 2.13, there exists a set $E_9 \subset (1,+\infty)$ with finite linear measure such that for $|z| = r \notin E_9$, we have for any given $\varepsilon(0 < 2\varepsilon < \sigma - \delta)$

$$|A_s(z)| \le \exp_{p+1}\left\{(\delta + \varepsilon)\log_q \varphi(r)\right\}. \tag{2.23}$$

Using (2.22) and (2.24), we get as $|z| = r \in G \setminus E_9, r \to +\infty$

$$\exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\} \le |A_s(z)| \le \exp_{p+1}\left\{(\delta + \varepsilon)\log_q \varphi(r)\right\}.$$

Hence

$$\sigma - \varepsilon \le \delta + \varepsilon$$

which is a contradiction with the fact that $0 < 2\varepsilon < \sigma - \delta$. Then $\rho_{[p,q]}(A_s,\varphi) = \delta \ge \sigma$. ■

Lemma 2.15 [23] *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$, $d(z)$ *are entire functions. If* $0 \le \rho_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi)$, *then* $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi)$ *and* $\rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi)$. *Moreover, if* $\rho_{[p,q]}(f,\varphi) = +\infty$, *then* $\rho_{[p+1,q]}(g,\varphi) = \rho_{[p+1,q]}(f,\varphi)$.

Proof. Case 1. $\rho_{[p,q]}(f,\varphi) < +\infty$. Using the definition of the $[p,q]-\varphi$ order, we can find an increasing sequence $\{r_n\}, (r_n \to +\infty)$ and a positive integer n_0 such that for all $n > n_0$ and for any given $\varepsilon \in \left(0, \frac{\rho_{[p,q]}(f,\varphi) - \rho_{[p,q]}(d,\varphi)}{2}\right)$ (because $0 \le \rho_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi) \le \rho_{[p,q]}(f,\varphi)$), we obtain

$$T(r_n,f) \ge \exp_p\left\{(\rho_{[p,q]}(f,\varphi) - \varepsilon)\log_q \varphi(r_n)\right\}, \tag{2.24}$$

and

$$T(r_n,d) \le \exp_p\left\{(\rho_{[p,q]}(d,\varphi) + \varepsilon)\log_q \varphi(r_n)\right\}. \tag{2.25}$$

Using the properties of the characteristic function we get

$$T(r,f) \le T(r,g) + T(r,d) + O(1). \tag{2.26}$$

By substituting (2.24), (2.25) in (2.26), for all sufficiently large n, we obtain

$$\begin{aligned} \exp_p\left\{(\rho_{[p,q]}(f,\varphi) - \varepsilon)\log_q \varphi(r_n)\right\} &\le T(r_n,g) \\ + \exp_p\left\{(\rho_{[p,q]}(d,\varphi) + \varepsilon)\log_q \varphi(r_n)\right\} &+ O(1). \end{aligned} \tag{2.27}$$

Using the fact that $\varepsilon \in \left(0, \frac{\rho_{[p,q]}(f,\varphi) - \rho_{[p,q]}(d,\varphi)}{2}\right)$, then (2.27) will be

$$(1 - o(1))\exp_p\left\{(\rho_{[p,q]}(f,\varphi) - \varepsilon)\log_q \varphi(r_n)\right\} \le T(r_n,g) + O(1),$$

for all sufficiently large n. Then

$$\rho_{[p,q]}(f,\varphi) \leq \rho_{[p,q]}(g,\varphi). \tag{2.28}$$

From the other side, we have

$$T(r,g) \leq T(r,f) + T(r,d),$$

and from

$$\rho_{[p,q]}(d,\varphi) < \rho_{[p,q]}(f,\varphi)$$

we get

$$\rho_{[p,q]}(g,\varphi) \leq \rho_{[p,q]}(f,\varphi). \tag{2.29}$$

By (2.28) and (2.29), it results then that

$$\rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi).$$

Similarly, we will prove that $\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi)$.

Using the definition of the lower $[p,q]-\varphi$ order, we can find an increasing sequence $\{r_n\}, (r_n \to +\infty)$ and a positive integer n_1 such that for all $n > n_1$ and for any given $\varepsilon \in \left(0, \frac{\mu_{[p,q]}(f,\varphi)-\mu_{[p,q]}(d,\varphi)}{2}\right)$ (because $0 \leq \mu_{[p,q]}(d,\varphi) < \rho_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi)$), we obtain

$$T(r_n,f) \geq \exp_p\left\{(\mu_{[p,q]}(f,\varphi)-\varepsilon)\log_q \varphi(r_n)\right\}, \tag{2.30}$$

and

$$T(r_n,d) \leq \exp_p\left\{(\mu_{[p,q]}(d,\varphi)+\varepsilon)\log_q \varphi(r_n)\right\}. \tag{2.31}$$

Using the properties of the characteristic function we get

$$T(r,f) \leq T(r,g) + T(r,d) + O(1), \tag{2.32}$$

by substituting (2.30), (2.31) in (2.32) we obtain

$$\begin{aligned} \exp_p\left\{(\mu_{[p,q]}(f,\varphi)-\varepsilon)\log_q \varphi(r_n)\right\} &\leq T(r_n,g) \\ +\exp_p\left\{(\mu_{[p,q]}(d,\varphi)+\varepsilon)\log_q \varphi(r_n)\right\} &+ O(1). \end{aligned} \tag{2.33}$$

Using the fact that $\varepsilon \in \left(0, \frac{\mu_{[p,q]}(f,\varphi)-\mu_{[p,q]}(d,\varphi)}{2}\right)$, (2.33) becomes

$$(1-o(1))\exp_p\left\{(\mu_{[p,q]}(f,\varphi)-\varepsilon)\log_q \varphi(r_n)\right\} \leq T(r_n,g) + O(1),$$

for all sufficiently large n. Hence

$$\mu_{[p,q]}(f,\varphi) \leq \mu_{[p,q]}(g,\varphi). \tag{2.34}$$

From the other side, we have

$$T(r,g) \leq T(r,f) + T(r,d),$$

and from $\mu_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi)$, we get

$$\mu_{[p,q]}(g,\varphi) \leq \mu_{[p,q]}(f,\varphi). \tag{2.35}$$

From (2.34) and (2.35), it results then that

$$\mu_{[p,q]}(g,\varphi)=\mu_{[p,q]}(f,\varphi).$$

Case 2. $\rho_{[p,q]}(f,\varphi)=+\infty$. Through the absurd. We suppose that $\rho_{[p,q]}(g,\varphi)\neq\rho_{[p,q]}(f,\varphi)$. Note first that by Lemma 2.4, the inequality $\rho_{[p,q]}(g,\varphi)>\rho_{[p,q]}(f,\varphi)$ is impossible. Assuming that $\rho_{[p,q]}(g,\varphi)<\rho_{[p,q]}(f,\varphi)$. Using the definition of the $[p,q]-\varphi$ order, there exist an increasing sequence r_n, $(r_n\to+\infty)$ and a positive integer n_0 such that for all $n>n_0$ and for any given $\varepsilon>0$

$$\begin{aligned} T(r_n,g) &\leq \exp_p\left\{\left(\rho_{[p,q]}(g,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\},\\ T(r_n,d) &\leq \exp_p\left\{\left(\rho_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\}.\end{aligned}$$

From the fact that

$$T(r_n,f)\leq T(r_n,g)+T(r_n,d)+O(1)$$

we obtain, for all sufficiently large n,

$$\rho_{[p,q]}(f,\varphi)\leq\max\left\{\rho_{[p,q]}(g,\varphi),\rho_{[p,q]}(d,\varphi)\right\},$$

and this contradicts what we had assumed.

Similarly, we prove $\mu_{[p,q]}(g,\varphi)=\mu_{[p,q]}(f,\varphi)$. Note first that the inequality $\mu_{[p,q]}(g,\varphi)>\mu_{[p,q]}(f,\varphi)$ is impossible because by Lemma 2.5 we have

$$\mu_{[p,q]}(g,\varphi)=\mu_{[p,q]}(fd,\varphi)\leq\max\left\{\rho_{[p,q]}(f,\varphi),\mu_{[p,q]}(d,\varphi)\right\},$$

and the fact that $\mu_{[p,q]}(d,\varphi)\leq\rho_{[p,q]}(d,\varphi)<\mu_{[p,q]}(f,\varphi)$ gives $\mu_{[p,q]}(g,\varphi)\leq\mu_{[p,q]}(f,\varphi)$. Assuming that $\mu_{[p,q]}(g,\varphi)<\mu_{[p,q]}(f,\varphi)$. Using the definition of the lower $[p,q]-\varphi$ order there exists an increasing sequence r_n, $(r_n\to+\infty)$ and a positive integer n_0' such that for all $n>n_0'$ and for any given $\varepsilon>0$

$$\begin{aligned} T(r_n,g) &\leq \exp_p\left\{\left(\mu_{[p,q]}(g,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\}\\ T(r_n,d) &\leq \exp_p\left\{\left(\mu_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\}.\end{aligned}$$

From the fact that $T(r_n,f)\leq T(r_n,g)+T(r_n,d)+O(1)$ we obtain, for all sufficiently large n,

$$T(r_n,f)\leq\exp_p\left\{\left(\mu_{[p,q]}(g,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\}+\exp_p\left\{\left(\mu_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q\varphi(r_n)\right\}+O(1),$$

then $\mu_{[p,q]}(f,\varphi)\leq\max\left\{\mu_{[p,q]}(g,\varphi),\mu_{[p,q]}(d,\varphi)\right\}$. This is a contradiction with our assumption.

Case 3. $\mu_{[p,q]}(f,\varphi)<+\infty$ and $\rho_{[p,q]}(f,\varphi)=+\infty$. We can prove case 3 by using the similar

method we used to prove Cases 1 and 2.

At last, we will prove $\rho_{[p+1,q]}(g,\varphi)=\rho_{[p+1,q]}(f,\varphi)$. We assume that $\rho_{[p,q]}(f,\varphi)=+\infty$. There exists an increasing sequence $\{r_n\}$, $(r_n\to+\infty)$, such that we have

$$\rho_{[p+1,q]}(f,\varphi)=\lim_{n\to+\infty}\frac{\log_{p+1}T(r_n,f)}{\log_q\varphi(r_n)}.$$

Using $\rho_{[p,q]}(d,\varphi) < \mu_{[p,q]}(f,\varphi)$ and the definitions of the $[p,q]-\varphi$ order and the lower $[p,q]-\varphi$ order, we obtain

$$\lim_{n\to+\infty}\frac{T(r_n,d)}{T(r_n,f)}=0,$$

hence

$$T(r_n,d)\le\frac{1}{2}T(r_n,f).$$

Therefore, there exists a positive integer N, such that $n>N$

$$T(r_n,f)\le 2T(r_n,g)+O(1).$$

It results then that $\rho_{[p+1,q]}(f,\varphi)\le\rho_{[p+1,q]}(g,\varphi)$. By using the same arguments as in the proof of **Case 1**, from $T(r,g)\le T(r,f)+T(r,d)$, we can find a positive integer N, such that for all $n>N$

$$T(r_n,g)\le 2T(r_n,f).$$

Then, $\rho_{[p+1,q]}(g,\varphi)\le\rho_{[p+1,q]}(f,\varphi)$. Thus $\rho_{[p+1,q]}(f,\varphi)=\rho_{[p+1,q]}(g,\varphi)$. ■

Lemma 2.16 [23] *Let $A_j(z)(j=0,1,\cdots,k)$, $A_k(z)(\not\equiv 0)$, $F(z)(\not\equiv 0)$ be meromorphic functions and let f be a meromorphic solution of (2.21) of infinite $[p,q]-\varphi$ order satisfying the following condition*

$$b=\max\{\rho_{[p+1,q]}(F,\varphi),\rho_{[p+1,q]}(A_j,\varphi)(j=0,1,\cdots,k)\}<\rho_{[p+1,q]}(f,\varphi),$$

then

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)=\lambda_{[p+1,q]}(f,\varphi)=\rho_{[p+1,q]}(f,\varphi).$$

Proof. Assuming that f is a meromorphic solution of (2.21) that has infinite $[p,q]-\varphi$ order. The first thing to notice is that by definition we have

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)\le\lambda_{[p+1,q]}(f,\varphi)\le\rho_{[p+1,q]}(f,\varphi),$$

we need then to demonstrate that

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)\ge\lambda_{[p+1,q]}(f,\varphi)\ge\rho_{[p+1,q]}(f,\varphi).$$

We can rewrite (2.21) as

$$\frac{1}{f}=\frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z)\right). \tag{2.36}$$

From Lemma 2.8 and (2.36) we get for $|z|=r$ outside a set $E_4\subset(0,+\infty[$ of finite linear measure

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\le m\left(r,\frac{1}{F}\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k}m\left(r,A_j\right)+O(1)\\ &\le m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}m(r,A_j)+O(\log rT(r,f)). \end{aligned} \tag{2.37}$$

By noticing from (2.21) that if f has a zero at z_0 of order $\alpha(\alpha>k)$, and $A_0,A_1,\cdots,A_k$ are all analytic at z_0, then F must have a zero at z_0 of order at least $\alpha-k$, we obtain

$$n\left(r,\frac{1}{f}\right)\le k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}n\left(r,A_j\right),$$

and

$$N\left(r,\frac{1}{f}\right)\leq k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}N\left(r,A_j\right). \tag{2.38}$$

Combining (2.37) and (2.38) we get for sufficiently large $r\notin E_4$

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right)+O(1) \\ &\leq T(r,F)+\sum_{j=0}^{k}T\left(r,A_j\right)+k\overline{N}\left(r,\frac{1}{f}\right)+O(\log rT(r,f)), \end{aligned} \tag{2.39}$$

For sufficiently large r, we have

$$O(\log rT(r,f))\leq\frac{1}{2}T(r,f). \tag{2.40}$$

From the definition of the $[p,q]-\varphi$ order, for sufficiently large r and for any given $\varepsilon(0<2\varepsilon<\rho_{[p+1,q]}(f,\varphi)-b)$, we have

$$T(r,F)\leq\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\}, \tag{2.41}$$

$$T(r,A_j)\leq\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\}, j=0,1,\cdots,k. \tag{2.42}$$

By substituting (2.40), (2.41) and (2.42) in (2.39), for all sufficiently large $r\notin E_4$, we obtain

$$T(r,f)\leq 2k\overline{N}\left(r,\frac{1}{f}\right)+(k+2)\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(r)\right\}. \tag{2.43}$$

Using Lemma 2.7, from (2.43), for any given $\nu>1$ there exists a $r_1=r_1(\nu)$ and sufficiently large $r>r_1$, we get

$$T(r,f)\leq 2k\overline{N}\left(\nu r,\frac{1}{f}\right)+(k+2)\exp_{p+1}\left\{(b+\varepsilon)\log_q\varphi(\nu r)\right\}$$

which gives

$$\rho_{[p+1,q]}(f,\varphi)\leq\overline{\lambda}_{[p+1,q]}(f,\varphi),$$

therefore

$$\rho_{[p+1,q]}(f,\varphi)\leq\overline{\lambda}_{[p+1,q]}(f,\varphi)\leq\lambda_{[p+1,q]}(f,\varphi).$$

And from $\overline{\lambda}_{[p+1,q]}(f,\varphi)\leq\lambda_{[p+1,q]}(f,\varphi)\leq\rho_{[p+1,q]}(f,\varphi)$, it results that

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)=\lambda_{[p+1,q]}(f,\varphi)=\rho_{[p+1,q]}(f,\varphi).$$

■

Lemma 2.17 [23] *Let* $G\subset(1,+\infty)$ *be a set with a positive upper logarithmic density (or infinite logarithmic measure), and let* $A_j(z)(j=0,1,\cdots,k)$ *with* $A_k(z)(\not\equiv 0)$ *and* $F(z)(\not\equiv 0)$ *be meromorphic functions with finite* $[p,q]-\varphi$ *order. If there exist a positive constant* $\sigma>0$ *and an integer s,* $0\leq s\leq k$*, such that for sufficiently large* $\varepsilon>0$*, we have* $|A_s(z)|\geq\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\}$ *as* $|z|=r\in G$*,* $r\to+\infty$ *and*

$$\max\left\{\rho_{[p,q]}(A_j,\varphi)(j\neq s),\rho_{[p,q]}(F,\varphi)\right\}<\sigma,$$

then every transcendental meromorphic solution f of equation (2.21) *satisfies* $\rho_{[p,q]}(f,\varphi)\geq\sigma$.

Proof. Through the absurd, suppose that f is a transcendental meromorphic solution of equation (2.21) such that $\rho_{[p,q]}(f,\varphi)<\sigma$. By (2.21), we get

$$A_s=\frac{F}{f^{(s)}}-\sum_{\substack{j=0\\ j\neq s}}^{k}A_j\frac{f^{(j)}}{f^{(s)}}. \tag{2.44}$$

From the hypothesis of Lemma 2.17, we have $\max\left\{\rho_{[p,q]}(A_j,\varphi)(j\neq s),\rho_{[p,q]}(F,\varphi)\right\}<\sigma$ and our assumption $\rho_{[p,q]}(f,\varphi)<\sigma$, then we get from (2.44) by using Lemma 2.6

$$\begin{aligned}\rho_2 &= \rho_{[p,q]}(A_s,\varphi)\\ &\leq \max\left\{\rho_{[p,q]}(A_j,\varphi)(j\neq s),\rho_{[p,q]}(F,\varphi),\rho_{[p,q]}(f,\varphi)\right\}<\sigma.\end{aligned}$$

Using Lemma 2.13 for any given $\varepsilon(0<2\varepsilon<\sigma-\rho_2)$, we can find a set $E_9\subset(1,+\infty)$ that has a finite linear measure such that

$$|A_s(z)|\leq \exp_{p+1}\left\{(\rho_{[p,q]}(A_s,\varphi)+\varepsilon)\log_q\varphi(r)\right\}=\exp_{p+1}\left\{(\rho_2+\varepsilon)\log_q\varphi(r)\right\}, \tag{2.45}$$

holds for all z satisfying $|z|=r\notin E_9$. By the hypotheses of Lemma 2.17, there exists a set G satisfying $\overline{\log dens}G>0$ (or $m_l(G)=+\infty$) such that

$$|A_s(z)|\geq \exp_{p+1}\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\} \tag{2.46}$$

holds for all z satisfying $|z|=r\in G$, $r\to+\infty$. Combining (2.45) and (2.46) it results that for all z verifying $|z|=r\in G\setminus E_9$, $r\to+\infty$

$$\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q\varphi(r)\right\}\leq|A_s(z)|\leq\exp_{p+1}\left\{(\rho_2+\varepsilon)\log_q\varphi(r)\right\},$$

hence

$$\sigma-\varepsilon\leq\rho_2+\varepsilon.$$

This contradicts the fact that $0<2\varepsilon<\sigma-\rho_2$. Consequently, any transcendental meromorphic solution f of equation (2.21) satisfies $\rho_{[p,q]}(f,\varphi)\geq\sigma$. ∎

Lemma 2.18 [23] *Let* $A_0,A_1,\cdots,A_k\not\equiv 0$, $F\not\equiv 0$ *be finite* $[p,q]-\varphi$ *order meromorphic functions. If f is a meromorphic solution of the equation* (2.21) *with* $\rho_{[p,q]}(f,\varphi)=+\infty$ *and* $\rho_{[p+1,q]}(f,\varphi)=\rho<+\infty$, *then*

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\rho_{[p,q]}(f,\varphi)=+\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)=\lambda_{[p+1,q]}(f,\varphi)=\rho_{[p+1,q]}(f,\varphi)=\rho.$$

Proof. Assuming that f is a meromorphic solution of (2.21) that has infinite $[p,q]-\varphi$ order and $\rho_{[p+1,q]}(f,\varphi)=\rho<+\infty$. The equation (2.21) can be rewritten as

$$\frac{1}{f}=\frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0(z)\right). \tag{2.47}$$

From Lemma 2.8 and (2.47) we get for $|z|=r$ outside a set E_4 of finite linear measure and any $\varepsilon>0$

$$\begin{aligned}m\left(r,\frac{1}{f}\right) &\leq m\left(r,\frac{1}{F}\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k}m\left(r,A_j\right)+O(1)\\ &\leq m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k}m(r,A_j)+O(\log rT(r,f)).\end{aligned} \tag{2.48}$$

On the other, from (2.21), if f has a zero at z_0 of order $\alpha(\alpha > k)$, and $A_0, A_1, \cdots, A_k$ are all analytic at z_0, then F must have a zero at z_0 of order at least $\alpha - k$. Then

$$n\left(r,\frac{1}{f}\right) \le k\overline{n}\left(r,\frac{1}{f}\right) + n\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} n\left(r, A_j\right)$$

and

$$N\left(r,\frac{1}{f}\right) \le k\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} N\left(r, A_j\right). \quad (2.49)$$

By (2.48) and (2.49) we get for sufficiently large $r \notin E_4$ and any given $\varepsilon > 0$

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right) + O(1) \\ &\le T(r,F) + \sum_{j=0}^{k} T\left(r, A_j\right) + k\overline{N}\left(r,\frac{1}{f}\right) + O(\log r T(r,f)). \end{aligned} \quad (2.50)$$

From the hypothesis of Lemma 2.18, we have

$$\rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(F,\varphi) \quad \text{and} \quad \rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(A_j,\varphi), \quad j = 0,1,\ldots,k.$$

Using Lemma 2.9, we can find a set $E_5 \subset [1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_5$ we have

$$\max\left\{\frac{T(r,F)}{T(r,f)}, \frac{T(r,A_j)}{T(r,f)}, j = 0,1,\ldots,k\right\} \to 0, r \to +\infty$$

hence

$$T(r,F) = o(T(r,f)), \quad T(r,A_j) = o(T(r,f)) \quad j = 0,1,\ldots,k. \quad (2.51)$$

Since

$$\frac{\log T(r,f)}{T(r,f)} \to 0 \quad \text{for} \quad r \to +\infty,$$

we have for sufficiently large r

$$\log(T(r,f)) = o(T(r,f)). \quad (2.52)$$

Substituting (2.51), (2.52) in (2.50) we get for $r \in E_5 \setminus E_4$

$$T(r,f) \le k\overline{N}\left(r,\frac{1}{f}\right) + o(T(r,f)) + O\left(\log r\right).$$

Hence

$$(1 - o(1))T(r,f) \le k\overline{N}\left(r,\frac{1}{f}\right) + O\left(\log r\right) \quad (2.53)$$

Therefore, by making use of Proposition 1.5, Lemma 2.7, Definition 1.12, Remark 1.5 and (2.53) we get for any f with $\rho_{[p,q]}(f,\varphi) = +\infty$ and $\rho_{[p+1,q]}(f,\varphi) = \rho$

$$+\infty = \rho_{[p,q]}(f,\varphi) \le \overline{\lambda}_{[p,q]}(f,\varphi), \quad \rho_{[p+1,q]}(f,\varphi) \le \overline{\lambda}_{[p+1,q]}(f,\varphi)$$

Hence

$$\rho_{[p+1,q]}(f,\varphi) \le \overline{\lambda}_{[p+1,q]}(f,\varphi) \le \lambda_{[p+1,q]}(f,\varphi).$$

On the other hand, we know that by definition we have

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) \le \lambda_{[p+1,q]}(f,\varphi) \le \rho_{[p+1,q]}(f,\varphi),$$

it results then

$$\overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) = \rho.$$

■

Lemma 2.19 [23] *Assume that* $k \geq 2$ *and* $A_0, A_1, \cdots, A_k (\not\equiv 0), F$ *are meromorphic functions. Let* $\rho_1 = \max\{\rho_{[p,q]}(A_j, \varphi)(j=0,1,\cdots,k), \rho_{[p,q]}(F,\varphi)\} < \infty$ *and let* f *be a meromorphic solution of infinite* $[p,q]-\varphi$ *order of equation* (2.21) *with* $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. *Then,* $\rho_{[p+1,q]}(f,\varphi) \leq \rho_1$.

Proof. We suppose that f is a meromorphic solution of (2.21) with infinite $[p,q]-\varphi$ order and $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. By using the Hadamard factorization theorem, f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions such that

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \leq \rho_{[p,q]}(f,\varphi) = \rho_{[p,q]}(g,\varphi) \leq +\infty,$$

and

$$\lambda_{[p,q]}(d,\varphi) = \rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu.$$

By making use of Lemma 2.13, we can find a set $E_9 \subset (1,+\infty)$ of r of finite linear measure such that for any $\varepsilon(0 < 2\varepsilon < \mu_{[p,q]}(f,\varphi) - \rho_{[p,q]}(d,\varphi))$ and all $|z| = r \notin E_9$, and by using the hypothesis, we get

$$\begin{aligned} |A_j(z)| &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(A_j,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\} \\ &\leq \exp_{p+1}\left\{\left(\rho_1+\varepsilon\right)\log_q \varphi(r)\right\}, \quad j=0,1,\cdots,k-1, \end{aligned} \tag{2.54}$$

$$\begin{aligned} |A_k(z)| &\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(A_k,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\right\} \\ &\geq \exp\left\{-\exp_p\left\{\left(\rho_1+\varepsilon\right)\log_q \varphi(r)\right\}\right\}, \end{aligned} \tag{2.55}$$

and

$$|F(z)| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(F,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\} \leq \exp_{p+1}\left\{\left(\rho_1+\varepsilon\right)\log_q \varphi(r)\right\}. \tag{2.56}$$

From the definition of the $[p,q]-\varphi$ order, the lower $[p,q]-\varphi$ order and (2.56) for all z satisfying $|z| = r \notin E_9$ with $|g(z)| = M(r,g)$ and any $\varepsilon(0 < 2\varepsilon < \mu_{[p,q]}(f,\varphi) - \rho_{[p,q]}(d,\varphi))$, we obtain

$$\begin{aligned} \left|\frac{F(z)}{f(z)}\right| &= \frac{|F(z)|}{|g(z)|}|d(z)| \\ &\leq \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(d,\varphi)+\varepsilon\right)\log_q \varphi(r)\right\}\exp_{p+1}\left\{\left(\rho_1+\varepsilon\right)\log_q \varphi(r)\right\}}{\exp_{p+1}\left\{\left(\mu_{[p,q]}(f,\varphi)-\varepsilon\right)\log_q \varphi(r)\right\}} \qquad (2.57) \\ &\leq \exp_{p+1}\left\{\left(\rho_1+\varepsilon\right)\log_q \varphi(r)\right\}. \qquad (2.58) \end{aligned}$$

From Lemma 2.10, we can find a set $E_6 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_6)$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)), j=0,1,\cdots,k. \tag{2.59}$$

By (2.21) we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{|A_k(z)|}\left(|A_0(z)| + \left|\frac{F(z)}{f(z)}\right| + \sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right|\right). \tag{2.60}$$

Replacing (2.54), (2.55), (2.57) and (2.59) in (2.60) we obtain

$$\left|\frac{\nu_g(r)}{z}\right|^k |1+o(1)| \leq \frac{1}{\exp\left\{-\exp_p\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}\right\}} \times$$

$$\left(\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} + \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}\left\{1+\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)|\right\}\right)$$

$$\leq \quad (k+1)\frac{\left|\nu_g(r)\right|^{k-1}}{r^{k-1}}|1+o(1)|\exp\left\{2\exp_p\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}\right\}.$$

Therefore

$$\left|\nu_g(r)\right| |1+o(1)| \leq (k+1) r |1+o(1)| \exp\left\{2\exp_p\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\}\right\} \tag{2.61}$$

holds for all z verifying $|z| = r \notin ([0,1] \cup E_6 \cup E_9)$ and $|g(z)| = M(r,g)$, $r \to +\infty$. From (2.61) we obtain

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_g(r)}{\log_q \varphi(r)} \leq \rho_1 + \varepsilon. \tag{2.62}$$

Using the fact that $\varepsilon > 0$ is arbitrary, by Lemma 2.3 and Lemma 2.7 we obtain from (2.62)

$$\rho_{[p+1,q]}(g,\varphi) \leq \rho_1.$$

Since $\rho_{[p,q]}(d,\varphi) \leq \mu_{[p,q]}(f,\varphi)$, then by using Lemma 2.15 we get

$$\rho_{[p+1,q]}(g,\varphi) = \rho_{[p+1,q]}(f,\varphi).$$

Hence

$$\rho_{[p+1,q]}(f,\varphi) \leq \rho_1.$$

■

Chapter 3

On the growth of solutions of LDE with meromorphic coefficients with finite $[p,q]-\varphi$ order

1 Introduction

In this chapter, we investigate the growth of meromorphic solutions to higher order homogeneous and non-homogeneous linear differential equations with meromorphic coefficients of finite $[p,q]-\varphi$ order. We obtain some results about $[p,q]-\varphi$ order and the $[p,q]-\varphi$ convergence exponent of solutions for such equations. In [20], Liu, Tu and Zhang studied the growth and zeros of solutions of equations

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f'+A_0f=0 \tag{3.1}$$

et

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_1f'+A_0f=F(z), \tag{3.2}$$

where $A_0(z)(\not\equiv 0), A_0(z),...,A_{k-1}(z)$ and $F(z)(\not\equiv 0)$ are entire functions of $[p,q]-\varphi$ order and they obtained the following results.

Theorem 3.1 [20] *Let $A_j(z)(j=0,1,...,k-1)$ be entire functions satisfying*

$$\max\{\rho_{[p,q]}(A_j,\varphi), j=1,...,k-1\}<\rho_{[p,q]}(A_0,\varphi)<\infty.$$

Then every solution $f\not\equiv 0$ of equation (3.1) satisfies $\rho_{[p,q]}(f,\varphi)=\rho_{[p,q]}(A_0,\varphi)$.

And they obtained the following results in the case of non-homogeneous equations.

Theorem 3.2 [20] *Let $A_j(z)(j=0,1,...,k-1)$ and $F(z)(\not\equiv 0)$ be entire functions and let f be a solution of equation (3.2) satisfying*

$$\max\{\rho_{[p,q]}(A_j,\varphi), \rho_{[p,q]}(F,\varphi)\, j=1,...,k-1\}<\rho_{[p,q]}(A_0,\varphi)<\infty.$$

Then $\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\rho_{[p,q]}(f,\varphi)$.

Theorem 3.3 [20] *Let $A_j(z)(j=0,1,...,k-1)$ and $F(z)(\not\equiv 0)$ be entire functions satisfying*

$$\max\{\rho_{[p,q]}(A_j,\varphi), \rho_{[p+1,q]}(F,\varphi)\, j=1,...,k-1\}<\rho_{[p,q]}(A_0,\varphi)<\infty.$$

Then every solution $f\not\equiv 0$ of equation (3.2) satisfies $\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\rho_{[p,q]}(f,\varphi)=\rho_{[p,q]}(A_0,\varphi)$.

After this, Saidani and Belaïdi studied some of the properties of the solution of the higher order linear differential equation

$$A_k f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = 0 \tag{3.3}$$

et

$$A_k f^{(k)} + A_{k-1} f^{(k-1)} + \ldots + A_1 f' + A_0 f = F(z), \tag{3.4}$$

and they obtained the following results.

Theorem 3.4 [22] *Let* $H \subset (1, +\infty)$ *be a set with a positive upper logarithmic density (or* $m_l(H) = +\infty$*) and let* $A_j(z)$ $\left(j = 0, 1, \cdots, k\right)$ *with* $A_k(z)(\not\equiv 0)$ *be meromorphic functions with finite*$[p,q]$*-order. If there exist a positive constant* $\sigma > 0$ *and an integer* s, $0 \le s \le k$, *such that for all sufficiently small* $\epsilon > 0$, *we have* $|A_s(z)| \ge \exp_{p+1}\left\{(\sigma - \epsilon)\log_q r\right\}$ *as* $|z| = r \in H$, $r \to +\infty$ *and* $\rho = \max\left\{\rho_{[p,q]}\left(A_j\right)\ (j \ne s)\right\} < \sigma$, *then every non-transcendental meromorphic solution* $f \not\equiv 0$ *of* (3.3) *is a polynomial with* $\deg f \le s-1$ *and every transcendental meromorphic solution* f *of* (3.3) *with* $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}\left(f\right)$ *satisfies*

$$\rho_{[p,q]}\left(f\right) = \mu_{[p,q]}\left(f\right) = +\infty,$$

$$\sigma \le \rho_{[p+1,q]}\left(f\right) \le \rho_{[p,q]}\left(A_s\right).$$

Theorem 3.5 [22] *Let* $H \subset (1, +\infty)$ *be a set with a positive upper logarithmic density (or* $m_l(H) = +\infty$*) and let* $A_j(z)$ $\left(j = 0, 1, \cdots, k\right)$ *with* $A_k(z)(\not\equiv 0)$ *and* $F(z)(\not\equiv 0)$ *be meromorphic functions with finite*$[p,q]$*-order. If there exist a positive constant* $\sigma > 0$ *and an integer* s, $0 \le s \le k$, *such that for all sufficiently small* $\epsilon > 0$, *we have* $|A_s(z)| \ge \exp_{p+1}\left\{(\sigma - \epsilon)\log_q r\right\}$ *as* $|z| = r \in H$, $r \to +\infty$ *and* $\rho = \max\left\{\rho_{[p,q]}\left(A_j\right), \rho_{[p,q]}(F)\ (j \ne s)\right\} < \sigma$, *then every non-transcendental meromorphic solution* $f \not\equiv 0$ *of* (3.4) *is a polynomial with* $\deg f \le s-1$ *and every transcendental meromorphic solution* f *of* (3.4) *with* $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \min\left\{\sigma, \mu_{[p,q]}\left(f\right)\right\}$ *satisfies*

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty$$

and

$$\sigma \le \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}\left(f\right) \le \rho_{[p,q]}\left(A_s\right).$$

So we ask: What about the growth of meromorphic solutions of equations (3.3) and (3.4) with meromorphic coefficients of finite $[p,q]-\varphi$ order when the dominant fixed coefficient is the arbitrary coefficient A_s?

2 Main results

The main purpose of this work is examine the above question. We now present our main results, so for the homogeneous differential equation (3.3), we obtain the following result.

Theorem 3.6 [23] *Let* $G \subset (1, +\infty)$ *be a set with a positive upper logarithmic density (or* $m_l(G) = +\infty$*) and let* $A_j(z)$ $(j = 0, 1, \cdots, k)$ *with* $A_k(z) \not\equiv 0$ *be meromorphic functions with finite* $[p,q]-\varphi$ *order. If there exists a positive constant* $\sigma > 0$ *and an integer* s, $0 \le s \le k$ *such that for sufficiently small* $\varepsilon > 0$, *we have* $|A_s(z)| \ge \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q \varphi(r)\right\}$ *as* $|z| = r \in G$,

$r \to +\infty$ and $\rho = \max\{\rho_{[p,q]}(A_j,\varphi)(j \neq s)\} < \sigma$, then every non-transcendental meromorphic solution $f \not\equiv 0$ of (3.3) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (3.3) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ satisfies

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty, \sigma \leq \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).$$

Remark 3.1 *Putting $\varphi(r) = r$ in Theorem 3.6, we obtain Theorem 3.4.*

Corollary 3.1 [23] *Under the hypothesis of Theorem 3.6, suppose further that ψ is a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Then, every transcendental meromorphic solution f of equation (3.3) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ satisfies*

$$\begin{aligned} \sigma &\leq \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi) \\ &= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi). \end{aligned}$$

Considering non-homogeneous linear differential equation (3.4), we obtain the following results.

Theorem 3.7 [23] *Let $G \subset (1,+\infty)$ be a set with a positive upper logarithmic density (or $m_l(G) = +\infty$) and let $A_j(z)$ $(j = 0,1,\cdots,k)$ with $A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p,q]-\varphi$ order. If there exists a positive constant $\sigma > 0$ and an integer s, $0 \leq s \leq k$ such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geq \exp_{p+1}\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}$ as $|z| = r \in G$, $r \to +\infty$ and $\rho_1 = \max\{\rho_{[p,q]}(A_j,\varphi) \quad (j \neq s), \rho_{[p,q]}(F,\varphi)\} < \sigma$, then every non-transcendental meromorphic solution $f \not\equiv 0$ of (3.4) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (3.4) with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \min\{\sigma, \mu_{[p,q]}(f,\varphi)\}$ satisfies*

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty$$

and

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi).$$

Remark 3.2 *Putting $\varphi(r) = r$ in Theorem 3.7, we obtain Theorem 3.5.*

Corollary 3.2 [23] *Let $A_j(z)$ $(j = 0,1,\ldots,k)$, $F(z)$, G satisfy all the hypothesis of Theorem 3.7, and let ψ be a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Then, every transcendental meromorphic solution f with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ of equation (3.4) satisfies*

$$\begin{aligned} \sigma &\leq \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi) \\ &= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \leq \rho_{[p,q]}(A_s,\varphi). \end{aligned}$$

3 Proofs of the main results

3.1 Proof of Theorem 3.6

Let $f \not\equiv 0$ be a rational solution of equation (3.3). At the beginning, we will prove that f must be a polynomial with $\deg f \leq s-1$. If either f is a rational function, which has a pole at z_0 of degree $m \geq 1$ or f is a polynomial with deg $f \geq s$, then $f^{(s)}(z) \not\equiv 0$. From equation (3.3) we have

$$A_s(z) f^{(s)} = -\sum_{\substack{j=0 \\ j \neq s}}^{k} A_j(z) f^{(j)}.$$

By Lemma 2.4, and Lemma 2.14 we obtain

$$\begin{aligned}
\sigma \leq \rho_{[p,q]}(A_s, \varphi) &= \rho_{[p,q]}(A_s f^{(s)}, \varphi) \\
&= \rho_{[p,q]}\left(-\left(\sum_{\substack{j=0 \\ j \neq s}}^{k} A_j(z) f^{(j)} \right), \varphi \right) \\
&\leq \max_{j=0,1,\cdots,k, j \neq s} \left\{ \rho_{[p,q]}(A_j, \varphi) \right\},
\end{aligned}$$

and this contradicts the fact that $\rho = \max\left\{\rho_{[p,q]}(A_j, \varphi)(j \neq s)\right\} < \sigma$. Hence, f must be a polynomial with deg $f \leq s-1$.
Assuming now that f is a transcendental meromorphic solution of (3.3) that satisfies $\lambda_{[p,q]}\left(\frac{1}{f}, \varphi\right) < \mu_{[p,q]}(f, \varphi)$. Using Lemma 2.13, we can find a set $E_9 \subset (1, +\infty)$ of finite linear measure (and so of finite logarithmic measure) such that for any ε $(0 < 2\varepsilon < \sigma - \rho)$ we have

$$|A_j(z)| \leq \exp_{p+1}\{(\rho + \varepsilon) \log_q \varphi(r)\} \quad j = 0, 1, \cdots, k, j \neq s \tag{3.5}$$

holds for all z verifying $|z| = r \notin E_9$. By making use of Lemma 2.11, we can find a set $E_7 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin ([0,1] \cup E_7)$ and $|g(z)| = M(r, g)$ and for sufficiently large r we have

$$\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq r^{2s}, (s \geq 1 \text{ is an integer}). \tag{3.6}$$

From Lemma 2.1, we can find a set $E_1 \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ such that for all z verifying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B\left[T(2r, f)\right]^{k+1}, \quad j = 1, 2, \cdots, k, j \neq s. \tag{3.7}$$

According to the hypothesis of Theorem 3.6, we can find a set $G \subset (1, +\infty)$ with a positive upper logarithmic density (or $m_l(G) = +\infty$) , such that for all z verifying $|z| = r \in G$, $r \to +\infty$ and sufficiently small $\varepsilon > 0$, we have

$$|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon) \log_q \varphi(r)\right\}. \tag{3.8}$$

By (3.3) we have

$$|A_s| \leq \left| \frac{f}{f^{(s)}} \right| \left(|A_0| + \sum_{\substack{j=1 \\ j \neq s}}^{k} |A_j| \left| \frac{f^{(j)}}{f} \right| \right). \tag{3.9}$$

Replacing (3.5), (3.6), (3.7) and (3.8) in (3.9) we get for all z verifying $|z| = r \in G \setminus ([0,1] \cup E_1 \cup E_7 \cup E_9)$, $r \to +\infty$

$$\exp_{p+1}\left\{(\sigma - \varepsilon) \log_q \varphi(r)\right\} \leq Bkr^{2s} \exp_{p+1}\left\{(\rho + \varepsilon) \log_q \varphi(r)\right\} \left[T(2r, f)\right]^{k+1}.$$

From $0 < 2\varepsilon < \sigma - \rho$, we obtain

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}\right\} \le Bkr^{2s}\left[T(2r,f)\right]^{k+1}. \tag{3.10}$$

Using Lemma 2.7 and (3.10) for any given $\nu > 1$ there exists an $r_1 = r_1(\nu)$ and sufficiently large $r > r_1$, we get

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}\right\} \le Bk(\nu r)^{2s}\left[T(2\nu r,f)\right]^{k+1}.$$

By making use of Definition 1.12 and Remark 1.5, we get

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty,$$

and

$$\sigma \le \rho_{[p+1,q]}(f,\varphi).$$

Therefore

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty, \quad \sigma \le \rho_{[p+1,q]}(f,\varphi). \tag{3.11}$$

From Lemma 2.14, we have

$$\max\left\{\rho_{[p,q]}(A_j,\varphi) : j = 0,1,\cdots,k\right\} = \rho_{[p,q]}(A_s,\varphi) = \beta < +\infty.$$

Making use of Lemma 2.19, and the fact that f is a meromorphic solution of equation (3.3) of infinite $[p,q]-\varphi$ order such that $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$ leads to

$$\rho_{[p+1,q]}(f,\varphi) \le \max\left\{\rho_{[p,q]}(A_j,\varphi) : j = 0,1,\cdots,k\right\} = \rho_{[p,q]}(A_s,\varphi). \tag{3.12}$$

Thus (3.11) and (3.12) yield $\mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$ and $\sigma \le \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi)$.

3.2 Proof of Corollary 3.1

Let ψ be a transcendental meromorphic function with $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Putting $\eta = f - \psi$. By Lemma 2.4, we obtain $\rho_{[p+1,q]}(\eta,\varphi) = \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi)$. By making use of Theorem 3.6, we obtain $\sigma \le \rho_{[p+1,q]}(\eta,\varphi) \le \rho_{[p,q]}(A_s,\varphi)$. Replacing $f = \eta + \psi$ into (1.3) gives

$$\begin{aligned} &A_k(z)\eta^{(k)} + A_{k-1}(z)\eta^{(k-1)} + \ldots + A_1(z)\eta' + A_0(z)\eta \\ &= -\left(A_k(z)\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \ldots + A_1(z)\psi' + A_0(z)\psi\right) = U(z). \end{aligned} \tag{3.13}$$

Since $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$, then according to Theorem 3.6, we can see that ψ is not a solution of equation (3.3), hence the right side U of equation (3.13) is non-zero. Furthermore, by Lemma 2.4, and Lemma 2.6 we get

$$\rho_{[p+1,q]}(U,\varphi) \le \max\left\{\rho_{[p+1,q]}(\psi,\varphi), \rho_{[p+1,q]}(A_j,\varphi) \quad (j = 0,1,\cdots,k)\right\} < \sigma.$$

As a consequence

$$\max\left\{\rho_{[p+1,q]}(U,\varphi), \rho_{[p+1,q]}(A_j,\varphi) \quad (j = 0,1,\cdots,k)\right\} < \sigma \le \rho_{[p+1,q]}(\eta,\varphi).$$

From Lemma 2.16 we get

$$\begin{aligned} \sigma &\le \overline{\lambda}_{[p+1,q]}(\eta,\varphi) = \lambda_{[p+1,q]}(\eta,\varphi) \\ &= \rho_{[p+1,q]}(\eta,\varphi) = \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi), \end{aligned}$$

which gives

$$\begin{aligned} \sigma &\le \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi) \\ &= \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi). \end{aligned}$$

3.3 Proof of Theorem 3.7

Let $f \not\equiv 0$ be a rational solution of (3.4). To begin with, we will prove that f must be a polynomial with $\deg f \le s-1$. If either f is a rational function, which has a pole at z_0 of degree $m \ge 1$, or f is a polynomial with $\deg f \ge s$, then $f^{(s)}(z) \not\equiv 0$. From (3.4), we have

$$A_s(z) f^{(s)} = F - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z) f^{(j)}.$$

By Lemma 2.4, and Lemma 2.14, and the fact that $\rho_{[p,q]}(f,\varphi) = 0$ (because f is a non-transcendental so $T(r,f) = O(\log r)$) we obtain

$$\begin{aligned} \sigma \le \rho_{[p,q]}(A_s,\varphi) &= \rho_{[p,q]}(A_s f^{(s)},\varphi) \\ &= \rho_{[p,q]}\left(\left(F - \sum_{\substack{j=0 \\ j\neq s}}^{k} A_j(z) f^{(j)}\right),\varphi\right) \\ &\le \max_{j=0,1,\cdots,k,\, j\neq s}\left\{\rho_{[p,q]}(A_j,\varphi), \rho_{[p,q]}(F,\varphi)\right\}, \end{aligned}$$

and this contradicts the fact that $\rho_1 = \max\left\{\rho_{[p,q]}(A_j,\varphi) \quad (j \neq s), \rho_{[p,q]}(F,\varphi)\right\} < \sigma$. Hence, f must be a polynomial with $\deg f \le s-1$.

Assuming now that f is a transcendental meromorphic solution of (3.4) that satisfies $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$. By Lemma 2.17 we have $\rho_{[p,q]}(f,\varphi) \ge \sigma$. Since $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}(f,\varphi)$, then by Hadamard factorization theorem, there exists entire functions $g(z)$ and $d(z)$ such that f can be written as $f(z) = \frac{g(z)}{d(z)}$, and

$$\mu_{[p,q]}(g,\varphi) = \mu_{[p,q]}(f,\varphi) = \mu \le \rho_{[p,q]}(g,\varphi) = \rho_{[p,q]}(f,\varphi),$$

$$\rho_{[p,q]}(d,\varphi) = \lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) = \beta < \min\left\{\sigma, \mu_{[p,q]}(f,\varphi)\right\}.$$

From the definition of the $[p,q]-\varphi$-order and the lower $[p,q]-\varphi$-order, we obtain

$$\begin{aligned} |g(z)| &= M(r,g) \ge \exp_{p+1}\left\{\left(\mu_{[p,q]}(g,\varphi) - \varepsilon\right)\log_q \varphi(r)\right\}, \\ |d(z)| &\le M(r,d) \le \exp_{p+1}\left\{\left(\rho_{[p,q]}(d,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\}. \end{aligned} \tag{3.14}$$

Let

$$\rho_1 = \max\left\{\rho_{[p,q]}(A_j,\varphi) \quad (j \neq s), \rho_{[p,q]}(F,\varphi)\right\} < \sigma.$$

From (3.14) and Lemma 2.13, for any ε verifying

$$0 < 2\varepsilon < \min\left\{\sigma - \rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\right\},$$

there exists a set $E_9 \subset (1,+\infty)$ of finite logarithmic measure such that for any given z verifying $|z| = r \notin E_9$ at which $|g(z)| = M(r,g)$, we have

$$\begin{aligned} \left|\frac{F(z)}{f(z)}\right| &= \frac{|F(z)|}{|g(z)|}|d(z)| \\ &\le \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(d,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\} \exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q \varphi(r)\right\}}{\exp_{p+1}\left\{\left(\mu_{[p,q]}(g,\varphi) - \varepsilon\right)\log_q \varphi(r)\right\}} \\ &\le \exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q \varphi(r)\right\}. \end{aligned} \tag{3.15}$$

Using the similar way in proving Theorem 3.6, for all z satisfying $|z| = r \in G\setminus([0,1]\cup E_1\cup E_7\cup E_9)$, $r \to +\infty$ at which $|g(z)| = M(r,g)$, and for all ε $\left(0 < 2\varepsilon < \min\left\{\sigma - \rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\right\}\right)$ we obtain (3.6), (3.7), (3.8) and

$$|A_j(z)| \le \exp_{p+1}\{\rho_1 + \varepsilon)\log_q \varphi(r)\}, \quad j = 0, 1, \cdots, \quad k, j \ne s. \tag{3.16}$$

The equation (3.4) gives

$$|A_s| \le \left|\frac{f}{f^{(s)}}\right| \left(|A_0| + \sum_{\substack{j=1\\ j\ne s}}^{k} |A_j| \left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right| \right). \tag{3.17}$$

Replacing (3.6), (3.7), (3.8), (3.15) and (3.16) in (3.17), for all z such that $|z| = r \in G\setminus([0,1]\cup E_1\cup E_7\cup E_9)$, $r \to +\infty$, at which $|g(z)| = M(r,g)$, and for all ε satisfying

$$0 < 2\varepsilon < \min\left\{\sigma - \rho_1, \mu_{[p,q]}(g,\varphi) - \rho_{[p,q]}(d,\varphi)\right\},$$

we get

$$\begin{aligned}
\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\} &\le r^{2s}(\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} \\
&+ \sum_{\substack{j=1\\ j\ne s}}^{k} \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} B\left[T(2r,f)\right]^{k+1}) \\
&+ \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} \\
&\le B(k+1) r^{2s} \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q \varphi(r)\right\} \left[T(2r,f)\right]^{k+1}.
\end{aligned} \tag{3.18}$$

The fact that $0 < 2\varepsilon < \sigma - \rho_1$ gives

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}\right\} \le B(k+1) r^{2s}\left[T(2r,f)\right]^{k+1}. \tag{3.19}$$

Using Lemma 2.7 with equation (3.19) for any given $\nu > 1$ there exists an $r_2 = r_2(\nu)$ and sufficiently large $r > r_2$, we get

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q \varphi(r)\right\}\right\} \le B(k+1)(\nu r)^{2s}\left[T(2\nu r,f)\right]^{k+1}. \tag{3.20}$$

By making use of Definition 1.12 and Remark 1.5, we get

$$\rho_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = +\infty, \quad \rho_{[p+1,q]}(f,\varphi) \ge \sigma. \tag{3.21}$$

According to Lemma 2.14, and the hypothesis of Theorem 3.7 we get

$$\max\left\{\rho_{[p,q]}(A_j,\varphi) \quad (j = 0,1,\ldots,k), \rho_{[p,q]}(F,\varphi)\right\} = \rho_{[p,q]}(A_s,\varphi) = \beta < +\infty.$$

Using Lemma 2.19, and the fact that f is a meromorphic solution of (1.4) of $[p,q]-\varphi$-order with $\lambda_{[p,q]}\left(\frac{1}{f},\varphi\right) < \mu_{[p,q]}\left(f,\varphi\right)$, we obtain

$$\rho_{[p+1,q]}(f,\varphi) \le \max\left\{\rho_{[p,q]}(A_j,\varphi) \quad (j = 0,1,\ldots,k), \rho_{[p,q]}(F,\varphi)\right\} = \rho_{[p,q]}(A_s,\varphi). \tag{3.22}$$

From Lemma 2.18, and since $F \not\equiv 0$, we obtain

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty \tag{3.23}$$

and

$$\sigma \le \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi). \tag{3.24}$$

It results from (3.22), (3.23), and (3.24) that

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \lambda_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi) = \rho_{[p,q]}(f,\varphi) = +\infty$$

and

$$\sigma \le \overline{\lambda}_{[p+1,q]}(f,\varphi) = \lambda_{[p+1,q]}(f,\varphi) = \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi).$$

3.4 Proof of Corollary 3.2

Let ψ be a transcendental meromorphic function with $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$. Putting $\vartheta = f - \psi$. Since $\rho_{[p+1,q]}(f,\varphi) \ge \sigma$ and $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$ then, $\rho_{[p+1,q]}(\psi,\varphi) < \rho_{[p+1,q]}(f,\varphi)$. Hence, by Lemma 2.4 we obtain $\rho_{[p+1,q]}(\vartheta,\varphi) = \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi)$. By making use of Theorem 3.7 we obtain $\sigma \le \rho_{[p+1,q]}(\vartheta,\varphi) \le \rho_{[p,q]}(A_s,\varphi)$. Replacing $f = \vartheta + \psi$ into (3.4) gives

$$\begin{aligned} & A_k(z)\vartheta^{(k)} + A_{k-1}(z)\vartheta^{(k-1)} + \ldots + A_1(z)\vartheta' + A_0(z)\vartheta \\ = {} & F(z) - \left(A_k(z)\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \ldots + A_1(z)\psi' + A_0(z)\psi\right) = V(z). \end{aligned} \tag{3.25}$$

Since $\rho_{[p+1,q]}(\psi,\varphi) < \sigma$, then according to Theorem 3.7, ψ is not a solution of equation (3.4), hence the right side $V(z)$ of equation (3.25) is non-zero. Furthermore, by Lemma 2.4, and Lemma 2.6 we have

$$\begin{aligned} \rho_{[p+1,q]}(V,\varphi) &\le \max\left\{\rho_{[p+1,q]}(\psi,\varphi), \rho_{[p+1,q]}(A_j,\varphi) \quad (j = 0,1,\cdots,k), \rho_{[p+1,q]}(F,\varphi)\right\} \\ &= \rho_{[p+1,q]}(\psi,\varphi) < \sigma. \end{aligned}$$

As a consequence

$$\max\left\{\rho_{[p+1,q]}(V,\varphi), \rho_{[p+1,q]}(A_j,\varphi) \quad (j = 0,1,\cdots,k)\right\} < \sigma \le \rho_{[p+1,q]}(\vartheta,\varphi).$$

From Lemma 2.16, we get

$$\begin{aligned} & \sigma \le \overline{\lambda}_{[p+1,q]}(\vartheta,\varphi) = \lambda_{[p+1,q]}(\vartheta,\varphi) \\ = {} & \rho_{[p+1,q]}(\vartheta,\varphi) = \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi), \end{aligned}$$

which gives

$$\begin{aligned} & \sigma \le \overline{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi) \\ = {} & \rho_{[p+1,q]}(f-\psi,\varphi) = \rho_{[p+1,q]}(f,\varphi) \le \rho_{[p,q]}(A_s,\varphi). \end{aligned}$$

The proof of the Corollary is finished.

Conclusion and Perspectives

Throughout this work, by using a generalized concept of order called φ-order, we have discussed the possibility of extending some results about the growth of meromorphic solutions to linear differential equations of the form:

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{3.26}$$

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{3.27}$$

where A_j and F are meromorphic functions of finite $[p,q]-\varphi$ order.
We have obtained the relationship between the solutions and the meromorphic coefficients in terms of φ-order, estimations about the $[p,q]-\varphi$ order and the $[p,q]-\varphi$ convergence exponent of the solutions to such equations.

Now, some open questions and problems are proposed.
Problem 1. Can we get the similar result using the (α,β,ν) -order defined in [3]? In other words what can be said about the growth of solutions of the differential equations (3.26) and (3.27) if the coefficients are meromorphic functions of (α,β,ν) -order?

Problem 2. What are the hypothesis on the dominant coefficient that guarantee that the solutions of the above equations have a finite (α,β,ν) -order?

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