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Abstract

This thesis deals with the study of the geometry of curves on Three dimensional Lorentzian manifolds. The aim of this thesis is to classify f-biharmonic curves in three dimensional Lorentzian manifolds. The f-Biharmonic curves are generalizations of harmonic curves and biharmonic curves. The motivation to study f-biharmonic curves in pseudo-Riemannian manifolds are it helps to bridge the gap between modern di–erential geometry and the mathematical physics of general relativity. We obtain some descriptions of f-biharmonic curves in a three dimensional generalized symmetric spaces. Also we classify f-biharmonic and biharmonic curves in three dimensional generalized symmetric spaces.

Keywords: harmonic maps, biharmonic maps, f-biharmonic curve, Generalized symmetric spaces,

Résumé

Cette thèse porte sur l'étude de la géométrie des courbes dans les variétés lorentziennes de dimension trois. Le but de cette thèse est de classer les courbes $f-$biharmoniques dans les variétés lorentziennes de dimension trois. Les courbes $f-$biharmoniques sont des généralisations des courbes harmoniques et des courbes biharmoniques. La motivation pour étudier les courbes $f-$biharmoniques dans les variétés pseudo-riemanniennes est qu'elles aident á combler le fossé entre la géométrie différentielle moderne et la physique mathématique de la relativité générale. Nous obtenons quelques descriptions de courbes $f-$biharmoniques dans des espaces symétriques généralisés de dimension trois. Nous classons également les courbes $f-$biharmoniques et biharmoniques dans des espaces symétriques généralisés de dimension trois.

Mots cles: courbes harmoniques, courbes biharmoniques, espaces symétrique généralisées

Introduction

"...$\mathcal{T}$hus, in a sense, mathematics has been most advanced by those who distinguished themselves by intuition rather than by rigorous proofs". $\mathcal{F}$elix Klein

The concept of harmonicity extends beyond the typical understanding in mappings between Euclidean spaces, with notable instances such as geodesic curves. Harmonic and biharmonic maps are solutions to second and fourth-order nonlinear elliptic systems of equations, respectively. These equations pose significant challenges in solving due to their complexity in both cases. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two pseudo-Riemannian manifolds of dimension m and n respectively. We will call $e(\varphi)_x = \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i))$ the energy density of φ at $x \in M$ for any $\{e_i\}_{i=1}^m$ orthonormal basis of the tangent space T_xM can then be integrated over M, The energy or Dirichlet integral of φ over $\Omega \subset M$ is defined as

$$E(\varphi) = \frac{1}{2}\int_\Omega |d\varphi|^2 dv_g. \tag{1}$$

where dv_g is the volume measure on M defined by the metric g and $|d\varphi|^2$ the Hilbert-Schmidt norm of $d\varphi$.

A harmonic map φ is a critical point of the energy functional E. The map φ being harmonic means that

$$\frac{d}{dt}E(\varphi_t)\Big|_{t=0} = 0,$$

holds for arbitrary smooth variation φ_t of φ, and denote the tension field $\tau(\varphi)$ of φ by

$$\tau(\varphi) = tr_g\nabla d\varphi = \sum_{i=1}^{m} \epsilon_i\Big(\nabla^\varphi_{e_i}e_i - d\varphi(\nabla_{e_i}e_i)\Big), \tag{2}$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$.

In 1986 G.Y Jiang [48] introduced the concept of biharmonic maps $\varphi : (M^m, g) \longrightarrow (N^n, h)$ defined as critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_\Omega h(\tau(\varphi), \tau(\varphi))dv_g. \tag{3}$$

The Euler-Lagrange equation attached to the bienergy is

$$\tau_2(\varphi) = 0, \tag{4}$$

where $\tau_2(\varphi)$ is the bitension field given by

$$\tau_2(\varphi) = -(\Delta^{\varphi}\tau(\varphi) + tr_g R^N(\tau(\varphi), d\varphi)d\varphi), \tag{5}$$

Numerous researchers have delved into the study of such harmonic maps, among them notable figures like W.J.LU [59],S.Degla [26], and R.Caddeo [20],[21].

Certainly, every harmonic map is also biharmonic, establishing harmonic maps as a subset within the broader category of biharmonic maps. It's noteworthy that, excluding maps between Euclidean spaces where any polynomial map with a degree lower than four qualifies as biharmonic there is a limited discovery of proper (non-harmonic) biharmonic maps between Riemannian manifolds.

In 2021 M. Belarbi, L.Belarbi and H. Elhendi [?] classified the biharmonic maps in three-dimensional generalized symmetric spaces and $Sol3$ became a particular consequence.

In 2024 [2]we classified the f-biharmonic curves in three-dimensional generalized symmetric spaces.

Let $f : M \to \mathbb{R}$ be a smooth positive function on M. The f-energy functional of the map φ is given by

$$E_f(\varphi) = \frac{1}{2}\int_{\Omega} f h(d\varphi(e_i), d\varphi(e_i))d\nu_g, \tag{6}$$

A map φ is called f-harmonic if it is a critical point of the energy functional E_f. The Euler-Lagrange equation attached to the f-energy is

$$\tau_f(\varphi) = 0,$$

where $\tau_f(\varphi)$ is the f-tension field given by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(gradf).$$

On the other hand the f-bienergy functional of the map φ is defined by

$$E_{2,f}(\varphi) = \frac{1}{2}\int_{\Omega} f h(\tau(\varphi), \tau(\varphi))dv_g. \tag{7}$$

A map φ is called f-biharmonic if it is a critical point of the energy functional $E_{2,f}$. The Euler-Lagrange equation attached to the f-bienergy is

$$\tau_{2,f}(\varphi) = 0, \tag{8}$$

where $\tau_{2,f}(\varphi)$ is the f-bitension field given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \triangle f\tau(\varphi) + 2\nabla^{\varphi}_{gradf}\tau(\varphi). \tag{9}$$

In this theses, we study the f-biharmonicity curves in three-dimensional generalized symmetric spaces and we give the necessary and sufficient conditions for $f-$biharmonic curves in $\mathbb{M}_3$; We will consider the metric

$$g_{\varepsilon,\lambda} = \varepsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{10}$$

where $\varepsilon_i = \pm 1$, and $\lambda \neq 0$ also we consider a curve $\gamma : I \subset \mathbb{R} \to (\mathbb{M}_3, g)$ and proof the following theorems and lemmas

Theorem 1. *the f-biharmonic equation* (9) *reduces to the system*

$$\begin{cases} -3\varepsilon f\kappa\kappa' - 2\varepsilon f'\kappa^2 = 0 \\ \\ f\kappa'' - f\kappa^3 - f\varepsilon\varepsilon_1\tau^2\kappa + f''\kappa + 2f'\kappa' - \dfrac{2f\kappa B_3^2}{\lambda} + \dfrac{f\kappa}{\lambda} = 0 \\ \\ f\varepsilon(\tau'\kappa + 2\tau\kappa') + 2\varepsilon f'\kappa\tau + f\varepsilon_1\kappa\dfrac{2}{\lambda}\big(N_3B_3\big) = 0, \end{cases} \tag{11}$$

here, κ denote the geodesic curvature, τ the torsion of the curve, and N_3 and B_3 the components of the normal and bi-normal vector fields of the Frenet frame field, respectively.

A discussion follows to derive necessary and sufficient conditions for proper f-harmonicity.

Theorem 2. *γ is a proper f-biharmonic curve if and only if one of the following cases holds*

1) γ has zero torsion, and f is given by $f = c_1\kappa^{-\frac{3}{2}}$. Its curvature satisfies the following differential equation:

$$3(\kappa')^2 - 2\kappa''\kappa = 4\kappa^2\left(\kappa^2 + \frac{2B_3^2}{\lambda} + \frac{1}{\lambda}\right)$$

2) γ has non-zero torsion with $\dfrac{\tau}{\kappa} = \dfrac{1}{c_1^2}\exp\left(\displaystyle\int -\frac{2\varepsilon N_3B_3}{|\lambda|\tau}\,ds\right)$, and f is given by $f = c_1\kappa^{-\frac{3}{2}}$. Its curvature satisfies the following differential equation

$$3(\kappa')^2 - 2\kappa''\kappa = 4\kappa^2\left(\kappa^2 + \frac{\kappa}{c_1^2}\exp\left(\int -\frac{2\varepsilon N_3B_3}{|\lambda|\tau}\,ds\right) + \frac{2B_3^2}{\lambda} + \frac{1}{\lambda}\right)$$

The thesis is divided into four chapters, each serving a distinct purpose

Chapter 1 In this inaugural chapter, we lay the foundational groundwork by defining key concepts such as topological manifolds, differentiable manifolds, maps between smooth manifolds and tangent spaces. Additionally, we introduce fundamental notions of vector fields, Lie brackets, and tensor analysis,especially the metric tensor, providing a solid framework for the subsequent discussions.

Chapter 2 The second chapter focuses on the detailed exploration of Riemannian and pseudo Riemannain manifolds, tangent and cotangent bundles associated with Riemannian manifolds. We define these bundles, offer illustrative examples showcasing their importance, and delve into the intricacies of curvature and connection within these bundles in the context of Riemannian geometry.

Chapter 3 In the third chapter, we broaden our scope by introducing a significant generalization of harmonic maps known as f-bi-harmonic maps. These mappings serve as critical points of the f-bi-energy functional, which we define and analyze. We derive the Euler-Lagrange equation governing the behavior of the f-bi-energy functional, characterized by the vanishing of the f-bi-tension field.

Chapter 4 In this final chapter, we consolidate our investigation, leveraging the results and properties established in the preceding chapters. Our focus is on studying the f-biharmonicity of a curve defined within a generalized symmetric space equipped with its metric. Furthermore, we present proofs for existence theorems, relying on intrinsic properties of the curve.

Notation

- (U, ϕ): Chart on a manifold M.
- $\{(U_i, \phi_i)\}_{i \in I}$: Atlas on a manifold M.
- $\mathcal{C}^\infty(M)$: Smooth functions on M.
- e_i: Unitary base vectors in a general coordinate system.
- $\Gamma(TM)$: Set of vector fields on M.
- $\Gamma(T^*M)$: Set of differential forms on M.
- $\otimes$: Tensor product (of vectors or forms).
- g: Metric tensor on M.
- ∇: Linear connection on (M, g).
- Γ^j_{ki}: Components of the connection (Christoffel symbols).
- R_{lijk}: Components of the Riemann curvature tensor.
- $\mathcal{L}_X$: Lie derivative.

- f^*: Pull back by f.
- $<,>$: Euclidean scalar product.
- ω_g: Volume form $\omega_g = \sqrt{|g_{ij}|}\, dx^1 \wedge \ldots \wedge dx^m$.
- $[\ ,\ ]$: Lie bracket of two vector fields.
- ∂_k: Derivation with respect to the k-th variable.

Chapter 1

Preliminaries

"$\mathcal{O}$ne geometry cannot be more true than another;
it can only be more convenient" $\mathcal{H}$enri $\mathcal{P}$oincare, La science & l'hypothese

The primary objective of this chapter is to present the fundamental concepts of a differentiable manifold, including tangent spaces, where its elements are interpreted as first-order linear differential operators.These operators annihilate constants and act on real-valued functions locally defined around a point on the manifold.The focus of this chapter lies on vector fields and operations associated with them, which will be elaborated upon in the third section.

The fourth section delves into tensor analysis, particularly introducing the concept of the metric tensor. Another section is dedicated to exploring the significant class of metric properties. Subsequent sections of the chapter address various topics such as differential forms. The material of this chapter borrows from many sources, including [3],[6],[7],[54],[60],[[66]]

1.1 Differential manifolds

For many problems both in mathematics and beyond, manifolds are the natural class of underlying spaces with which to work. Apart from its own intrinsic interest, a knowledge of differentiable manifolds has become useful even mandatory in an ever-increasing number of areas of mathematics and of its applications. This is not too surprising, since differentiable manifolds are the underlying, if unacknowledged, objects of study in much of advanced calculus and analysis.
Generally speaking, a manifold of dimension n is a topological space locally homeomorphic [1] to $\mathbb{R}^n$.The idea of a differentiable manifold had its genesis in the nineteenth century with the work of C. F. Gauss and of B. Riemann.Gauss was interested in surveying and cartography, which led him to develop the tools of calculus on curved surfaces. His famous theorema egregium, or remarkable theorem, revealed that one could consider the intrinsic properties of a surface independently of the way in which it was embedded in three-dimensional space, and this led him, Riemann, and others, to abstract these concepts even further. Their ideas have had far reaching applications in many areas of mathematics and the natural sciences.

[1] **Homeomorphism** from Greek roots meaning similar shape, named by Henri Poincare', are continuous map with a continuous inverse that preserve all topological properties.

Definition 1. *[Topological manifold]*

Let M be a Hausdorff [2] *and seconde countable topological space* [3]*. Let m be a positive integer. If, for every x in M, there exists an open neighborhood U of x such that U is homeomorphic to some open subset of Euclidean space $\mathbb{R}^m$, then we say that M is an m-dimensional topological manifold.*

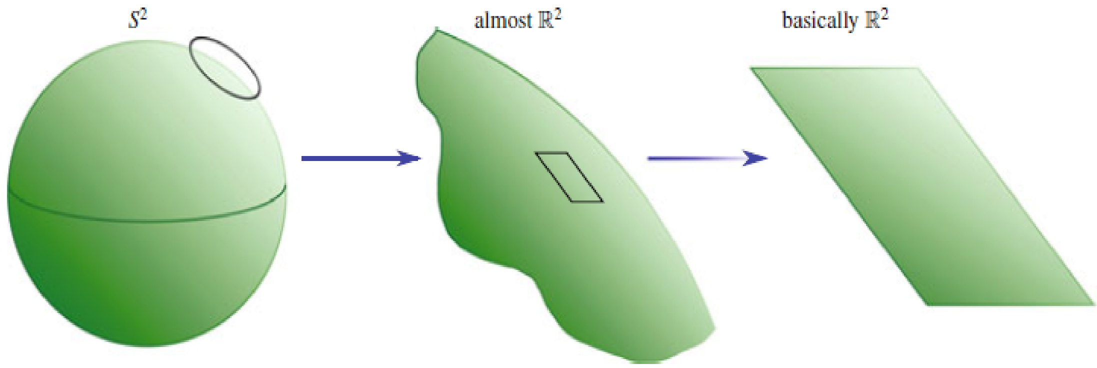

Figure 1.1: The sphere S^2 looks locally as $\mathbb{R}^2$

The word manifold comes from German. Specifically, Riemann used the term Mannigfaltigkeit in his PhD thesis to describe a certain generalization of surfaces. This was translated to manifoldness by Clifford. Prior. Mathematicians had classically studied geometry, first Euclidean, then spherical and hyperbolic. Surfaces were studied in depth, including by Riemann. It is known the first systematic account of the field of topology was in Analysis situs by Poincare', and the first definition he wrote down was of what he called a manifold.

Definition 2. *[Local charts]*

Let M be an n dimensional topological manifold. A local chart (the term chart is inspired from Cartography) on M is a pair (U, φ) such that U is a subset of M called the domain of the chart, and φ is a one-to-one map from U onto some open subset of $\mathbb{R}^n$.

Remark 1. *1) Defining a chart (U, φ) on M amounts to labeling each point $p \in U$ by means of n real numbers, since $\varphi(p) = (x_1(p), x_2(p), \cdots, x_n(p)) \in \mathbb{R}^n$. This relation defines the n functions $x_1, x_2, \ldots, x_n$, which will be called the coordinate functions or simply local coordinates, associated with the chart (U, φ). The fact that φ is a one-to-one mapping ensures that two different points of U differ, at least, in the value of one of the coordinates.*
2)The coordinates associated with any chart (U, φ) must be functionally independent among themselves, since the definition of a chart requires that $\varphi(U)$, denoted as $\{\varphi(p) \mid p \in U\}$, be an open subset of $\mathbb{R}^n$. If, for instance, the coordinate x_n could be expressed as a function of $x_1, x_2, \ldots, x_{n-1}$, then the points $\varphi(p)$ $(p \in U)$ would lie in a hypersurface of $\mathbb{R}^n$, which is not an open subset of $\mathbb{R}^n$.

Some points of M may belong to the domain of more than one chart. In order to have a well-defined meaning of the differentiability of the mappings between manifolds we shall

[2] The Hausdorff property says that any two distinct points can be separated by disjoint open neighborhoods.
[3] [40] Analyses what happens when one of these properties fails to hold.

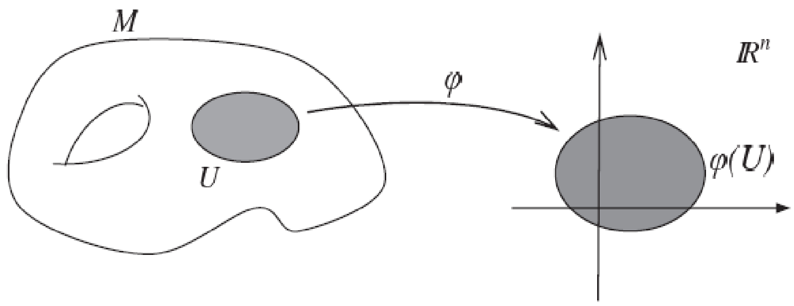

Figure 1.2: The image of U under φ must be an open subset of some $\mathbb{R}^n$

demand that the admissible coordinate systems be related to one another by means of differentiable functions.

Definition 3. *[Compatibility of charts]*
Let M be an n dimensional topological manifold.Two charts (U, φ) and (V, ψ) of M are called C^p-compatible $p \geq 1$, if one of the two conditions is satisfied

a- $U \cap V = \emptyset$.

b- *The change of charts map called also a transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a C^p-diffeomorphism as a maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.This means that all the partial derivatives up to and including the order p exist and are continuous; Moreover, these functions are invertible, and their inverses enjoy the same differentiability properties.*

The definition above says that if two coordinate systems overlap, they are related to each other in a sufficiently smooth way. Without this condition, a differentiable function in one coordinate system may not be differentiable in the other system.

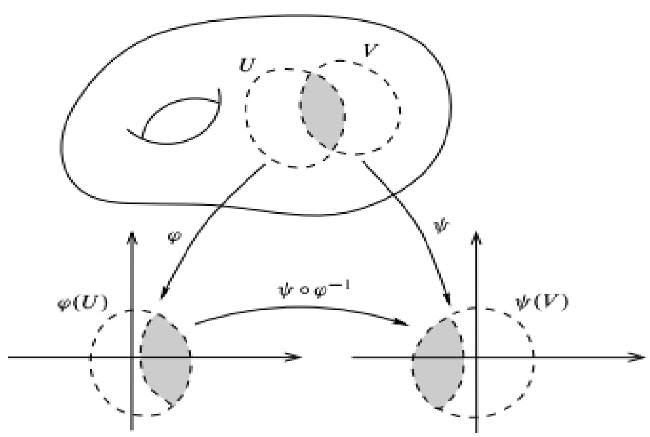

Figure 1.3: Transition maps between charts

Remark 2. 1. *Since the maps ψ and φ are homeomorphisms, the composition map $\varphi\circ\psi^{-1}$ is also a homeomorphism, and hence continuous. Therefore, any two charts on a topological manifold are C^0-compatible and it is by considering these changes of coordinates that we can enhance the structure of a topological manifold and introduce a differential structure on M. In this way, we can extend the familiar differential calculus known on $\mathbb{R}^n$ to this context.*

2. *In principle, the dimensions of the charts have not been fixed. Thus, one could have $\varphi(U) \subseteq \mathbb{R}^n$ and $\psi(V) \subseteq \mathbb{R}^m$ with $m \neq n$. However, if $U \cap V \neq \emptyset$, the fact that φ and φ^{-1} are diffeomorphisms implies that the two charts must be of the same dimension.*

3. *Let M be a manifold. A subset A of M is said to be open if for any chart (U, φ) belonging to the atlas of M, the set $\varphi(A \cap U)$ is open in $\mathbb{R}^n$.*

Definition 4. *[Atlas]*

Let M be an n dimensional topological manifold. An atlas of dimension n for M is a set $A = \{(U_\alpha, \phi_\alpha)\}$ of n-dimensional charts such that

(i) *The open sets U_α cover M ie $M = \bigcup_{\alpha\in I} U_\alpha$;*

(ii) *All charts in A are pairwise compatible.*

An atlas thus allows the definition of local coordinates everywhere on M. Two atlases are said to be equivalent if their union is still an atlas, i.e., $A = \{(U_\alpha, \phi_\alpha)\}$ and $A' = \{(V_\beta, \psi_\beta)\}$ are equivalent if all charts (U_α, ϕ_α) and (V_β, ψ_β) are pairwise compatible.

Definition 5. *[Maximal atlas]*

A $\mathcal{C}^p$-atlas for $p \geq 1$ is said to be a maximal atlas (also called a C^p-differentiable structure) if for every $(U, \phi) \in A$, we have $(V, \varphi) \in A$ for all (V, φ) charts that are $\mathcal{C}^p$-compatible with (U, ϕ). A maximal atlas, thus, contains all its compatible atlases. A is maximal in the sense that it is not strictly contained in a larger n-dimensional C^p atlas on M.

Remark 3. *To work effectively, it is important to choose the best possible atlases, that is, atlases with the fewest possible charts and the simplest possible transition functions-structures.*

It was H. Weyl in 1913 who first provided an intrinsic definition of a differential manifold which we will present below, followed by Whitney in 1936 for an abstract definition of a Riemannian manifold,which we will present in the next chapter.

Definition 6. *[Differentiable manifolds]*

An n-dimensional differentiable manifold of class C^p is an n-dimensional topological manifold M together with a maximal atlas of class C^p. For $p = 0$, one recovers the topological manifold. The C^∞ case delivers a smooth manifold, or simply a manifold.

Remark 4. *The provided definition of a manifold can be extended and generalized, particularly in the complex case. In this context, a complex manifold of dimension n is characterized by coordinate systems composed of complex numbers $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. The transition functions within this complex manifold are described as bi-holomorphic functions, meaning they are holomorphic, bijective, and possess holomorphic inverses.*

Definition 7. *[Orientation]*

An n-dimensional manifold M is orientable if it can be covered with an atlas such that all the transition functions preserve the orientation of $\mathbb{R}^n$. When the manifold is of class C^p with $p \geq 1$, orientability can be expressed analytically in terms of the Jacobian determinant of the transition functions. In other words, M is orientable if the Jacobian of the transformation $(x_i \to y_j)$ is positive i.e $\det\left(\frac{Dy_j}{Dx_i}\right) > 0$, where (x_i) and (y_j) are two coordinate systems on an open set $U \subset M$. shortly, in an oriented manifold, only those coordinate transformations that preserve the orientation are permitted.

The concept of orientation generalizes the notion of "direction of traversal" on a curve. Unlike curves, where there is always an orientation, in the case of manifolds of higher dimension, the existence of an orientation is a property.

Example 1. *Here we present some classical differentiable manifolds*

- *The Euclidean space $\mathbb{R}^n$ is the most trivial example, where a single chart covers the whole space and ϕ may be the identity map.*
- *Connected manifolds of dimension 1 are called curves.*
- *Any regular surface S in $\mathbb{R}^3$ equipped with the atlas associated with a family of regular parametrization is a differentiable manifold of dimension 2.*

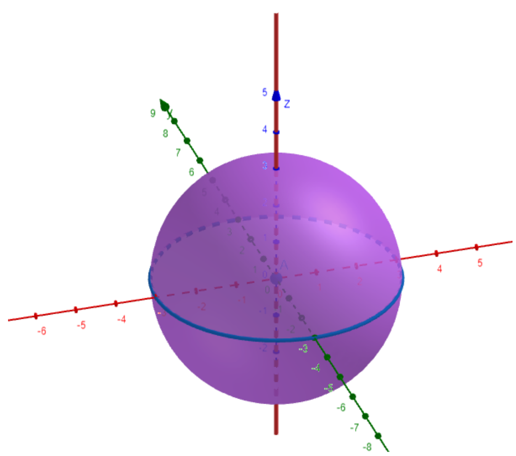

Figure 1.4: The 2-sphere S^2 as a surface of R^3

Example 2. *Consider the 2-sphere $S^2 \subset \mathbb{R}^3$ defined as $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. We will show that the sphere S^2 is a 2-dimensional manifold by considering the six hemispheres: $O_x^{\pm} = \{\pm x > 0\}$, $O_y^{\pm} = \{\pm y > 0\}$, and $O_z^{\pm} = \{\pm z > 0\}$. Each hemisphere is an open set on the sphere and is homeomorphic to a unit disk in $\mathbb{R}^2$ denoted by D. For instance, consider O_x^+. Define*

$$p_+ : O_x^+ \subset S^2 \to D \subset \mathbb{R}^2$$

$$(x, y, z) \to (y, z)$$

Then p_+ is a homeomorphism with inverse $(y,z) \mapsto (\sqrt{1-(y^2+z^2)}, y, z)$. Similarly, we can define $p_- : O_x^- \to D$ with inverse $(y,z) \mapsto (-\sqrt{1-(y^2+z^2)}, y, z)$. The six hemispheres equipped with the maps p_+ and p_- form an atlas on the sphere. Now, consider, for example, the intersection of hemispheres O_x^+ and O_y^+. The coordinate change corresponds to the map $(0,y,z) \mapsto (x,0,z)$. Viewed as a map on $\mathbb{R}^3$, this change of coordinates corresponds to a rotation, with the axis of rotation being the x-axis and an angle of $\frac{\pi}{2}$, in the y-z plane. This map is smooth ($C^\infty(\mathbb{R}^3)$), and thus, the sphere is a differentiable manifold.

This next theorem provide a alternative way other than the coordinates one to construct a manifold.

Theorem 3. *[3]*

Let $F : U \to \mathbb{R}^m$ be a C^∞ function on an open set $U \subset \mathbb{R}^{n+m}$ and take $c \in \mathbb{R}^m$. Assume that for each $a \in F^{-1}(c)$, the derivative

$$DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$$

is surjective. Then $F^{-1}(c)$ has the structure of an n-dimensional manifold which is Hausdorff and has a countable basis of open sets.

1.1.1 Differential maps on manifolds

Functions are the first "elementary objects" associated with a manifold: at a point $p \in M$, a number $f(p)$ is associated. We are familiar with the notions of differentiability and diffeomorphisms for applications between vector spaces.In this section, we will define these notions for applications between manifolds. The principle is always the same: we will say that an application between manifolds is differentiable (or is a diffeomorphism) if, when read in a chart, it is differentiable. Let's formalize this definition.

Definition 8. *[Differentiable function]*

Let M be an n dimensional smooth manifold and $p \in M$. A function $f : M \to \mathbb{R}$ is differentiable at $p \in M$ if for some chart (U, ϕ) at p, the function

$$f : \phi(U) \to \mathbb{R}$$

$$p \to f \circ \phi^{-1}(p)$$

is differentiable at $\phi(p)$.

The composition $f \circ \phi^{-1}(p)$ is a real-valued function defined on an open subset of $\mathbb{R}^n$, which may be differentiable or not . The differentiability of this composition does not depend on the chart chosen, since the charts of the atlas of M are C^k-related (for some $k \geq 1$).

For purposes of differentiation, a differentiable manifold locally has the structure of Euclidean space. Thus, the differentiability of a map can be tested in local coordinates. The diffeomorphism requirement for the chart transitions then guarantees that differentiability defined in this manner is a consistent notion, i.e. independent of the choice of a chart.

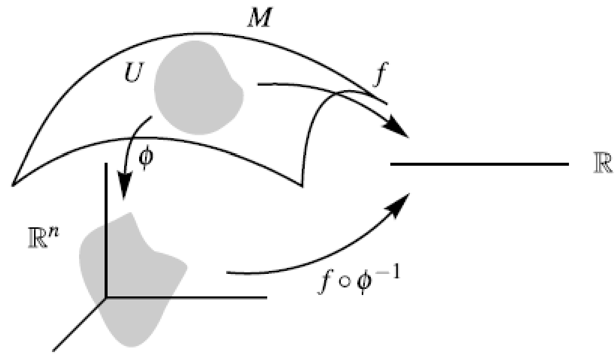

Figure 1.5: The representation of f in local coordinates

Definition 9. *[Differential maps between manifolds]*

Let M^m and N^n two differentiable manifolds, a mapping $f : M \longrightarrow N$ is said to be differentiable (or C^∞-differentiable), if for every chart (U, ϕ) of M and every chart (V, ψ) of N such that $f(U) \subset V$, the mapping $\tilde{f} = \psi \circ f \circ \phi^{-1} : \phi(U) \longrightarrow \psi(V)$ is differentiable as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.The map $\tilde{f}$ which is defined between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, is called the local representation of f in the charts (U, ϕ) and (V, ψ). The space of smooth maps from M to N is denoted by $C^\infty(M, N)$. If $N = \mathbb{R}$ (as a smooth manifold), we simply write $C^\infty(M) := C^\infty(M, \mathbb{R})$.

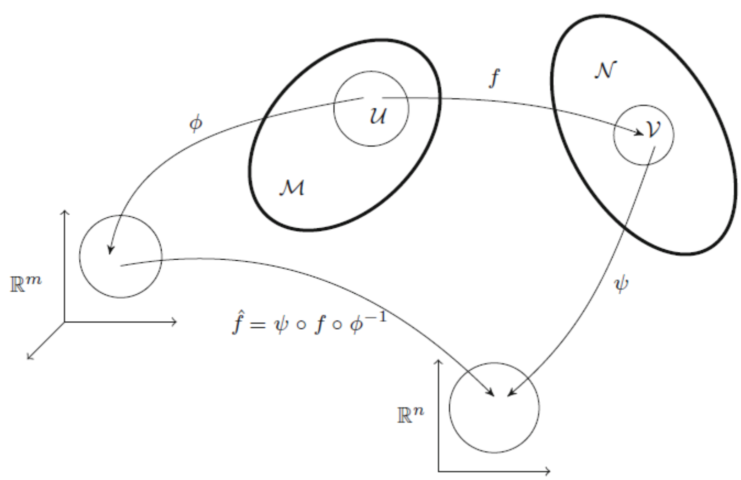

Figure 1.6: Representing a map between manifolds in local charts

Proposition 1. *Let M and N be smooth manifolds. Then*

- *$f \in C^k(M, \mathbb{R})$ if for every chart (U, ϕ) of M, the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^k by composition of diffeomorphisms law.*

- *$f : M \to N$ is a C^k-diffeomorphism if f is a homeomorphism, and both f and f^{-1} are C^k-maps, in this case M and N are diffeomorphic .Clearly, diffeomorphic manifolds have the same dimension.*

- *If M is a C^k manifold and N is a C^l manifold, a map ψ from M into N is differentiable of class C^r (with $r \leq \min\{k,l\}$).*

Definition 10. *[Germ of function]*
Let M be a smooth n dimensional manifold and $p \in M$. A germ of smooth function at p is an equivalence class of smooth functions defined in a neighborhood of p, under the following equivalence relation

$$f \sim g \quad \text{if} \quad f = g \text{ on an open neighborhood of } p$$

We denote by $\mathcal{F}_p$ the set of germs of smooth functions at p.The set $\mathcal{F}_p$ naturally carries the structure of a commutative algebra over $\mathbb{R}$ and it is a real vector space of infinite dimension.

Definition 11. *[Curves]*
A curve in an m-dimensional manifold M is a map $c : (a,b) \to M$ where (a,b) is an open interval such that $a < 0 < b$. c is locally a map from an open interval to M. On a chart (U,φ), a curve $c(t)$ has the coordinate presentation $x = \varphi \circ c : (a,b) \to \mathbb{R}^m$.

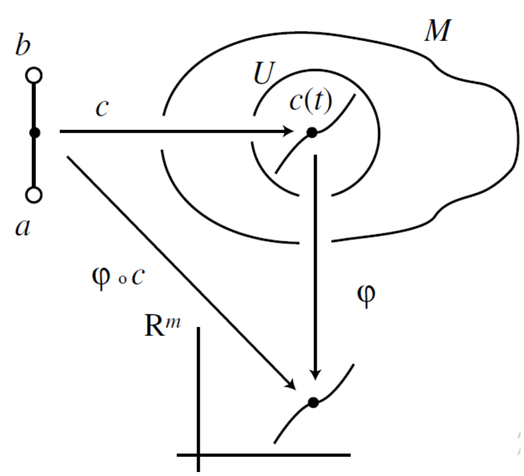

Figure 1.7: A curve c in M and its coordinate presentation

Definition 12. *[Pull-back]*
Let $f : M \to N$ be a smooth map of manifolds. Given a smooth map $g : N \to \mathbb{R}$, we define a new map $f^*g : M \to \mathbb{R}$ by $f^*g = g \circ f$. The map $f^* : C^\infty(N) \to C^\infty(M)$ is applied to functions defined on N to produce functions defined on M; hence the name pullback for f^*.

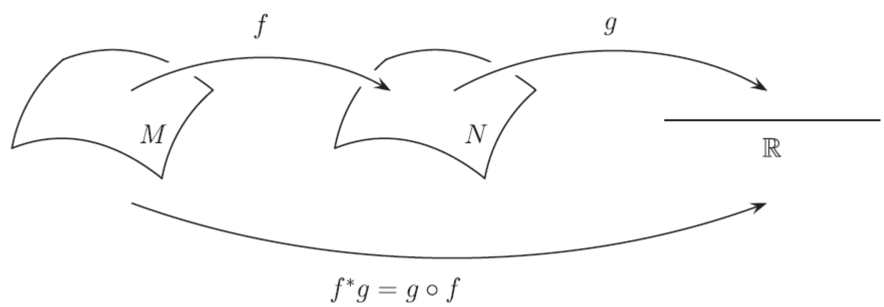

Figure 1.8: Pullback of the function g by f

1.2 Tangent spaces

A differentiable map between Euclidean spaces $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is locally approximated by a linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$, the differential of f at x. In order to obtain an analogous approximation on manifolds, we first need to associate a vector space to each point of the manifold. Tangent spaces will be the domains and ranges of the differentials. While in real analysis, the differentials are defined between the constant spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, the tangent spaces of manifolds vary with respect to their reference points. The collection of all tangent spaces on a smooth manifold is a new smooth manifold of double dimension constituting the prototype of a vector bundle. The tangent bundle is the appropriate space to define the total differentials of smooth maps between smooth manifolds.See [75] for more details and proofs.

The most intuitive method to define tangent vectors is to use curves.Recall from elementary vector calculus that a vector $\tilde{v} \in \mathbb{R}^3$ is said to be tangent to a surface $S \subset \mathbb{R}^3$ at a point $p \in S$ if there exists a differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to S$, such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \tilde{v}$. The set T_pS of all these vectors is a 2-dimensional vector space, called the tangent space to S at p, and can be identified with the plane in $\mathbb{R}^3$ that is tangent to S at p. To extend this concept to an abstract n-dimensional manifold, we must derive a representation of $\tilde{v}$ that isn't reliant on the surrounding Euclidean space $\mathbb{R}^3$. To achieve this, we recognize that the elements of $\tilde{v}$ can be expressed as

$$\tilde{v}_i = \frac{d(x_i \circ \gamma)}{dt}(0),$$

where $x_i : \mathbb{R}^3 \to \mathbb{R}$ is the ith coordinate function. If we ignore the ambient space, $x_i : S \to \mathbb{R}$ is just a differentiable function, and

$$\tilde{v}_i = \tilde{v}(x_i),$$

where, for any differentiable function $f : S \to \mathbb{R}$, we define

$$\tilde{v}(f) := \frac{d(f \circ \gamma)}{dt}(0).$$

This allows us to see $\tilde{v}$ as an operator $\tilde{v} : C^\infty(S) \to \mathbb{R}$, and it is clear that this operator completely determines the vector $\tilde{v}$. It is this new interpretation of a tangent vector that will be used to define tangent spaces for manifolds.

Definition 13. *[Tangent vector at a curve]*

Let M be a smooth n dimensional manifold,and let $\gamma : I \subset \mathbb{R} \to M$ be a curve The tangent vector V to the curve at the point $\gamma(0)$ is the mapping

$$V : C^\infty(M) \to \mathbb{R}$$
$$f \mapsto V(f) := \left.\frac{df(\gamma(t))}{dt}\right|_{t=0}.$$

For the moment, it is worth pointing out that $df(\gamma(t))/dt|_{t=0}$ makes sense [4] *because it is a numerical derivative and intuitively it is the velocity of the curve at the point p.*

Definition 14. *Let M be a differentiable manifold.Two C^k local paths at a point $p \in M$*

$$\gamma_1 : I \subset \mathbb{R} \to M, \quad \gamma_2 : J \subset \mathbb{R} \to M$$

such that the two curves passe through p ie $\gamma_1(0) = \gamma_2(0) = p$. we say that they have the same germ at p if and only if, by composing with a local chart (U, ϕ) on the neighborhood of p, we obtain two paths with the same derivative vector at p

$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$$

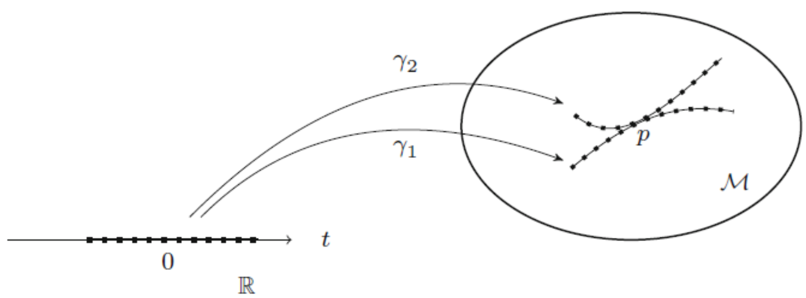

Figure 1.9: Two parameterized curves tangent at a point p.

Corollary 1. *If γ_1, γ_2 are tangent at p, we write $\gamma_1 \sim_p \gamma_2$ and the tangency of curves at $x \in M$ is an equivalence relation.*

Proof: *Reflection and symmetry of $\gamma_1 \sim_p \gamma_2$ are elementary . Assume that $\gamma_1 \sim_p \gamma_2$ and $\gamma_2 \sim_p \gamma_3$. The first implies that $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. for any chart (U, ϕ) with $p \in U$. Similarly, $(\phi \circ \gamma_2)'(0) = (\phi \circ \gamma_3)'(0)$. because of the above definition Therefore, $\gamma_1 \sim_p \gamma_3$, which implies the transitivity of the relation.*

Proposition 2. *[67] Let $p \in M$. A tangent vector to M at p denoted X_p,and it fulfill the following properties:*

$$X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$$

[4] It's because $\frac{d\gamma}{dt}|_{t=0} = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t}$ doesn't make sense because the points $\gamma(t)$ do not necessarily add up, and therefore $\gamma(t) - \gamma(0)$ does not have a clear meaning.

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$$

for $f, g \in C^\infty(M)$, $\alpha, \beta \in \mathbb{R}$.

Remark 5. *For a constant function,* $f(p) = c$ *for all* $p \in M$*, we have*

$$X_p(f) = X_p(c) = c\,X_p(1) = c\,X_p(1 \cdot 1) = c \cdot 1\,X_p(1) + c1 \cdot X_p(1) = 2c\,X_p(1) = 2\,X_p(c);$$

therefore,

$$X_p(c) = 0.$$

Definition 15. *[Tangent space]*

Let M *be a smooth manifold of dimension* n*.The set of equivalence classes of all smooth curves on* M*, passing through* p*, is called the tangent space of* M *at* p *and will be denoted by* T_pM*. The elements of the vector space* T_pM *are called tangent vectors at* p *which represents all velocities of curves passing through* p*.The vectors* $\dfrac{\partial}{\partial x_i}$, $i = 1 \dots n$*, form a basis for* T_pM *on which we define the operations given by by*

$$(X_p + W_p)(f) = X_p(f) + W_p(f),$$

$$(\alpha X_p)(f) = \alpha \cdot X_p(f),$$

for $X_p, W_p \in T_pM$, $f \in C^\infty(M)$*, and* $\alpha \in \mathbb{R}$*. Hence,* 0_p*, the zero vector of* T_pM*, satisfies* $0_p(f) = 0$ *for* $f \in C^\infty(M)$*.*

Proposition 3. *The tangent space of* $\mathbb{R}^m$ *at* $p \in \mathbb{R}^m$ *can be canonically identified as* $T_p\mathbb{R}^m \cong \mathbb{R}^m$*. Similarly for* M *a smooth manifold of dimension* $m = dimM$ *and let* $p \in M$*. Then* $\dim T_pM = \dim M$*.*

Proposition 4. *Let* $X \in T_pM$ *be a tangent vector, and let* (U, ϕ) *be a chart containing* p*. If*

$$X = X^i \frac{\partial}{\partial x^i}\bigg|_p,$$

then the real numbers $X^1, \dots, X^{\dim M}$ *are called the components of* X *with respect to the tangent space basis induced by the chart* (U, ϕ)*. The basis* $\dfrac{\partial}{\partial x^i}\bigg|_p$ *is also called a coordinate basis*

Example 3. *On the plane* $\mathbb{R}^2$ *with Euclidean coordinates* (x, y)*, consider the parametric curve* $\gamma(t) = (\cos(t), \sin(t))$*. The tangent vector to* γ *at* $t = 0$ *is*

$$X_p = \frac{d\gamma}{dt}\bigg|_{t=0} = \begin{bmatrix} \frac{dx}{dt}\big|_{t=0} \\ \frac{dy}{dt}\big|_{t=0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial y},$$

Hence, its components are $(X_1, X_2) = (0, 1)$ *in the basis* $\left\{\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right\}$.

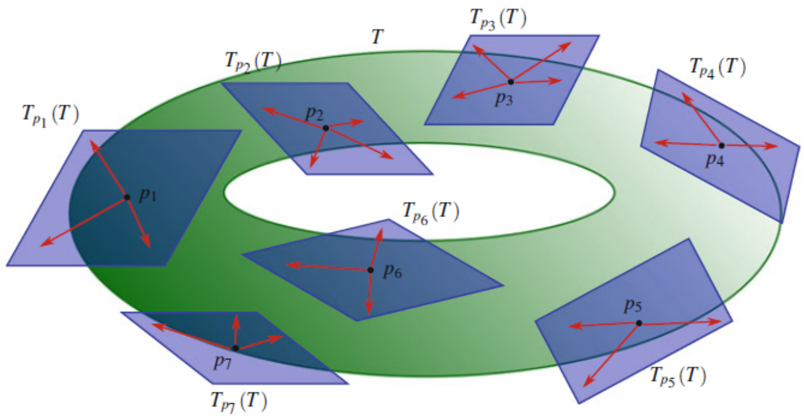

Figure 1.10: Tangent vectors spaces at differnet points on the torus.

The concept of differentiability of a map has been presented in Sect. 1.1.1. A map between smooth manifolds is differentiable if its representation in local charts is differentiable, Now we are going to give the definition of the differential of f as a linear map between the tangent spaces T_pM and T_qN that maps tangent vector at p to tangent vectors at $q = f(p)$.

Definition 16. *[The differential of a smooth map]*
Let $f : M \to N$ be a smooth map between manifolds. (U, φ) is a chart around p, and (V, ψ) is a chart around $f(p)$. The differential of f at $p \in M$ is the linear map

$$d_pf : T_pM \to T_{f(p)}N$$

given by

$$d_pf(v) = d_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})(v),$$

Alternative notations for this map are Df and Tf, and it is also known as the tangent map.

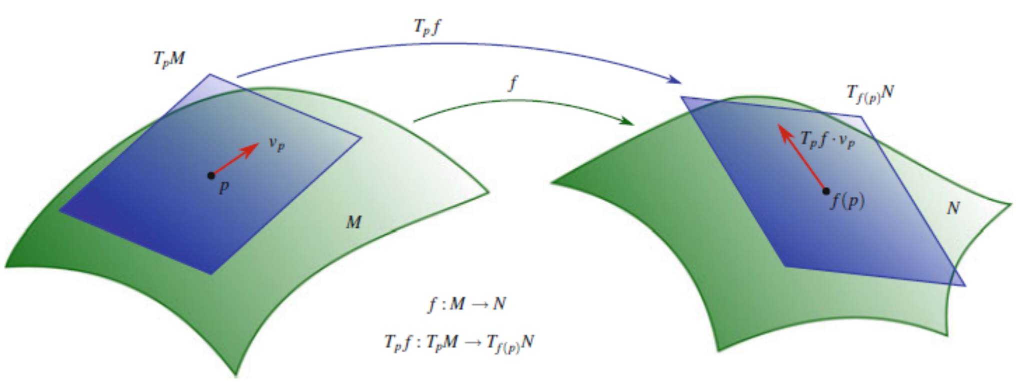

Figure 1.11: differential of a map defined on manifolds

Definition 17. *[Derivatives]*

A k-derivation of M is a map $D : C^{k+1}(M, \mathbb{R}) \to C^k(M, \mathbb{R})$ such that, for all f, g in $C^{k+1}(M, \mathbb{R})$

- *D is an $\mathbb{R}$-linear map.*
- *(Leibniz rule) for all $f, g \in C^\infty(M)$ we have $D(fg) = D(f)g(p) + f(p)D(g)$.*

we denote $D_pC^\infty(M)$ the vector space of derivations at p.

Proposition 5. *Another way to think about tangent vectors is as directional derivatives. Given a vector v in $\mathbb{R}^n$, one defines the corresponding directional derivative at a point $x \in \mathbb{R}^n$ by*

$$\forall f \in C^\infty(\mathbb{R}^n): \quad (D_v f)(x) = \frac{d}{dt}[f(x+tv)]\bigg|_{t=0} = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(x).$$

This map is naturally a derivation at x. Furthermore, every derivation at a point in $\mathbb{R}^n$ is of this form. Hence, there is a one-to-one correspondence between vectors (thought of as tangent vectors at a point) and derivations at a point.

If we think of v as the initial velocity of a differentiable curve γ initialized at x, i.e., $v = \gamma'(0)$, then instead, define D_v by:

$$\forall f \in C^\infty(\mathbb{R}^n): \quad (D_v f) = (f \circ \gamma)'(0).$$

Theorem 4. *Let M be an n dimensional differentiable manifold and $p \in M$. The map*

$$T_pN \longrightarrow D_pC^\infty(M)$$

$$X \mapsto D_pf(X)$$

is an isomorphism of vector spaces meaning that they have the same vector structure.

Definition 18. *[One-forms]*

Given that T_pM represents a vector space, its dual space exists, comprising linear functions mapping elements of T_pM to $\mathbb{R}$. This dual space is referred to as the cotangent space at point p, denoted as T_p^*M. An element $\omega : T_pM \to \mathbb{R}$ belonging to T_p^*M is termed a dual vector, cotangent vector, or, within the realm of differential forms, a one-form. A prime illustration of a one-form is the differential df of a function $f \in \mathcal{C}^\infty(M)$. When a vector V acts upon f, denoted as $V[f]$, it yields $V^i \partial f / \partial x_i \in \mathbb{R}$. Consequently, the action of $df \in T_p^*M$ on $V \in T_pM$ is denoted by

$$\langle df, V \rangle = V[f] = V^i \frac{\partial f}{\partial x_i} \in \mathbb{R}.$$

Observing that df is expressed in terms of the coordinate $x = \phi(p)$ as

$$df = \left(\frac{\partial f}{\partial x_i}\right) dx^i,$$

it becomes intuitive to regard $\{dx^i\}$ as a basis of T_p^*M. Moreover, this constitutes a dual basis, as evidenced by

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x_j} = \delta^i_j.$$

Representing an arbitrary one-form ω as

$$\omega = \omega_i dx^i,$$

where the ω_μ denote its components. by considering a vector $V = V^i\partial/\partial x_i$ and a one-form $\omega = \omega_j dx^j$. The inner product $\langle\cdot,\cdot\rangle : T_p^*M \times T_pM \to \mathbb{R}$, extending the notion of dot product to an arbitrary vector space, is defined as

$$\langle\omega, V\rangle = \omega_i V^j dx^i\left(\frac{\partial}{\partial x_j}\right) = \omega_i V^j \delta^i_j = \omega_i V^j.$$

1.2.1 The tangent & cotangent bundle

When we glue together the tangent spaces at all points on a manifold M, we get a set that can be thought of both as a union of vector spaces and as a manifold in its own right. Meaning that one can organize the set of tangent vectors of a manifold M into a manifold TM.

Definition 19. *[Tangent bundle]*
Let M be an n dimensional smooth manifold. The tangent bundle, denoted by TM, is defined to be:

$$TM = \bigcup_{p\in M}(\{p\} \times T_pM)$$

elements in TM can be written as (p, X) where $p \in M$ and $X \in T_pM$.

Similarly, the cotangent bundle, denoted by T^*M, is defined to be:

$$T^*M = \bigcup_{p\in M}(\{p\} \times T_p^*M)$$

elements in T^*M can be written as (p, ω) where $p \in M$ and $\omega \in T_p^*M$.

It is essential to emphasize that this union is disjoint: elements X_p and $Y_{p'}$ belonging to different tangent spaces cannot be added.The tangent bundle TM of a manifold M comes with the natural projection map $\pi : TM \to M$, defined by $(p, v) \mapsto p$. For any subset $U \subseteq M$, we define

$$TU := \pi^{-1}(U) = \{(p, v) \mid p \in U, v \in T_pM\}.$$

The smooth structure on TM allows to define smooth vector fields.

Proposition 6. *[75] The tangent bundle (resp cotangent bundle) has the structure of a differentiable manifold of dimension $2dimM$ induced by the structure of M in a natural way, where each coordinate system on M, (U, ϕ), induces a coordinate system on TM, $(\pi^{-1}(U), \phi)$.*

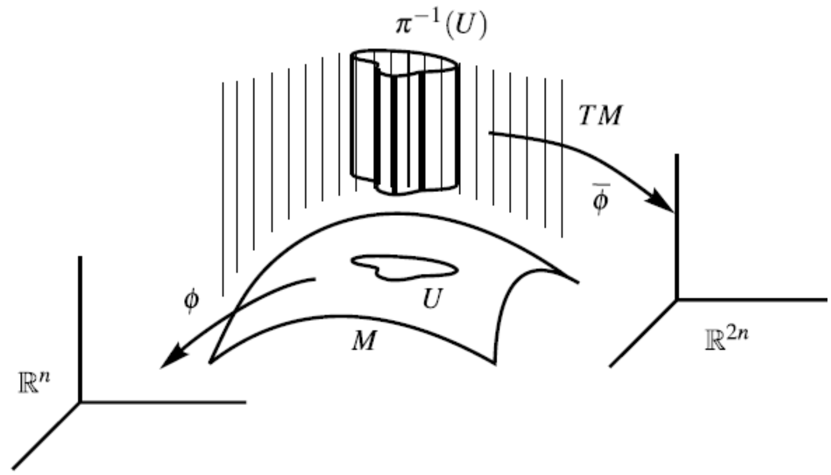

Figure 1.12: Tangent bundle of a manifold M as a $2n$-dimensional manifold

1.3 Vector fields

A vector field is a mathematical construct that assigns a vector to each point in a given space. It represents quantities of both magnitude and direction, such as velocity or force. Vector fields are used in various fields, like fluid dynamics and electromagnetism, and play a vital role in vector calculus.

Here we will define vector fields as a family of tangent vectors, one for each point on the manifold $X_p \in T_pM$, depending smoothly on the base point $p \in M$. This allows us to study dynamics on the manifold, demonstrating the utility of the concept of tangent spaces. Another advantage of this concept is that now we have a vector space associated with each point on the manifold. This allows us to extend many of the natural constructions of linear algebra to manifolds see [[6],[23],[42]] for more details on the subjects treated in the next sections.

Definition 20. *[Vector fields]*

Let M be a smooth manifold of dimension n. A vector field X on M is a section of the tangent bundle TM, i.e, $X : M \to TM$ such that $\pi \circ X(p) = p$ for every $p \in M$. In other words, a vector field on M is a map X which assigns to each point $p \in M$ a tangent vector $X(p) = X_p \in T_p(M)$. Let (U, ϕ) be a local chart of M, and $p \in U$, then

$$X(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x_i}\bigg|_p . \tag{1.1}$$

The real-valued functions $X^i : U \to \mathbb{R}, 1 \le i \le n$, are called the components of X related to the local chart.

Remark 6. *A source full of examples comes from the following fact: for every multi-variable function f, one can associate a vector field - its gradient ∇f. This vector field points in the direction where the variation of the function is maximal.*

Example 4. *Consider the simplest case,as shown in the figure , e_1, which is the vector field generating the x-axis given by $e_1 = \dfrac{\partial}{\partial x}$(figure on the right), and e_2, the vector field generating the y-axis given by $e_2 = \dfrac{\partial}{\partial y}$ (figure in the middle). Since these two vector fields represent a basis for $\mathbb{R}^2$, any other vector field can be expressed as a linear combination of them. Specifically, consider the vector field $V = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ (figure on the left).*

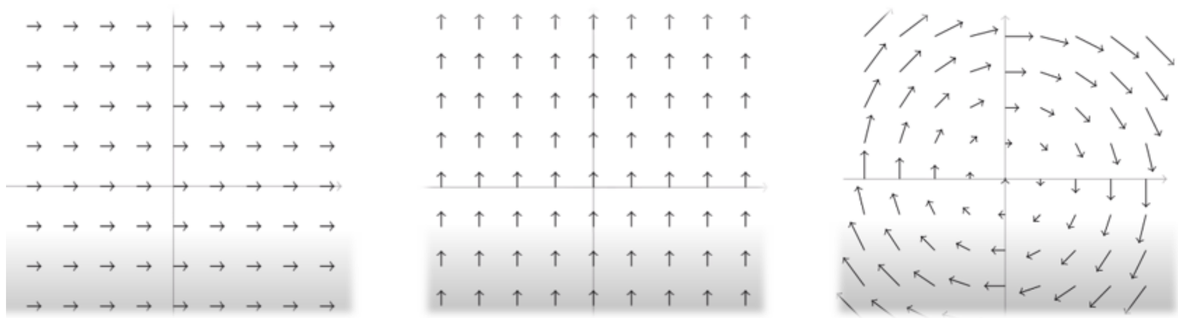

Figure 1.13: The representation of the vector field $V = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$

Definition 21. *[Smooth vector field]*

A vector field X is differentiable (of class C^∞) if X is a smooth section of the tangent bundle TM, i.e., X is smooth as a map. meaning that X is smooth if and only if its components are smooth for all charts in some atlas for M.Or equivalently, for all $f \in C^\infty(M)$, the function $X(f)$ also belongs to $C^\infty(M)$.

Definition 22. *[Integral curves]*

A vector field is called smooth if it is possible to find curves on the manifold such that all the vectors in the vector field are tangent to the curves. These curves are called the integral curves of the vector field, and finding them is essentially what differential equations are all about. Figure below shows several integral curves of a smooth vector field in $\mathbb{R}^2$

Remark 7. *A vector field on a manifold such as $\mathbb{R}^2$ or $\mathbb{R}^3$ gives us a vector at each point of the manifold. But a vector v_p at a point p of a manifold M is an element of the tangent space at that point T_pM. So being given a vector field is the same thing as being given an element of each tangent space of M. This is often called a section of the tangent bundle TM*

Proposition 7. *[42] Let X be a vector field on M. Then the following assertions are equivalent*

(a) *X is smooth.*

(b) *For every coordinate system $(U, x^1, \ldots, x^n)$ of M, the functions X_i defined in(1.1) are smooth.*

(c) *For every open set V of M and $f \in C^\infty(V)$, the function $X(f) \in C^\infty(V)$.*

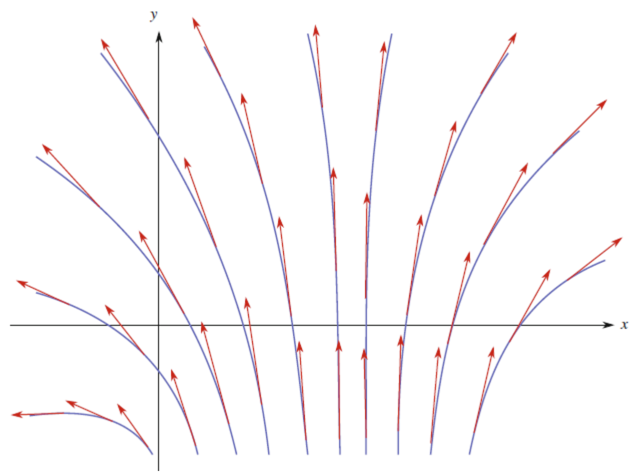

Figure 1.14: Integral curves of $\mathbb{R}^2$

properties 1. *Denote by $\Gamma(TM)$ the set of all smooth vector fields on M. We have the following algebraic structure on $\Gamma(TM)$*

For $X_1, X_1 \in \Gamma(TM)$, $\alpha \in \mathbb{R}$, and $f \in C^\infty(M)$

1. $X_1 + X_2 \in \Gamma(TM)$, *i.e.*, $(X_1 + X_2)(p) = X_1(p) + X_2(p)$;
2. $\alpha X_1 \in \Gamma(TM)$, *i.e.*, $(\alpha X_1)(p) = \alpha X_1(p)$;
3. $f X_1 \in \Gamma(TM)$, *i.e.*, $(f X_1)(p) = f(p) X_1(p)$.

1.3.1 Lie bracket of smooth vector fields

In a smooth n dimensional manifold M. Given two vector fields $X, Y \in \Gamma(TM)$, the composition $X \circ Y$ is not a vector field. Taking the simplest example, if $X = Y = \dfrac{\partial}{\partial x}$ as vector fields on $\mathbb{R}$, then $XY = \dfrac{\partial^2}{\partial x^2}$ is a second-order derivative, which is not a vector field[5]. However, the commutator also called the Lie bracket turns out to be a vector field with which $\Gamma(TM)$ becomes a Lie algebra over $\mathbb{R}$.

Definition 23. *[The commutator]*

Let M be an n dimensional smooth manifold; For any two vector fields $X, Y \in \Gamma(TM)$, the commutator

$$[X, Y] := X \circ Y - Y \circ X$$

called the Lie bracket of X and Y indeed defines a derivation. In other words, it is a vector field on $\Gamma(TM)$.It is instructive to see how this works in local coordinates. In any chart around the point $p \in M$, if

$$X = X^i \frac{\partial}{\partial x_i}, \quad Y = Y^i \frac{\partial}{\partial x_i},$$

[5] The reason is that the composition $X \circ Y$ and $Y \circ X$ does not en general satisfy the Leibnitz rule

with coefficient functions $X^i, Y^i \in C^\infty(M)$, the composition XY is a second-order differential operator on functions $f \in C^\infty(M)$:

$$\begin{aligned} X(Y(f)) &= X\left(\sum_{j=1}^{n} Y_j \frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} Y_j \frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i,j=1}^{n} X_i \left(\frac{\partial Y_j}{\partial x_i}\frac{\partial f}{\partial x_j} + Y_j \frac{\partial^2 f}{\partial x_j \partial x_i}\right). \end{aligned}$$

Subtracting a similar expression for YX,

$$\begin{aligned} Y(X(f)) &= Y\left(\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}\right) \\ &= \sum_{i,j=1}^{n} Y_j \frac{\partial}{\partial x_j}\left(X_i \frac{\partial f}{\partial x_i}\right) \\ &= \sum_{i,j=1}^{n} \left(\frac{\partial Y_j}{\partial x_i}\frac{\partial X_i}{\partial x_j}\frac{\partial f}{\partial x_i} + X_i Y_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right). \end{aligned}$$

the terms involving second derivatives cancel, and we obtain

$$[X,Y](f) = \sum_{i,j=1}^{n} \left(X_i \frac{\partial}{\partial x_i}(Y_j) - Y_i \frac{\partial}{\partial x_i}(X_j)\right)\frac{\partial}{\partial x_j}(f)$$

properties 2. *[67] Let X, Y, Z be C^∞ vector fields on M, and let f, g be in $C^\infty(M,\mathbb{R})$. Then the Lie brackets $[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ has the following properties*

(1)*Antisymmetric* $[X,Y] = -[Y,X]$;
(2) *Bilinearity* $[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$;
(3) *Jacobi identity* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$;
(4) $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X, \qquad \forall f, g \in C^\infty(M)$.

Example 5. *In $\mathbb{R}^3$ we consider the two vectors fields given by Let $X = \dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial z}$ and $Y = \dfrac{\partial}{\partial y}$. Then, the Lie bracket $[X,Y]$ is given by*

$$[X,Y] = \frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial z} - \frac{\partial X}{\partial y}.$$

Calculating the derivatives,yields

$$[X,Y] = -\frac{\partial}{\partial z}.$$

Remark 8. *The commutator is useful for the following reasons(among others)*

- ♣ *first the Lie bracket measures the extent to which the derivatives in directions X and Y do not commute meaning that the vector field $[X, Y] := XY - YX$ represents the infinitesimal non-commutation of the flows generated by X and Y.*

- ♣ *Second reason is that once we have a chart, we can use $\dfrac{\partial}{\partial x_i}$ as a basis for vector fields in a neighborhood. Any set of n linearly independent vector fields may be chosen as a basis, but they need not to form a coordinate system. In a coordinate system,*

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

because partial derivatives commute. So, n vector fields will form a coordinate system only if they commute, i.e, have vanishing commutators with one another.

- ♣ *Third reason in terms of group theory, the composition operators form an infinite-dimensional Lie group (non-commutative). Vector fields are thus the generators. The space of vector fields is a Lie algebra (with the Lie bracket). Historically, Sophus Lie[6] studied this group and algebra to subsequently develop the theory of Lie groups and algebras.*

1.4 Differential forms

Definition 24. *Let $f \in C^\infty(M)$; the differential of f at the point $p \in M$, denoted by df_p, is defined by*

$$df_p(X_p) := X_p(f), \text{ for } X_p \in T_pM.$$

The map df_p is a linear transformation from T_pM to $\mathbb{R}$, since if $X_p, Y_p \in T_pM$ and $a, b \in \mathbb{R}$,

$$df_p(aX_p + bY_p) = (aX_p + bY_p)(f) = aX_p(f) + bY_p(f) = a\, df_p(X_p) + b\, df_p(Y_p).$$

so that it can be canonically identified with an element of the dual space given by

$$T_p^*M := \mathcal{L}(T_pM, \mathbb{R}).$$

The differentials of smooth functions are naturally smooth sections of a remarkable vector bundle, the cotangent bundle.

[6]A Norwegian mathematician, known for advancing the theory of continuous symmetries and creating Lie algebra and Lie groups. He was arrested in France during the war, suspected of espionage, and continued his research on geometric transformations while imprisoned.

Proposition 8. *[Construction of the cotangent bundle]*

Let's now present the explicit construction of the cotangent bundle T^*M based on an atlas on M. Each chart U, $\phi = (x_1, \dots, x_n)$ determines, on U, a basis $(\partial x_1, \dots, \partial x_n)$ of the tangent bundle $TU = TM|_U$. The dual basis, denoted by $(dx^1, \dots, dx^n)$, is defined by the identities

$$dx^i \partial x_j = \delta^i_j, \quad i, j = 1, \dots, n,$$

and it provides a basis for the cotangent bundle $T^*M|_U$. The dual basis at a point $p \in U$ is denoted by $(dx^1(p), \dots, dx^n(p))$. Any chart (U, ϕ) thus determines a linear bijection on the fibers

$$U \times \mathbb{R}^n \longrightarrow T^*M|_U,$$

$$(p, (\alpha_1, \dots, \alpha_n)) \longmapsto \sum_{i=1}^{n} \alpha_i \, dx^i|_p.$$

Corollary 2. *Let M be a smooth manifold of dimension n.The differential of a smooth function $f : M \to \mathbb{R}$ is a smooth section of the cotangent bundle T^*M.*

En fact,Let $U, \phi = (x_1, \dots, x_n)$ be a local chart on M. The following identity holds on U:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, dx^i.$$

Remark 9. *the differential d has a single definition, but it is used in several different settings that are not related in an immediately obvious way.the following table illustrate the various uses of the differential on manifolds.*

Construct	Argument	Other Names	Other Symbols
$d\phi : TM \mapsto TN$	$\phi : M \mapsto N$	Tangent Mapping, pushforward	$T\phi, \phi_*$
$df : TM \mapsto \mathbb{R}$	$f : M \mapsto \mathbb{R}$	Directional derivative	$v(f), d_v(f), \nabla_v(f)$
$dx^\alpha : TM \mapsto \mathbb{R}$	$x^\alpha : M \mapsto \mathbb{R}$	Dual frame to ∂x_α	β^α

Definition 25. *A smooth section of T^*M is called a differential 1-form on M. The vector space of differential 1-forms is denoted by*

$$\Omega^1(M) := \Gamma(T^*M).$$

Definition 26. *[Orientation]*

In order to perform integration of a differential form over a smooth manifold M, it is necessary for M to be orientable. Therefore, we begin by defining the orientation of a manifold. Consider a connected m-dimensional differentiable manifold M. At any point $p \in M$, the tangent space T_pM is defined by the basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$, where x^μ represents the local coordinates on the chart U_i to which p belongs. Suppose there exists another chart U_j such that $U_i \cap U_j \neq \emptyset$, with local coordinates y^α. If $p \in U_i \cap U_j$, the tangent space T_pM can be *represented by either* $\{e_\mu\}$ *or* $\{\tilde{e}_\alpha\} = \{\partial/\partial y^\alpha\}$. *The transition between these bases is governed by*

$$(\tilde{e}_\alpha) = \left(\frac{\partial x^\mu}{\partial y^\alpha}\right) e_\mu. \tag{5.97}$$

When the determinant $J = \det(\partial x^\mu/\partial y^\alpha)$ *is greater than* 0 *on* $U_i \cap U_j$, *the bases* $\{e_\mu\}$ *and* $\{\tilde{e}_\alpha\}$ *are said to define the same orientation on* $U_i \cap U_j$. *Conversely, if* $J < 0$, *they define opposite orientations.*

Definition 27. *[Oriented Manifolds]*
Let M *be a connected manifold covered by* $\{U_i\}$. *The manifold* M *is orientable if, for any overlapping charts* U_i *and* U_j, *there exist local coordinates* $\{x^\mu\}$ *for* U_i *and* $\{y^\alpha\}$ *for* U_j *such that* $J = \det(\partial x^\mu/\partial y^\alpha) > 0$.

When an m-dimensional manifold M *is orientable, it possesses an m-form* ω *that never equals zero. This particular m-form, termed a volume element, serves as a measure when integrating a function* $f \in \mathcal{C}^\infty(M)$ *across* M.

Definition 28. *Let* M *be a manifold and* $\alpha \in \Omega^1(M)$ *be a 1-form. For any smooth curve* $\gamma : I \subset \mathbb{R} \to M$, *the integral of* α *along* γ *is defined as*

$$\int_\gamma \alpha := \int_I \alpha(\gamma(t))||\dot{\gamma}(t)||\, dt,$$

this integral is independent of the parametrization of the curve γ

Theorem 5. *[Theorem of Stokes [7]]*
Let M^n *be a manifold of dimension* $n \geq 1$ *oriented with boundary, and let* $\omega \in \Omega^{n-1}(M)$ *be a compactly supported* $(n-1)$*-form. We have the equality*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{1.2}$$

Here, ∂M *denotes the boundary of* M *equipped with the induced orientation,* $d\omega \in \Omega^n(M)$ *is the exterior derivative of* ω, *and in the right-hand side, it is understood that we integrate the restriction of* ω *to* ∂M.

In the statement of the theorem, the support of ω *denotes, unsurprisingly, the closed set*

$$supp(\omega) := \{p \in M \mid \omega(p) \neq 0\}.$$

When M *is compact, the compactness assumption on the support is redundant.*

1.5 Tensor analysis

Tensor [8] calculus is a technique that can be regarded as a follow-up on linear algebra. It is a generalization of this discipline.Tensor algebra and tensor analysis were developed by Riemann, Christoffel, Ricci, Levi-Civita and others in the nineteenth century. The special theory

[7]In dim 2 the theorem became formula of Green-Riemann and in dim 3 formula of Ostrogradski]

[8]The word tensor= [ten.sor] was introduced in 1846 by William Rowan Hamilton.

of relativity, as propounded by Einstein in 1905, was elegantly expressed by Minkowski in terms of tensor fields in a flat space-time.For a more rigorous treatment see, e.g, [1],[23],[25]]

In a letter to Tullio Levi-Civita, a co-inventor of tensor calculus, Einstein expressed his admiration for the subject in the following words:

> *"I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot".*

1.5.1 Tensors & tensor spaces

Definition 29. *Let E be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. The dual vector space to E is The space of linear maps from E to K denoted as $E^* := \{f : E \to K, | f \text{ is linear}\}$; where K is considered as a vector space over itself. The elements of the dual vector space are variously called linear functionals, covectors, or one-forms on E.*

A dual vector is a linear object that maps a vector to a scalar. This may be generalized to multilinear objects called tensors, which map several vectors and dual vectors to a scalar. A tensor $\mathcal{T}$ of type (p, q) is a multilinear map that maps p dual vectors and q vectors to $\mathbb{R}$.

Definition 30. *[Tensor]*

Let E be a vector space over K. A type (p, q)-tensor $\mathcal{T}$ on E is a multilinear map

$$\mathcal{T} : \underbrace{E^* \times E^* \times \ldots \times E^*}_{p\ copies} \times \underbrace{E \times E \times \ldots \times E}_{q\ copies} \to K.$$

and we write the space of (p, q)-tensors on E. $\mathcal{T}^p_q(E)$ as

$$\mathcal{T}^p_q(E) := \underbrace{E \otimes E \otimes \ldots \otimes E}_{p\ copies} \otimes \underbrace{E^* \otimes E^* \otimes \ldots \otimes E^*}_{q\ copies}.$$

The set $\mathcal{T}^p_q(E)$ can be equipped with a K-vector space structure by defining the following operations

- ***Tensor sum*** $+$

$$+ : \mathcal{T}^p_q(E) \times \mathcal{T}^p_q(E) \to \mathcal{T}^p_q(E)$$
$$(\mathcal{T}, \mathcal{S}) \mapsto \mathcal{T} + \mathcal{S}$$

- ***Scalar multiplication*** $\cdot$**:**

$$\cdot : K \times \mathcal{T}^p_q(E) \to \mathcal{T}^p_q(E)$$
$$(\alpha, \mathcal{T}) \mapsto \alpha \cdot \mathcal{T}$$

These operations define a K-vector space structure on $\mathcal{T}^p_q(E)$making it a $\mathbb{R}$ vector space of dimension n^{p+q}

Proposition 9. *Let $\mathcal{T} \in \mathcal{T}^p_q(E)$ and $\mathcal{S} \in \mathcal{T}^r_s(E)$. The tensor product of $\mathcal{T}$ and $\mathcal{S}$ is the tensor $\mathcal{T} \otimes \mathcal{S} \in \mathcal{T}^{p+r}_{q+s}(E)$ defined by:*

$$(\mathcal{T} \otimes \mathcal{S})(\omega_1, \ldots, \omega_{p+r}; v_1, \ldots, v_{q+s}) = \mathcal{T}(\omega_1, \ldots, \omega_p; v_1, \ldots, v_q)\mathcal{S}(\omega_{p+1}, \ldots, \omega_{p+r}; v_{q+1}, \ldots, v_{q+s}).$$

where $\omega_i \in E^*$ and $v_i \in E$.

Theorem 6. *Let $\mathcal{T}^r_s(E)$ be the space of tensors of type (r, s). Let $\{e_1, \ldots, e_n\}$ be a basis for E, and $\{e^1, \ldots, e^n\}$ be the dual basis for E^*. Then*

$$\{e^{j_1} \otimes \ldots \otimes e^{j_r} \otimes e_{j_{r+1}} \otimes \ldots \otimes e_{j_{r+s}} \mid 1 \leq j_i \leq r + s\}$$

is a basis for $\mathcal{T}^r_s(E)$. Hence, An element of $\mathcal{T}^r_s(E)$ is written in terms of the bases described earlier as

$$\mathcal{T} = \mathcal{T}^{j_1 \ldots j_r}{}_{j_{r+1} \ldots j_{r+s}} \frac{\partial}{\partial x^{j_1}} \ldots \frac{\partial}{\partial x^{j_r}} dx^{j_{r+1}} \ldots dx^{j_{r+s}}$$

Example 6. *Let E be a vector space then*

- *Vectors can be seen as functions $E^* \to \mathbb{R}$, so vectors are contravariant tensors.*
- *Linear functionals $f : E \to E$ are covariant tensors.*
- *Inner products are functions from $E \times E \to \mathbb{R}$, so covariant tensors.*
- *The determinant of a matrix is a multilinear function of the columns (or rows) of a square matrix, so it is a covariant tensor.*

Definition 31. *[The Einstein's summation convention]*
At this point it is convenient to introduce the Einstein summation convention. This makes it possible to indicate sums without dots or a summation symbol.

Example 7. *Consider the sum of the series $S = a_1 x + a_2 x^2 + \ldots + a_n x^n$.*
By using summation convention, drop the sigma sign and write it as

$$S = a_i x^i \quad \text{or} \quad S = \sum_{i=1}^{n} a_i x^i$$

This convention is called Einstein's summation convention and is stated as

> *If a suffix occurs twice in a term, once in the lower position and once in the upper position then that suffix implies sum over the defined range. If the range is not given, then assume that the range is from 1 to n.*

Definition 32. *[Lie derivative]*

Let X be a vector field on a smooth manifold M. We can now define the Lie derivative $\mathcal{L}_X$ applied to a tensor field $\mathcal{T}$. For a tensor field of type (r, s), denoted as $\mathcal{T}$, we associate a tensor field of the same type, $\mathcal{L}_X\mathcal{T}$, according to the following rules

1. $\mathcal{L}_X f = Xf$ *for any scalar function* f,
2. $\mathcal{L}_X Y = [X, Y]$ *for any vector field* Y,
3. $\mathcal{L}_X(\mathcal{T}_1 + \mathcal{T}_2) = \mathcal{L}_X\mathcal{T}_1 + \mathcal{L}_X\mathcal{T}_2$ *for all tensor fields* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *of type* (r, s),
4. $\mathcal{L}_X(\mathcal{T}_1 \otimes \mathcal{T}_2) = \mathcal{L}_X\mathcal{T}_1 \otimes \mathcal{T}_2 + \mathcal{T}_1 \otimes \mathcal{L}_X\mathcal{T}_2$ *for all tensor fields* $\mathcal{T}_1$ *and* $\mathcal{T}_2$,
5. $\mathcal{L}_X(f\mathcal{T}) = f\mathcal{L}_X\mathcal{T} + (Xf)\mathcal{T}$ *for any tensor field* $\mathcal{T}$ *and any scalar function* f.

These rules define the action of the Lie derivative on tensor fields. The Lie derivative preserves the type of the tensor field.

Remark 10. *Note that the Lie derivative $\mathcal{L}_X T|_p$ of a tensor of type (r, s) depends not only on the direction of the vector field X at point p but also on the direction of X in a neighborhood of p. This notion of derivation seems too limited to serve as a generalization of the concept of partial derivative in $\mathbb{R}^n$. To achieve such a generalization, an additional structure on the manifold need to be introduced, we will deal with it latter on in the few next sections.*

1.5.2 Metric tensor

The metric tensor serves as a fundamental instrument enabling the measurement of vector magnitudes, angles between vectors, and distances between points on a manifold, all without reliance on an external space. This concept carries significant implications in theoretical physics and mathematics, notably in the development of general relativity and the exploration of geometric characteristics inherent to curved manifolds.

Definition 33. *[Metric tensor]*

A metric tensor on a vector space E is a function $g : E \times E \to \mathbb{R}$ $(g \in (E^2)^*)$ which is

1. *Bilinear*

$$g(\alpha v_1 + \beta v_2, w) = \alpha g(v_1, w) + \beta g(v_2, w)$$
$$g(v, \alpha w_1 + \beta w_2) = \alpha g(v, w_1) + \beta g(v, w_2)$$

   $\alpha, \beta \in \mathbb{R}$ *and* $v_1, v_2, w_1, w_2 \in E$ *i.e., g is a (0,2) tensor;*

2. *Symmetric*

$$g(v, w) = g(w, v).$$

3. *Non-degenerate*

$$g(v, w) = 0 \quad \forall w \implies v = 0.$$

Example 8. *Poincare Half-Plane: defined as* $\mathcal{H} = \{(x, y) \in \mathbb{R}^2, y > 0\}$ *also sometimes referred to as Lobachevsky space or Bolyai Lobachevsky space with the metric defined as*

$$\begin{aligned} g_{\mathcal{H}} :& E \times E \to \mathbb{R} \\ (x, y) \to & g_{\mathcal{H}}(x, y) = \frac{1}{y^2}(dx^2 + dy^2) \end{aligned}$$

1.5.3 The first fundamental form

Measurements in $M \subset \mathbb{R}^n$ *only depend on the restriction of the Euclidean scalar product of the ambient space* R^n *to the tangent spaces* T_pM*. The first fundamental form gets to the heart of this fact.*

Definition 34. *Consider a surface* X *mapping from a subset* $U \subset \mathbb{R}^2$ *to* $\mathbb{R}^3$*. Let* $(u(t), v(t))$ *belong to* U*, and let* $\alpha : I \subset \mathbb{R} \to U$ *be a regular curve on* U *defined as* $\alpha(t) = X(u(t), v(t))$*. Then, the coefficients of the first fundamental form of the surface* X *are given by*

$$E(u, v) = g(X_u(u, v), X_u(u, v)) = \tilde{g}_{11}$$

$$F(u, v) = g(X_u(u, v), X_v(u, v)) = \tilde{g}_{12}$$
$$G(u, v) = g(X_v(u, v), X_v(u, v)) = \tilde{g}_{22}$$

or, in short,

$$E = g(X_u, X_u), \quad F = g(X_u, X_v) = g(X_v, X_u), \quad G = g(X_v, X_v).$$

The first fundamental form of X *is*

$$I(du, dv) = E(du)^2 + 2Fdudv + G(dv)^2,$$

or represented on its matrix form

$$\tilde{g_{ij}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix}$$

Here the tensor metric I *is known as the metric tensor of the surface and* $\tilde{g_{ij}}$ *its representation.*

Example 9. *The helicoid is parameterized by*

$$\phi : (u, v) \in \mathbb{R}^2 \mapsto (v \cos u, v \sin u, au) \in \mathbb{R}^3,$$

where $a > 0$ *is a fixed parameter. Here,* $\phi_u = (-v \sin u, v \cos u, a)$ *and* $\phi_v = (\cos u, \sin u, 0)$ *then, the coefficients of the first fundamental form are easily calculated:*

$$E = g(\phi_u, \phi_u) =< \phi_u, \phi_u >= v^2 + a^2, \quad F = 0, \quad and \quad G = 1.$$

Thus,

$$\tilde{g}_{ij} = \begin{pmatrix} v^2 + a^2 & 0 \\ \\ 0 & 1 \end{pmatrix}$$

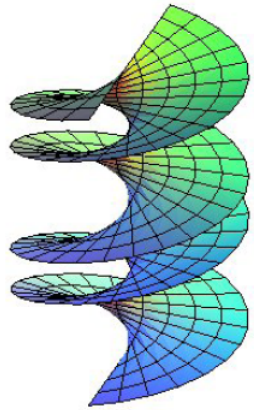

Figure 1.15: The helicoid

Example 10. *The cylinder C defined by $f(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 = 1$ admits the parametrization*

$$\phi : (u, v) \in \mathbb{R}^2 \mapsto \phi(u, v) = (\cos u, \sin u, v) \in \mathbb{R}^3.$$

Here, $\phi_u = (-\sin u, \cos u, 0)$ and $\phi_v = (0, 0, 1)$, resulting in the coefficients of the first fundamental form

$$E = G = 1, \quad F = 0.$$

Thus,

$$\tilde{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is noteworthy that the plane and the right cylinder share the same first fundamental form (in these parameterizations). This implies that they are locally isometric.

Proposition 10. *[49]*

Let $X : U \to \mathbb{R}^3$ be a regular surface .For a curve $\gamma(t) = X(u(t), v(t))$ on the surface X, we have

1. *The length of the curve from $t = t_1$ to $t = t_2$ is given by*

$$\begin{aligned} L(\gamma(t)) &= \int_{t_1}^{t_2} |\gamma'(s)|_g ds \\ &= \int_{t_1}^{t_2} |\frac{\partial X}{\partial u}\frac{du}{ds} + \frac{\partial X}{\partial v}\frac{dv}{ds}|_g dt \\ &= \int_{t_1}^{t_2} \sqrt{g\left(\frac{\partial X}{\partial u}\frac{du}{ds} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial s}, \frac{\partial X}{\partial u}\frac{du}{ds} + \frac{\partial X}{\partial v}\frac{dv}{ds}\right)} dt \\ &= \int_{t_1}^{t_2} \sqrt{Edu^2 + 2Fdudv + Gdv^2} dt \end{aligned}$$

2. *The energy of $\gamma(t)$ is given by the formula*

$$E(\gamma(t)) = \frac{1}{2}\int_{t_1}^{t_2} |\gamma'(s)|_g^2 ds = \frac{1}{2}\int_{t_1}^{t_2} Edu^2 + 2Fdudv + Gdv^2 dt$$

3. *Let θ be the angle between the curves γ_1 and γ_2 at the point of their intersection $\gamma_1(t_1) = \gamma_2(t_2)$. Then*

$$\cos\theta = \frac{g(\gamma_1', \gamma_2')}{\sqrt{g(\gamma_1', \gamma_1')}\sqrt{g(\gamma_2', \gamma_2')}}$$

4. *The area of an open connected set $X(U) \subset \mathbb{R}^3$ in a surface is*

$$\int\int_U |X_u \times X_v|\, dudv.$$

Hence, The area of an open connected set in a surface is

$$\int\int_U \sqrt{EG - F^2}\, dudv.$$

This comes from a vector product identity:

$$|X_u \times X_v|^2 = g(X_u, X_u)g(X_v, X_v) - (g(X_u, X_v))^2 = EG - F^2.$$

All the metric evaluations over the surface are obtained from the first quadratic form. In other words, they are intrinsic characteristics of the surface and can be evaluated by anyone living on S without referring to the exterior space in which the surface is embedded.

Definition 35. *[Christoffel symbols]*

Let $e_i \in \{e_p\}\, p \in \{1, ..., m\}$ be one of m covariant basis vectors spanning an m-dimensional space E, then $\frac{\partial e_i}{\partial x_j}$ is a vector within that same m-dimensional space and therefore can be expressed as a linear combination of the m basis vectors; The Christoffel symbols of the second kind, Γ_{ij}^k, are the components of the vector $\frac{\partial e_i}{\partial x_j}$ relative to the basis e_k. Thus,

$$\frac{\partial e_i}{\partial x_j} = \Gamma_{ij}^k e_k \tag{1.3}$$

Remark 11. *To get an expression for Γ_{ij}^l by itself, we take the dot product of equation* (1.3) *with e_l to get:*

$$\frac{\partial e_i}{\partial x_j} \cdot e_l = \Gamma_{ij}^k e_k \cdot e_l = \Gamma_{ij}^l \delta_k^l = \Gamma_{ij}^l$$

Γ is not a tensor, and so the Christoffel symbols do not exhibit the usual transformation behavior of a tensor field under the change of charts. The next statement, in particular, shows how to calculate the Christoffel symbols from the metric.

Proposition 11. *Let* $(\phi = (U, x_1, \ldots, x_n))$ *be a chart of* (M, g)*. Then we have*

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2} g^{km} \Gamma_{kij} \\ &= \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right),\end{aligned}$$

Example 11. *The metric on* $\mathbb{R}^2$ *in polar coordinates is* $g = dr^2 + r^2 d\phi^2$*. The non-vanishing components of the Levi-Civita connection coefficients are*

$$\begin{aligned}\Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = r^{-1} \\ \Gamma^r_{\phi\phi} &= -r\end{aligned}$$

Example 12. *In the Poincaré half-plane defined in example*(8) *as the space* $\mathcal{H} = \{(x, y) \in \mathbb{R}^2, y > 0\}$ *with the metric*

$$g_{\mathcal{H}}(x, y) = \frac{1}{y^2}(dx^2 + dy^2)$$

The non-vanishing components of the Levi-Civita connection coefficients are

$$\begin{aligned}\Gamma^x_{xy} &= \Gamma^x_{yx} = -y^{-1} \\ \Gamma^y_{yy} &= -y^{-1} \\ \Gamma^y_{xx} &= y^{-1}\end{aligned}$$

Chapter 2

Pseudo-Riemannain geometry

"There are things that appear incredible to most people who have not studied mathematics." Archimedes

In the preceding chapter, we explored the concept of a manifold as a topological space that locally resembles $\mathbb{R}^n$. The ability to perform calculus on a manifold relies on the existence of smooth coordinate systems. However, a manifold can acquire additional structure when equipped with a metric tensor, extending the notion of inner product from $\mathbb{R}^n$ to arbitrary manifolds. With this added structure, we define an inner product between vectors in a tangent space T_pM, enabling comparisons between vectors at different points via the "connection" concept [65].

While the metric properties of $\mathbb{R}^n$, such as distances and angles, are inherently determined by the canonical Cartesian coordinates, no such preferred coordinates exist in a general differentiable manifold. To define distances and angles on such a manifold, additional structure is required, typically achieved by introducing a special tensor field known as a Riemannian metric. This concept, introduced by Riemann in his 1854 habilitation lecture "On the hypotheses which underlie geometry," emerged in response to the discovery of non-Euclidean geometry by Gauss, Bolyai, and Lobachevsky in the 1830s. The notion of a Riemannian metric has since proven highly fruitful, notably contributing to the development of Einstein's general theory of relativity. For further details and proofs, interested readers may refer to the following references [[54],[63], [49], [58], [29],[41], [42]].

2.1 Riemannian manifolds

In this section, we consider a smooth n-dimensional manifold denoted by M. The tangent space of M at a point $p \in M$ is denoted by T_pM, and the tangent bundle of M, denoted by TM, is the union of all such tangent spaces over all points in M. Similarly, the dual space to T_pM, denoted T_p^*M, and the cotangent bundle of M, denoted by T^*M, are defined. The set of sections of TM, denoted $\Gamma(TM)$, refers to the collection of vector fields defined on M. Likewise, $\Gamma(T^*M)$ denotes the set of sections of T^*M, representing the 1-forms defined on M.

Definition 36. *[Riemannian metric]*

A Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \mathcal{T}_0^2(M)$ that is symmetric (i.e., $g(X, Y) = g_p(Y, X)$) and positive definite (i.e., $g_p(X, X) \geq 0$ where the equality holds only when $X = 0$. Owing to the properties of g, we can introduce in any tangent space T_pM a scalar product of two arbitrary tangent vectors X_p, Y_p ,

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

$$X_p.Y_p = g_p(X_p, Y_p)$$

If $x : U \to \mathbb{R}^n$ is a local chart, according to (Theorem 6), we have

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j \tag{2.1}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right). \tag{2.2}$$

And the dx^i's denote the differentials of the coordinate functions $x^1, \ldots, x^n$.

Take an infinitesimal displacement $dx^i \dfrac{\partial}{\partial x^i} \in T_pM$ and plug it into g using both (2.1) and (2.2) to find

$$ds^2 = g\big(dx^i \frac{\partial}{\partial x^i}, dx^j \frac{\partial}{\partial x^j}\big) = dx^i dx^j g\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big) = g_{ij} dx^i dx^j.$$

Remark 12. *1)We also call the quantity $ds^2 = g_{ij}\, dx^i\, dx^j$ a metric, although in a strict sense the metric is a tensor $g = g_{ij}\, dx^i \otimes dx^j$.*

Theorem 7. *[49]*
Each differentiable manifold may be equipped with a Riemannian metric.

Proposition 12. *[**?**] Let (M, g) be a Riemannian manifold of dimension n, with(U, ϕ) and (V, ψ) being two charts on M. If g_{ij} (resp $\tilde{g_{kl}}$) denotes the components of g with respect to the chart (U, ϕ)(resp (V, ψ)), then for every $x \in (U \cap V)$, the coordinate transformation is given by*

$$y = y(x) = (y_1, \ldots, y_n),$$

where

$$g_{ij} = \sum_{k,l=1}^{n} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \tilde{g}_{kl},$$

and $\left(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\right)$ and $\left(\dfrac{\partial}{\partial y_1}, \ldots, \dfrac{\partial}{\partial y_n}\right)$ denote the bases of vector fields associated with the charts (U, ϕ) and (V, ψ) respectively. For the proof, note that for every $i = 1, \ldots, n$,

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k}.$$

Hence,

$$\begin{aligned} g_{ij} &= \sum_{i,j=1}^{n} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\ &= \sum_{k,l=1}^{n} g\left(\frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}, \frac{\partial y_l}{\partial x_j}\frac{\partial}{\partial y_l}\right) \\ &= \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j} \sum_{k,l=1}^{n} g\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right) \\ &= \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j} \tilde{g}_{kl}. \end{aligned}$$

Definition 37. *[Riemannian manifold]*

An m-dimensional Riemannian manifold is a smooth m-dimensional manifold M each of whose tangent spaces T_xM is an inner-product space. The inner product is assumed to depend smoothly on $x \in M$.In simpler words, Riemannian manifold (M, g) is n- dimensional smooth manifold M together with a given Riemannian metric g.

Definition 38. *[Pull-back of a metric]*

Let M and N be two manifolds, $\phi \in C^\infty(M, N)$, and $\mathcal{T}$ a $(0, 2)$ tensor field on N. The pull-back tensor of $\mathcal{T}$ is the $(0, 2)$ tensor $\phi^*(\mathcal{T})$ on M, defined as follows: for any $p \in M$ and $v, w \in T_pM$,

$$(\phi^*\mathcal{T})_p(u, w) = \mathcal{T}_p(d_p\phi(v), d_p\phi(w)),$$

where $d_p\phi : TM \to T_{\phi(p)}N$ is the differential of ϕ at p.

Notice that the pull-back can be defined for every covariant tensor.

Proposition 13. *Let (N, g) be a Riemannian manifold and $f : M \to N$ an immersion [1]. Then f^*g is a Riemannian metric on M called the induced metric.*

Proof *We just have to prove that f^*g is symmetric and positive definite. Let $p \in M$ and $v, w \in T_pM$. Since g is symmetric,*

$$\begin{aligned} (f^*g)(v, w) &= g(df(v), df(w)) && \textit{(Definition of the pullback)} \\ &= g(df(w), df(v)) && \textit{(Symmetry of } g\textit{)} \\ &= (f^*g)(w, v), \end{aligned}$$

which shows that f^*g is symmetric.

[1] A C^∞-class function map $f : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is called an immersion at $a \in U$ if df_a is injective.In this case we have $p \geq n$.

To prove that f^*g is positive definite, let $v \in T_pM$. Since f is an immersion, $df(v) \neq 0$, and by the positive definiteness of g on N, we have

$$(f^*g)(v,v) = g(df(v), df(v)) > 0.$$

Therefore, f^*g is positive definite. Hence, f^*g is a Riemannian metric on M, and it is called the induced metric whose local expression is given by

$$\begin{aligned}
(f^*g)_{ij} &= f^*g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\
&= g\left(df\left(\frac{\partial}{\partial x_i}\right), df\left(\frac{\partial}{\partial x_j}\right)\right) \\
&= \left(\frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j} g(f)\left(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta}\right)\right) \\
&= \frac{\partial f^\alpha}{\partial x_i}\frac{\partial f^\beta}{\partial x_j} g_{\alpha\beta}(f)
\end{aligned}$$

Remark 13. *While this formula always gives something well-defined, the result is not guaranteed to actually be a Riemannian metric even if g is unless we require that df_p is injective. If df_p is noninjective, then $df_p(v) = 0$ for some $v \neq 0$, and subsequently $f^*g(v,v) = 0$, violating the positivity requirement.*

Definition 39. *[Isometries]*

Let (M,g) and (N,h) be two Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an isometry if $f^*h = g$. This definition means that the following equality holds

$$g_p(X,Y) = h_{f(p)}(df_p(X), df_p(Y)), \quad \forall X, Y \in T_pM, \forall p \in M \tag{2.3}$$

or equivalently, df is a linear isometry between T_pM and $T_{f(p)}N$.

Remark 14. *In the same way that differentiable manifolds are equivalent if they are related by a diffeomorphism, Riemannian manifolds are equivalent if they are related by an isometry, a diffeomorphism that preserves the metric.*

properties 3. *[75] the metric must satisfy the following properties for all $p \in M$*

- *Just as in Euclidean geometry, if p is a point in a Riemannian manifold (M,g), we define the length or norm of any tangent vector $X \in T_pM$ to be $||X|| := g(X,X)^{1/2}$.*
- *For a smooth curve $\gamma : [a,b] \to M$ the length of the curve is given by*

  $$L(\gamma) = \int_a^b (g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}}\, dt;$$

  A geodesic can be defined as the stationary point of the action $L(\gamma(t))$.
- *The energy of the curve $\gamma : [a,b] \to M$ $\gamma(t)$ is given by $E(\gamma) = \dfrac{1}{2}\displaystyle\int_a^b g(\gamma'(t), \gamma'(t))^2\, dt$*

- *We define the angle between two nonzero vectors $X, Y \in T_pM$ to be the unique $\theta \in [0, \pi]$ satisfying*
$$\cos\theta = \frac{g(X,Y)}{g(X,X)^{1/2}g(Y,Y)^{1/2}}.$$
- *We say that X and Y are orthogonal if $g(X,Y) = 0$. Vectors $e_1, \ldots, e_p$ are called orthonormal if they are of length 1 and pairwise orthogonal,i.e if $g(e_i, e_j) = \delta^i_j$.*

Example 13. *One obvious example of a Riemannian manifold is $\mathbb{R}^n$ with its Euclidean metric g, which is just the usual inner product on each tangent space $T_x\mathbb{R}^n = \mathbb{R}^n$. In Cartesian coordinates [2], this can be written as*

$$g = dx^i\, dx^i = (dx^i)^2$$

The matrix of g in these coordinates is thus $g_{ij} = \delta^i_j$.

Definition 40. *[The Riemannian volume form][60]*
Let (M, g) be an oriented Riemannian n-manifold with or without boundary. There exists a unique n-form dV_g on M,which vanishes nowhere, called the Riemannian volume form, characterized by any one of the following three equivalent properties

(a) If $\{\varepsilon^1, ..., \varepsilon^n\}$ is any local oriented orthonormal coframe for T^*M, then $dV_g = (\varepsilon_1 \wedge, ..., \wedge\varepsilon_n$.

(b) If $\{e^1, ..., e^n\}$ is any local oriented orthonormal frame for TM, then $dV_g(e_1, ..., e_n) = 1$.

(c) If $\{x^1, \ldots, x^n\}$ are any oriented local coordinates, then $dV_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$.

Definition 41. *[Conform metrics]*
Two metrics g and g_0 on a manifold M are said to be conformal with each other if there exists a positive function $f \in C^\infty(M)$ such that $g = f \cdot g_0$. Two Riemannian manifolds (M, g) and (M_0, g_0) are said to be conformally equivalent if there exists a diffeomorphism $\phi : M \to M_0$ such that $\phi^* g_0$ is conformal to g.
In fact, two metrics are conformal if and only if they define the same angles but not necessarily the same lengths.

Definition 42. *[Distance]*
Let (M, g) be a Riemannian manifold, the distance between two points $p \in M$ and $q \in M$ can be defined as

$$d(p, q) := \inf\{L(\gamma) : \gamma : [a, b] \to M \text{ piecewise smooth curve with } \gamma(a) = p, \gamma(b) = q\}$$

The distance function satisfies the usual axioms

(a) $d(p, q) \geq 0$ for all p, q, and $d(p, q) > 0$ for all $p \neq q$.

(b) $d(p, q) = d(q, p)$.

[2]En general the metric g takes diffrente formes depending on the coordinates system used

(c) $d(p,q) \leq d(p,r) + d(r,q)$ *(triangle inequality) for all* $p, q,$ *and* $r \in M$.

Definition 43. *[Non-degenerated metrics]*

The metric is said to be non-degenerate at p *if there is no non-zero vector* $X \in T_pM$ *such that* $g(X,Y) = 0$ *for all vectors* $Y \in T_pM$. *In terms of components, the metric is non-degenerate if the matrix* (g_{ij}) *of components of* g *is non-singular. Then we can define a unique symmetric tensor of type* $(2,0)$ *with components* g^{ij} *with respect to the basis* $\{e_a\}$ *dual to the basis* $\{e^a\}$, *by the relations* $g_{ij}g^{jk} = \delta_i^k$.

i.e. the matrix (g^{ij}) *of components is the inverse of the matrix* (g_{ij}). *It follows that the matrix* (g^{ij}) *is also non-singular, so the tensors* g^{ij}, g_{ij} *can be used to give an isomorphism between any covariant tensor argument and all contravariant arguments, or to raise and lower indices. Thus, if* X^i *are the components of a contravariant vector, then* X_i *are the components of a uniquely associated covariant vector, where* $X_i = g_{ij}X^j$ *and* $X^i = g^{ij}X_j$.

2.2 Pseudo-Riemannian Metrics

There are alternative methods for quantifying the lengths of tangent vectors on smooth manifolds by a relaxation of certain conditions found in the definition of a Riemannian metric one can obtain other types of metrics. Specifically, a pseudo-Riemannian metric is derived by relaxing the stipulation that the metric must be positive.

Definition 44. *[Pseudo-Riemannian metrics]*

A pseudo-Riemannian metric on a smooth manifold M *is a symmetric 2-tensor field* g *that is nondegenerate at each point* $p \in M$. *This means that the only vector orthogonal to everything is the zero vector. More formally,* $g(X,Y) = 0$ *for all* $Y \in T_pM$ *if and only if* $X = 0$. *Every Riemannian metric is also a pseudo-Riemannian metric; but in general pseudo-Riemannian metrics need not be positive.*

Note that in the indefinite case, it is possible for a nonzero vector to be orthogonal to itself, and thus to have norm zero. Thus $||v||$ *is not technically a norm in the usual sense, but it is customary to call it the norm of* v *anyway.*

Definition 45. *[Index of a metric]*

Given a pseudo-Riemannian metric g *and a point* $p \in M$, *by a simple extension of the Gram-Schmidt algorithm([60]Proposition 2.63), one can construct a basis* $(e_1, \ldots, e_n)$ *for* T_pM *in which* g *has the expression*

$$g = -(x_1)^2 - \ldots - (x_s)^2 + (x_{s+1})^2 + \ldots + (x_n)^2$$

for some integer $0 \leq s \leq n$. *This integer* s, *called the index of* g, *is equal to the maximum dimension of any subspace of* T_pM *on which* g *is negative definite. Therefore, the index is independent of the choice of basis.*

Definition 46. *[Another approach of index]*

Since (g_{ij}) *is a symmetric matrix, the eigenvalues are real. If* g *is Riemannian, all the eigenvalues are strictly positive, and if* g *is pseudo-Riemannian, some of them may be negative.*

If there are r positive and s negative eigenvalues, the pair (r, s) is called the index of the metric. If $s = 1$, the metric is called a Lorentz metric.

By far the most important pseudo-Riemannian metrics are the Lorentz metrics, which are pseudo-Riemannian metrics of index 1 ie the signature of the metric is $(-, +, ..., +)$ We adopt the "mostly plus" MTW[3] convention.

Definition 47. *[Lorentzian manifolds]*
A differentiable manifold M with a pseudo-Riemannian metric g of index 1 is called Riemannian Lorentzian manifold.

Definition 48. *A Lorentzian vector space (E, g) is an n-dimensional vector space E equipped with a Lorentzian scalar product $\langle , \rangle$, characterized as a non-degenerate symmetric bilinear form with index 1. This implies the existence of a basis $\{e_1, ..., e_n\}$ of E such that:*

$$\begin{cases} \langle e_i, e_j \rangle = 1, & \textit{for } i = j, \\ \langle e_i, e_j \rangle = -1, & \textit{for } i \neq j, \\ \langle e_i, e_j \rangle = 0, & \textit{otherwise}, \end{cases}$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

Proposition 14. *Let g be a Lorentzian metric on M. For every $p \in M$, there exists a neighborhood thereof with a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$ at p.*

When a metric is diagonalized using an appropriate orthogonal matrix, it becomes straightforward to adjust all its diagonal elements to either ± 1 by scaling the basis vectors with positive numbers accordingly. Starting with a Riemannian metric results in the Euclidean metric $\delta = diag(1, \ldots, 1)$, while starting with a Lorentz metric yields the Minkowski metric $\eta = diag(-1, 1, \ldots, 1)$.

Example 14. $g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *it reduces to the Minkowski metric , since* $g = S\tilde{g}S^{-1}$
where: $S = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ *and* $S^{-1} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$ *while* $\tilde{g} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Remark 15. *Lorentzian Differential Geometry, in turn, is interesting not only for Mathematics itself, but it reaches Physics as well, being the mathematical language of General Relativity. Einstein's theory gave us the mathematical tools to explain and predict physical phenomena in the theory of black holes and cosmology. It combines space and time into a single entity, represented by a spacetime manifold, and introduces a curved Lorentzian metric on it to represent the effects of gravity imposed by matter energy.*

[3] Misner-Thorne-Wheeler

Proposition 15. *With the Lorentz metric, we can classify tangent vectors at $p \in M$ into three categories, this tripartition is called causal character of vectors. Given a vector $v \in T_pM$, we say it is*

1. *Timelike, if $g_p(v,v) < 0$,*
2. *Spacelike, if $g_p(v,v) > 0$ or $v = 0$,*
3. *Lightlike, if $g_p(v,v) = 0$ and $v \neq 0$,*
4. *Causal, if v is timelike or lightlike.*

Correspondingly, a vector field $X \in \Gamma(TM)$ is called timelike, spacelike, lightlike, orcausal if $X(p)$ is timelike, spacelike, lightlike, or causal, for all $p \in M$. similarly for a smooth curve $\gamma : I \to M$ is called timelike, spacelike, lightlike, orcausal, if all its tangent vectors $\gamma'(t)$ are timelike, spacelike, lightlike, or causal, for all $t \in I$.

Since the metric is smooth, there is a boundary between the set of spacelike vectors and the set of timelike vectors. This boundary will form a double cone and is called the light cone [4] *or the set of vectors X where $g(X,X) = 0$.*

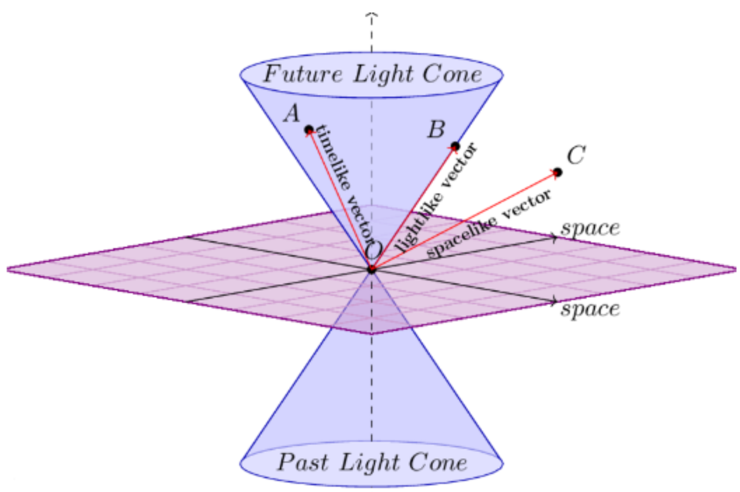

Figure 2.1: The light cone

Example 15. *The Minkowski* [5] *space $(M,g) = (\mathbb{R}^{1+n}, g_{Mink})$, where g_{Mink} denotes the Minkowski metric, is a Lorentzian manifold. In Cartesian coordinates of $\mathbb{R}^{1+n}$, g is given by*

$$g_{Mink} = -dt^2 + dx_1^2 + \ldots + dx_n^2$$

In Minkowski space $(\mathbb{R}^{1+n}, g_{Mink})$, vectors $(t,x) \in \mathbb{R}^{1+n}$ are of

[4] The reason for this naming is due to the well-known postulate in relativity stating that information cannot travel faster than light

[5] "Space itself, time itself, are condemned to fade away like mere shadows, and only a kind of union of the two preserves an independent reality" *Hermann Minkowski, 1908*

1. *space type if* $|t| < ||x||$,
2. *time type if* $|t| > ||x||$,
3. *null if* $|t| = ||x||$.

2.3 Connections & covariant derivative

We want to generalize the derivative in order to operate on abstract manifolds however, partial derivative is too limited in the sense that it depends on a direction at the point of interest, and the Lie derivative depends on the direction of the vector field X *at some point as well as at neighboring points. By using connections, we introduce an extra structure onto the manifold that allows us to achieve the desired generality.*

The tangent space to a differentiable manifold M *at a point* p*, denoted as* T_pM*, is a vector space distinct from the tangent space to* M *at any other point* q*, denoted as* T_qM*. Generally, there is no natural way to relate* T_pM *to* T_qM *if* $p \neq q$*. This implies that for two tangent vectors at different points, for example,* $v \in T_pM$ *and* $w \in T_qM$*, there is no natural way to compare or combine them. However, in many cases, it is possible to define the parallel transport of a tangent vector from one point to another point on the manifold along a curve.*

Definition 49. *[Affine connection]*
Lest (M, g) *be a pseudo Riemannian manifold.A connection (also called covariant derivative of* Y *with respect to* X*).* ∇ *is an operator that maps two vector fields* $X \in \Gamma(TM)$ *and* $Y \in \Gamma(TM)$ *to a third vector field* $\nabla_X Y \in \Gamma(TM)$ *such that the following conditions are satisfied*

$$\begin{aligned}
\nabla_{(X_1+X_2)}Y &= \nabla_{X_1}Y + \nabla_{X_2}Y \\
\nabla_{fX}Y &= f\nabla_X Y \\
\nabla_X(Y_1+Y_2) &= \nabla_X Y_1 + \nabla_X Y_2 \\
\nabla_X(fY) &= f\nabla_X Y + X(f)Y
\end{aligned}$$

Here, $f : M \to \mathbb{R}$ *denotes a real differentiable function, and* $X(f)$ *refers to the directional derivative of* f *in the direction of* X*.*

Proposition 16. *[29] Let* $(\phi = U, (x_1, \dots, x_n))$ *be a chart of* (M, g)*. The Christoffel symbols (of the second kind) with respect to* ϕ *are the* C^∞*-functions* $\Gamma^k_{ij} : U \to \mathbb{R}$ *defined by*

$$\nabla_{e_i} e_j =: \Gamma^k_{ij} e_k \quad (1 \leq i, j \leq n).$$

Where $e_i = \partial/\partial x^i$ *be the coordinate basis in* T_pM*. The connection coefficients specify how the basis vectors change from point to point. Once the action of* ∇ *on the basis vectors is defined, we can calculate the action of* ∇ *on any vectors .*

Lemma 1. *Let* ∇ *be an affine connection on* M*,* $U \subset M$*, and* $X \in \Gamma(TM), Y \in \Gamma(TM)$ *in the local coordinate system associated with* U*. We have:*

$$X = X^i \frac{\partial}{\partial x_i}, \quad Y = Y^i \frac{\partial}{\partial x_i}$$

on this set, we have

$$\begin{aligned}
\nabla_X Y &= X^i \nabla \left(\frac{\partial}{\partial x_i} \left(Y^j \frac{\partial}{\partial x_j} \right) \right) \\
&= X^i Y^j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) + X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x_j} \\
&= X^i Y^j \Gamma^k_{ij} \frac{\partial}{\partial x_k} + X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x_j} \\
&= \left(\frac{\partial Y^k}{\partial x_i} X^i + \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial x_k}. \\
&= \left(X Y_k + \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial x_k}.
\end{aligned}$$

$\nabla_X Y$ *is independent of the derivative of* X *, unlike the Lie derivative* $\mathcal{L}_V W = [V, W]$.

Definition 50. *[Torsion tensor]*

Let (M, g) *be a Pseudo-Riemannian manifold of dimension* m*, and* ∇ *be a linear connection on* M*. the torsion tensor associated with the connection* ∇ *is the* $(1,2)$*tensor given by the map*

$$\begin{aligned}
T &: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \\
T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y]
\end{aligned}$$

Properties

1. *From the* $\mathbb{R}$*-bilinearity of* ∇ *and* $[,]$*, we can conclude that* T *is* $\mathbb{R}$*-bilinear. Let* $f \in C^\infty(M)$*. We have*

$$\begin{aligned}
T(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\
&= f \nabla_X Y - Y(f) X - f \nabla_Y X + Y(f) X - f[X, Y] \\
&= f \nabla_X Y - f \nabla_Y X - f[X, Y] \\
&= f T(X, Y).
\end{aligned}$$

2. *Since* T *is antisymmetric (i.e.,* $T(X, Y) = -T(Y, X)$*), we then have*

$$\begin{aligned}
T(X, fY) &= -T(fY, X) \\
&= -T(Y, fX) \\
&= f T(X, Y).
\end{aligned}$$

Theorem 8. *[Levi-Civita connection]*

Let (M, g) *be a pseudo Riemannian manifold. Then there exists one and only one connection* ∇ *on* M *such that besides the defining properties written above we have for all* $X, Y, Z \in \Gamma(TM)$

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad \textit{(torsion-free condition)} \tag{2.4}$$

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \textit{(metric property).} \tag{2.5}$$

The map ∇ is called the Levi-Civita connection of (M, g) and is uniquely determined by the so-called Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y). \tag{2.6}$$

One of the remarkable things about Levi-Civita connections is that there always exists a unique such connection on a manifold M with metric g.

Definition 51. *[Killing fields]*
A vector field X qualifies as a Killing field when the Lie derivative of the metric g with respect to X vanishes $\mathcal{L}_X g = 0$; Where

$$\begin{aligned}
\mathcal{L}_X g(Y, Z) &= X \cdot g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&= X \cdot g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y, Z) + g(\nabla_Y X, Z) - g(Y, \nabla_X Z) + g(Y, \nabla_Z X) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X)
\end{aligned}$$

Hens, in the context of the Levi-Civita connection, this can be expressed as
$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \qquad \forall X, Y, Z \in \Gamma(TM)$

Example 16. *A vector field defined on a circle that consistently points counterclockwise with uniform magnitude at every point qualifies as a Killing vector field. This is because displacing each point along this vector field results in a pure rotation of the circle without distorting its shape.*

Remark 16. *Killing fields serve as the infinitesimal creators of isometries, we want to consider the idea that a metric may be invariant when every point in spacetime is systematically shifted by some infinitesimal amount ,meaning the flows they generate produce continuous transformations that preserve distances on the manifold. Put plainly, these flows generate symmetries such that shifting every point on an object by the same amount in the direction of the Killing vector maintains the object's distance properties intact.*

2.3.1 Inverse tangent bundle

Definition 52. *[Inverse tangent bundle]*
Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Pseudo- Riemannian manifolds. The inverse tangent bundle is defined by

$$\phi^{-1}TN = \{(p, V) \mid p \in M, V \in T_{\phi(p)}N\},$$

and the set $\Gamma(\phi^{-1}TN) = \{X : M \to TN | \forall p \in M, X_p \in T_{\phi(p)N}\}$ denote the set of all sections on $\phi^{-1}TN$.

Definition 53. *[Induced connection on the inverse tangent bundle]*
Let M and N be two Pseudo Riemannain manifolds, $\phi : M \to N$ a smooth map, and ∇^N a linear connection on N. We define the pull-back connection on the inverse tangent bundle $\phi^{-1}TN$ by

$$\nabla^\phi : \Gamma(TM) \times \Gamma(\phi^{-1}TN) \to \Gamma(\phi^{-1}TN)$$

$$X, Y \to \nabla^{\phi}_X Y$$

defined by $\nabla^{\phi}_X Y = \nabla^{\phi}_X (V \circ \phi) = \nabla^N_{d\phi(X)} V, \quad \forall X \in \Gamma(TM), \forall V \in \Gamma(TN)$

Remark 17. *Let $\phi : M \to N$ be a differentiable map between two differentiable manifolds.*

1. *For every $Y \in \Gamma(TN)$, $Y \circ \phi : M \to TN$ is a section over $\phi^{-1}(TN)$.*
2. *For every $X \in \Gamma(TM)$, $d\phi(X)$ is a section over $\phi^{-1}(TN)$.*

Proposition 17. *Let $X \in \Gamma(TM)$ and $V \in \Gamma(\phi^{-1}TN)$, Locally, we have*

$$X = X^i \frac{\partial}{\partial x_i} \quad \text{and} \quad V = V^j \frac{\partial}{\partial y_j} \circ \phi, \quad i = 1, ..., m; \quad j = 1, ..., n$$

where X^i, V^j are smooth function, and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ resp. $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ are the basic fields on M ,resp on N. Therefor

$$\begin{aligned}
\nabla^{\phi}_X V &= X^i \left(\nabla^{\phi}_{\frac{\partial}{\partial x_i}} V^j \frac{\partial}{\partial y_j} \circ \phi \right) \\
&= X^i \left(\frac{\partial V^j}{\partial x_i} \frac{\partial}{\partial y_j} \circ \phi + V^j \nabla^{\phi}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y_j} \circ \phi \right) \\
&= X^i \left(\frac{\partial V^j}{\partial x_i} \frac{\partial}{\partial y_j} \circ \phi + V^j \nabla^N_{d\phi(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial y_j} \right) \\
&= X^i \left(\frac{\partial V^j}{\partial x_i} \frac{\partial}{\partial y_j} \circ \phi + V^j \nabla^N_{(\frac{\partial \phi^{\alpha}}{\partial x_i} \frac{\partial}{\partial y^{\alpha}} \circ \phi)} \frac{\partial}{\partial y_j} \right) \\
&= X^i \left(\frac{\partial V^j}{\partial x_i} \frac{\partial}{\partial y_j} \circ \phi + V^j \frac{\partial \phi^{\alpha}}{\partial x_i} \nabla^N_{\frac{\partial}{\partial y^{\alpha}}} \frac{\partial}{\partial y_j} \circ \phi \right) \\
&= X^i \left(\frac{\partial V^j}{\partial x_i} \frac{\partial}{\partial y_j} \circ \phi + V^j \frac{\partial \phi^{\alpha}}{\partial x_i} \Gamma^k_{\alpha j} \frac{\partial}{\partial y_k} \circ \phi \right) \\
&= X^i \left(\frac{\partial V^k}{\partial x_i} + V^j \frac{\partial \phi^{\alpha}}{\partial x_i} \Gamma^k_{\alpha j} \right) \frac{\partial}{\partial y_k} \circ \phi.
\end{aligned}$$

Proposition 18. *Let $\phi : M \to N$ be a differentiable map between pseudo Riemannain manifolds.*

- *If ∇^N is a linear connection compatible with a metric h on N, then the linear connection ∇^{ϕ} is compatible with the metric h_{ϕ} on $\phi^{-1}TN$. That is, for all $X \in \Gamma(TM)$ and $V, W \in \Gamma(\phi^{-1}TN)$, we have*

$$X(h_{\phi}(V, W)) = h_{\phi}(\nabla^{\phi}_X V, W) + h_{\phi}(V, \nabla^{\phi}_X W).$$

- *If ∇^N is a torsion-free connection on N. Then,*

$$\nabla^\phi_X d\phi(Y) = \nabla^\phi_Y d\phi(X) + d\phi([X,Y]),$$

for all $X, Y \in \Gamma(TM)$.

Definition 54. *[Second fundamental form]*

Let (M,g) and (N,h) be two Pseudo Riemannian manifolds, and $\phi : (M,g) \to (N,h)$ a differentiable map of class C^∞. The second fundamental form of the map ϕ is the covariant derivative of the vector 1-form $d\phi$, defined by

$$\nabla d\phi(X,Y) = \nabla d\phi(Y,X) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla^M_X Y),$$

for all $X, Y \in \Gamma(TM)$.

Proposition 19. *Let $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\}$ resp. $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\}$ be the basic vectors fields on$M$,resp on N.Hence,*

$$\begin{aligned}
\nabla d\phi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= \nabla^\phi_{\frac{\partial}{\partial x_i}} d\phi\left(\frac{\partial}{\partial x_j}\right) - d\phi\left(\nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right) \\
&= \nabla^\phi_{\frac{\partial}{\partial x_i}} \frac{\partial \phi^\alpha}{\partial x_j} \frac{\partial}{\partial y^\alpha} \circ \phi - d\phi \tilde{\Gamma}^k_{ij} \frac{\partial}{\partial x_k} \\
&= \frac{\partial^2 \phi^\alpha}{\partial x_j x_i} \frac{\partial}{\partial y^\alpha} \circ \phi + \frac{\partial \phi^\alpha}{\partial x_j} \nabla^\phi_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y^\alpha} \circ \phi - \tilde{\Gamma}^k_{ij} \frac{\partial \phi^\beta}{\partial x_k} \frac{\partial}{\partial y_\beta} \circ \phi \\
&= \frac{\partial^2 \phi^\alpha}{\partial x_j x_i} \frac{\partial}{\partial y^\alpha} \circ \phi + \frac{\partial \phi^\alpha}{\partial x_j} \nabla^N_{d\phi(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial y^\alpha} - \tilde{\Gamma}^k_{ij} \frac{\partial \phi^\beta}{\partial x_k} \frac{\partial}{\partial y_\beta} \circ \phi \\
&= \frac{\partial^2 \phi^\alpha}{\partial x_j x_i} \frac{\partial}{\partial y^\alpha} \circ \phi + \frac{\partial \phi^\alpha}{\partial x_j} \nabla^N_{(\frac{\partial \phi^k}{\partial x_i} \frac{\partial}{\partial_k} \circ \phi)} \frac{\partial}{\partial y^\alpha} - \tilde{\Gamma}^k_{ij} \frac{\partial \phi^\beta}{\partial x_k} \frac{\partial}{\partial y_\beta} \circ \phi \\
&= \frac{\partial^2 \phi^\alpha}{\partial x_j x_i} \frac{\partial}{\partial y^\alpha} \circ \phi + \frac{\partial \phi^\alpha}{\partial x_j} \frac{\partial \phi^k}{\partial x_i} \Gamma^\mu_{k\alpha} \frac{\partial}{\partial y^\mu} \circ \phi - \tilde{\Gamma}^k_{ij} \frac{\partial \phi^\beta}{\partial x_k} \frac{\partial}{\partial y_\beta} \circ \phi \\
&= \left(\frac{\partial^2 \phi^\nu}{\partial x_j x_i} + \frac{\partial \phi^\alpha}{\partial x_j} \frac{\partial \phi^k}{\partial x_i} \Gamma^\nu_{k\alpha} - \tilde{\Gamma}^k_{ij} \frac{\partial \phi^\nu}{\partial x_k} \right) \frac{\partial}{\partial y_\nu} \circ \phi
\end{aligned}$$

Where Γ and $\tilde{\Gamma}$ are respectivlly the christoffel symbols corresponding to the connexion of M and N.

The second fundamental form of a map between pseudo-Riemannian manifolds is used largely in the analysis of certain properties of maps between manifolds.

Definition 55. *Let (M,g) and (N,h) be two Pseudo Riemannian manifolds. We say that a map $\phi : (M,g) \to (N,h)$ is totally geodesic if $\nabla d\phi = 0$.*

Definition 56. *[Tension field]*

Let $\phi : (M, g) \to (N, h)$ be a smooth map between pseudo-Riemannain manifolds. The trace of the second fundamental form of the map ϕ is called the tension field of the map ϕ, denoted by

$$\tau(\phi) = \operatorname{tr}_g \nabla d\phi.$$

With respect to an orthonormal basis $\{e_i\}$ on M, we have

$$\tau(\phi) = \nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla^{M}_{e_i} e_i).$$

Parallel transport & geodesics

When employing the Levi-Civita connection, we can straightforwardly compute connection coefficients, Riemann tensors, and various related quantities. Additionally, the Levi-Civita connection provides an interpretation of geodesics as the shortest paths connecting two given points, adding to its simplicity. In Newtonian mechanics, the trajectory of a free particle corresponds to the straightest and shortest possible curve:a straight line. Einstein proposed that this principle should extend to general relativity; if gravity is considered a component of spacetime geometry, freely falling particles should follow the straightest and shortest possible paths.

Now, to construct a vector field on a curve γ defined on a pseudo-Riemannian manifold endowed with a connection ∇ that remains unchanged as we move along the curve: Let I be an interval of $\mathbb{R}$ containing 0, and $\gamma : I \to M$ be a curve. Suppose $X \in \Gamma(TM)$ is a vector field over M. We aim to calculate $\nabla_{\dot\gamma(t)} X$ in a coordinate system. We decompose X as $X = X^i \dfrac{\partial}{\partial x^i}$, and proceed as follows

$$\begin{aligned}
\nabla_{\gamma'(t)} X_{\gamma(t)} &= \nabla_{\frac{d(x^i\circ\gamma)}{dt}\frac{\partial}{\partial x^i}} X^j \frac{\partial}{\partial x^j} \\
&= \frac{d(x^i\circ\gamma)}{dt}\left(\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^j \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)
\end{aligned}$$

This shows that $\nabla_{\gamma'} X$ depends only on the values of X taken along the curve.

Definition 57. *[Parallel transport]*

A section $V \in \Gamma(TM)$ is said to be parallel with respect to the connection ∇ if

$$\nabla_X V = 0$$

for all $X \in \Gamma(TM)$.

Definition 58. *A vector field Y on a manifold M equipped with an affine connection ∇ is uniform if*

$$\nabla_X Y = 0$$

for every vector field $X \in \Gamma(TM)$.

A case of particular interest is that in which the vector tangent to a curve is parallel-transported along it. If this is the case, the curve is called a geodesic of the linear connection.

Definition 59. *[Geodesics]*

Let (M,g) be a pseudo-Riemmannian manifold and Let ∇ be a connection on the tangent bundle TM . A curve $\gamma : I \subset \mathbb{R} \to M$ is called autoparallel or geodesic with respect to ∇ if

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0 \tag{2.7}$$

i.e., if the tangent field of $\dfrac{d\gamma}{dt}$ is parallel along the curve γ. In a local chart(U, x_i), equation (2.7) *is given by:*

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

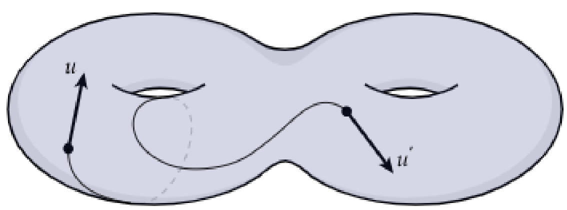

Figure 2.2: a representation of a parallel transport of a vector u along a curve

Definition 60. *The geodesic curvature of a curve $\gamma : (a,b) \to M$ of class C^2 naturally parameterized (i.e., with constant speed $||\dot{\gamma}(t)|| = 1$ for all $t \in (a;b)$ on a Riemannian manifold (M,g) is the norm of its covariant acceleration. We denote this curvature by $\kappa(t) = ||\nabla_{\dot{\gamma}}\dot{\gamma}||$. It is clear that the curve is geodesic if and only if its geodesic curvature is identically zero.*

Intuitively, geodesics are lines that "curve as little as possible"; they are the "straightest possible lines" one can draw in a curved geometry. Given a derivative operator, ∇, we define a geodesic to be a curve whose tangent vector is parallel propagated along itself. The most fundamental fact about geodesics, which is proved in [60]Chapter 4 , is that given any point $p \in M$ and any vector v tangent to M at p, there is a unique geodesic starting at p with initial velocity v.

2.4 Curvatures on pseudo-Riemannian manifolds

As with surfaces, the basic geometric invariant is curvature, but curvature becomes a much more complicated quantity in higher dimensions because a manifold may curve in so many different directions. The curvature of a manifold M is a measure of the deviation from the usual Euclidean geometry in the tangent space of M at various points. It quantifies how vectors change when transported along infinitesimal closed curves.If we consider a closed curve passing through a point p and parallel transport along a tangent vector X from T_pM, we generally do not return to X. This situation is different from the case of $\mathbb{R}^n$ and is due to the fact that covariant derivatives do not generally commute. To measure this non-commutation, we introduce the curvature tensor.See[[18],[60]]

2.4.1 Curvature tensor

Definition 61. *Let (M, g) be a pseudo-Riemannian manifold of dimension m, and ∇ a Levi-Civita connection. Then we define the function:*

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \forall X, Y, Z \in \Gamma(TM),$$

is a tensor of type $(1,3)$ on M, called a curvature tensor of Riemann or simply curvature tensor.

When a tangent vector is transported parallelly to itself along a closed curve, the resulting vector upon returning to the initial point may not necessarily align with the original tangent vector. Curvature tensor measures the deference between the initial position of the vector and the final one.

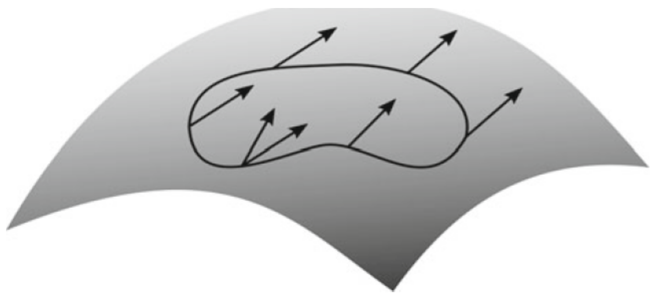

Figure 2.3: When a tangent vector is parallelly transported along a closed curve, it may not return to its initial state.

Definition 62. *The curvature tensor R can be expressed as a function of the Christoffel symbols*

$$R(\partial_i, \partial_j)\partial_k = \sum_{s=1}^{m} R^s_{ijk} \partial_s,$$

where $\{\partial_i\}$ represents a local basis of the vector fields on M. With the condition $[\partial_i, \partial_j] = 0$, we derive

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \frac{\partial \Gamma^l_{jk}}{\partial x_i}\partial_l + \Gamma^l_{jk}\nabla_{\partial_i}\partial_l - \frac{\partial \Gamma^l_{ik}}{\partial x_j}\partial_l + \Gamma^l_{ik}\nabla_{\partial_j}\partial_l \\
&= \frac{\partial \Gamma^l_{jk}}{\partial x_i}\partial_l + \Gamma^l_{jk}\Gamma^s_{il}\partial_s - \frac{\partial \Gamma^l_{ik}}{\partial x_j}\partial_l + \Gamma^l_{ik}\Gamma^s_{jl}\partial_s \\
&= \left\{ \frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j} + \Gamma^l_{jk}\Gamma^s_{il} - \Gamma^l_{ik}\Gamma^s_{jl} \right\} \partial_s.
\end{aligned}$$

Therefore, the components of the curvature tensor R are given by

$$R^s_{ijk} = \Gamma^l_{jk}\Gamma^s_{il} - \Gamma^l_{ik}\Gamma^s_{jl} + \frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j}.$$

Proposition 20. *Let (M, g) be a pseudo-Riemannian manifold. For all $X, Y, Z, W \in \Gamma(TM)$ we have*

1. $R(X, Y)Z = -R(Y, X)Z$.
2. $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$.
3. $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$.
4. *R verified Bianchi's identity algebraic:*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

5. *R verified Bianchi's identity differential:*

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0.$$

2.4.2 Sectional curvature

We now come to the sectional curvature $K_p(\Pi)$ of the 2-plane $\Pi \subset T_pM$ spanned by two vectors X andY in T_pM. It is defined by

$$K_p(\Pi) = \frac{g_p(R(X, Y)Y, X)}{g_p(X, X)g(Y, Y) - g(X, Y)^2} \tag{2.8}$$

where R is the Riemannian curvature tensor and $g_p(X, Y)$ denotes the metric tensor at the point p on the manifold M.

The right-hand side of formula (2.8) is, in fact, independent of the choice of a basis for Π. Clearly, if $\{X, Y\}$ is an orthonormal basis for Π, then

$$K_p(\Pi) = R(X, Y, X, Y) = g_p(R(X, Y)Y, X)$$

Example 17.

1. *The space $\mathbb{R}^n$ has curvature 0.*
2. *The sphere $\mathbb{S}^n$ has constant sectional curvature $+1$.*
3. $\mathcal{H}^2 = \{(x, y) \in \mathbb{R}^2,\ y > 0\}$ *the hyperbolic space with the metric* $g = \dfrac{dx^2 + dy^2}{y^2}$, *has constant sectional curvature -1.*

Remark 18. *The sectional curvature measures how the intrinsic geometry of the manifold deforms in the tangent plane determined by vectors X and Y. It indicates how vectors evolve when parallel transported along closed curves in this plane.*

- ♣ *Positive sectional curvature means that vectors come closer to each other after parallel transport, corresponding to "convex" local curvature of the manifold in the tangent plane.*
- ♣ *Negative sectional curvature means that vectors move away from each other after parallel transport, indicating "concave" local curvature.*
- ♣ *Zero sectional curvature indicates that vectors maintain the same relative distance after parallel transport, as in a flat Euclidean space.*

2.4.3 Ricci curvature

Ricci curvature is a powerful mathematical tool that describes how spacetime curves in the presence of matter and energy in general relativity. Its application in Einstein's equations and the prediction of gravitational phenomena make it a key concept for understanding the geometry of the universe on a large scale.

Definition 63. *The Ricci curvature of pseudo-Riemannian manifold (M, g), of dimension m is a tensor of type $(0, 2)$ defined by:*

$$\begin{aligned} Ric(X,Y) &= trace(Z \longmapsto R(Z,X)Y) \\ &= \varepsilon_i g(R(e_i,X)Y,e_i), \end{aligned}$$

for all $X, Y \in \Gamma(TM)$, where $\{e_i\}$ is an orthonormal frame on M and $\varepsilon_i = g(e_i, e_i)$.

Proposition 21. *The Ricci curvature is symmetrical. Indeed*

$$\begin{aligned} Ric(X,Y) &= \varepsilon_i g(R(e_i,X)Y,e_i) \\ &= \varepsilon_i g(R(Y,e_i)e_i,X) \\ &= \varepsilon_i g(R(e_i,Y)X,e_i) \\ &= Ric(Y,X) \qquad . \end{aligned}$$

Definition 64. *The Ricci tensor of a pseudo-Riemannian manifold (M, g), of dimension n is a tensor of type $(1, 1)$ defined by:*

$$Ricci(X) = \varepsilon_i R(X,e_i)e_i, \quad \forall X \in \Gamma(TM).$$

Remark 19. *For all $X, Y \in \Gamma(TM)$ we have*

$$Ric(X,Y) = g(Ricci(X),Y).$$

Definition 65. *We call scalar curvature of a pseudo-Riemannian manifold (M, g), of dimension m, the function defined on M by:*

$$S = trace_g Ric = \varepsilon_i \epsilon_j g(R(e_i,e_j)e_j,e_i).$$

A Riemannian or pseudo-Riemannian metric is said to be an Einstein metric if its Ricci tensor is a constant multiple of the metric gthat is,

$$Ric = \lambda g, \text{ for some constant } \lambda$$

2.4.4 Scalar curvature

The scalar curvature R of a Riemannian manifold (M^m, g) is defined as the trace of the Ricci tensor

$$R = g^{ij} Ric_{ij},$$

where g^{ij} are the components of the inverse tensor of the Riemannian metric g, and Ric_{ij} are the components of the Ricci tensor.

properties 4. *Here are some important properties of scalar curvature R*

1. *Scalar curvature is a global invariant of the Riemannian manifold and does not depend on the choice of local coordinates.*

2. *Scalar curvature is additive: for a Riemannian manifold formed by the union of several disjoint manifolds, the total scalar curvature is the sum of the scalar curvatures of the submanifolds.*

2.5 Operators on Riemannian manifold

2.5.1 Gradient operator

The gradient operator usually have been introduced based on its expression in Cartesian coordinates; this is by far the simplest way to introduce it. However, its definition does not rely on any particular coordinate system; it acts on a scalar field intrinsically. It applies to a scalar field and yields a vector field denoted as $grad(f)$

Definition 66. *[Gradient]*

Let (M, g) be a pseudo-Riemannian manifold of dimension m.The gradient operator $grad(f)$ of a function $f : M \to \mathbb{R}$ is defined as the unique tangent vector such that for any tangent vector $X \in \Gamma(TM)$

$$g(grad(f), X) = X(f) = df(X) \tag{2.9}$$

Where $df = \dfrac{\partial f}{\partial x^i} dx^i \in \Gamma(T^*M)$ is the differential of f In local coordinates $(U, x^1, \ldots, x^m)$, the gradient operator for all $f \in C^1(M)$ is given by

$$grad(f)|_U = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Remark 20. *The equation (2.9) shows that $grad(f)$ is orthogonal to the level surfaces $f =$ const because if X is tangent to one of these surfaces, then $X(f) = 0$ and therefore $g(gradf, X) = 0$, which implies that $grad(f)$ is orthogonal to the vector fields tangent to the level surfaces of f .*

properties 5. *The gradient has the following properties*

1. *The gradient is linear: $grad(\alpha f + \beta g) = \alpha grad(f) + \beta grad(g)$, where α and β are constants.*

2. *The gradient satisfies the product rule:* $grad(fg) = ggrad(f) + fgrad(g)$.

3. *The gradient of a constant function is zero:* $grad(c) = 0$.

Example 18. *Let's consider a real value function* $f : M = \mathbb{R}^3 \to \mathbb{R}$ *then we have*

1. *In Cartesian coordinates, we have* $grad(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

2. *In polar coordinates,* $grad(f) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$.

2.5.2 Divergence operator for vector fields

Let (M, g) *be a pseudo-Riemannain manifold.The divergence operator* $div(X)$ *of a vector field* $X \in \Gamma(TM)$ *is defined as the real-valued function given by*

$$div(X) = \nabla^i X_i,$$

where ∇^i *are the contravariant components of the connection operator.*

properties 6. *The divergence has the following properties*

1. *the divergence is linear:* $div(\alpha X + \beta Y) = \alpha\, div(X) + \beta\, div(Y)$*, where* α *and* β *are constants.*

2. *The divergence of the gradient is the Laplacian:* $div(grad(f)) = \Delta f$.

Proposition 22. *Consider a semi-Riemannian manifold* (M, g) *of dimension* n*. Then, the divergence of a vector field* $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \in \Gamma(TM)$ *is given by*

$$\operatorname{div} X = \sum_{i=1}^{n} \left(\frac{\partial X_i}{\partial x_i} + X_j \Gamma_{ij}^k\right),$$

where $\{\partial/\partial x_i\}$ *forms a local basis of vector fields on* M.

Proposition 23. *For a semi-Riemannian manifold* (M, g)*, the following properties hold*

1. $\operatorname{div}(X + Y) = \operatorname{div} X + \operatorname{div} Y$.

2. $\operatorname{div}(fX) = f \operatorname{div} X + X(f)$.

for all $X, Y \in \Gamma(TM)$ *and* $f \in C^\infty(M)$.

Lemma 2. *On a Riemannian manifold* (M, g)*, the following expression holds*

$$\frac{\partial}{\partial x_k}\left(\sqrt{\det(g_{ij})}\right) = \sqrt{\det(g_{ij})} \sum_{i=1}^{n} \Gamma_{ij}^k.$$

Proposition 24. *For a Riemannian manifold* (M, g)*, the divergence of a vector field* $X \in \Gamma(TM)$ *is given by*

$$\operatorname{div} X = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_k}\left(\sqrt{\det(g_{ij})} X^k\right).$$

2.5.3 Laplacian operator

The Laplacian plays a fundamental role in physics, as it appears in almost every physical discipline that involves partial differential equations (EDP). It is defined intrinsically (independent of the coordinate system) as the second-order differential operator:

$$\Delta = div\vec{\nabla} = \nabla^2$$

Calculating the Laplacian of a field involves first computing its gradient and then the divergence of that gradient. The Laplacian can be calculated for scalar fields f, vector fields X, or tensor fields $\mathcal{T}$.

Let (M, g) be a Riemannian manifold with or without boundary, and let x^i be any smooth local coordinates on an open set $U \subseteq M$. The coordinate representations of the divergence and Laplacian are as follows:

The divergence of X is given by

$$div(X) = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}\,X^i\right),$$

where $X = X^i\frac{\partial}{\partial x^i}$ is a vector field and $\det(g)$ is the determinant of the metric tensor.

hence, the Laplacian of a function f is given by

$$\Delta u = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}\,g^{ij}\frac{\partial f}{\partial x^j}\right),$$

Definition 67. *Let (M, g) be a Pseudo-Riemannian manifold, we define the Laplacian operator note $\triangle$, on M by*

$$\begin{array}{rcl} \triangle : C^\infty(M) & \longrightarrow & C^\infty(M) \\ f & \longmapsto & \triangle(f) = div(grad\, f). \end{array}$$

Remark 21. ***Laplace's Problem***

A function f that satisfies the problem

$$\Delta f = 0$$

in a domain D is said to be a solution to Laplace's problem. It is also stated that f is a harmonic function.

Proposition 25. *Let (M, g) be a Pseudo-Riemannian manifold, then*

1. $\triangle(f + h) = \triangle(f) + \triangle(h)$,
2. $\triangle(fh) = h\triangle(f) + f\triangle(h) + 2g(grad\, f, grad\, h)$,

for all $f, h \in C^\infty(M)$.

Proposition 26. *Let (M, g) be a Pseudo-Riemannian manifold, then the expression of the Laplacian in local coordinates is given by:*

$$\triangle(f) = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right), \quad for\ all\ f \in C^\infty(M).$$

In effect, let $f \in C^\infty(M)$, then

$$\begin{aligned}
\triangle(f) &= div(grad\, f) \\
&= g^{ij} g(\nabla_{\frac{\partial}{\partial x_i}} grad\, f, \frac{\partial}{\partial x_j}) \\
&= g^{ij}(\frac{\partial}{\partial x_i} g(grad\, f, \frac{\partial}{\partial x_j}) - g(grad\, f, \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j})) \\
&= g^{ij}(\frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_j}) - \Gamma^k_{ij} g(grad\, f, \frac{\partial}{\partial x_k})) \\
&= g^{ij}(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial f}{\partial x_k}).
\end{aligned}$$

Theorem 9. *[Divergence theorem]*

Let (M, g) be a Riemannian manifold, and let D be a compact domain with boundary on M. Suppose ω is a 1-form and X is a vector field defined on a neighborhood in D, then

$$\int_D \operatorname{div} X \, \mathrm{v}_g = \int_{\partial D} g(X, n)\, v_{\partial D}$$

and

$$\int_D \operatorname{div} \omega \, \mathrm{vol}_g = \int_{\partial D} \omega(n)\, v_{\partial D},$$

where ∂D is the boundary of D and $n \approx n(x)$ is the unit normal at a point $x \in \partial D$.

Corollary 1.9.1

Let X be a vector field (resp. ω a 1-form) with compact supports in a domain D, then

$$\int_D \operatorname{div} X \, \mathrm{vol}_g = 0 \quad \text{and} \quad \int_D \operatorname{div} \omega \, \mathrm{vol}_g = 0.$$

Chapter 3

Harmonic maps between Riemannian manifolds

"Si vous voulez trouver les secrets de l'univers, pensez en termes d'énergie, de fréquence, d'information et de vibration."Nikola Tesla-Ingénieur, Inventeur, Physicien, Scientifique (1856 - 1943)

Introduction

This chapter explores the development of harmonic and biharmonic maps between Riemannian manifolds. Harmonic maps were introduced by Sampson and Nash in the 1950s, with significant implications in differential geometry. Eells and Sampson collaborated on groundbreaking research in 1964, further advancing the field. Lichnerowicz extended this concept to f-harmonic maps in 1970, followed by Ara's pioneering work in 1999. Jiang's introduction of biharmonic maps in 1986 sparked considerable interest, with numerous contributors advancing the study

The subject of harmonic maps is vast and has found many applications,We first consider relevant aspects of harmonic functions on Euclidean space; then we give a general introduction to harmonic maps.More information on harmonic maps can be found in the following articles and books [[43], [53], [57], [59]]

3.1 Harmonic functions on Euclidean spaces

Harmonic functions on an open domain $\Omega \subset \mathbb{R}^m$ *are solutions of the Laplace equation* $\Delta f = 0$, *where*

$$\Delta := \frac{\partial^2}{(\partial x^1)^2} + \ldots + \frac{\partial^2}{(\partial x^m)^2} \quad for \quad (x^1, \ldots, x^m) \in \mathbb{R}^m. \tag{3.1}$$

The operator Δ *is called the Laplace operator or Laplacian after P.S. Laplace. Equation* (3.1) *and the Poisson equation* $-\Delta f = g$ *play a fundamental role in mathematical physics*

- ♣ *The Laplacian occurs in Newton's law of gravitation :the gravitational potential* U *obeys the law* $-\Delta U = -4\pi G\rho$, *where* ρ *is the mass density.*

♣ *Electromagnetism :the electric potential* V *is a solution of* $-\epsilon_0 \Delta V = \rho$*, where* ρ *is the electric charge distribution.*

♣ *Fluid mechanics the right-hand side term in the Navier-Stokes system*[1]

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = \nu \Delta u_i$$

models the effect of viscosity,

♣ *The heat equation* $\dfrac{\partial f}{\partial t} = \Delta f.$

Harmonic functions serve as critical points, also known as extremal, for the Dirichlet functional

$$E(f) = \frac{1}{2} \int_\Omega \left(\frac{\partial f(x)}{\partial x^i} \right)^2 d^m x = \frac{1}{2} \int_\Omega \left| \frac{df}{dx}(x) \right|^2 d^m x,$$

where $d^m x := dx^1 \cdot \ldots \cdot dx^m$ *represents the volume element of* $\mathbb{R}^m$*.*

The variational formulation (G. Green, 1833; K.F. Gauss, 1837; W. Thomson, 1847; B. Riemann, 1853) highlights that the Laplace operator is dependent on the canonical metric on $\mathbb{R}^m$*, as* $|df(x)|$ *represents the Euclidean norm of* $df(x) \in (\mathbb{R}^m)^* \cong \mathbb{R}^m$*.*

3.2 The concepts of energy & tension

In the following paragraph, $M = (M, g)$ *and* $N = (N, h)$ *represent smooth pseudo-Riemannian manifolds, both of arbitrary (finite) dimensions* m *and* n *respectively. The Levi-Civita connections of* M *and* N *are denoted by* ∇^M *and* ∇^N *respectively. We define a compact domain* D *of* M *as a non-empty connected open subset of* M*. Throughout, we utilize the Einstein summation convention, implying summation over repeated subscript-superscript pairs.*

Before introducing the generalization of harmonic maps to pseudo-Riemannian manifolds, we first introduce some related concepts, namely Energy and Tension Field. We will let $(x^1, \ldots, x^m)$ *denote local coordinates on* M *in a neighborhood of a point* $p \in M$ *, and* $(y^1, \ldots, y^n)$ *local coordinates on* N*. Thus, we can write the Riemannian metrics* g *and* h *in these local coordinates as*

$$ds^2 = g_{ij} dx^i dx^j, \quad \tilde{ds}^2 = h_{\alpha\beta} dy^\alpha dy^\beta,$$

Definition 68. *[Density of a map]*

Let $\phi : (M, g) \to (N, h)$ *be a* C^∞*-class map between two Pseudo-Riemannian manifolds of dimensions* m *and* n*, respectively. The density of* ϕ *is the application defined for all* $p \in M$ *by*

$$e(\phi) : M \to \mathbb{R}^+$$
$$e(\phi)(p) = \frac{1}{2} \|d_p \phi\|^2$$

[1] The Navier-Stokes equations describe how a fluid flows. They are derived by applying Newton's laws of motion to the flow of an incompressible fluid.They dictate not position but rather velocity.

where $\|d\phi\|$ *is the Hilbert-Schmidt norm of the differential* $d\phi$ *at the point p.that is the trace of the pull back of the metric on the tangent space by the map* ϕ *If* $\{e_i\}_{1\le i\le m}$ *is an orthonormal basis of* T_pM*, then*

$$\|d\phi\|^2 = tr_g(\phi^* h) = h(d\phi(e_i), d\phi(e_i)).$$

Definition 69. *[Energy of a map]*

Let $\phi : (M, g) \to (N, h)$ *be a* C^∞*-class map between two Pseudo-Riemannian manifolds of dimensions* m *and* n*, respectively. The energy functional* $E(\phi)$ *on a compact* [2] *domain* $D \in M$ *is defined as*

$$E(\phi) = \int_D e(\phi(t))dv_g = \frac{1}{2}\int_D \|d\phi\|^2\, dv_g$$

In a local coordinate representation, we have

$$E(\phi) = \int_D g^{ij}\frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi^\beta}{\partial x_j}h_{\alpha\beta}\sqrt{\det(|g_{ij}|)}\, dx^1\wedge\cdots\wedge dx^m.$$

3.2.1 First variation of energy

Theorem 10. *[57]*

Let $\phi : (M, g) \longrightarrow (N, h)$ *be a smooth map between two pseudo-Riemannian manifolds of dimensions* m *and* n *respectively ;and let* (ϕ_t) *be a smooth variation of* ϕ *supported in D. Then:*

$$\frac{d}{dt}E(\phi_t)\big|_{t=0} = -\int_D h(v, \tau(\phi))\, v_g,$$

where $v = \dfrac{d\phi_t}{dt}\big|_{t=0}$ *denotes the variation vector field of* $\{\phi_t\}$*,*

$$\tau(\phi) = \mathrm{trace}_g\nabla d\phi = \varepsilon_i\big(\nabla^\phi_{e_i}d\phi(e_i) - d\phi(\nabla^M_{e_i}e_i)\big), \tag{3.2}$$

is called tension field of ϕ *where* $\{e_1, ..., e_m\}$ *is an orthonormal frame on* (M, g) *and* $\varepsilon_i = g(e_i, e_i) = \pm 1$.

proof *Let's consider* $\varphi : M \times (-\delta, \delta) \to N$ *by* $\varphi(x, t) = \phi_t(x)$*, let* ∇^ϕ *denote the pull-back connection on* $\phi^{-1}TN$*. We have* $[\partial_t, X] = 0$.

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[2] The importance of the concept of compactness lies in the fact that it allows us to reduce problems of seemingly infinite complexity to the study of a finite number of cases.

$$\begin{aligned}
\frac{d}{dt}E(\phi_t)\big|_{t=0} &= \frac{1}{2}\frac{d}{dt}\int_D h(d\phi_t(e_i), d\phi_t(e_i))\, v_g\big|_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\int_D h(d\tau(e_i,0), d\tau(e_i,0))\, v_g\big|_{t=0} \\
&= \frac{1}{2}\int_D \frac{\partial}{\partial t} h(d\tau(e_i,0), d\tau(e_i,0))\, v_g\big|_{t=0} \\
&= \int_D h(\nabla^{\tau}_{(0,\frac{d}{dt})} d\tau(e_i,0), d\tau(e_i,0))\, v_g\big|_{t=0} \\
&= \int_D h(\nabla^{\tau}_{(e_i,0)} d\tau(0,\frac{d}{dt}), d\tau(e_i,0))\, v_g\big|_{t=0} \\
&= \int_D h(\nabla^{N}_{d\phi(e_i)} v, d\phi(e_i))\, v_g \\
&= \int h(\nabla^{\phi}_{e_i} v, d\phi(e_i))\, v_g. \qquad (3.3)
\end{aligned}$$

Define an 1-form on M by

$$\omega(X) = h(v, d\varphi(X)), \quad X \in \Gamma(TM).$$

We have:

$$\begin{aligned}
\mathrm{div}^M \omega &= \varepsilon_i(\nabla_{e_i}\omega)(e_i) \\
&= \varepsilon_i\big(e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\big) \\
&= \big(h(\nabla^{\phi}_{e_i} v, d\phi(e_i)) + h(v, \nabla^{\phi}_{e_i} d\phi(e_i)) - h(v, d\phi(\nabla^M_{e_i} e_i))\big) \\
&= h(\nabla^{\phi}_{e_i} v, d\phi(e_i)) + h(v, \tau(\phi)), \qquad (3.4)
\end{aligned}$$

according to formulas (3.3), (4.5), and the divergence theorem we obtain:

$$\frac{d}{dt}E(\phi_t)\big|_{t=0} = -\int_D h(v, \tau(\phi)) v_g.$$

Theorem 11. *[Harmonic map]*

A smooth map $\phi : (M, g) \to (N, h)$ between two Pseud-Riemannian manifolds is harmonic if and only if:

$$\tau(\phi) = trace_g \nabla d\phi = 0.$$

If $(x^i)_{1\le i\le m}$ and $(y^\alpha)_{1\le\alpha\le n}$ denote local coordinates on M and N respectively, then respectively the equation $\tau(\phi) = 0$ takes the form:

$$\tau(\phi)^k = \varepsilon_i(\Delta\phi^k + g^{ij\,N}\Gamma^k_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j})\frac{\partial}{\partial y^\alpha} \circ \phi = 0. \qquad (3.5)$$

where $\Delta\phi^k = \dfrac{1}{\sqrt{|g|}}\dfrac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\dfrac{\partial\phi^k}{\partial x^j})$ is the Laplace operator on (M, g) and ${}^N\Gamma^k_{\alpha\beta}$ are the Christoffel symbols on N.

Example 19. *If $M = I \subset \mathbb{R}$, then a map $\phi : I \longrightarrow (N^n, h)$ is harmonic if*

$$\frac{d^2\phi^\alpha}{dt^2} + \Gamma^\alpha_{\beta\delta}\frac{d\phi^\beta}{dt}\frac{d\phi^\delta}{dt} = 0,$$

therefore, ϕ is harmonic if and only if it is a geodesic.

Example 20. *The second fundamental form of the identity mapping, Id_M, from the Riemannian manifold (M, g) to itself is zero. This implies that Id_M is totally geodesic, hence, by definition, it is also harmonic.*

Example 21. *Let (M, g) be a Riemannian manifold and let $f : M \to \mathbb{R}$ be a smooth function. Then,*

$$\begin{aligned}
\tau(f) &= trace\nabla df \\
&= \nabla df(e_i, e_i) \\
&= \nabla^f_{e_i} df(e_i) - df(\nabla^M_{e_i} e_i) \\
&= e_i(e_i(f)) - (\nabla^M_{e_i} e_i)(f) \\
&= g(\nabla_{e_i} gradf, e_i) \\
&= div\ gradf \\
&= \Delta(f)
\end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on M,

Remark 22. *Any totally geodesic map is a harmonic map, the reverse is not always true.*

Example 22. *Consider the mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\varphi(x, y) = x^2 - y^2$. Since $\Delta\varphi = 0$, φ is harmonic .However,*

$$\begin{aligned}
(\nabla d\varphi)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) &= \nabla^\varphi_{\frac{\partial}{\partial x}} d\varphi - d\varphi(\nabla^{\mathbb{R}^2}_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}) \\
&= \nabla^\varphi_{\frac{\partial}{\partial x}} d\varphi(\frac{\partial}{\partial x}) \\
&= \frac{\partial^2}{\partial x^2} \\
&= 2.
\end{aligned}$$

Then φ is not totaly geodesic.

Remark 23. *The composition of two harmonic maps is not in general a harmonic map. In particular if ϕ is harmonic and ψ is totally geodesic (i.e. $\nabla\psi = 0$), then $\psi \circ \phi$ is harmonic.*

Example 23. *We define the maps φ and ψ by*

$$\begin{aligned}
\varphi : \mathbb{R}^2 &\to \mathbb{R} \\
(x, y) &\mapsto x^2 - y^2
\end{aligned}$$

$$\begin{aligned}\psi :&\mathbb{R} \to \mathbb{R}^2\\ x &\mapsto (x, 0)\end{aligned}$$

We have

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 2 - 2 = 0;$$

then, φ is harmonic, and

$$\Delta\psi = (\Delta\psi_1, \Delta\psi_2) = (0, 0) = 0.$$

Then, ψ is harmonic.

$$\begin{aligned}\varphi \circ \psi :&\mathbb{R} \to \mathbb{R}\\ x &\mapsto x^2\end{aligned}$$

$$\Delta(\varphi \circ \psi) = 2 \neq 0.$$

Then, $\varphi \circ \psi$ is not harmonic.

Example 24. *Let S be a surface in Euclidean space $\mathbb{R}^3$, and let $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a local parametrization of S, where Ω is an open subset of $\mathbb{R}^2$, such that:*

$$\left|\frac{\partial\varphi}{\partial x}\right|^2 = \left|\frac{\partial\varphi}{\partial y}\right|; \quad \langle\frac{\partial}{\partial x}, \frac{\partial\varphi}{\partial y}\rangle_{\mathbb{R}^3} = 0;$$

Let the unit normal vector be given by

$$\mathbf{N} = \frac{\frac{\partial\varphi}{\partial x} \wedge \frac{\partial\varphi}{\partial y}}{\left|\frac{\partial\varphi}{\partial x} \times \frac{\partial\varphi}{\partial y}\right|}.$$

Define:

$$E = \left|\frac{\partial\varphi}{\partial x}\right|^2; \quad F = \langle\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\rangle_{\mathbb{R}^3}; \quad G = \left|\frac{\partial\varphi}{\partial y}\right|^2$$

and

$$e = \langle\mathbf{N}, \frac{\partial^2\varphi}{\partial x^2}\rangle_{\mathbb{R}^3}; \quad f = \langle\mathbf{N}, \frac{\partial^2\varphi}{\partial x\partial y}, \frac{\partial\varphi}{\partial y}\rangle_{\mathbb{R}^3}; \quad g = \langle\mathbf{N}, \frac{\partial^2\varphi}{\partial y^2}\rangle_{\mathbb{R}^3}.$$

The mean curvature of S is given by:

$$H = \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}.$$

We have:

$$\begin{aligned}\langle\frac{\partial\varphi}{\partial x}, \tau(\varphi)\rangle_{\mathbb{R}^3} &= \langle\frac{\partial\varphi}{\partial x}, \frac{\partial^2\varphi}{\partial x^2}\rangle_{\mathbb{R}^3} + \langle\frac{\partial\varphi}{\partial x}, \frac{\partial^2\varphi}{\partial y^2}\rangle_{\mathbb{R}^3}\\ &= \langle\frac{\partial\varphi}{\partial x}, \frac{\partial^2\varphi}{\partial x^2}\rangle_{\mathbb{R}^3} + \langle\frac{\partial^2\varphi}{\partial x\partial y}, \frac{\partial\varphi}{\partial y}\rangle_{\mathbb{R}^3}\\ &= \frac{1}{2}\langle\frac{\partial\varphi}{\partial x}, \left|\frac{\partial\varphi}{\partial x}\right|^2\rangle_{\mathbb{R}^3} - \frac{1}{2}\langle\frac{\partial\varphi}{\partial x}, \left|\frac{\partial\varphi}{\partial y}\right|^2\rangle_{\mathbb{R}^3}\\ &= 0.\end{aligned}$$

In the same way, $\langle \frac{\partial\varphi}{\partial x}, \tau(\varphi)\rangle_{\mathbb{R}^3} = 0$*. Therefore,* $\tau(\varphi)$ *is normal on the surface* S*, and we have:*

$$H = \frac{e+g}{2E} = \frac{\langle \mathbf{N}, \tau(\varphi)\rangle_{\mathbb{R}^3}}{2E}.$$

Then, S *is minimal if and only if* φ *is harmonic.*

3.2.2 First variation of bienergy

Let $\phi : M \to N$ *be a differentiable map between pseudo-Riemannain manifolds. The bienergy of the map* ϕ *over a compact domain* D *in* M *is defined by*

$$E_2(\phi) = \frac{1}{2}\int_D |\tau(\phi)|^2 \, v_g,$$

where $\tau(\phi)$ *is the tension field of the map* ϕ *defined above.*

Definition 70. *[Biharmonic Map]*
The map $\varphi : M \to N$ *is said to be biharmonic if*

$$\frac{d}{dt}E_2(\phi_t)\Big|_{t=0} = 0$$

for every compact domain D *in* M *and for every variation* (φ_t) *with support included in* D*.*

Theorem 12. *Let* $\phi : M \to N$ *be a differentiable map between pseudo-Riemannain manifolds, and* $\{\phi_t\}_{t\in I}$*, where* $I =]-\delta, \delta[$*, be a variation of* ϕ *with support included in* D*. Then,*

$$\frac{d}{dt}E_2(\phi_t)\Big|_{t=0} = \int_D h(v, \tau_2(\phi)) \, v_g,$$

where $v(x) = \frac{d\phi_t}{dt}\big|_{t=0}$*, and*

$$\tau_2(\phi) = -\mathrm{tr}_g(\nabla^{\phi^2}\tau(\phi) - \mathrm{tr}_g R^N(\tau(\phi), d(\phi))d\phi \tag{3.6}$$

is the bitension field of the map ϕ*. Here,* R *denotes the curvature tensor of* N*.*

proof *Define* $\varphi : M \times (-\delta, \delta) \longrightarrow N$ *by* $\varphi(x,t) = \phi_t(x)$*.*
First note that

$$\frac{d}{dt}E_2(\phi_t)\big|_{t=0} = \int_D h\left(\nabla^{\tau}_{(0,\frac{d}{dt})}\nabla d\varphi((e_i,0),(e_i,0)), \nabla d\varphi((e_i,0),(e_i,0))\right) v_g\Big|_{t=0}. \tag{3.7}$$

Calculating in a normal frame at $p \in M$ *we have*

$$\begin{aligned}
\nabla^{\tau}_{(0,\frac{d}{dt})} d\varphi(e_i,0) &= \nabla^{\tau}_{(e_i,0)} d\varphi(0,\frac{d}{dt}) + d\varphi([(0,\frac{d}{dt}),(e_i,0)]) \\
&= \nabla^{\tau}_{(e_i,0)} d\varphi(0,\frac{d}{dt}). \\
\nabla^{\tau}_{(0,\frac{d}{dt})} d\varphi(\nabla^M_{e_i} e_i,0) &= \nabla^{\tau}_{(\nabla^M_{e_i} e_i,0)} d\varphi(0,\frac{d}{dt}). \\
\nabla^{\tau}_{(0,\frac{d}{dt})} \nabla d\varphi((e_i,0),(e_i,0)) &= \nabla^{\tau}_{(0,\frac{d}{dt})} \nabla^{\tau}_{(e_i,0)} d\varphi(e_i,0) - \nabla^{\tau}_{(0,\frac{d}{dt})} d\varphi\left(\nabla^{M\times(-\delta,\delta)}_{(e_i,0)}(e_i,0)\right) \\
&= R^N(d\varphi(0,\frac{d}{dt}),d\varphi(e_i,0))d\varphi(e_i,0) + \nabla^{\tau}_{(e_i,0)} \nabla^{\tau}_{(0,\frac{d}{dt})} d\varphi(e_i,0) \\
&+ \nabla^{\tau}_{[(0,\frac{d}{dt}),(e_i,0)]} d\varphi(e_i,0) - \nabla^{\tau}_{(0,\frac{d}{dt})} d\varphi(\nabla^M_{e_i} e_i,0). \\
&= R^N(d\varphi(0,\frac{d}{dt}),d\varphi(e_i,0))d\varphi(e_i,0) + \nabla^{\tau}_{(e_i,0)} \nabla^{\tau}_{(e_i,0)} d\varphi(0,\frac{d}{dt}) \\
&- \nabla^{\tau}_{(\nabla^M_{e_i} e_i,0)} d\varphi(0,\frac{d}{dt}). \qquad (3.8)
\end{aligned}$$

From where

$$\begin{aligned}
h\big(\nabla^{\varphi}_{(0,\frac{d}{dt})} \nabla d\varphi((e_i,0),(e_i,0)),\, \nabla d\varphi((e_i,0),(e_i,0))\big)\Big|_{t=0} &= h(R^N(v,d\phi(e_i))d\phi(e_i),\tau(\phi)) \\
&+ h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} v,\tau(\phi)) \\
&- h(\nabla^{\phi}_{\nabla^M_{e_i} e_i} v,\tau(\phi)). \qquad (3.9)
\end{aligned}$$

Let $\omega \in \Gamma(T^*M)$, be a 1-form on support in D, defined by

$$\omega(X) = h\big(\nabla^{\phi}_X v,\tau(\phi)\big), \qquad X \in \Gamma(TM).$$

We calculate the divergence of ω

$$\begin{aligned}
\operatorname{div}^M \omega &= \big(e_i(\omega(e_i)) - \omega(\nabla^M_{e_i} e_i)\big) \\
&= \big(e_i(h(\nabla^{\phi}_{e_i} v,\tau(\phi))) - h(\nabla^{\phi}_{\nabla^M_{e_i} e_i} v,\tau(\phi))\big) \\
&= \big(h(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} v,\tau(\phi)) + h(\nabla^{\phi}_{e_i} v,\nabla^{\phi}_{e_i}\tau(\phi)) - h(\nabla^{\phi}_{\nabla^M_{e_i} e_i} v,\tau(\phi))\big). \qquad (3.10)
\end{aligned}$$

From the formulas (3.9) *and* (3.10), *we obtain*

$$\begin{aligned}
h\big(\nabla^{\varphi}_{(0,\frac{d}{dt})} \nabla d\varphi((e_i,0),(e_i,0)),\nabla d\varphi((e_i,0),(e_i,0))\big)\big|_{t=0} &= h\big(R^N(v,d\phi(e_i))d\phi(e_i),\tau(\phi)\big) \\
&- h\big(\nabla^{\phi}_{e_i} v,\nabla^{\phi}_{e_i}\tau(\phi)\big) + \operatorname{div}^M\omega. \quad (3.11)
\end{aligned}$$

Let $\tilde{w} \in \Gamma(T^*M)$, be an 1-form on support in D, given by

$$\tilde{w}(X) = h\big(v,\nabla^{\phi}_X\tau(\phi)\big), \qquad X \in \Gamma(TM).$$

We calculate the divergence of $\tilde{w}$:

$$\begin{aligned}
\mathrm{div}^M \tilde{w} &= \big(e_i(\eta(e_i)) - \eta(\nabla^M_{e_i} e_i)\big) \\
&= \big(e_i(h(v, \nabla^\phi_{e_i}\tau(\phi))) - h(v, \nabla^\phi_{\nabla^M_{e_i} e_i}\tau(\phi)) \\
&= h\big(\nabla^\phi_{e_i} v, \nabla^\phi_{e_i}\tau(\phi)) + h(v, \nabla^\phi_{e_i}\nabla^\phi_{e_i}\tau(\phi)\big) - h\big(v, \nabla^\phi_{\nabla^M_{e_i} e_i}\tau(\phi))\big).
\end{aligned} \tag{3.12}$$

Substituting (3.12) *in* (3.11), *we obtain*

$$\begin{aligned}
h\big(\nabla^\varphi_{(0,\frac{d}{dt})}\nabla d\varphi((e_i,0),(e_i,0)),\ \ \nabla d\varphi((e_i,0),(e_i,0))\big)\big|_{t=0} &= h\big(R^N(\tau(\phi), d\phi(e_i))d\phi(e_i), v\big) \\
&+ h\big(v, \nabla^\phi_{e_i}\nabla^\phi_{e_i}\tau(\phi)\big) - h\big(v, \nabla^\phi_{\nabla^M_{e_i} e_i}\tau(\phi)\big) \\
&+ \mathrm{div}^M\omega - \mathrm{div}^M\tilde{w}.
\end{aligned} \tag{3.13}$$

From the formulas (3.7), (3.13) *and according and if*

$$\int_D div(\omega) v_g = 0, \tag{3.14}$$

we obtain

$$\frac{d}{dt}E_2(\phi_t)\big|_{t=0} = -\int_D h\left(-R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) - \nabla^\phi_{e_i}\nabla^\phi_{e_i}\tau(\phi) + \nabla^\phi_{\nabla^M_{e_i} e_i}\tau(\phi), v\right) v_g.$$

Theorem 13. *Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between two pseudo-Riemannian manifolds of dimensions m and n respectively, then φ is said biharmonic if and only if*

$$\tau_2(\phi) = -\mathrm{trace}_g R^N(\tau(\phi), d\phi)d\phi - \mathrm{trace}_g(\nabla^\phi)^2\tau(\phi) = 0. \tag{3.15}$$

Let M and N be two pseudo-Riemannian manifolds with the coordinates (x^i) and (y^α) respectively, then, in the neighborhood of the points $p \in M$ and $\phi(p) \in N$ we have

$$\begin{aligned}
\tau_2(\phi) &= g^{ij}\Bigg(\frac{\partial^2\tau^\sigma}{\partial x^i\partial x^j} + 2\frac{\partial\tau^\sigma\partial\tau^\beta}{\partial x^j\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} + \tau^\alpha\frac{\partial^2\phi^\beta}{\partial x^i\partial x^j}{}^N\Gamma^\sigma_{\alpha\beta} \\
&\quad + \tau^\alpha\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial^N\Gamma^\sigma_{\alpha\beta}}{\partial x^j} + \tau^\alpha\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\rho}{\partial x^j}{}^N\Gamma^\upsilon_{\alpha\beta}{}^N\Gamma^\sigma_{\upsilon\rho} \\
&\quad - {}^M\Gamma^k_{ij}\big(\frac{\partial\tau^\sigma}{\partial x^k} + \tau^\alpha\frac{\partial\phi^\beta}{\partial x^k}{}^N\Gamma^\sigma_{\alpha\beta}\big) - \tau^\upsilon\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}{}^N R^\sigma_{\beta\alpha\upsilon}\Bigg)\frac{\partial}{\partial y^\sigma}\circ\phi,
\end{aligned}$$

where $\tau^\gamma = g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} + \frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}{}^N\Gamma^\gamma_{\alpha\beta}\circ\phi - \frac{\partial\phi^\gamma}{\partial x^k}{}^M\Gamma^k_{ij}\right)$ *and* ${}^N R^\sigma_{\beta\alpha\upsilon}$ *designate the components of the curvature tensor of* (N, h).

Example 25. *A smooth map $\varphi : (M^m, g) \to (\mathbb{R}^n, g_{eucl})$ is biharmonic if and only if $\Delta^M(\Delta^M\varphi_i) = 0$ for all $i = 1, \ldots, n$.*

Proposition 27. *Every map satisfying the harmonic condition is necessarily biharmonic, although the reverse statement is not universally valid.*

Proof: *A map is harmonic if it satisfies the condition $\tau(\phi) = trace_g \nabla d\phi = 0$. Consequently, it automatically fulfills the biharmonicity condition. The second part of the proof is illustrated by the following examples:*

Example 26. *1. The inverse of the stereographic projection $\varphi : \mathbb{R}^n \to S^n$, defined by $x \mapsto \varphi(x) = \dfrac{1}{\|x\|^2+1}(2x, \|x\|^2 - 1)$, is biharmonic but non-harmonic unless $n = 4$.*

2. Polynomials of degree 3 on $\mathbb{R}^m$ serve as examples of biharmonic maps that are not harmonic.

The proposition further elucidates this connection by indicating that while all harmonic maps are indeed biharmonic, not every biharmonic map qualifies as harmonic, thus highlighting the intricate relationship between these mappings.

3.3 $f-$harmonic maps & $f-$biharmonic maps

3.3.1 $f-$harmonic maps

Let (M, g) and (N, h) be two pseudo-Riemannian manifolds, and let D be a compact domain M. Consider $f : M \to \mathbb{R}_+^*$ that is C^∞-class function.

The f-energy of a C^∞ class map $\phi : M \to N$ over the domain D is defined as follows:

$$E_f(\phi) = \frac{1}{2}\int_D f(x)|d\phi|^2\, v_g,$$

Definition 71. *[f-harmonic map]*

An application $\phi : (M, g) \to (N, h)$ of class C^∞ is said to be f-harmonic if it is a critical point of the energy functional f-energy $E_f(\phi)$ for every compact domain $D \subset M$, i.e.

$$\frac{d}{dt}E_f(\phi_t)\Big|_{t=0} = 0. \tag{3.16}$$

where ϕ_t is a variation of ϕ with support in D.

Definition 72. *[f-tension field]*

Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between two pseudo-Riemannian manifolds of dimensions m and n respectively. The f-tension of ϕ is a section $\tau_f \in \Gamma(\phi^{-}TN)$ defined by

$$\begin{aligned}\tau_f &= \mathrm{tr}_g \nabla d\phi \\ &= \nabla^{\phi}_{ei} f d\phi(e_i) - f d\phi(\nabla^M_{ei} e_i) \\ &= f\tau(\phi) + d\phi(grad^M f)\end{aligned}$$

where $\{e_1, \ldots, e_m\}$ is an orthonormal basis on M.

Theorem 14. *First variation of the f-energy [?]*

Let $\phi : (M, g) \to (N, h)$ be a C^∞-class mapping, and let $\{\phi_t\}$ be a C^∞ variation of ϕ with support in D. Then, the first variation of the f-energy is given by

$$\frac{d}{dt}E_f(\phi_t)\Big|_{t=0} = -\int_D h(v, \tau_f(\phi)\, v_g,$$

where $v = \frac{d\phi_t}{dt}\big|_{t=0}$ denotes the associated vector field of the variation $\{\phi_t\}$.

proof

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis on M. Consider the mapping $\varphi : M \times (-\delta, \delta) \to N$ defined by $\varphi(x, t) = \phi_t(x)$, where ϕ_t is given by the variation. Then, we have the first variation of the f-energy as

$$\begin{aligned} \frac{d}{dt}E_f(\phi_t)\Big|_{t=0} &= \int_D f\, h\left(\nabla^\varphi_{(0,\frac{d}{dt})} d\varphi(e_i, 0), d\varphi(e_i, 0)\right)\, v_g \\ &= \int_D f\, h\left(\nabla^\varphi_{(e_i,0)} d\varphi(0, \frac{d}{dt}), d\varphi(e_i, 0)\right)\, v_g \\ &= \int_D f\, h\left(\nabla^\phi_{e_i} v, d\phi(e_i, 0)\right)\, v_g \end{aligned} \tag{3.17}$$

If w is the differential 1-form with support in D, given by

$$w(X) = h(v, f d\phi(X)), \quad X \in \Gamma(TM) \tag{3.18}$$

Then, the divergence of ϕ on M is given by

$$div^M w = \big(e_i(w(e_i)) - w\big(\nabla^M_{e_i}\big)$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame for TM. This can be further expressed as:

$$div^M w = h(\nabla^\phi_{e_i} v, f d\phi(e_i)) + h(v, \nabla^\phi_{e_i} f d\phi(e_i)) - h(v, f d\phi(\nabla^M_{e_i} e_i) \tag{3.19}$$

According to formulas (3.17) and (3.19), together with Stokes' theorem, we obtain

$$\frac{d}{dt}E_f(\phi)\Big|_{t=0} = -\int_D h\big(v, \nabla^\phi_{e_i} f d\phi(e_i) - f d\phi\big(\nabla^M_{e_i} e_i\big)\big) v_g \tag{3.20}$$

In the case where $v = \tau_f(\phi)$, we have the following theorem

Theorem 15. *A map $\phi : (M, g) \to (N, h)$ of class C^∞ is f-harmonic if and only if*

$$\tau_f(\phi) = tr_g \nabla f d\phi = f\tau(\phi) + d\phi(grad^M f) = 0 \tag{3.21}$$

It is easily seen that an f-harmonic map with $f = cst$ is nothing but a harmonic map

3.3.2 f-biharmonic map

Let (M,g) and (N,h) be two pseudo-Riemannian manifolds, and let D be a compact domain in M. The f-bi-energy functional of a mapping $\phi:(M,g)\to(N,h)$ of class C^∞ is defined by Here's the code with τ in place of ϑ:

$$E_{2,f}(\phi)=\frac{1}{2}\int_D |\tau_f(\phi)|^2\, v_g \tag{3.22}$$

Definition 73. *[f-bi-tension field]*
Let $\phi:(M,g)\longrightarrow(N,h)$ be a smooth map between two pseudo-Riemannian manifolds of dimensions m and n respectively. The f-bi-tension of ϕ is a section $\tau_{2f}\in\Gamma(\phi^{-}TN)$ defined by

$$\begin{aligned}\tau_{2f}(\phi) &= -\Big(f\mathrm{tr}_g R^N\big(\tau_f(\phi),d\phi\big)d\phi+\mathrm{tr}_g\big(\nabla^\phi f\nabla^\phi\tau(\phi)-f\nabla^\phi_{\nabla^M}\tau_f(\phi)\big)\Big)\\ &= -\Big(fR^N\big(\tau_f(\phi),d\phi(e_i)\big)d\phi(e_i)+\big(\nabla^\phi_{e_i}f\nabla^\phi_{e_i}\tau(\phi)-f\nabla^\phi_{\nabla^M_{e_i}e_i}\tau_f(\phi)\big)\end{aligned}$$

where $\{e_1,\ldots,e_m\}$ is an orthonormal basis on M.

Definition 74. *[f-biharmonic map]*
A mapping $\phi:(M,g)\to(N,h)$ of class C^∞ is said to be f-biharmonic if it is a critical point of the f-bi-energy functional for every compact domain $D\subseteq M$

$$\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0}=0. \tag{3.23}$$

Theorem 16. *First variation of the f-bienergy*
Let $\phi:(M,g)\to(N,h)$ be a C^∞-class mapping, and let $\{\phi_t\}$ be a C^∞ variation of ϕ with support in D. Then, the first variation of the f-bi-energy is given by

$$\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0}=-\int_D h(v,\tau_{2,f}(\phi)\, v_g,$$

where $v=\dfrac{d\phi_t}{dt}\big|_{t=0}$ denotes the associated vector field of the variation $\{\phi_t\}$.

Proof
Let $\{e_1,\ldots,e_m\}$ be an orthonormal basis on M. Consider the mapping $\varphi:M\times(-\delta,\delta)\to N$ defined by $\varphi(x,t)=\phi_t(x)$, where ϕ_t is given by the variation. Then, we have the first variation of the f-bi-energy as

$$\begin{aligned}\frac{d}{dt}E_{2,f}(\phi_t)\Big|_{t=0} &= \int_D h\Big(\nabla^{\varphi}_{(0,\frac{d}{dt})}\nabla f d\varphi\big((e_i,0),(e_i,0)\big),\nabla f d\varphi\big((e_i,0),(e_i,0)\big)\Big)\ v_g \qquad (3.24)\\ &= \int_D h\left(\nabla^{\varphi}_{(e_i,0)}d\varphi(0,\frac{d}{dt}),d\varphi(e_i,0)\right)\ v_g\\ &= \int_D h\Big(fR^N(d\varphi(0,\frac{d}{dt}),d\varphi(e_i,0))d\phi(e_i,0)+\nabla^{\varphi}_{(e_i,0)}f\nabla^{\varphi}_{(e_i,0)}d\varphi(0,\frac{d}{dt})\\ &- f\nabla^{\varphi}_{(\nabla^M_{e_i}e_i,0)}d\varphi(0,\frac{d}{dt}),\nabla f d\varphi\big((e_i,0),(e_i,0)\big)\Big)\\ &= h\Big(fR^N\big(v,d\phi(e_i))d\phi(e_i),\tau_f(\phi)\big)+h\big(\nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}v,\tau(\phi)\big)-h\big(f\nabla^{\phi}_{\nabla^M_{e_i e_i}}v,\tau_f(\phi)\big)\end{aligned}$$

If w denotes the 1-form differential with support in D, defined by

$$w(X)=h\big(f\nabla\phi_X v,\tau_f(\phi)\big),\quad X\in\Gamma(TM)$$

Then

$$\begin{aligned}div^M w &= e_i\big(w(e_i)\big)-w\big(\nabla^M_{e_i}e_i\big) \qquad (3.25)\\ &= e_i\big(h(f\nabla^{\phi}_{e_i}v,\tau_f(\phi)\big)-h(f\nabla^{\phi}_{\nabla^M_{e_i}e_i}v,\tau_f(\phi))\big)\\ &= h\big(\nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}v,\tau_f(\phi)\big)+h\big(f(\nabla^{\phi}_{e_i}v,\nabla^{\phi}_{e_i}\tau_f(\phi)\big)-h\big(f\nabla^{\phi}_{\nabla^M_{e_i}e_i}v,\tau_f(\phi)\big)\end{aligned}$$

By substituting Equation (3.25) into Equation (3.24), we obtain

$$\begin{aligned}&h\Big(\nabla^{\varphi}_{(0,\frac{d}{dt})}\nabla f d\varphi\big((e_i,0),(e_i,0)\big),\nabla f d\varphi\big((e_i,0),(e_i,0)\big)\Big) \qquad (3.26)\\ &= h\Big(fR^N\big(v,d\phi(e_i))d\phi(e_i),\tau_f(\phi)\Big)+div^M w-div^M\tilde{w}+h\big(v,\nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}\tau_f(\phi)\big)\\ &- h\big(v,f\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau_f(\phi)\big).\end{aligned}$$

If $\tilde{w}$ denotes the 1-form differential with support in D, defined by

$$\tilde{w}(X)=h\big(v,f\nabla^{\phi}_X\tau_f(\phi)\big),\quad X\in\Gamma(TM)$$

Then

$$\begin{aligned}div^M\tilde{w} &= e_i\big(\tilde{w}(e_i)\big)-\tilde{w}\big(\nabla^M_{e_i}e_i\big) \qquad (3.27)\\ &= e_i\big(h(v,f\nabla^{\phi}_{e_i}\tau_f(\phi)\big)-h(v,f\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau_f(\phi))\big)\\ &= h\big(v,\nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}\tau_f(\phi)\big)+h\big(v,\nabla^{\phi}_{e_i}f\nabla^{\phi}_{e_i}\tau_f(\phi)\big)-h\big(v,f\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau_f(\phi)\big)\end{aligned}$$

Combining (3.28) and (3.27) we obtain

$$\begin{aligned} & h\Big(\nabla^{\varphi}_{(0,\frac{d}{dt})}\nabla f d\varphi\big((e_i,0),(e_i,0)\big),\nabla f d\varphi\big((e_i,0),(e_i,0)\big)\Big) \\ & = h\Big(fR^N\big(v,d\phi(e_i)\big)d\phi(e_i),\tau_f(\phi)\Big) + div^M w - div^M \tilde{w} + h\big(v,\nabla^{\phi}_{e_i} f\nabla^{\phi}_{e_i}\tau_f(\phi)\big) \\ & - h\big(v,f\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau_f(\phi)\big). \end{aligned} \tag{3.28}$$

By employing formulas (3.24), (3.28), and the divergence theorem, we obtain

$$\frac{d}{dt}E_{2,f}(\phi_t)\big|_{t=0} = -\int_D h\Big(-fR^N(\tau_f(\phi),d\phi(e_i))d\phi(e_i) - \nabla^{\phi}_{e_i}\tau_f(\phi) + f\nabla^{\phi}_{\nabla^M_{e_i}e_i}\tau(\phi)v\Big)v_g \tag{3.29}$$

Theorem 17. *An application $\phi:(M,g)\to(N,h)$ of class C^∞ is f-biharmonic if and only if*

$$\tau_{2,f} = -f\mathrm{tr}_g R^N(\tau_f(\phi),d\phi)d\phi - \mathrm{tr}_g\big(\nabla^{\phi} f\nabla^{\phi}\tau_f(\phi) - f\nabla^{\phi}_{\nabla^M}\tau_f(\phi)\big) \tag{3.30}$$

Chapter 4

f-biharmonic maps in the three-dimensional generalized symmetric spaces

"The essence of mathematics lies in its freedom." George Cantor

Introduction

In previous works,R. Caddeo, Montaldo, and Piu (2001) explored biharmonic curves on surfaces, later extending their study to provide a classification of biharmonic curves in Cartan-Vranceanu 3-dimensional spaces in articles [22]and[20]. Karaca, Fatma, and O, Cihan (cited in [3]) investigated f-biharmonic curves in Sol spaces, Cartan-Vranceanu 3-dimensional spaces, and homogeneous contact 3-manifolds. They particularly focused on analyzing non-geodesic f-biharmonic curves in these spaces. Djaa et al. discussed the generalization of harmonic and biharmonic maps in their article. Elhendi and Belarbi (2020) published a paper on the Generalized Bi-f-harmonic Map Equations on Singly Warped Product Manifolds. Subsequently, in 2022, the same authors along with M. Belarbi et all investigated Biharmonic curves in Three-Dimensional Generalized Symmetric Spaces.

Drawing inspiration from these previous studies, our paper delves into the f-biharmonicity condition within Three-Dimensional Generalized Symmetric Spaces. We establish the fundamental criteria for identifying curves in these spaces that demonstrate f-biharmonic behavior.

4.1 The geometry of three-dimensional generalized symmetric space

Following [24], any proper (that is, non-symmetric) three-dimensional generalized symmetric space (M, g) *is of order* 4*. Moreover, it is given by the space* $\mathbb{R}^3(x, y, t)$ *with the pseudo-Riemannian metric:*

$$g_{\varepsilon,\lambda} = \varepsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{4.1}$$

where $\varepsilon = \pm 1$ *and* $\lambda \neq 0$ *is a real constant. Depending on the values of* ε *and* λ*, these metrics attain any possible signature:* $(3, 0)$, $(0, 3)$, $(2, 1)$, $(1, 2)$.

Let $(\mathbb{M}_3, g)$ be a three-dimensional generalized symmetric space as descirbed above, and denote by ∇, R and Ric the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor of M, respectively.

A left-invariant orthonormal frame $\{e_1, e_2, e_3\}$ in the three-dimensional generalized symmetric space is given by:

$$e_1 = e^{-t}\frac{\partial}{\partial x}, \qquad e_2 = e^{t}\frac{\partial}{\partial y}, \qquad e_3 = \frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}. \tag{4.2}$$

The Lie brackets of this frame can be computed as

$$[e_2, e_3] = e^{t}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}\right) - \frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}\left(e^{t}\frac{\partial}{\partial y}\right) = -\frac{1}{\sqrt{|\lambda|}}e_2, \tag{4.3}$$

$$[e_1, e_3] = e^{-t}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}\right) - \frac{1}{\sqrt{|\lambda|}}\frac{\partial}{\partial t}\left(e^{-t}\frac{\partial}{\partial x}\right) = \frac{1}{\sqrt{|\lambda|}}e_1, \tag{4.4}$$

Now,we use the formula of Kozule to calculate the Levi-Civita connexion

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

Considering the Lie brackets we have just computed, we arrive at the following outcomes

1. *Let* $\nabla_{e_1}e_1 = \alpha e_1 + \beta e_2 + \delta e_3$

$$2g(\nabla_{e_1}e_1, e_1) = 0, \quad 2g(\nabla_{e_1}e_1, e_2) = 0, \quad 2g(\nabla_{e_1}e_1, e_3) = -2g([e_1, e_3], e_1)$$
$$2g(\nabla_{e_1}e_1, e_3) = -2g\left(\frac{1}{\sqrt{|\lambda|}}e_1, e_1\right) = -\frac{2\varepsilon}{\sqrt{|\lambda|}}$$
$$g(\nabla_{e_1}e_1, e_3) = -\frac{\varepsilon}{\sqrt{|\lambda|}} = \varepsilon_1\delta$$

Hence, $\nabla_{e_1}e_1 = -\dfrac{\varepsilon_1\varepsilon}{\sqrt{|\lambda|}}e_3$

2. *let* $\nabla_{e_2}e_2 = \alpha e_1 + \beta e_2 + \delta e_3$

$$2g(\nabla_{e_2}e_2, e_1) = 0, \quad 2g(\nabla_{e_2}e_2, e_2) = 0, \quad 2g(\nabla_{e_2}e_2, e_3) = -2g([e_2, e_3], e_2)$$
$$2g(\nabla_{e_2}e_2, e_3) = 2g\left(\frac{1}{\sqrt{|\lambda|}}e_2, e_2\right) = \frac{2\varepsilon}{\sqrt{|\lambda|}}$$
$$g(\nabla_{e_2}e_2, e_3) = \frac{\varepsilon}{\sqrt{|\lambda|}} = \delta\varepsilon_1 \Rightarrow \delta = \frac{\varepsilon_1\varepsilon}{\sqrt{|\lambda|}}$$

Hence, $\nabla_{e_2}e_2 = \dfrac{\varepsilon_1\varepsilon}{\sqrt{|\lambda|}}e_3$

3. *Let* $\nabla_{e_1}e_3 = \alpha e_1 + \beta e_2 + \delta e_3$

$$2g(\nabla_{e_1}e_3, e_1) = -2g([e_3, e_1], e_1), \quad 2g(\nabla_{e_1}e_3, e_2) = 0, \quad 2g(\nabla_{e_1}e_3, e_3) = 0$$
$$2g(\nabla_{e_1}e_3, e_1) = 2g\left(\frac{1}{\sqrt{|\lambda|}}e_1, e_1\right) = \frac{2\varepsilon}{\sqrt{|\lambda|}}$$
$$g(\nabla_{e_1}e_3, e_1) = \frac{\varepsilon}{\sqrt{|\lambda|}} = \alpha\varepsilon \Rightarrow \alpha = \frac{1}{\sqrt{|\lambda|}}$$

Hence, $\nabla_{e_1} e_3 = \frac{1}{\sqrt{|\lambda|}} e_1$

4. *Let* $\nabla_{e_2} e_3 = \alpha e_1 + \beta e_2 + \delta e_3$

$$2g(\nabla_{e_2} e_3, e_2) = 2g([e_2, e_3], e_2), \quad 2g(\nabla_{e_2} e_3, e_1) = 0, \quad 2g(\nabla_{e_2} e_3, e_3) = 0$$
$$2g(\nabla_{e_2} e_3, e_2) = -2g(\frac{1}{\sqrt{|\lambda|}} e_2, e_2) = \frac{-2\varepsilon}{\sqrt{|\lambda|}}$$
$$g(\nabla_{e_1} e_3, e_1) = \frac{-\varepsilon}{\sqrt{|\lambda|}} = \beta\varepsilon \Rightarrow \beta = \frac{-1}{\sqrt{|\lambda|}}$$

Hence, $\nabla_{e_2} e_3 = \frac{-1}{\sqrt{|\lambda|}} e_2$

Ultimately, concerning the provided orthonormal basis, the non-vanishing Levi-Civita connections are

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} e_3, & \nabla_{e_1} e_3 &= \frac{1}{\sqrt{|\lambda|}} e_1, \\ \nabla_{e_2} e_2 &= \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} e_3, & \nabla_{e_2} e_3 &= -\frac{1}{\sqrt{|\lambda|}} e_2, \end{aligned} \tag{4.5}$$

where $\varepsilon_1 = \frac{\lambda}{|\lambda|}$.

Subsequently, we employ the following notation and sign convention for the Riemannian curvature operator

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4.6}$$

By utilizing the results derived from both equations (4.3) *and* (4.6), *as well as from equation*(4.5), *we obtain:*

$$R(e_1, e_2)e_1 = \nabla_{e_1} \nabla_{e_2} e_1 - \nabla_{e_2} \nabla_{e_1} e_1 - \nabla_{[e_1,e_2]} e_1 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} \nabla_{e_2} e_3 = \frac{-\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} \frac{1}{\sqrt{|\lambda|}} e_2 = \frac{-\varepsilon\varepsilon_1}{\sqrt{|\lambda|}^2} e_2 = \frac{-\varepsilon}{\lambda} e_2$$
$$R(e_1, e_2)e_2 = \nabla_{e_1} \nabla_{e_2} e_2 - \nabla_{e_2} \nabla_{e_1} e_2 - \nabla_{[e_1,e_2]} e_2 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} \nabla_{e_1} e_3 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} \frac{1}{\sqrt{|\lambda|}} e_1 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}^2} e_1 = \frac{\varepsilon}{\lambda} e_1$$
$$R(e_1, e_3)e_1 = \nabla_{e_1} \nabla_{e_3} e_1 - \nabla_{e_3} \nabla_{e_1} e_1 - \nabla_{[e_1,e_3]} e_1 = \frac{-1}{\sqrt{|\lambda|}} \nabla_{e_1} e_1 = \frac{1}{\sqrt{|\lambda|}} \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} e_3 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}^2} e_3 = \frac{\varepsilon}{\lambda} e_3$$
$$R(e_1, e_3)e_3 = \nabla_{e_1} \nabla_{e_3} e_3 - \nabla_{e_3} \nabla_{e_1} e_3 - \nabla_{[e_1,e_3]} e_3 = \frac{-1}{\sqrt{|\lambda|}} \nabla_{e_1} e_3 = \frac{-1}{\sqrt{|\lambda|}} \frac{1}{\sqrt{|\lambda|}} e_1 = \frac{-1}{\sqrt{|\lambda|}^2} e_1 = \frac{-1}{|\lambda|} e_1$$
$$R(e_2, e_3)e_2 = \nabla_{e_2} \nabla_{e_3} e_2 - \nabla_{e_3} \nabla_{e_2} e_2 - \nabla_{[e_2,e_3]} e_2 = \frac{1}{\sqrt{|\lambda|}} \nabla_{e_2} e_2 = \frac{1}{\sqrt{|\lambda|}} \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}} e_3 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}^2} e_3 = \frac{\varepsilon}{\lambda} e_3$$
$$R(e_2, e_3)e_3 = \nabla_{e_2} \nabla_{e_3} e_3 - \nabla_{e_3} \nabla_{e_2} e_3 - \nabla_{[e_2,e_3]} e_3 = \frac{1}{\sqrt{|\lambda|}} \nabla_{e_2} e_3 = \frac{-1}{\sqrt{|\lambda|}} \frac{1}{\sqrt{|\lambda|}} e_2 = \frac{\varepsilon\varepsilon_1}{\sqrt{|\lambda|}^2} e_2 = \frac{-1}{|\lambda|} e_2$$

By exploiting the symmetries inherent in the tensor, we derive the following outcomes

$$\begin{aligned}
R(e_1,e_2)e_1 &= -R(e_2,e_1)e_1 = -\frac{\varepsilon}{\lambda}e_2, & R(e_1,e_2)e_2 &= -R(e_2,e_1)e_2 = \frac{\varepsilon}{\lambda}e_1,\\
R(e_1,e_3)e_1 &= -R(e_3,e_1)e_1 = \frac{\varepsilon}{\lambda}e_3 & R(e_1,e_3)e_3 &= -R(e_3,e_1)e_3 = -\frac{1}{|\lambda|}e_1,\\
R(e_2,e_3)e_2 &= -R(e_3,e_2)e_2 = \frac{\varepsilon}{\lambda}e_3, & R(e_2,e_3)e_3 &= -R(e_3,e_2)e_3 = -\frac{1}{|\lambda|}e_2,
\end{aligned} \tag{4.7}$$

The expression for the Riemannian curvature tensor is presented as

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W). \tag{4.8}$$

Furthermore, we introduce the following notation

$$R_{123} = R(e_1,e_2)e_3, \qquad R_{1234} = R(e_1,e_2,e_3,e_4). \tag{4.9}$$

By directly computing using (4.5), (4.3), (4.6), (4.8), *and* (4.9), *we determine the non-zero components of the Riemannian curvature for the three-dimensional generalized symmetric space relative to the orthonormal basis* $\{e_1,e_2,e_3\}$ *(omitting those obtainable from the symmetric properties of curvature):*

$$\begin{aligned}
R(e_1,e_2,e_1,e_2) &= -g(R(e_1,e_2)e_1,e_2) = -g(-\frac{\varepsilon}{\lambda}e_2,e_2) = \frac{1}{\lambda}\\
R(e_1,e_2,e_2,e_1) &= -g(R(e_1,e_2)e_2,e_1) = -g(\frac{\varepsilon}{\lambda}e_1,e_1) = \frac{-1}{\lambda}\\
R(e_1,e_3,e_1,e_3) &= -g(R(e_1,e_3)e_1,e_3) = -g(\frac{\varepsilon}{\lambda}e_3,e_3) = -\frac{\varepsilon\varepsilon_1}{\lambda}\\
R(e_1,e_3,e_3,e_1) &= -g(R(e_1,e_3)e_3,e_1) = -g(-\frac{1}{|\lambda|}e_1,e_1) = \frac{\varepsilon}{|\lambda|}\\
R(e_2,e_3,e_2,e_3) &= -g(R(e_2,e_3)e_2,e_3) = -g(\frac{\varepsilon}{\lambda}e_3,e_3) = \frac{-\varepsilon}{|\lambda|}\\
R(e_2,e_3,e_3,e_2) &= -g(R(e_2,e_3)e_3,e_2) = -g(-\frac{1}{|\lambda|}e_2,e_2) = \frac{\varepsilon}{|\lambda|}
\end{aligned}$$

From this point forward, we will employ the following abbreviation for the curvature tensor

$$\begin{aligned}
R_{1212} &= -R_{2112} = \frac{1}{\lambda} &,\quad R_{1221} &= -R_{2121} = -\frac{1}{\lambda},\\
R_{1313} &= -R_{3113} = -\frac{\varepsilon}{|\lambda|}, & R_{1331} &= -R_{3131} = \frac{\varepsilon}{|\lambda|},\\
R_{2323} &= R_{3223} = -\frac{\varepsilon}{|\lambda|}, & R_{2332} &= -R_{3232} = \frac{\varepsilon}{|\lambda|}.
\end{aligned} \tag{4.10}$$

The computation of the Ricci curvature components $\{Ric_{ij}\}$ *is performed using the formula*

$$Ric_{11} = Ric_{12} = Ric_{13} = Ric_{23} = Ric_{22} = 0, \;\; Ric_{33} = -\frac{2}{|\lambda|}. \tag{4.11}$$

Ultimately, the scalar curvature τ *of the three-dimensional generalized symmetric spaces is defined as*

$$\tau = trRic = \sum_{i=1}^{3} g(e_i,e_i)Ric(e_i,e_i) = -\frac{2}{|\lambda|}. \tag{4.12}$$

4.2 $f-$biharmonic curves in 3-dimensional generalized symmetric spaces

Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{M}_3, \gamma = \gamma(s)$ be any curve within $(\mathbb{M}_3, g)$. This curve is categorized as spacelike, timelike, or lightlike depending on whether all of its velocity vectors $\dot{\gamma}(s)$ are spacelike, timelike, or lightlike, respectively. If γ is spacelike or timelike, it can be reparameterized such that $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \varepsilon$, where $\varepsilon = 1$ for spacelike curves and $\varepsilon = -1$ for timelike curves respectively. Such a reparameterization renders $\gamma(s)$ to be unit speed or arc length-parameterized in this context.

Definition 75. *[Frenet Frame [54]]*

The Frenet frame $\{F_i\}_{i=1}^n$ associated with a curve $\gamma : I \subset \mathbb{R} \to (N^n, h)$ parameterized by arc length is obtained through the orthonormalization of the $(n+1)$-tuple $\{\nabla^{(k)}_{\partial_t} d\gamma(\partial t)\}_{k=0}^n$, as follows

$$\begin{aligned}
F_1 &= d\gamma \frac{\partial}{\partial t}, \\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} F_1 &= k_1 F_2, \\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, 3, \ldots, n-1, \\
\nabla^{\gamma}_{\frac{\partial}{\partial t}} F_n &= -k_{n-1} F_{n-1},
\end{aligned}$$

where the functions $\{k_1, k_2, \ldots, k_{n-1}\}$ represent the curvatures of γ, and $\nabla(\gamma)$ denotes the connection on the pull-back bundle $\gamma^{-1}(TN)$. Notably, $F_1 = T = \gamma'$ serves as the unit tangent vector field along the curve.

Let $\{T, N, B\}$ be the orthonormal frame field tangent to $\mathbb{M}_3$ along γ and defined as follows: T is the unit vector field tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ normal to γ and $B = T \times_{\mathbb{M}_3} N$.
The pseudo-vector product operation $\times_{\mathbb{M}_3}$ is related to the determinant function by

$$det(u, v, w) = g(u \times_M v, w).$$

With respect to the orthonormal basis e_1, e_2, e_3, we express

$$\begin{aligned}
T &= T_1 e_1 + T_2 e_2 + T_3 e_3 = T_1 e^{-t} \frac{\partial}{\partial x} + T_2 e^t \frac{\partial}{\partial y} + T_3 \frac{1}{\sqrt{p|\lambda|}} \frac{\partial}{\partial t} \\
N &= N_1 e_1 + N_2 e_2 + N_3 e_3 = N_1 e^{-t} \frac{\partial}{\partial x} + N_2 e^t \frac{\partial}{\partial y} + N_3 \frac{1}{\sqrt{p|\lambda|}} \frac{\partial}{\partial t} \\
B &= B_1 e_1 + B_2 e_2 + B_3 e_3 = B_1 e^{-t} \frac{\partial}{\partial x} + B_2 e^t \frac{\partial}{\partial y} + B_3 \frac{1}{\sqrt{p|\lambda|}} \frac{\partial}{\partial t}.
\end{aligned}$$

and The following Frenet formulas are applicable

$$\begin{aligned} \nabla_T T &= \varepsilon\kappa N \\ \nabla_T N &= -\varepsilon\kappa T + \varepsilon_1 \tau B \\ \nabla_T B &= -\varepsilon\tau N, \end{aligned} \tag{4.13}$$

where $g(T,T) = \varepsilon, g(N,N) = \varepsilon, g(B,B) = \varepsilon_1$. *Here* $\kappa = |\nabla_T T|$ *is the geodesic curvature of* γ *and* τ *is the geodesic torsion.*

Additionally, we have: :

$$T \times N = \varepsilon_1 B \quad , B \times T = \varepsilon N \quad , N \times B = \varepsilon T$$

Theorem 18. *The f-biharmonic equation* (3.3) *reduces to the system*

$$\begin{cases} -3\varepsilon f\kappa\kappa' - 2\varepsilon f'\kappa^2 = 0 \\ \\ f\kappa'' - f\kappa^3 - f\varepsilon\varepsilon_1\tau^2\kappa + f''\kappa + 2f'\kappa' - \dfrac{2f\kappa B_3^2}{\lambda} + \dfrac{f\kappa}{\lambda} = 0 \\ \\ f\varepsilon(\tau'\kappa + 2\tau\kappa') + 2\varepsilon f'\kappa\tau + f\varepsilon_1\kappa\dfrac{2}{\lambda}\big(N_3 B_3\big) = 0, \end{cases} \tag{4.14}$$

Proof *First we rewrite the f-biharmonic equation*

$$\tau_{2,f}(\gamma) = f\tau_2(\gamma) + \Delta f\tau(\gamma) + 2\Delta^{\gamma}_{grad\ f}\tau \tag{4.15}$$

In order to obtain all the quantities in the equation (4.15), we will start by computing $\tau(\gamma)$ *where* $\gamma = \gamma(s)$ *is a curve parameterized by arc length s. Let* $\mathbf{e}_1 = \frac{\partial}{\partial s}$ *be an orthonormal frame on* $I \subset \mathbb{R}$. *Then we have*

$$\tau(\gamma) = tr(\nabla d\gamma) = \nabla^{\gamma}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) - d\gamma\left(\nabla^{\gamma}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s}\right) = \nabla^{\dot\gamma}_{\dot\gamma(s)} = \nabla_T T = \epsilon\kappa N \tag{4.16}$$

Next, let's determine the equation of biharmonicity given by $\tau_2(\gamma)$.

$$\tau_2(\gamma) = \nabla^3_T T - R(\nabla_T T, T)T \tag{4.17}$$

Utilizing the Serret-Frenet formulas (4.13), through direct computations, we obtain:

$$\begin{aligned}
\nabla_T^3 T &= \nabla_T(\nabla_T(\nabla_T T)) \\
&= \nabla_T(\nabla_T \varepsilon\kappa N) \\
&= \varepsilon(\nabla_T(\nabla_T \kappa N)) \\
&= \varepsilon(\nabla_T(\kappa' N) + \nabla_T(\kappa \nabla_T N)) \\
&= \varepsilon(\kappa'' N + \kappa' \nabla_T N + \nabla_T(-\varepsilon\kappa^2 T + \varepsilon_1 \tau\kappa B)) \\
&= \varepsilon(\kappa'' N - \varepsilon\kappa\kappa' T + \varepsilon_1\tau\kappa' B - 2\varepsilon\kappa'\kappa T - \varepsilon\kappa^2\varepsilon\kappa N + \varepsilon_1\tau'\kappa B + \varepsilon_1\tau\kappa' B - \varepsilon_1\tau\kappa\varepsilon\tau N) \\
&= \varepsilon\kappa'' N - \varepsilon\varepsilon\kappa\kappa' T + \varepsilon\varepsilon_1\tau\kappa' B - \varepsilon 2\varepsilon\kappa'\kappa T - \varepsilon\varepsilon\kappa^2\varepsilon\kappa N + \varepsilon\varepsilon_1\tau'\kappa B + \varepsilon\varepsilon_1\tau\kappa' B - \varepsilon\varepsilon_1\tau\kappa\varepsilon\tau N \\
&= \varepsilon\kappa'' N - \kappa\kappa' T + \varepsilon\varepsilon_1\tau\kappa' B - 2\kappa'\kappa T - \kappa^2\varepsilon\kappa N + \varepsilon\varepsilon_1\tau'\kappa B + \varepsilon\varepsilon_1\tau\kappa' B - \varepsilon_1\tau\kappa\tau N \\
&= -3\kappa\kappa' T + (\varepsilon\kappa'' - \varepsilon\kappa^3 - \varepsilon_1\tau^2\kappa)N + \varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa')B
\end{aligned}$$

Therefore, the biharmonicity equation can be expressed by the formula

$$\tau_2(\gamma) = -3\kappa\kappa' T + (\varepsilon\kappa'' - \varepsilon\kappa^3 - \varepsilon_1\tau^2\kappa)N + \varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa')B - \varepsilon\kappa R(N,T)T \tag{4.18}$$

The second part involves finding the expression for $\nabla^\gamma_{gradf}\tau(\gamma)$.

$$\begin{aligned}
\nabla^\gamma_{gradf}\tau(\gamma) &= \nabla^\gamma_{gradf}(\varepsilon\kappa N) = f'\nabla_T(\epsilon\kappa N) \\
&= \varepsilon f'\big(\kappa' N + \kappa\nabla_T N\big) = \varepsilon f'\big(\kappa' N + \kappa\big(-\varepsilon\kappa T + \varepsilon_1\tau B\big) \\
&= -f'\kappa^2 T + \varepsilon f'\kappa' N + \varepsilon\varepsilon_1 f'\kappa\tau B
\end{aligned} \tag{4.19}$$

Finally, by combining equations (4.16), (4.18), and (4.19), along with other calculations pertaining to the formula for the biharmonicity of f, the equation (4.15) can be expressed as

$$\begin{aligned}
\tau_{2,f}(\gamma) &= f\big(-3\kappa\kappa' T + (\varepsilon\kappa'' - \varepsilon\kappa^3 - \varepsilon_1\tau^2\kappa)N + \varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa')B - \varepsilon\kappa R(N,T)T\big) \\
&+ f''\epsilon\kappa N + 2\big(\varepsilon f'\big(\kappa' N - \varepsilon\kappa^2 T + \kappa\varepsilon_1\tau B\big) \\
&= -f3\kappa\kappa' T + \big(f\varepsilon\kappa'' - f\varepsilon\kappa^3 - f\varepsilon_1\tau^2\kappa\big)N + f\varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa')B - f\varepsilon\kappa R(N,T)T \\
&+ f''\epsilon\kappa N + 2\varepsilon f'\kappa' N - 2\varepsilon\kappa^2 T + 2\kappa\varepsilon_1\tau B \\
&= T\big(-f3\kappa\kappa' - 2f'\kappa^2\big) + N\big(f\varepsilon\kappa'' - f\varepsilon\kappa^3 - f\varepsilon_1\tau^2\kappa + f''\epsilon\kappa + 2\varepsilon f'\kappa'\big) \\
&+ B\big(f\varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa'\big) + 2\kappa\varepsilon_1\tau) - f\varepsilon\kappa R(N,T)T
\end{aligned}$$

where

$$\Delta f\tau(\gamma) = f''\epsilon\kappa N. \tag{4.20}$$

Therefore, we deduce the following outcome:

$$\tau_{2,f}(\gamma) = T\big(-3f\kappa\kappa' - 2f'\kappa^2\big) + N\big(f\varepsilon\kappa'' - f\varepsilon\kappa^3 - f\varepsilon_1\tau^2\kappa + f''\epsilon\kappa + 2\varepsilon f'\kappa'\big) + B\big(f\varepsilon\varepsilon_1(\tau'\kappa + 2\tau\kappa'\big) + 2f'\kappa\varepsilon_1\tau) - f\varepsilon\kappa R(N,T)T \tag{4.21}$$

The condition for the biharmonicity of f is the vanishing of the equation (4.21)*. Thus, taking the scalar product of the equation with T, N, and B yields the following:*

$$g(\tau_{2,f}(\gamma), T) = -3\varepsilon f\kappa\kappa' - 2\varepsilon f'\kappa^2 - f\varepsilon\kappa R(N,T,T,T) \tag{4.22}$$
$$g(\tau_{2,f}(\gamma), N) = f\kappa'' - f\kappa^3 - f\varepsilon\varepsilon_1\tau^2\kappa + f''\kappa + 2f'\kappa' - f\varepsilon\kappa R(N,T,T,N) \tag{4.23}$$
$$g(\tau_{2,f}(\gamma), B) = f\varepsilon(\tau'\kappa + 2\tau\kappa') + 2\varepsilon f'\kappa\tau - f\varepsilon\kappa R(N,T,T,B) \tag{4.24}$$

Now, let's compute $R(N,T,T,T)$, $R(N,T,T,N)$, and $R(N,T,T,B)$ and plug them in the above equations to get the system (4.14)

Initially, due to the symmetries of the curvature tensor, we ascertain that $R(N,T,T,T)$ completely vanishes. In this scenario, we calculate both $R(N,T,T,N)$ and $R(N,T,T,B)$ utilizing the following decompositions

$R(N,T,T,T) = R(N_ie_i, T_ie_j, T_ke_k, N_le_l)$ and $R(N,T,T,B) = R(N_ie_i, T_ie_j, T_ke_k, B_le_l)$, $j, j, k, l \in \{1,2,3\}$

$$\begin{aligned}
R(N,T,T,N) &= N_iT_jT_kN_lR_{i,j,k,l} \\
&= \frac{1}{\lambda}\big(N_2T_1(N_1T_2 - N_2T_1) + T_2N_1(N_2T_1 - N_1T_2)\big) + \frac{\varepsilon}{|\lambda|}\big(N_3T_1(N_3T_1 - N_1T_3) \\
&+ \frac{\varepsilon}{|\lambda|}(N_1T_3(N_1T_3 - N_3T_1) + N_3T_2(N_3T_2 - N_2T_3) + N_2T_3(N_2T_3 - N_3T_2)\big) \\
&= \frac{-B_3\varepsilon_1}{\lambda}\big(T_2N_1 - N_2T_1\big) + \frac{\varepsilon\varepsilon_1}{|\lambda|}\big(N_3T_1(-B_2) + N_1T_3(B_2) + N_3T_2(B_1) + N_2T_3(-B_1)\big) \\
&= \frac{-B_3\varepsilon_1}{\lambda}\big(-\varepsilon_1B_3\big) + \frac{\varepsilon}{\lambda}\big(B_2(N_1T_3 - N_3T_1) + B_1(N_3T_2 - N_2T_3)\big) \\
&= \frac{1}{\lambda}\big(2B_3^2 - 1\big)
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
R(N,T,T,B) &= N_iT_jT_kB_lR_{i,j,k,l} \\
&= \frac{1}{\lambda}\big(-\varepsilon_1T_1B_2B_3 + T_2B_1\varepsilon_1B_3\big) + \frac{\varepsilon}{|\lambda|}\big(T_1B_3\varepsilon_1B_2 - T_3B_1\varepsilon_1B_2 + T_2B_3\varepsilon_1B_1 - T_3B_2\varepsilon_1B_1\big) \\
&= \frac{\varepsilon_1B_3}{\lambda}\big(B_1T_2 - T_1B_2\big) + \frac{\varepsilon_1\varepsilon}{|\lambda|}\big(B_2(T_1B_3 - T_3B_1) + B_1(T_2B_3 - T_3B_2)\big) \\
&= \frac{\varepsilon_1B_3}{\lambda}\big(\varepsilon N_3\big) + \frac{\varepsilon_1\varepsilon}{|\lambda|}\big(B_2(\varepsilon N_2) + B_1(-\varepsilon N_1)\big) \\
&= \frac{\varepsilon\varepsilon_1B_3N_3}{\lambda} + \frac{\varepsilon_1}{|\lambda|}\big(B_2(N_2) - B_1(N_1)\big) \\
&= \frac{-2}{\lambda}\big(N_3B_3\big)
\end{aligned} \tag{4.26}$$

4.3 $f-$biharmonicity conditions of the curve in $\mathbb{M}_3$

In this section we will discuss some spacial cases and give conditions for the f-biharmonicity of the curve according to these cases

Corollary 3. *If γ defines a trajectory with constant non-zero curvature, then it is evident that γ is biharmonic because in this case, f is also constant.*

proof *If γ is non-geodesic curve. By using the first equation of the system, we have* (4.14) *we get*

$$-3\varepsilon f\kappa\kappa' - 2\varepsilon f'\kappa^2 = 0 \Rightarrow -2\varepsilon f'\kappa^2 = 0$$

Since $\kappa \neq 0$ thus $f' = 0$.

Corollary 4. *If γ defines a trajectory with constant non-zero torsion and satisfies the condition $N_3B_3 = 0$, then γ is biharmonic.*

Proof:

Let's consider the first equation from system (4.14)*:*

$$3f\kappa'\kappa + 2f'\kappa^2 = 0 \Rightarrow \frac{\kappa'}{\kappa} = -\frac{2}{3}\frac{f'}{f}$$

Now, from the third equation of system (4.14)*, and suppose that the condition $N_3B_3 = 0$ holds, then we get:*

$$2f\varepsilon\tau\kappa' + 2\varepsilon f'\kappa\tau = 0 \Rightarrow \tau(f\kappa' + f'\kappa) = 0 \Rightarrow \frac{\kappa'}{\kappa} = \frac{f'}{f}$$

Therefore, f is constant.

Corollary 5. *If the torsion of γ remains constant and non-zero, then the value of f can be expressed as follows:*

$$f = \exp\left(\int -3\varepsilon\varepsilon_1\frac{(N_3B_3)}{\lambda\tau}ds\right)$$

Proof *Let's start by examining the equation $\frac{\kappa'}{\kappa} = -\frac{2}{3}\frac{f'}{f}$. Substituting this expression into the third equation of the system yields:*

$$2f\varepsilon(\tau\kappa') + 2\varepsilon f'\kappa\tau + f\varepsilon_1\kappa\frac{2}{\lambda}(N_3B_3) = 0$$

$$2f\varepsilon\tau\frac{\kappa'}{\kappa} + 2\varepsilon f'\tau + f\varepsilon_1\frac{2}{\lambda}(N_3B_3) = 0$$

$$2f\varepsilon\tau(-\frac{2}{3}\frac{f'}{f}) + 2\varepsilon f'\tau + f\varepsilon_1\frac{2}{\lambda}(N_3B_3) = 0$$

$$2\varepsilon\tau(-\frac{2}{3}\frac{f'}{f}) + 2\varepsilon\frac{f'}{f}\tau = -\varepsilon_1\frac{2}{\lambda}(N_3B_3)$$

$$-\frac{2}{3}\frac{f'}{f}+\frac{f'}{f}=-\varepsilon\varepsilon_1\frac{\big(N_3B_3\big)}{\lambda\tau}$$
$$\frac{f'}{f}=-3\varepsilon\varepsilon_1\frac{\big(N_3B_3\big)}{\lambda\tau}$$

Finally taking the integrals of both sides we obtain the explicit expression of the function f under the condition innocence earlier

$$\int\frac{f'}{f}=\int-3\varepsilon\varepsilon_1\frac{\big(N_3B_3\big)}{\lambda\tau}$$
$$ln(f)=\int-3\varepsilon\varepsilon_1\frac{\big(N_3B_3\big)}{\lambda\tau}ds$$
$$f=\exp\Big(\int-3\varepsilon\varepsilon_1\frac{\big(N_3B_3\big)}{\lambda\tau}ds\Big)$$

Corollary 6. *If γ has zero torsion, then the curve is f-biharmonic if and only if the following systems is satisfied:*

$$\begin{cases}\kappa^3f^2 & =\exp(c)\\ (f\kappa)'' & =f\kappa(\kappa^2+\dfrac{2B_3^2}{\lambda}-\dfrac{1}{\lambda})\\ N_3B_3 & =0\end{cases}\tag{4.27}$$

Proof

For $\tau=0$, integrating the first equation of the first system gives

$$\begin{cases}3f\kappa'\kappa+2f'\kappa^2 & = & 0\\ 3f^2\kappa'\kappa^2+2ff'\kappa^3 & = & 0\\ \dfrac{3\kappa'\kappa^2}{\kappa^3}+\dfrac{2ff'}{f^2} & = & 0\\ \ln(\kappa^3f^2) & = & c\\ \kappa^3f^2 & = & \exp(c)\end{cases}$$

The second equation of the system (4.14) results in

$$f\kappa''-f\varepsilon\varepsilon_1\tau^2\kappa+f''\kappa+2f'\kappa'=f\kappa^3+\frac{2f\kappa B_3^2}{\lambda}+\frac{f\kappa}{\lambda}$$

Hence,

$$(f\kappa)''=f\kappa^3+\frac{2f\kappa B_3^2}{\lambda}-\frac{f\kappa}{\lambda}$$

The third equation of the system (4.14) yields

$$f\varepsilon_1\kappa\frac{2}{\lambda}\big(N_3B_3\big)=0,$$

Corollary 7. *If γ is non geodesic curve and has constant torsion, then it is f-biharmonic if and only if the following system is satisfied*

$$\begin{cases} \kappa^3 f^2 &= \exp(c) \\ (f\kappa)'' &= f\kappa(\kappa^2 + \varepsilon\varepsilon_1\tau + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\ f^2\kappa^2\tau &= \exp\big(\displaystyle\int -\frac{2\varepsilon N_3 B_3}{|\lambda|\tau}\big) \end{cases} \tag{4.28}$$

Proof

For the first and second conditions, we have already demonstrated the respective results when considering $\kappa = const \neq 0$, thus they remain unchanged. Regarding the second result, it does not depend on any derivative of τ. Consequently, it becomes

$$(f\kappa)'' = f\kappa(\kappa^2 + \varepsilon\varepsilon_1\tau + \frac{2B_3^2}{\lambda} + \frac{1}{\lambda})$$

From the third equation of the system, we find

$$\begin{aligned} f\varepsilon(\tau'\kappa + 2\tau\kappa') + 2\varepsilon f'\kappa\tau &= -f\varepsilon_1\kappa\frac{2}{\lambda}\big(N_3B_3\big) \\ \frac{f^2\kappa(\tau'\kappa + 2\tau\kappa')) + 2f'f\kappa^2\tau}{f\kappa} &= -f\varepsilon_1\varepsilon\kappa\frac{2}{\lambda}\big(N_3B_3\big) \\ \frac{f^2\kappa(\tau'\kappa + 2\tau\kappa')) + 2f'f\kappa^2\tau}{f^2\kappa^2} &= -\varepsilon_1\varepsilon\frac{2}{\lambda}\big(N_3B_3\big) \\ \frac{f^2\kappa(\tau'\kappa + 2\tau\kappa')) + 2f'f\kappa^2\tau}{f^2\kappa^2\tau} &= -\frac{2\varepsilon}{|\lambda|\tau}\big(N_3B_3\big) \\ \frac{(f^2\kappa^2\tau)'}{f^2\kappa^2\tau} &= -\frac{2\varepsilon}{|\lambda|\tau}\big(N_3B_3\big) \\ \ln(f^2\kappa^2\tau) &= \int -\frac{2\varepsilon}{|\lambda|\tau}\big(N_3B_3\big) \\ f^2\kappa^2\tau &= \exp\left(\int -\frac{2\varepsilon}{|\lambda|\tau}\big(N_3B_3\big)\right) \end{aligned}$$

Theorem 19. *γ is a proper f-biharmonic curve if and only if:*

1) γ has zero torsion and f is given by $f = c_1\kappa^{-\frac{3}{2}}$, and its curvature satisfies the following differential equation

$$3(\kappa')^2 - 2\kappa''\kappa = 4\kappa^2(\kappa^2 + \frac{2B_3^2}{\lambda} + \frac{1}{\lambda})$$

2)γ has non zero torsion with $\frac{\tau}{\kappa} = \frac{1}{c_1^2}\exp\big(\int -\frac{2\varepsilon N_3 B_3}{|\lambda|\tau}\big)$, the function f is given by $f = c_1\kappa^{-\frac{3}{2}}$, and its curvature satisfies the following differential equation

$$3(\kappa')^2 - 2\kappa''\kappa = 4\kappa^2(\kappa^2 + \frac{\kappa}{c_1^2}\exp\big(\int -\frac{2\varepsilon N_3 B_3}{|\lambda|\tau}\big) + \frac{2B_3^2}{\lambda} + \frac{1}{\lambda})$$

Proof *For the first case ,we will substitute* $f = c_1\kappa^{-\frac{3}{2}}$ *obtained from the equation* $\kappa^3 f^2 = c_1$ *into the second equation of the system* (4.27)

$$
\begin{cases}
(c_1\kappa^{-\frac{3}{2}}\kappa)'' &= c_1\kappa^{-\frac{3}{2}}\kappa(\kappa^2 + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
(\kappa^{-\frac{1}{2}})'' &= \kappa^{-\frac{1}{2}}(\kappa^2 + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
-\dfrac{1}{2}(\kappa''\kappa^{-\frac{3}{2}} - \dfrac{3}{2}(\kappa')^2\kappa^{\frac{-5}{2}}) &= \kappa^{-\frac{1}{2}}(\kappa^2 + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
3(\kappa')^2 - 2\kappa''\kappa &= 4\kappa^2(\kappa^2 + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda})
\end{cases}
$$

Next, For the seconde case ,we will substitute $f = c_1\kappa^{-\frac{3}{2}}$ *obtained from the equation* $\kappa^3 f^2 = c_1$ *into the third equation of the system* (4.28)

$$c_1^2\kappa^{-3}\kappa^2\tau = \exp\Big(\int -\frac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big)$$

$$\frac{\tau}{\kappa} = \frac{1}{c_1^2}\exp\Big(\int -\frac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big) \tag{4.29}$$

Finally we substitute the values $f = c_1\kappa^{-\frac{3}{2}}$ *and* (4.29) *in the seconde equation of the system* (4.28) *we obtain*

$$
\begin{cases}
(c_1\kappa^{-\frac{3}{2}}\kappa)'' &= c_1\kappa^{-\frac{3}{2}}\kappa(\kappa^2 + \dfrac{\kappa}{c_1^2}\exp\Big(\displaystyle\int -\dfrac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big) + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
(\kappa^{-\frac{1}{2}})'' &= \kappa^{-\frac{1}{2}}(\kappa^2 + \dfrac{\kappa}{c_1^2}\exp\Big(\displaystyle\int -\dfrac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big) + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
-\dfrac{1}{2}(\kappa''\kappa^{-\frac{3}{2}} - \dfrac{3}{2}(\kappa')^2\kappa^{\frac{-5}{2}}) &= \kappa^{-\frac{1}{2}}(\kappa^2 + \dfrac{\kappa}{c_1^2}\exp\Big(\displaystyle\int -\dfrac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big) + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}) \\
3(\kappa')^2 - 2\kappa''\kappa &= 4\kappa^2\left(\kappa^2 + \dfrac{\kappa}{c_1^2}\exp\Big(\displaystyle\int -\dfrac{2\varepsilon N_3 B_3}{|\lambda|\tau}\Big) + \dfrac{2B_3^2}{\lambda} + \dfrac{1}{\lambda}\right)
\end{cases}
$$

Conclusion

In conclusion, this study focused on the classification of f biharmonic curves in the generalized symmetric space, a Lorentzian manifold of dimension 3. The results obtained have highlighted a significant correlation between the geometry of curves and their biharmonic properties, particularly concerning curvature and torsion. The research of f-biharmonic curve on the three-dimensional generalized symmetric space$(\mathbb{M}_3, g)$ follows a systematic approach that includes several crucial steps. Here is a summary of the main steps followed in this thesis:

1. ***Calculation of Lie Brackets****; Initially, Lie brackets are computed for the basis vectors of $\mathbb{M}_3$. These brackets offer vital insights into the behavior of the basis vectors, which are essential for comprehending the intrinsic geometric properties of the generalized symmetric space.*

2. ***Computation of Levi-Civita Connections****; Subsequently, the Levi-Civita connections are determined. These connections illustrate how a basis vector changes in the direction of other basis vectors, defined by e_1, e_2, and e_3. They serve as a foundation for examining the deformation of the bases in the generalized symmetric space.*

3. ***Calculation of Riemann Curvatures****; Riemann curvatures are then calculated. These curvatures describe the intrinsic curvature of the generalized symmetric space.*

4. ***Use of Frenet Frames****; Finally, Frenet frames are used to establish the equation of f-biharmonicity. These Frenet frames provide a natural basis for studying the curve and enable the derivation of an equation that characterizes f-biharmonic curves in the generalized symmetric space.*

Through these steps, the thesis establishes a thorough framework for examining f-biharmonic curves in generalized symmetric spaces, facilitating a deep understanding of these complex geometric entities. These findings hold particular significance across various fields, such as mathematical physics and the theory of special relativity, where precise characterization of trajectories in Lorentzian spaces is essential.

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