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by

Hafida MOURI

Dissertation committee :
President : Dr. BOUZIT Hamid Associate professor at UMAB
Supervisor : Dr. BELAÏDI Benharrat Full professor at UMAB
Examiner: Dr. FETTOUCH Houari Associate professor at UMAB

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MASTER

Résumé:

Dans ce mémoire de master, on s'intéresse à la croissance des solutions de l'équation différentielle linéaire complexe suivante $f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_0(z)f=0,$ où $A_0(z),...,A_{k-1}(z)$ sont des fonctions analytiques dans $\overline{\mathbf{C}}\setminus\{z_0\}$, $z_0\in\mathbf{C}$. Certaines estimations de la borne inférieure de croissance des solutions de l'équation différentielle sont obtenues en utilisant le concept du $[p,q]$-ordre inférieur.

Abstract :

In this master thesis, we are interested about the growth of the solutions of the following complex linear differential equation $f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_0(z)f=0,$ where $A_0(z),...,A_{k-1}(z)$ are analytic functions in $\overline{\mathbf{C}}\setminus\{z_0\}$, $z_0\in\mathbf{C}$. Some estimates of the lower bound of the growth of the solutions of the differential equation are obtained using the concept of $[p,q]$-lower order.

الملخص :

في هذه الأطروحة الماستر، نحن مهتمون بنمو حلول المعادلة التفاضلية الخطية المعقدة التالية $f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_0(z)f=0$ ، حيث $A_0(z),...,A_{k-1}(z)$ دوال تحليلية في $\overline{\mathbf{C}}\setminus\{z_0\}$, $z_0\in\mathbf{C}$. يتم الحصول على تقديرات معينة للحد الأدنى لنمو حلول المعادلة التفاضلية باستخدام مفهوم $[p,q]$- ترتيب أقل.

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INTRODUCTION

The study of solutions to linear differential equations is very important in the field of complex analysis, by using Nevanlinna's theory.

We concentrate specifically on the study of the growth of solutions to higher-order homogeneous linear differential equations near a singular point.

The Nevanlinna's value distribution theory of meromorphic functions key tool for meromorphic functions was created in 1929 by the Finnish mathematician Rolf Nevanlinna [13]. The Nevanlinna characteristic function $T(r; f)$ is a measure of a function's growth and its associated counting function estimate how often certain values are taken. Nevanlinna theory has many applications in complex analysis and in the theory of functions; in particular, it plays an important role in the theory of complex differential equations. Using this tool, as well as other forms of modern complex analysis. Several researchers have used this theory to study the properties of solutions of linear differential equations in the complex plane, including growth, oscillation, fixed point, and behavior of meromorphic functions.

Beginning in 1942, H. Wittich [14] was the first to do systematic research on Nevanlinna theory applications to complex differential equations. The previous topic is very important in complex analysis.

Recently, many authors have studied the properties of solutions near a singular point. In 2016, Fettouch and Hamouda investigated the growth of solutions around an isolated essential singularity point [5]. After Long and Zeng improved the result of Fettouch-Hamouda, they obtained some estimations on the $[p, q]$-order of growth of solutions [11]. Dahmani and Belaïdi generalized Long-Zeng's results and found a number of results on the $[p, q]$-order and on the lower $[p, q]$-order [4]. In 2020, Liu, Long, and Zeng investigated the growth of solutions of second-order linear differential equations of the following type [10];

$$f'' + A(z) f' + B(z) f = 0, \tag{0.0.1}$$

where the $A(z)$ and $B(z)$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$.

This thesis focuses at the higher-order homogeneous linear differential equation of the following form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0,\ (k \geq 2), \tag{0.0.2}$$

where $A_j(z)(j = 0, ..., k-1)$, are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \mathbb{C}$.
In this work, we study the growth of solutions of the equation (0.0.2) around a singular point by using the concept of $[p, q]$-order of growth, which is an improvement and a generalization of the paper of Liu, Long, and Zeng [10].
However, the structure of the work is as follows: an introduction, two chapters, and a conclusion. The first chapter of this master's thesis introduces the essential ideas and results of the Nevanlinna value distribution theory of meromorphic functions on the complex plane $\mathbb{C}$ and in $\overline{\mathbb{C}} - \{z_0\}$, which are applicable to the second chapter.
In Chapter 2, we study the growth of nontrivial solutions of complex linear differential equations under various hypotheses about coefficients. We first recall some existence results for Liu, Long, and Zeng. After we generalize what we mentioned earlier, where we applied the concepts of $[p, q]$-order and lower $[p, q]$-type, we give some auxilliary lemmas that we need to demonstrate our results.

Chapter 1

Some elements of the theory of Nevanlinna for meromorphic functions

Some background is included in this chapter, primarily from the Nevanlinna theory [1, 6, 8, 9, 15]. The Jensen's formula, counting function, proximity function, characteristic function with its properties, and more will be presented.

1.1 Jensen formula

Theorem 1.1.1 ([9]) *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and $a_1, a_2, ...$(resp. $b_1, b_2, ...$), its zeros (resp. its poles), each taken into account according to its multiplicity. Then*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + \sum_{|b_j|<r} \log\frac{r}{|b_j|} - \sum_{|a_j|<r} \log\frac{r}{|a_j|}.$$

Proof. We give the proof for the case that f has no zeros and no poles on the circle $|z| = r$. Consider the function

$$g(z) = f(z)\frac{\prod\limits_{|a_j|<r} \dfrac{r^2 - \overline{a_j}z}{r(z-a_j)}}{\prod\limits_{|b_j|<r} \dfrac{r^2 - \overline{b_j}z}{r(z-b_j)}}.$$

Then, $g \neq 0, \infty$ in the disc $|z| \leqslant r$, hence $\log|g(z)|$ is a harmonic function. By the mean formula of harmonic functions, we have

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g\left(re^{i\varphi}\right)\right| d\varphi. \tag{1.1.1}$$

On the other hand,

$$|g(0)| = |f(0)| \frac{\prod\limits_{|a_j|<r} \frac{r}{|a_j|}}{\prod\limits_{|b_j|<r} \frac{r}{|b_j|}},$$

from which

$$\log|g(0)| = \log|f(0)| + \sum_{|a_j|<r} \log\frac{r}{|a_j|} - \sum_{|b_j|<r} \log\frac{r}{|b_j|}. \tag{1.1.2}$$

For $z = re^{i\varphi}$, we have for all a_j and b_j

$$\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right| = \left|\frac{z\bar{z} - \overline{a_j}z}{r(z-a_j)}\right| = \left|\frac{z\left(\overline{z-a_j}\right)}{r(z-a_j)}\right| = 1 = \left|\frac{r^2 - \overline{b_j}z}{r(z-b_j)}\right|.$$

Hence

$$\left|g\left(re^{i\varphi}\right)\right| = \left|f\left(re^{i\varphi}\right)\right|. \tag{1.1.3}$$

Applying (1.1.2) and (1.1.3) to (1.1.1), we obtain the Jensen formula. □

1.2 Characteristic function of Nevanlinna

Definition 1.2.1 ([9], [15]) *Let x be a positive real number. The truncated logarithm $\log^+$ is defined by*

$$\log^+ x \quad = \quad \max\{\log x\ , 0\} \quad = \quad \begin{cases} \log x & if \quad x > 1. \\ 0 & if \quad 0 \leqslant x \leqslant 1. \end{cases}$$

Notice that the truncated logarithm defined above is a continuous function and nonnegative on $(0, \infty)$.

Lemma 1.2.1 ([6], [9]) *Let α,β,α_i positive real numbers. So we have the following properties:*

$$(1)\, \log\alpha \leq \log^+\alpha,$$

$$(2)\, \log^+\alpha \leq \log^+\beta \textit{ for } \alpha \leq \beta,$$

$$(3)\, \log\alpha = \log^+\alpha - \log^+(\frac{1}{\alpha}),$$

$$(4)\, |\log\alpha| = \log^+\alpha + \log^+(\frac{1}{\alpha}),$$

$$(5)\, \log^+\left(\prod_{i=1}^{n}\alpha_i\right) \leq \sum_{i=1}^{n}\log^+(\alpha_i), (1 \leq i \leq n),$$

$$(6)\,\log^+\left(\sum_{i=1}^{n}\alpha_i\right)\leq\sum_{i=1}^{n}\log^+(\alpha_i)+\log n.$$

Proof. **(3)** We have ([1])

$$\begin{aligned}\log^+\alpha-\log^+\frac{1}{\alpha} &= \max\{\log\alpha,0\}-\max\left\{\log\frac{1}{\alpha},0\right\}\\ &= \max\{\log\alpha,0\}+\min\left\{-\log\frac{1}{\alpha},0\right\}\\ &= \max\{\log\alpha,0\}+\min\{\log\alpha,0\}\\ &= \log\alpha.\end{aligned}$$

(4) We have ([1])

$$\begin{aligned}\log^+\alpha+\log^+\frac{1}{\alpha} &= \max\{\log\alpha,0\}+\max\left\{\log\frac{1}{\alpha},0\right\}\\ &= \max\{\log\alpha,0\}+\max\{-\log\alpha,0\}\\ &= \max\{\log\alpha,0\}-\min\{\log\alpha,0\}\\ &= |\log\alpha|\,.\end{aligned}$$

(5) • If $\prod_{i=1}^{n}\alpha_i\leqslant 1$, then the inequality holds trivially.

• If $\prod_{i=1}^{n}\alpha_i>1$, then

$$\log^+\left(\prod_{i=1}^{n}\alpha_i\right)=\log\left(\prod_{i=1}^{n}\alpha_i\right)=\sum_{i=1}^{n}\log\alpha_i\underset{(by\ \ 1))}{\leqslant}\sum_{i=1}^{n}\log^+\alpha_i.$$

(6) By (2) and (5) above

$$\log^+\left(\sum_{i=1}^{n}\alpha_i\right)\leqslant\log^+\left(n\max_{1\leqslant i\leqslant n}\alpha_i\right)\leqslant\log^+ n+\log^+\left(\max_{1\leqslant i\leqslant n}\alpha_i\right)\leqslant\log^+ n+\sum_{i=1}^{n}\log^+\alpha_i.$$

□

Definition 1.2.2 ([9]) *(Unintegrated counting function) Let f be a meromorphic function, not being identically equal to $a \in \mathbb{C}$. We denote by $n(r,a,f)$ the number of the roots of $f(z)=a$ in the disc $|z|<r$, each root according to its multiplicity. Similarly $\bar{n}(r,a,f)$ counts the number of the distinct roots of $f(z)=a$ in the disc $|z|<r$. And we denote by $n(r,\infty,f)$ the number of the poles of f in the disc $|z|<r$, each pole according to its multiplicity. Similarly $\bar{n}(r,\infty,f)$ counts the number of the distinct poles of f in the disc $|z|<r$.*

Example 1.2.1 $f(z)=\frac{-1}{\cos z}$, $n(r,f)=n(r,\infty,f)=2\left[\frac{2r}{\pi}\right]$.

Example 1.2.2 $h(z)=\exp(z)$, $n(r,f)=0$, *because it is an entire function.*

Example 1.2.3 $f(z)=\frac{1}{\sin^2 z}$, *we have* $n(r,f)=2+4\left[\frac{r}{\pi}\right]$, $\overline{n}(r,f)=1+2\left[\frac{r}{\pi}\right]$ *because f has double poles at $z_k=k\pi$ $(k\in\mathbb{Z})$.*

Definition 1.2.3 ([9]) *Let f be a meromorphic function, we define the a-point function of f by*

$$N(r,a,f)=N\left(r,\frac{1}{f-a}\right):=\int_0^r\frac{n(t,a,f)-n(0,a,f)}{t}dt+n(0,a,f)\log r$$

If $f\not\equiv a\in\mathbb{C}$ and

$$N(r,\infty,f)=N\left(r,f\right):=\int_0^r\frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt+n(0,\infty,f)\log r.$$

Similary, we define the a-point distinct function of f by

$$\bar{N}(r,a,f)=\bar{N}\left(r,\frac{1}{f-a}\right):=\int_0^r\frac{\bar{n}(t,a,f)-\bar{n}(0,a,f)}{t}dt+\bar{n}(0,a,f)\log r$$

If $f\not\equiv a\in\mathbb{C}$ and

$$\bar{N}(r,\infty,f)=\bar{N}\left(r,f\right):=\int_0^r\frac{\bar{n}(t,\infty,f)-\bar{n}(0,\infty,f)}{t}dt+\bar{n}(0,\infty,f)\log r.$$

Example 1.2.4 $f(z)=\frac{1}{\sin^2 z}$, *we have* $n(r,f)=2+4\left[\frac{r}{\pi}\right]$, $n(0,f)=2$, *then,*

$$\begin{aligned}N(r,f)\quad &: \quad =\int_0^r\frac{4\left[\frac{t}{\pi}\right]}{t}dt+2\log r\\ &= \frac{4}{\pi}r+2\log r.\end{aligned}$$

Remark 1.2.1 *If f is an entire function, then $N\left(r,f\right)=\bar{N}\left(r,f\right)=0$.*

Example 1.2.5 $f(z) = \exp(z)$, *we have* $N(r, f) = 0$.

Lemma 1.2.2 ([9]) *Let f be a meromorphic function with a-points $\alpha_1, \alpha_2, ..., \alpha_m$ in the disc $|z| \leqslant r$ such that $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant ... \leqslant |\alpha_m| \leqslant r$, each counted according to its multiplicity. Then*

$$\int_0^r \frac{n(t,a,f)}{t} dt = \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t} dt = \sum_{0<|\alpha_j|\leqslant r} \log \frac{r}{|\alpha_j|}.$$

Proof. Denoting $r_j = |\alpha_j|$ $(j = 1, 2, ..., m)$. Then, we have

$$\begin{aligned}
\sum_{0<|\alpha_j|\leq r} \log \frac{r}{|\alpha_j|} &= \sum_{j=1}^{m} \log \frac{r}{r_j} \\
&= \log \frac{r^m}{r_1 \times r_2 \times ... \times r_m} \\
&= \log(\frac{r_2}{r_1} \times \frac{r_3^2}{r^2} \times ... \times \frac{r_m^{m-1}}{r_{m-1}^{m-1}} \times \frac{r^m}{r_m^m}) \\
&= \sum_{j=1}^{m-1} j(\log r_{j+1} - \log r_j) + m(\log r - \log r_m) \\
&= \sum_{j=1}^{m-1} j \int_{r_j}^{r_{j+1}} \frac{dt}{t} + m \int_{r_m}^{r} \frac{dt}{t} = \int_0^r \frac{n(t,a,f)}{t} dt.
\end{aligned}$$

□

Proposition 1.2.1 ([9]) *Let f be a meromorphic function with the Laurent expansion at the origin*

$$f(z) = \sum_{i=m}^{+\infty} c_i z^i, \qquad c_m \in \mathbb{C}^*, m \in \mathbb{Z}.$$

Then

$$\log|c_m| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + N(r, f) - N\left(r, \frac{1}{f}\right).$$

Proof. Consider the meromorphic function h

$$h(z) = f(z) z^{-m}, z \in \mathbb{C}$$

It is evident that $m = n(0, 0, f) - n(0, \infty, f)$ and $h(0) \neq 0, \infty$. If $m > 0$, then $n(0, \infty, f) = 0$ and $m = n(0, 0, f)$. If $m < 0$, then $n(0, 0, f) = 0$ and $n(0, \infty, f) = -m$. Finally, if $m = 0$, then

$n(0,0,f) = n(0,\infty,f) = 0$. So the functions h and f have the same poles and zeros in $0 < |z| \leq r$. By applying Jensen's formula and Lemma 1.2.2, we have

$$\begin{aligned}
\log|c_m| &= \log|h(0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right) r^{-m}\right| d\varphi + \sum_{0<|b_j|\leqslant r} \log\frac{r}{|b_j|} - \sum_{0<|a_j|\leqslant r} \log\frac{r}{|a_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi - \left[n(0,0,f) - n(0,\infty,f)\right]\log r \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right).
\end{aligned}$$

□

Definition 1.2.4 ([9]) *Let f be a meromorphic function, we define the proximity function of f by*

$$m(r,a,f) = m\left(r,\frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\varphi}) - a|} d\varphi \quad \text{if } f \not\equiv a \in \mathbb{C},$$

and

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi.$$

Example 1.2.6 *Let $f(z) = \exp z$. Then, we have*

$$\begin{aligned}
m(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\exp i\varphi)|\, d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\exp(r\exp i\varphi)|\, d\varphi \\
&= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \log\exp(r\cos\varphi)\, d\varphi \\
&= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (r\cos\varphi)\, d\varphi \\
&= \frac{r}{2\pi}\left[\sin\varphi\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{r}{\pi}.
\end{aligned}$$

Definition 1.2.5 ([9]) *Let f be a meromorphic function, the characteristic function of Nevanlinna of f will be defined as*

$$T(r,f) := m(r,f) + N(r,f).$$

Proposition 1.2.2 ([6], [9]) *Let $f_1, ..., f_n, f$ be a meromorphic functions and $a \in \mathbb{C}^*$, then*

(1) $m\left(r, \prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} m\left(r, f_i\right),$ $\quad (n \in \mathbb{N}^*)$,

(2) $m\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} m\left(r, f_i\right) + \log n,$ $\quad (n \in \mathbb{N}^*)$,

(3) $T\left(r, \prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} T\left(r, f_i\right),$ $\quad (n \in \mathbb{N}^*)$,

(4) $T\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} T\left(r, f_i\right) + \log n,$ $\quad (n \in \mathbb{N}^*)$,

(5) $T\left(r, f^n\right) = nT\left(r, f\right),$ $\quad (n \in \mathbb{N}^*)$,

(6) $m\left(r, a+f\right) = m\left(r, f\right) + O\left(1\right)$ *and* $m\left(r, af\right) = m\left(r, f\right) + O\left(1\right),$

(7) $T\left(r, a+f\right) = T\left(r, f\right) + O\left(1\right)$ *and* $T\left(r, af\right) = T\left(r, f\right) + O\left(1\right).$

Proof. **(1), (3)** We have

$$\begin{aligned} m\left(r, \prod_{i=1}^{n} f_i\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\prod_{i=1}^{n} f_i\left(re^{i\varphi}\right)\right| d\varphi \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^{n} \log^+ \left|f_i\left(re^{i\varphi}\right)\right| d\varphi \\ &= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi} \log^+ \left|f_i\left(re^{i\varphi}\right)\right| d\varphi \\ &= \sum_{i=1}^{n} m\left(r, f_i\right). \end{aligned}$$

If f_i has a pole of order $\lambda_i \geqslant 0$ at z_0, then it is a pole of order equal at most to $\sum\limits_{i=1}^{n} \lambda_i$ for the function $\prod\limits_{i=1}^{n} f_i$. Hence

$$N\left(r, \prod_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} N\left(r, f_i\right),$$

Therefore

$$\begin{aligned} T\left(r, \prod_{i=1}^{n} f_i\right) &= m\left(r, \prod_{i=1}^{n} f_i\right) + N\left(r, \prod_{i=1}^{n} f_i\right) \\ &\leqslant \sum_{i=1}^{n} m\left(r, f_i\right) + \sum_{i=1}^{n} N\left(r, f_i\right) = \sum_{i=1}^{n} T\left(r, f_i\right). \end{aligned}$$

(2), (4) We have

$$\begin{aligned} m\left(r, \sum_{i=1}^{n} f_i\right) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{i=1}^{n} f_i\left(re^{i\varphi}\right) \right| d\varphi \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{i=1}^{n} \log^+ \left| f_i\left(re^{i\varphi}\right) \right| + \log n \right) d\varphi \\ &= \sum_{i=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f_i\left(re^{i\varphi}\right) \right| d\varphi + \log n \\ &= \sum_{i=1}^{n} m\left(r, f_i\right) + \log n. \end{aligned}$$

If f_i has a pole of order $\lambda_i \geqslant 0$ at z_0, then it is a pole of order equal at most to $\max\limits_{1 \leqslant i \leqslant n} \lambda_i \leqslant \sum\limits_{i=1}^{n} \lambda_i$ for the function $\prod\limits_{i=1}^{n} f_i$. Hence

$$N\left(r, \sum_{i=1}^{n} f_i\right) \leqslant \sum_{i=1}^{n} N\left(r, f_i\right),$$

Therefore

$$\begin{aligned} T\left(r, \sum_{i=1}^{n} f_i\right) &= m\left(r, \sum_{i=1}^{n} f_i\right) + N\left(r, \sum_{i=1}^{n} f_i\right) \\ &\leqslant \sum_{i=1}^{n} m\left(r, f_i\right) + \log n + \sum_{i=1}^{n} N\left(r, f_i\right) = \sum_{i=1}^{n} T\left(r, f_i\right) + \log n. \end{aligned}$$

(5) We have : $|f^n| = |f|^n \leqslant 1 \Longleftrightarrow |f| \leqslant 1$.

- If $|f| \leqslant 1$, then

$$T\left(r, f^n\right) = N\left(r, f^n\right) = nN\left(r, f\right) = nT\left(r, f\right).$$

• If $|f| > 1$, then

$$\begin{aligned}
m(r, f^n) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f^n\left(re^{i\varphi}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log \left|f^n\left(re^{i\varphi}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\cdot n\int_0^{2\pi} \log \left|f\left(re^{i\varphi}\right)\right| d\varphi \\
&= n\cdot\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi \\
&= nm(r, f).
\end{aligned}$$

Hence

$$\begin{aligned}
T(r, f^n) &= m(r, f^n) + N(r, f^n) \\
&= nm(r, f) + nN(r, f) \\
&= nT(r, f).
\end{aligned}$$

(6) We have

$$\begin{aligned}
|m(r, a+f) - m(r, f)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \left(\log^+ \left|f\left(re^{i\varphi}\right) + a\right| - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right) d\varphi\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\log^+ \left(\left|f\left(re^{i\varphi}\right)\right| + |a|\right) - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right| d\varphi \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\log^+ |a| + \log 2\right| d\varphi \leqslant \log^+ |a| + \log 2,
\end{aligned}$$

And

$$\begin{aligned}
|m(r, af) - m(r, f)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \left(\log^+ \left|af\left(re^{i\varphi}\right)\right| - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right) d\varphi\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\log^+ \left(|a|\left|f\left(re^{i\varphi}\right)\right|\right) - \log^+ \left|f\left(re^{i\varphi}\right)\right|\right| d\varphi \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|\log^+ |a|\right| d\varphi = \log^+ |a| = |\log |a|| - \log^+ \frac{1}{|a|} \\
&\leqslant |\log |a||.
\end{aligned}$$

Hence,

$$m(r, a+f) = m(r, f) + O(1) \quad \text{and} \quad m(r, af) = m(r, f) + O(1).$$

(7) From (6), we get

$$\begin{aligned} T(r, a+f) &= N(r, a+f) + m(r, a+f) \\ &= N(r, f) + m(r, f) + O(1) \\ &= T(r, f) + O(1), \end{aligned}$$

And

$$\begin{aligned} T(r, af) &= N(r, af) + m(r, af) \\ &= N(r, f) + m(r, f) + O(1) \\ &= T(r, f) + O(1). \end{aligned}$$

□

1.3 Characteristic function near singular point

In order to determine the characteristic functions near a singular point, we must first establish the proximity function and the counting function near a singular point.

Definition 1.3.1 ([3], [5]) *(Counting function). The counting function of f near z_0 is defined by*

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r,$$

where $n(t, f)$ denote the number of poles of $f(z)$ in the region$\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$ counting its multiplicities.

Example 1.3.1 *Let $f(z) = \exp\left(\frac{1}{z^n}\right)$ $(n \in \mathbb{N}^*)$, $n(t, f) = 0$ so $N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{0-0}{t} dt - 0 \cdot \log r =$* 0.

Definition 1.3.2 ([3], [5]) *(Proximity function). For a meromorphic function f, the proximity function of f near z_0 is defined by*

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(z_0 - re^{i\varphi}) \right| d\varphi.$$

Example 1.3.2 *Let* $f(z) = \exp\left(\frac{1}{z}\right)$, $z_0 = 0$ *be an essential singular point, so we have*

$$\begin{aligned}
m_{z_0}(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(z_0 - re^{i\varphi})\right| d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left(\frac{1}{-re^{i\varphi}}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\exp\left(-\frac{e^{-i\varphi}}{r}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(\exp\left(-\frac{\cos\varphi}{r}\right)\right) d\varphi \\
&= \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(-\frac{\cos\varphi}{r}\right)) d\varphi = \frac{1}{2\pi}\left[-\frac{\sin\varphi}{r}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{1}{\pi}\frac{1}{r}.
\end{aligned}$$

Definition 1.3.3 ([3], [5]) *(Characteristic function). The characteristic function of* $f(z)$ *near* z_0 *is defined by*

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Example 1.3.3 $f(z) = \exp\left(\frac{1}{z}\right)$, $N_{z_0}(r,f) = 0$, *so*

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f) = \frac{1}{\pi}\frac{1}{r}.$$

Theorem 1.3.1 ([9]) *(First Fundamental Theorem of Nevanlinna). Let* f *be a meromorphic function,* $a \in \mathbb{C}$ *and let*

$$f(z) - a = \sum_{i=m}^{+\infty} c_i z^i, \qquad c_m \in \mathbb{C}^*\ ,\ m \in \mathbb{Z},$$

be the Laurent expansion of $f - a$ *at the origin. Then*

$$T\left(r,a,f\right) = T\left(r, \frac{1}{f-a}\right) = T\left(r,f\right) - \log\left|c_m\right| + \varphi\left(r,a\right),$$

where $\quad \left|\varphi\left(r,a\right)\right| \leqslant \log 2 + \log^+\left|a\right|.$

Proof. Assume first $a = 0$, then by the Proposition 1.2.1 and Lemma 1.2.1 (3), we have

$$\begin{aligned}
\log\left|c_m\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\varphi}\right)\right| d\varphi + N\left(r,f\right) - N\left(r,\frac{1}{f}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\varphi}\right)\right| d\varphi - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\left|f\left(re^{i\varphi}\right)\right|} d\varphi + N\left(r,f\right) - N\left(r,\frac{1}{f}\right) \\
&= m\left(r,f\right) - m\left(r,\frac{1}{f}\right) + N\left(r,f\right) - N\left(r,\frac{1}{f}\right) \\
&= T\left(r,f\right) - T\left(r,\frac{1}{f}\right),
\end{aligned}$$

Hence

$$T\left(r,\frac{1}{f}\right)=T\left(r,f\right)-\log\left|c_m\right|,\qquad where\quad \varphi\left(r,0\right)\equiv 0. \tag{1.3.1}$$

Proceeding now to the general case $a\neq 0$, we pose $h:=f-a$. Then

$$N\left(r,\frac{1}{h}\right)=N\left(r,\frac{1}{f-a}\right),\quad N\left(r,f\right)=N\left(r,h\right)\quad et\quad m\left(r,\frac{1}{h}\right)=m\left(r,\frac{1}{f-a}\right).$$

Moreover

$$\begin{aligned}\log^+\left|h\right| &= \log^+\left|f-a\right|\leqslant\log^+\left|f\right|+\log^+\left|a\right|+\log 2,\\ \log^+\left|f\right| &= \log^+\left|h+a\right|\leqslant\log^+\left|h\right|+\log^+\left|a\right|+\log 2.\end{aligned}$$

By integrating these two inequalities, we find that

$$\begin{aligned}m\left(r,h\right) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|h\left(re^{i\varphi}\right)\right|d\varphi\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|f\left(re^{i\varphi}\right)\right|+\log^+\left|a\right|+\log 2\right)d\varphi\\ &= m\left(r,f\right)+\log^+\left|a\right|+\log 2,\end{aligned}$$

And

$$\begin{aligned}m\left(r,f\right) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\varphi}\right)\right|d\varphi\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\log^+\left|h\left(re^{i\varphi}\right)\right|+\log^+\left|a\right|+\log 2\right)d\varphi\\ &= m\left(r,h\right)+\log^+\left|a\right|+\log 2.\end{aligned}$$

We pose $\varphi\left(r,a\right):=m\left(r,h\right)-m\left(r,f\right)$. Then

$$-\left[\log^+\left|a\right|+\log 2\right]\leqslant m\left(r,h\right)-m\left(r,f\right)\leqslant\log^+\left|a\right|+\log 2\Longleftrightarrow\left|\varphi\left(r,a\right)\right|\leqslant\log^+\left|a\right|+\log 2.$$

Applying $(1,2,1)$ for h, we obtain

$$\begin{aligned}T\left(r,\frac{1}{h}\right) &= T\left(r,h\right)-\log\left|c_m\right|\\ &= m\left(r,h\right)+N\left(r,h\right)-\log\left|c_m\right|\\ &= m\left(r,f\right)+\varphi\left(r,a\right)+N\left(r,f\right)-\log\left|c_m\right|\\ &= T\left(r,f\right)-\log\left|c_m\right|+\varphi\left(r,a\right).\end{aligned}$$

□

Remark 1.3.1 *We can rewrite the first fundamental theorem of Nevanlinna as follows: for all $a \in \mathbb{C}$, we have*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \quad r \to \infty.$$

1.4 Growth of meromorphic functions near singular point

1.4.1 The order and the lower order of growth

We suppose that $f(z)$ is an analytic function in $\overline{\mathbb{C}}$ except a finite singular point z_0.

Similarly to the case of complex plane, we define $\rho(f, z_0)$, $\mu(f, z_0)$, and $\underline{\tau}(f, z_0)$ for analytic or meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$.

Definition 1.4.1 ([10]) *The order and the lower order of growth of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ near z_0 are defined respectively by*

$$\rho_T(f, z_0) = \limsup_{r\to 0} \frac{\log^+ T_{z_0}(r,f)}{\log \frac{1}{r}}.$$

and

$$\mu_T(f, z_0) = \liminf_{r\to 0} \frac{\log^+ T_{z_0}(r,f)}{\log \frac{1}{r}}.$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$, the order and the lower order of growth are defined respectively by

$$\rho_M(f, z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ M_{z_0}(r,f)}{\log \frac{1}{r}}$$

and

$$\mu_M(f, z_0) = \liminf_{r\to 0} \frac{\log^+ \log^+ M_{z_0}(r,f)}{\log \frac{1}{r}},$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$.

1.4.2 The hyper order of growth

Definition 1.4.2 ([10]) *The hyper-order of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ near z_0 is defined by*

$$\rho_{2,T}(f, z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ T_{z_0}(r,f)}{\log \frac{1}{r}}$$

and for an analytic function f in $\overline{\mathbb{C}} - \{z_0\}$, the hyper-order is defined by

$$\rho_{2,M}(f, z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ \log^+ M_{z_0}(r,f)}{\log \frac{1}{r}},$$

Where $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$.

1.4.3 The lower-type near z_0

Likewise, we define the lower type with a procedure equivalent to the complex plane:

Definition 1.4.3 ([10]) *If* $f(z)$ *is an analytic function in* $\overline{\mathbb{C}}-\{z_0\}$, *with* $0<\mu(f,z_0)=\mu<\infty$, *then its lower-type is defined by*

$$\underline{\tau}(f,z_0)=\liminf_{r\to 0}\frac{\log^+ M_{z_0}(r,f)}{(\frac{1}{r})^{\mu}},$$

where $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$.

Here, we introduce the concepts of $[p,q]$-order and $[p,q]$-type of growth, we add similarly the definition of the lower $[p,q]$-order and lower $[p,q]$-type of growth. Before that we must give some notations :

- For $r\in\mathbb{R}$, we have : $\exp_1 r=e^r$ and $\exp_{p+1} r=\exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We also define
- For all r sufficiently large in $(0,+\infty)$, $\log_1 r=\log r$ and $\log_{p+1} r=\log\left(\log_p r\right)$, $p\in\mathbb{N}$.
- Moreover, we denote $\exp_0 r=r=\log_0 r$, $\exp_{-1} r=\log_1 r$ and $\log_{-1} r=\exp_1 r$.

Proposition 1.4.1 ([2]) *Let* $x_i\in\mathbb{R}$ *such that* $x_i>1$ *and* $i=1,...,n$. *Then*

(i) $\log_p\left(\sum\limits_{i=1}^{n}x_i\right)\leqslant\sum\limits_{i=1}^{n}\log_p x_i+O\left(1\right),$

(ii) $\log_p\left(\prod\limits_{i=1}^{n}x_i\right)\leqslant\sum\limits_{i=1}^{n}\log_p x_i+O\left(1\right).$

Proof. For the proof, we use the principle of mathematical induction.

(i) • For $p=1$, we have $\log\left(\sum\limits_{i=1}^{n}x_i\right)\leqslant\sum\limits_{i=1}^{n}\log x_i+O\left(1\right).$

•We suppose that $\log_p\left(\sum\limits_{i=1}^{n}x_i\right)\leqslant\sum\limits_{i=1}^{n}\log_p x_i+O\left(1\right)$ is true and we prove that it holds at order

$p+1$. We have

$$\begin{aligned}\log_{p+1}\left(\sum_{i=1}^{n} x_i\right) &= \log\left(\log_p\left(\sum_{i=1}^{n} x_i\right)\right)\\ &\leqslant \log\left(\sum_{i=1}^{n}\log_p x_i + O(1)\right)\\ &\leqslant \sum_{i=1}^{n}\log_{p+1} x_i + O(1).\end{aligned}$$

Hence, $\log_p\left(\sum\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n}\log_p x_i + O(1)$.

(ii) • For $p=1$, we have $\log\left(\prod\limits_{i=1}^{n} x_i\right) = \sum\limits_{i=1}^{n}\log x_i$, then $\log\left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n}\log x_i + O(1)$.

•We suppose that $\log_p\left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n}\log_p x_i + O(1)$ is true and we prove that it holds at order $p+1$. We have

$$\begin{aligned}\log_{p+1}\left(\prod_{i=1}^{n} x_i\right) &= \log\left(\log_p\left(\prod_{i=1}^{n} x_i\right)\right)\\ &\leqslant \log\left(\sum_{i=1}^{n}\log_p x_i + O(1)\right)\\ &\leqslant \sum_{i=1}^{n}\log_{p+1} x_i + O(1).\end{aligned}$$

Hence, $\log_p\left(\prod\limits_{i=1}^{n} x_i\right) \leqslant \sum\limits_{i=1}^{n}\log_p x_i + O(1)$.

$\square$

1.4.4 The $[p,q]$-order and the lower $[p,q]$-order of growth

Definition 1.4.4 **(**[11]**)** *Let $f(z)$ be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ and p,q two integers $p \geq q \geq 1$. Then the $[p,q]-$order of growth and the lower $[p,q]-$order are defined respectively by*

$$\rho_{[p,q]}(f,z_0) = \limsup_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q(\frac{1}{r})}$$

and

$$\mu_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q(\frac{1}{r})}.$$

Definition 1.4.5 ([11]) *Let $f(z)$ be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ and p,q two integers $p \geq q \geq 1$. Then the $[p,q]-$order and the the lower $[p,q]-$order of growth are defined respectively by*

$$\rho_{M,[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_{p+1}^{+}M_{z_0}(r,f)}{\log_q(\frac{1}{r})}$$

and

$$\mu_{M,[p,q]}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_{p+1}^{+}M_{z_0}(r,f)}{\log_q(\frac{1}{r})},$$

where $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$.

Remark 1.4.1 *If we put $p=q=1$ we obtain the order of growth $\rho_{[1,1]}(f,z_0)=\rho(f,z_0)$, for $p=2$ and $q=1$ is just the hyper order of growth $\rho_{[2,1]}(f,z_0)=\rho_2(f,z_0)$, for $q=1$ is the iterated p-order.*

Example 1.4.1 *Let $f(z)=\exp_3\left(\cosh\frac{1}{z^2}\right)$ be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ such that*

$$M(r,f)=\max_{|z|=r}|f(z)|=\exp_3\left\{\cosh\left(\frac{1}{r^2}\right)\right\}.$$

Then

$$\begin{aligned}\rho_{M,[2,1]}(f) &= \limsup_{r\to 0}\frac{\log_3^{+}M(r,f)}{\log\left(\frac{1}{r}\right)}\\ &= \limsup_{r\to 0}\frac{\log_3\left(\exp_3\left\{\cosh\left(\frac{1}{r^2}\right)\right\}\right)}{\log\left(\frac{1}{r}\right)}\\ &= \limsup_{r\to 0}\frac{\cosh\left(\frac{1}{r^2}\right)}{\log\left(\frac{1}{r}\right)}=+\infty.\end{aligned}$$

Proposition 1.4.2 *Let $p\geqslant q\geqslant 1$ be integers, and let f be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ of $[p,q]$-order. Then*

$$\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f).$$

Proof. By the fundamental inequality (see $[8, p.\ 18]$), for $R=2r$ we obtain

$$T(r,f)\leqslant\log^{+}M(r,f)\leqslant 3T(2r,f).$$

It follows that

$$\log_p T(r,f)\leqslant\log_{p+1}^{+}M(r,f)\leqslant\log_p(3T(2r,f))\leqslant\log_p T(2r,f)+C,$$

$C > 0$ is a constant. Then

$$\limsup_{r\to 0}\frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{r}\right)} = \rho_{[p,q]}(f,z_0) \leqslant \limsup_{r\to 0}\frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{r}\right)} = \rho_{M,[p,q]}(f,z_0)$$

$$\leqslant \limsup_{r\to 0}\left(\frac{\log_p T(2r,f)}{\log_q\left(\frac{1}{r}\right)} + \frac{C}{\log_q\left(\frac{1}{r}\right)}\right) = \limsup_{r\to 0}\left(\frac{\log_p T(2r,f)}{\log_q(\frac{1}{2r})}\cdot\frac{\log_p(\frac{1}{2r})}{\log_q(\frac{1}{r})} + \frac{C}{\log_q\left(\frac{1}{r}\right)}\right)$$

$$= \limsup_{r\to 0}\frac{\log_p T(2r,f)}{\log_q\left(\frac{1}{2r}\right)} = \rho_{[p,q]}(f).$$

Hence

$$\rho_{M,[p,q]}(f) = \rho_{[p,q]}(f).$$

□

Example 1.4.2 *For the function* $f(z) = \exp_\alpha(\frac{1}{z})$, $a \in \mathbb{C}^*$, $\alpha \geq 1$, *we have*

$$\begin{aligned}\rho_{[p,1]}(f,z_0) &= \limsup_{r\to 0}\frac{\log_{p+1}^+ M(r,f)}{\log\left(\frac{1}{r}\right)}\\ &= \limsup_{r\to 0}\frac{\log_{p+1}^+ \exp_\alpha(\frac{1}{r})}{\log\left(\frac{1}{r}\right)}.\end{aligned}$$

Then

$$\rho_{[p,1]}(f,z_0) = \begin{cases} +\infty & \text{if } p < \alpha,\\ 1 & \text{if } p = \alpha,\\ 0 & \text{if } p > \alpha.\end{cases}$$

1.4.5 The $[p,q]$-type and the lower $[p,q]$-type of growth

Definition 1.4.6 ([11]) *Let* f *be a meromorphic function in* $\overline{\mathbb{C}} - \{z_0\}$ *with* $\rho = \rho_{[p,q]}(f,z_0) \in (0,\infty)$. *Then the* $[p,q]-$*type of* f *is defined by*

$$\tau_{[p,q]}(f,z_0) = \limsup_{r\to 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^\rho}$$

and the lower $[p,q]-$*type of* $f(z)$ *with* $\mu = \mu_{[p,q]}(f,z_0) \in (0,\infty)$ *is defined by*

$$\underline{\tau}_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^\mu}.$$

Definition 1.4.7 ([11]) *Let f be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho=\rho_{[p,q]}(f,z_0)\in(0,\infty)$. Then the $[p,q]-$type of $f(z)$ is defined by*

$$\tau_{M,[p,q]}(f,z_0)=\limsup_{r\to 0}\frac{\log^+_{p+1} M_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^\rho}$$

and the lower $[p,q]-$type of f with $\mu=\mu_{[p,q]}(f,z_0)\in(0,\infty)$ is defined by

$$\underline{\tau}_{M,[p,q]}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log^+_{p+1} M_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^\rho},$$

where $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$

Example 1.4.3 *We calculate the $[p,q]$-type of the following function $f(z)=\exp(\exp(\frac{1}{z}))$. We have $\rho_{[2,1]}(f)=1$, then*

$$\begin{aligned}\tau_{M,[2,1]}(f,z_0) &= \limsup_{r\to 0}\frac{\log^+_{p+1} M_{z_0}(r,f)}{\log_{q-1}(\frac{1}{r})^\rho}\\ &= \limsup_{r\to 0}\frac{\log^+_3 \exp\exp(\frac{1}{r})}{\frac{1}{r}}=0.\end{aligned}$$

1.5 Linear and logarithmic measure

Definition 1.5.1 *The linear measure of a set $E\subset[0,+\infty)$ is defined by*

$$m(E)=\int_0^{+\infty}\chi_E(t)dt,$$

where $\chi_E(t)$ is the characteristic function of the set E and the logarithmic measure of a set $F\subset[1,+\infty)$ is defined by

$$m_l(F)=\int_1^{+\infty}\frac{\chi_F(t)}{t}dt.$$

Example 1.5.1 *The linear measure of the set $E=[2,6]\cup[7,8]\subset[0,+\infty)$ is*

$$m(E)=\int_0^{+\infty}\chi_E(t)dt=\int_2^6 dt+\int_7^8 dt=5.$$

2) *The logarithmic measure of the set $F=[1;e^2]\subset(1,+\infty)$ is*

$$m_l(F)=\int_1^{+\infty}\frac{\chi_F(t)}{t}dt=\int_1^{e^2}\frac{dt}{t}=2.$$

Chapter 2

The [p,q]-Order of Growth of Solutions of Linear Differential Equations Near a Singular Point

2.1 Introduction and Some Results

Consider for $k \geq 2$ the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.1.1}$$

where $A_0, ..., A_{k-1}$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$.

In [10] Liu, Long, and Zeng treated the growth of solutions of the second-order linear differential equation (0.0.1) when the coefficients $A(z)$ and $B(z)$ are analytic functions of lower order in $\overline{\mathbb{C}} - \{z_0\}$, firstly when $B(z)$ is a dominant coefficient with lower order, nextly with the lower type. Finally, they asked the question: what happen when the coefficient $A(z)$ dominates in the concept of lower order? for these reasons they obtained the following results.

Theorem 2.1.1 ([10]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\mu(A, z_0) < \mu(B, z_0) < \infty$. Then, every non trivial solution $f(z)$ of (0.0.1), that is analytic in $\overline{\mathbb{C}} - \{z_0\}$, satisfies $\rho_2(f, z_0) \geq \mu(B, z_0)$.*

Theorem 2.1.2 ([10]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying the fol-*

lowing conditions:

$$(i)\ \mu(A,z_0)=\mu(B,z_0);$$

$$(ii)\ \underline{\tau}_M(A,z_0)<\underline{\tau}_M(B,z_0).$$

Then, every non trivial solution $f(z)$ of (0.0.1), that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho_2(f,z_0)\geq\mu(B,z_0)$.

Theorem 2.1.3 ([10]) *Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying $\mu(B,z_0)<\mu(A,z_0)<\infty$. Then, every non trivial solution $f(z)$ of (0.0.1), that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho(f,z_0)\geq\mu(A,z_0)$.*

In this work, we improve the results of Liu, Long, and Zeng for higher-order linear differential equations of the form (2.1.1) where most of the coefficients are of $[p,q]$-order. Firstly, we investigate the growth of solutions of (2.1.1) when $A_0(z)$ is a dominant coefficient with the concept of lower order.

Theorem 2.1.4 ([12]) *Let $p\geq q\geq 1$ be integers, and let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0):(j=2,...,k-1),\mu_{[p,q]}(A_1,z_0)\}<\mu_{[p,q]}(A_0,z_0).$$

Then every non trivial solution that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1.1) satisfies $\rho_{[p,q]}(f,z_0)=+\infty$ and $\rho_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0)$.

To prove the Theorem 2.1.4 we need the following lemmas.

Lemma 2.1.1 ([12]) *Let $f(z)$ be a nonconstant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\mu_{[p,q]}(f,z_0)=\mu$. Then for $\alpha>\mu$, there exist a set $E\subset(0,1)$ with $m_l(E)=+\infty$ such that for all $|z-z_0|=r\in E$, we have*

$$M_{z_0}(r,f)\leq\exp_p\left\{\log_{q-1}\left(\frac{1}{r}\right)^{\alpha}\right\}.$$

Proof. By the definition of $\mu_{[p,q]}(f,z_0)$, there exists a sequence $\{r_n\}_n^{\infty}$ tending to zero satisfying $r_{n+1}<\frac{n}{n+1}r_n$ and

$$\liminf_{n\to+\infty}\frac{\log_{p+1}M_{z_0}(r_n,f)}{\log_q(\frac{1}{r_n})}=\mu<\alpha.$$

Then, for any given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}^+$ such that for all $n \geq n_0$,

$$\log M_{z_0}(r_n, f) \leq \exp_p \left\{ \log_q \left(\frac{1}{r_n} \right)^{\alpha-\varepsilon} \right\} \tag{2.1.2}$$

For ε given above, there exists an $n_1 \in \mathbb{N}^+$ such that for all $n \geq n_1$ and $r \in \left[\frac{n}{n+1} r_n, r_n\right]$,

$$\frac{1}{(\frac{n}{n+1})^{\alpha-\varepsilon}} \leq \frac{1}{r^{\varepsilon}} \tag{2.1.3}$$

By 2.1.2 and 2.1.3, for all $n \geq n_2 = \max\{n_0, n_1\}$ and for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$,

$$\log M_{z_0}(r, f) \leq \log M_{z_0}\left(\frac{n}{n+1} r_n, f\right) \leq \exp_p \left\{ \log_q \left(\frac{1}{\frac{n}{n+1} r_n} \right)^{\alpha-\varepsilon} \right\}$$

$$\leq \exp_p \left\{ \log_q \frac{1}{\left(\frac{n}{n+1}\right)^{\alpha-\varepsilon} r_n^{\alpha-\varepsilon}} \right\} \leq \exp_p \left\{ \log_q \left(\frac{1}{r} \right)^{\alpha} \right\}.$$

This is implies

$$M_{z_0}(r, f) \leq \exp_p \left\{ \log_{q-1} \left(\frac{1}{r} \right)^{\alpha} \right\}.$$

Set $E = \bigcup_{n=n_2}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$m_l(E) = \int_E \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \log(1 + \frac{1}{n}) = \infty.$$

$\square$

Lemma 2.1.2 ([6]) *Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, let $\gamma > 1$, $\varepsilon > 0$ be given real constants and $k \in \mathbb{N}$. Then there exist a set $E_3 \subset (0, r_0]$, $(r_0 \in (0,1))$ having finite logarithmic measure and a constant $\lambda > 0$ that depends on γ and k such that for all $|z - z_0| = r \in (0, r_0] \setminus E_3$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \lambda \left[\frac{1}{r^2} T_{z_0}(\frac{1}{\gamma} r, f) \log T_{z_0}(r, f) \right]^k.$$

2.2 Proof of Theorem 2.1.4

Proof. Set $\rho = \max\{\rho_{[p,q]}(A_j, z_0) : (j = 2, ..., k-1), \mu_{[p,q]}(A_1, z_0)\} < \mu_{[p,q]}(A_0, z_0)$. For any given ε $(2\varepsilon < \mu_{[p,q]}(A_0, z_0) - \rho)$, there exists $r_1 \in (0,1)$, such that for all $|z - z_0| = r \in (0, r_1)$

$$|A_j(z)| \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho+\varepsilon} \right\}, \ j = 2, .., k-1 \tag{2.2.1}$$

and

$$|A_0(z)| \geq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon} \right\}. \tag{2.2.2}$$

By Lemma 2.1.1, there exists a set $E \subset (0,1)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E$,

$$|A_1(z)| \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_1,z_0)+\varepsilon} \right\} \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho+\varepsilon} \right\}. \tag{2.2.3}$$

We rewrite (2.1.1) as

$$|A_0(z)| \leq \left| \frac{f^{(k)}(z)}{f(z)} \right| + \left| \frac{f^{(k-1)}(z)}{f(z)} \right| |A_{k-1}(z)| + \cdots + \left| \frac{f'(z)}{f(z)} \right| |A_1(z)| . \tag{2.2.4}$$

By Lemma 2.1.2, there exist a set $E_2 \subset (0, r_0]$, $(r_0 \in (0,1))$ that has a finite logarithmic measure and a constant $\lambda > 0$ that depends on $\alpha > 1$ and $j = 1, 2, ..., k$ such that for all $r = |z - z_0|$ satisfying $r \in (0, r_0] \setminus E_2$, we obtain

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{1}{\alpha} r, f \right) \log T_{z_0} \left(\frac{1}{\alpha} r, f \right) \right]^j, (j = 1, 2, ..., k). \tag{2.2.5}$$

Substituing 2.2.1, 2.2.2, 2.2.3 and 2.2.5 in 2.2.4, we have

$$\begin{aligned} \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon} \right\} &\leq \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{1}{\alpha} r, f \right) \log T_{z_0} \left(\frac{1}{\alpha} r, f \right) \right]^k \\ &+ \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{1}{\alpha} r, f \right) \log T_{z_0} \left(\frac{1}{\alpha} r, f \right) \right]^{k-1} \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho+\varepsilon} \right\} \\ &+ \cdots + \lambda \left[\frac{1}{r^2} T_{z_0} \left(\frac{1}{\alpha} r, f \right) \log T_{z_0} \left(\frac{1}{\alpha} r, f \right) \right] \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho+\varepsilon} \right\}. \end{aligned} \tag{2.2.6}$$

By 2.2.6, we get

$$\exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon} \right\} \leq \lambda k \left[\frac{1}{r^2} T_{z_0} (\frac{1}{\alpha} r, f) \log T_{z_0} (\frac{1}{\alpha} r, f) \right]^k \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho+\varepsilon} \right\}$$

which implies

$$\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\leq k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k}\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho+\varepsilon}\right\}.$$

Since ε $(2\varepsilon<\mu_{[p,q]}(A_0,z_0)-\rho)$, we obtain

$$\exp_p\left\{(1-o\,(1))\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\leq k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k}.$$

It follow that

$$\begin{aligned}(1-o\,(1))\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}&\leq\log_p\left(k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k}\right)\\&\leq\log_p(k\lambda)+\log_p(\frac{1}{r})+\log_p(T_{z_0}(\frac{1}{\alpha}r,f)))+O(1)\end{aligned}$$

which implies that

$$(\mu_{[p,q]}(A_0,z_0)-\varepsilon)\log_q(\frac{1}{r})+\log\,(1-o\,(1))$$
$$\leq\log_{p+1}(k\lambda)+\log_{p+1}(\frac{1}{r})+\log_{p+1}(T_{z_0}(\frac{1}{\alpha}r,f))+O(1).$$

Hence $\mu_{[p,q]}(A_0,z_0)-\varepsilon\leq\rho_{[p+1,q]}(f,z_0)$, since $\varepsilon>0$ is arbitrary, we conclude that $\mu_{[p,q]}(f,z_0)=+\infty$ and $\mu_{[p,q]}(A_0,z_0)\leq\rho_{[p+1,q]}(f,z_0)$. $\square$

Theorem 2.2.1 ([12]) *Let $p\geq q\geq 1$ be integers, and let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0):(j=2,...,k-1)\}\leq\mu_{[p,q]}(A_1,z_0)=\mu_{[p,q]}(A_0,z_0)$$

and

$$\max\{\tau_{[p,q],M}(A_j,z_0):(j=2,...,k-1),\underline{\tau}_{[p,q],M}(A_1,z_0)\}<\underline{\tau}_{[p,q],M}(A_0,z_0).$$

Then every non trivial solution that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1.1) *satisfies $\rho_{[p,q]}(f,z_0)=+\infty$ and $\rho_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0)$.*

To prove the Theorem 2.2.1, we need the following lemma.

Lemma 2.2.1 ([12]) *Let $f(z)$ be a nonconstant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\mu_{[p,q]}(f,z_0)=\mu\in(0,\infty)$ and $\underline{\tau}_{[p,q],M}(f,z_0)=\underline{\tau}$. Then for any $\beta>\underline{\tau}$, there exits a set $E\in(0,1)$ with $m_l\,(E)=+\infty$ such that for all $|z-z_0|=r\in E$,*

$$M_{z_0}(r,f)\leq\exp_p\left\{\beta\log_q\left(\frac{1}{r}\right)^{\mu}\right\}.$$

Proof. We will use the definition of lower $[p,q]$ −type, there exists a sequence $\{r_n\}_n^{\infty}$ tending to zero satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and

$$\liminf \frac{\log_p M_{z_0}(r_n, f)}{\log_q \left(\frac{1}{r_n}\right)^{\mu}} < \beta.$$

Then, for any given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}^+$ such that for all $n > n_0$,

$$M_{z_0}(r_n, f) \le \exp_p \left\{ (\beta - \varepsilon) \log_q \left(\frac{1}{r_n} \right)^{\mu} \right\}. \tag{2.2.7}$$

For ε given above, there exists an $n_1 \in \mathbb{N}^+$ such that for all $n \ge n_1$ and $r \in \left[\frac{n}{n+1} r_n, r_n\right]$,

$$(\beta - \varepsilon) \log_q \left(\frac{1}{\frac{n}{n+1} r_n} \right)^{\mu} < \beta \log_q \left(\frac{1}{r} \right)^{\mu}. \tag{2.2.8}$$

Combinig 2.2.7 and 2.2.8, for all $n \ge n_2 = \max\{n_0, n_1\}$ and for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$,

$$\begin{aligned} M_{z_0}(r, f) &\le M_{z_0}(\frac{n}{n+1} r_n, f) \\ &\le \exp_p \left\{ (\beta - \varepsilon) \log_q \left(\frac{1}{\frac{n}{n+1} r_n} \right)^{\mu} \right\} \\ &\le \exp_p \left\{ \beta \log_q \left(\frac{1}{r} \right)^{\mu} \right\}. \end{aligned}$$

Set $E = \bigcup\limits_{n=n_2}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$m_l(E) = \int_E \frac{dt}{t} = \sum_{n=n_2}^{\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n>n_2}^{\infty} \log(1 + \frac{1}{n}) = \infty.$$

□

2.3 Proof of Theorem 2.2.1

Proof. Setting $\max\left\{\rho_{[p,q]}(A_j, z_0) : (j = 2, ..., k-1)\right\} = \rho$, $\max\left\{\tau_{[p,q],M}(A_j, z_0) : (j = 2, ..., k-1)\right\} = \tau$, $\underline{\tau} = \underline{\tau}_{[p,q],M}(A_0, z_0)$. For any given $\varepsilon (\varepsilon < \frac{\underline{\tau} - \tau}{2})$, by the definition of $\underline{\tau}_{[p,q]}(A_0, z_0)$, there exists $r_0 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_0)$ and $|A_0(z)| = M_{z_0}(r, A_0)$, we have

$$|A_0(z)| \ge \exp_p\{(\underline{\tau} - \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \} \tag{2.3.1}$$

and

$$|A_j(z)| \leq \exp_p\{(\tau+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\rho}\}$$
$$\leq \exp_p\{(\tau+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\}, (j=2,...,k-1). \tag{2.3.2}$$

By Lemma 2.2.1 to $A_1(z)$, there exists a set $E_2 \subset (0,1)$ having infinite logarithmic measure such that for all $|z-z_0| = r \in E_2$, we have

$$|A_1(z)| \leq \exp_p\{(\underline{\tau}_{[p,q]}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_1,z_0)}\}$$
$$\leq \exp_p\{(\underline{\tau}_{[p,q]}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\}. \tag{2.3.3}$$

By Lemma 2.1.2 there exist a set $E_1 \subset (0, r_0], (r_0 \in (0,1))$ that has a finite logarithmic measure and a constant $\lambda > 0$ that depends on $\alpha > 1$ and $j = 1, 2, ..., k$ such that for all $r = |z - z_0|$ satisfying $r \in (0, r_0] \setminus E_2$, we obtain 2.2.5 From 2.1.1 we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|\frac{f^{(k-1)}(z)}{f(z)}\right| |A_{k-1}(z)| + \cdots + \left|\frac{f'(z)}{f(z)}\right| |A_1(z)|. \tag{2.3.4}$$

Set $E_0 = E_2 \setminus E_1$, obviously, $m_l(E_0) = \infty$. Combining 2.3.1, 2.3.2, 2.3.3 and 2.2.5 into 2.3.4, we obtain

$$\exp\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\}$$
$$\leq k\lambda\left[\frac{1}{r^2}T_{z_0}(\frac{1}{\alpha}r,f)\log T_{z_0}(\frac{1}{\alpha}r,f)\right]^k \exp_p\{(\tau+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\}. \tag{2.3.5}$$

From 2.3.5, we get

$$\exp\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\} \leq k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r,f)\right]^{2k} \exp_p\{(\tau+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\}. \tag{2.3.6}$$

It follow that for all $|z-z_0| = r \in (0, r_0] \setminus E_1$, and $|A_0(z)| = M_{z_0}(r, A_0)$, where $\lambda > 0$ is a constant. We deduce that $\mu_{[p,q]}(A_0, z_0) \leq \rho_{[p+1,q]}(f, z_0)$. □

Theorem 2.3.1 ([12]) *Let $p \geq q \geq 1$ be integers, and let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j, z_0) : (j = 2, ..., k-1), \mu_{[p,q]}(A_0, z_0)\} < \mu_{[p,q]}(A_1, z_0).$$

Then every non trivial solution that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ *of* (2.1.1) *satisfies* $\mu_{[p,q]}(A_1, z_0) \leq \rho_{[p,q]}(f, z_0)$.

To prove the Theorem 2.3.1 we need the following lemmas.

Lemma 2.3.1 ([3]) *Let* $f(z)$ *be a nonconstant meromorphic function in* $\overline{\mathbb{C}} - \{z_0\}$. *Then the following statements hold:*

$$(i) \quad T_{z_0}\left(r, \frac{1}{f}\right) = T_{z_0}(r, f) + O(1),$$

$$(ii) \quad T_{z_0}(r, f') < O(T_{z_0}(r, f) + \log\frac{1}{r}),\ r \in (0, r_0] \setminus E,$$

where $E \subset (0, r_0]$ *with* $m_l(E) < \infty$.

Lemma 2.3.2 ([12]) *Let* $f_1(z)$ *be an analytic function in* $\overline{\mathbb{C}} - \{z_0\}$ *satisfying* $\mu_{[p,q]}(f_1, z) = \mu_1 > 0$, *and* $f_2(z)$ *be an analytic function in* $\overline{\mathbb{C}} - \{z_0\}$ *satisfying* $\rho_{[p,q]}(f_2, z) = \rho_2 < \infty$, $\rho_2 < \mu_1 < \infty$. *Then there exists a set* $E \subset (0, 1)$ *having infinite logarithmic measure such that for all* $|z - z_0| = r \in E$, $\lim\limits_{r\to 0} \frac{T_{Z_0}(r, f_2)}{T_{z_0}(r, f_1)} = 0$.

Proof. By the definition of $\mu_{[p,q]}(f, z_0) = \liminf\limits_{r\to 0} \frac{\log_p T_{z_0}(r,f)}{\log_q(\frac{1}{r})}$, for any given $\varepsilon \in (0, \frac{\mu_1 - \rho_2}{2})$, there exists $r_0 \in (0, 1)$, such that for all $|z - z_0| = r \in (0, r_0)$,

$$T_{z_0}(r, f_1) > \exp_p\{\log_q \left(\frac{1}{r}\right)^{\mu_1 - \varepsilon}\} \tag{2.3.7}$$

we also apply the definition of $\rho_{[p,q]}(f_2, z_0) = \rho_2$ we have,

$$T_{z_0}(r, f_2) < \exp_p\{\log_q \left(\frac{1}{r}\right)^{\rho_2 + \varepsilon}\}. \tag{2.3.8}$$

It follow from 2.3.7and 2.3.8 that for all $r \in E$, we obtain the result

$$0 < \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} \leq \frac{\exp_p\{\log_q \left(\frac{1}{r}\right)^{\rho_2 + \varepsilon}\}}{\exp_p\{\log_q \left(\frac{1}{r}\right)^{\mu_1 - \varepsilon}\}} \to 0,\ r \to 0.$$

□

2.4 Proof of Theorem 2.3.1

Proof. By (2.1.1) we have

$$m_{z_0}(r, A_1) \leq \sum_{j=0,\ j\neq 1}^{k} m_{z_0}\left(r, \frac{f^{(j)}(z)}{f'(z)}\right) + \sum_{j=0,\ j\neq 1}^{k-1} m_{z_0}(r, A_j(z)) + \log k. \tag{2.4.1}$$

By Lemma 2.3.1, for a constant $r_2 \in (0,1)$, there is a set $E_3 \subset (0, r_2]$ with $m_l(E_3) < +\infty$ such that for all $|z - z_0| = r \in (0, r_2] \setminus E_3$, we have

$$\sum_{j=0,\ j\neq 1}^{k} m_{z_0}\left(r, \frac{f^{(j)}(z)}{f'(z)}\right) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right). \tag{2.4.2}$$

By applying Lemma 2.3.2, for any $\varepsilon \in (0, \frac{1}{2(k-1)})$ there exists a set $E_4 \subset (0, r_2)$ with $m_l(E_4) = \infty$ such that for sufficiently small $|z - z_0| = r \in E_4$,

$$m_{z_0}(r, A_j(z)) \leq \varepsilon m_{z_0}(r, A_1(z)),\ \ j \neq 1. \tag{2.4.3}$$

Combining 2.4.1, 2.4.2 and 2.4.3, for $|z - z_0| = r \in E_4 \setminus E_3$,

$$m_{z_0}(r, A_1(z)) \leq O(T_{z_0}(r, f) + \log \frac{1}{r}) + \varepsilon\,(k-1)\, m_{z_0}(r, A_1(z)) + \log k. \tag{2.4.4}$$

This implies that

$$m_{z_0}(r, A_1(z)) \leq CT_{z_0}(r, f) + C \log \frac{1}{r} + C_1 \tag{2.4.5}$$

with $C > 0$ and $C_1 > 0$ two positive constants, which we can write

$$T_{z_0}(r, A_1(z)) \leq CT_{z_0}(r, f) + C \log \frac{1}{r} + C_1.$$

Hence $\mu_{[p,q]}(A_1, z_0) \leq \rho_{[p,q]}(f, z_0)$. $\square$

2.5 Examples

Example 2.5.1 $f(z) = \exp_3\left\{\frac{1}{1-z}\right\}$ *is a solution of the following equation*

$$f'' + A_1(z)f' + A_0(z)f = 0,$$

where

$$A_0(z) = -\frac{1}{(1-z)^4} \exp\left\{2\exp\left(\frac{1}{1-z} + \frac{2}{1-z}\right)\right\}$$

and

$$A_1(z) = \frac{1}{(1-z)^2}\exp\left\{\frac{1}{z-1}\right\} + \frac{1}{(1-z)^2} + \frac{2}{1-z}.$$

We have

$$\rho_{[2,1]}(A_1, 1) = 0 < \mu_{[2,1]}(A_0, 1) = 1.$$

Obviously, the conditions of Theorem 2.1.4 *are satisfied and we see that*

$$\rho_{[2,1]}(f, 1) = +\infty$$

and

$$\rho_{[3,1]}(f, 1) = \mu_{[2,1]}(A_0, 1) = 1.$$

Example 2.5.2 *Let* $f(z) = \frac{1}{z}\exp_2\left\{\frac{1}{z^2}\right\}$ *is a solution of the following linear differential equation*

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0,$$

where

$$\begin{aligned} A_0(z) &= -\frac{4}{z^9}\exp\left\{\frac{3}{z^2}\right\} - \left(\frac{20}{z^9} + \frac{30}{z^7}\right)\exp\left\{\frac{2}{z^2}\right\} \\ &\quad -\left(\frac{8}{z^9} + \frac{48}{z^7} + \frac{48}{z^5}\right)\exp\left\{\frac{1}{z^2}\right\} - \frac{6}{z^3} \\ A_1(z) &= -\frac{4}{z^4}\exp\left\{\frac{1}{z^2}\right\} \end{aligned} \tag{2.5.1}$$

and

$$A_2(z) = -\frac{1}{z^3}\exp\left\{\frac{1}{z^2}\right\}.$$

We have

$$\max\left\{\rho_{[1,1]}(A_2(z), z_0), \mu_{[1,1]}(A_1(z), z_0)\right\} = \max\{2, 2\} = \mu_{[1,1]}(A_0(z), z_0) = 2$$

and

$$\max\left\{\tau_{[1,1],M}(A_2(z), z_0), \underline{\tau}_{[1,1],M}(A_1(z), z_0)\right\} = 1 < \underline{\tau}_{[1,1],M}(A_0(z), z_0) = 3.$$

We see that the conditions of Theorem 2.2.1 *are satisfied, then*

$$\rho_{[1,1]}(f(z), z_0) = +\infty$$

and

$$\rho_{[2,1]}(f(z), z_0) = \mu_{[1,1]}(A_0(z), z_0) = 2.$$

CONCLUSION

Throughout this work, we investigated the growth of solution of the following linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \ (k \geq 2)$$

For this reason, we have been discussed the possibility of generalizing certain results related to second-order complex differential equations to the higher-order in analogous or different ways; at the same time, we studied the growth of the solutions of equation (1.1) in the neighborhood of a singular point using the concept of $[p, q]$-order of growth and extension for other results.

For example, we generalize the results of Lui, Long and Zeng: Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.1.3 to Theorems 2.1.4, 2.2.1 and 2.3.1.

From there, we hope to solve the following problem:

What can be said about the growth of the solutions of the above equation?

if we assume that the coefficients are all or most of lower $[p, q]$-order.

Other questions are raised about what happens to the growth of solutions for nonhomogeneous equations:

When the coefficients are meromorphic functions, are the results generalizable?

And under what conditions is this generalization valid?

Or, for non-homogeneous linear differential equations, what are the assumptions that ensure that every non-trivial solution is of infinite $[p, q]$ order?

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