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ABDELHAMID IBN BADIS UNIVERSITY OF MOSTAGANEM

Estimation of the growth and study of the oscillation of solutions of complex differential equations and complex difference equations

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Specialty: Functional Analysis

by

Abdelkader Dahmani

Publicly defended before the examining committee made up of:

Chairperson:	Berrabah BENDOUKHA	Professor at Univ. of Mostaganem;
Examiner:	Mouffak BENCHOHRA	Professor at Univ. of Sidi Bel Abbes;
Examiner:	Abdelkader SENOUCI	Professor at Univ. of Tiaret;
Examiner:	Karima HAMANI	Professor at Univ. of Mostaganem;
Examiner:	Saada HAMOUDA	Professor at Univ. of Mostaganem;
Supervisor:	Benharrat BELAÏDI	Professor at Univ. of Mostaganem.

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Abstract

Understanding the growth and oscillation of solutions to differential equations, difference equations and delay-differential equations, is crucial for predicting their behavior. Nevanlinna theory, with its deep insight into the value distribution of meromorphic functions, provides a powerful framework for analyzing the growth and oscillation of solutions to these equations. In this thesis, by using this theory, we present some results regarding the growth and oscillation of solutions of linear differential equations with analytic or meromorphic coefficients in the extended complex plane except at a finite isolated point, we also discuss some results on the growth of solutions of linear difference equations and linear delay-differential equations , in which the coefficients are meromorphic functions in the complex plane.

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Keywords and phrases: Nevanlinna theory, linear differential equations, linear difference equations, linear delay-differential equations, growth of solutions, oscillation of solutions, analytic function, meromorphic function, isolated point, logarithmic order, logarithmic type.

Résumé

Comprendre la croissance et l'oscillation des solutions aux équations différentielles, aux équations aux différences et aux équations différentielles retardées est crucial pour prédire leur comportement. La théorie de Nevanlinna, avec sa profonde compréhension de la distribution des valeurs des fonctions méromorphes, fournit un cadre puissant pour analyser la croissance et l'oscillation des solutions à ces équations. Dans cette thèse, en utilisant cette théorie, nous présentons certains résultats concernant la croissance et l'oscillation des solutions des équations différentielles linéaires avec des coefficients analytiques ou méromorphes dans le plan complexe étendu, sauf en un point isolé fini. Nous discutons également de certains résultats sur la croissance des solutions des équations aux différences linéaires et des équations différentielles linéaires retardées, dans lesquelles les coefficients sont des fonctions méromorphes dans le plan complexe.
Mots clés: Théorie de Nevanlinna, équations différentielles linéaires, équations aux différences linéaires, équations différentielles linéaires retardées, croissance des solutions, oscillation des solutions, fonction analytique, fonction méromorphe, ordre logarithmique, type logarithmique.

ملخص

فهم نمو وتذبذب الحلول للمعادلات التفاضلية ومعادلات الفروق والمعادلات التفاضلية التأخيرية أمر بالغ الأهمية لتوقع سلوكها. نظرية نيفانلينا، مع رؤيتها العميقة في توزيع قيم الدوال الميرومورفية، توفر إطارًا قويًا لتحليل نمو وتذبذب الحلول لهذه المعادلات. في هذه الرسالة، باستخدام هذه النظرية، نقدم بعض النتائج المتعلقة بنمو وتذبذب حلول المعادلات التفاضلية الخطية ذات العوامل التحليلية أو الميرومورفية في المستوى المركب الموسع باستثناء نقطة منعزلة محدودة، نناقش أيضًا بعض النتائج حول نمو حلول معادلات الفروق الخطية والمعادلات التفاضلية التأخيرية الخطية، حيث تكون المعاملات دوال ميرومورفية في المستوى المركب.

الكلمات المفتاحية: نظرية نيفانلينا، المعادلات التفاضلية الخطية، معادلات الفروق الخطية، المعادلات التفاضلية التأخيرية، نمو الحلول، تذبذب الحلول، الدوال التحليلية، الدوال الميرومورفية، نقطة منعزلة، الترتيب اللوغاريتمي، النوع الوغاريتمي.

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3. A. Dahmani and B. Belaïdi, Finite logarithmic order meromorphic solutions of complex linear delay-differential equations, *Pan-Amer. J. Math*, Vol. 2, 2023, pp. 2-14. DOI:10.28919/cpr-pajm/2-14.

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5. A. Dahmani and B. Belaïdi, Growth of solutions to complex linear differential equations with analytic or meromorphic coefficients in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order, *WSEAS Transactions on Mathematics*, Vol. 23, 2024, pp. 107-117. DOI:10.37394/23206.2024.23.13.

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List of communications

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Contents

Introduction

A word on the notations: Throughout this thesis, we use $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{R}$ and $\mathbb{C}$ to denote the set of all natural numbers, the set of all integers, the set of the real numbers and the set of complex numbers respectively, with $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by Δ. A function f is called meromorphic, if it is meromorphic in $\mathbb{C}$ entirely, while by saying f is meromorphic around (or near) a finite singular point $z_0 \in \mathbb{C}$, then we mean that f is meromorphic in the extended complex plane except z_0 (ie., $\overline{\mathbb{C}} - \{z_0\}$).
Let $f(r)$ be complex valued function and $g(r)$ be real and positive. We write $f(r) = O\big(g(r)\big)$, whenever there exist constants C and r_0 such that, $|f(r)| \leq C|g(r)|$, for all $r \geq r_0$. Also, we write $f(r) = o\big(g(r)\big)$ if $\frac{f(r)}{g(r)} \to 0$ as $r \to \infty$. We shall use the $\log^+$ notation as follows. For $x \geq 0$, we write $\log^+ x = \max\{0, \log x\}$. Moreover, we use the notation $\log_p r$ for the p-th iteration of the logarithmic function ($p \in \mathbb{N}$), such that for all $r \in (0,\infty)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$. Similarly, we use the notation $\exp_p r$ for the p-th iteration of the exponential function.
The linear measure of a set $E \subset [0,+\infty)$ is denoted by $m_\ell(E) = \int_E dt$, while we denote the logarithmic measure of a set $E \subset (0,1)$ by $m_{\log}(E) = \int_E \frac{dt}{t}$. For those sets with infinite linear or logarithmic measure, we reserve notation E_i $i \in \mathbb{N}$, while we use $\mathcal{F}_i$ $i \in \mathbb{N}$ for sets of finite linear or logarithmic measure. It is worth noting that adopting the same notation does not imply that the sets are identical.

Introduction

In 1929, the Finnish mathematician R. Nevanlinna introduced his theory [61], which was the complete form of the value distribution theory, whose origins trace back to the famous theorems of Sokhotskii-Casorati (1868), Weierstrass (1876), Picard (1879), Hadamard (1892), Borel (1897) and others, all of which have become part of Nevanlinna theory results, but in a more refined and elucidating form, let us say. (see [21, 34, 38, 41, 47, 60, 75, 76, 79]. Perhaps this gives a simple idea of how vast this theory is. Its vastness can also be observed from its numerous applications in various fields such as functional equations, differential equations, complex dynamics and Diophantine equations. Nevanlinna theory has been applied in complex differential equations to study the properties of their solutions (see e.g. [43, 46, 47], the pioneers in this regard were F. Nevanlinna [62], R. Nevanlinna [63] and K. Yosida [77], but the first systematic application of this theory to solutions of differential equations was carried out in the 1940s and 1950s by H. Wittich (see e.g. [68–70]), laying the groundwork for further research, which was embodied through the numerous and ongoing publications that generalize and extend H. Wittich results about the properties of meromorphic solutions of complex differential equations in general, and specifically on the growth and oscillation of their solutions, by relating the order of the coefficients with the order of the solutions or with the exponent of convergence of the sequence of their zeros, which was firstly made by Bank and Laine (see [1,2]). This was also reflected in Nevanlinna theory, which had to undergo some extensions to different domain, such as the unit disk Δ (see [38,40,46,66]), the extended complex plane except an isolated point $\overline{\mathbb{C}} - \{z_0\}$ [32,37]. The latter occupies an important part of our thesis topic.

Although Nevanlinna theory linked to difference equations field in the 1980s by Shimomura [65] and Yanagihara [73, 74], it did not gain that momentum of applications in this field, at least not to the extent seen in differential equations, which has been justified by the scarcity of necessary tools, such as those provided by Nevanlinna theory, notably, the logarithmic derivative lemma, for studying the differential equations. This problem was largely resolved after the establishment of the difference analogous of the logarithmic derivative by Halburd-Korhonen [35,36] and Chiang-Feng [22], independently. The numerous subsequent works, especially those related to the growth of solutions, are concrete evidence of this claim (see [16,50]).

With the significant and rapid advancement witnessed in differential equations as well as the difference equations through the application of Nevanlinna theory, it was not difficult to anticipate that the next step would be the difference-differential equations (also called delay-deferential equations), especially because the groundwork of studying their meromorphic solution was already laid by Naftalevich [57–59]. The delay-deferential equations involve both the difference operator and the derivative, this why they can be regarded as a generalization of both difference and differential equations, yet they have their own specific applications, which is the reason behind the growing interest to investigate them, particularly the growth and the value distribution of their solutions (see [50]). This thesis discusses some results that can be considered as examples regarding the application of Nevanlinna theory in studying the growth and oscillation of solutions to these three types of equations. Besides this introduction, this thesis contains six chapters.

In Chapter 1, we briefly recall some basic results derived from Nevanlinna theory to provide the necessary background. In Chapters 2, 3 and 4, we consider the complex linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z),$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $F(z)$ are analytic or meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$.

We obtain some results on the growth and oscillation of solutions, where we use the $[p,q]$-order and the logarithmic order as growth indicators. In Chapter 5, we continue making use of the logarithmic order to estimate the growth of the complex linear difference equation

$$A_k(z)f(z+c_k)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=F(z),$$

where $A_k(z),\ldots,A_0(z)$ and $F(z)$ are meromorphic functions of finite logarithmic order, $c_i(i=1,\ldots,k,k\in\mathbb{N})$ are distinct non-zero complex constants. Its homogeneous case is also considered.
The final Chapter 6 is devoted to considering the logarithmic order of meromorphic solutions of the homogeneous and non-homogeneous linear delay-differential equations

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=0,$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=F(z),$$

where $A_{ij}(z)$ $(i=0,1,\ldots,n,j=0,1,\ldots,m,n,m\in\mathbb{N})$ and $F(z)$ are meromorphic of finite logarithmic order, $c_i(i=0,\ldots,n)$ are distinct non-zero complex constants.

Chapter 1

Background

In this chapter, we briefly review some selected facts from Nevanlinna theory, chosen based on their relevance and use as an essential material in the following chapters. The body of this chapter is divided into three sections, the material of the first section can be found in any classic book on Nevanlinna theory (see e.g. [3, 38, 47, 60]), while the content of the next two sections is mostly taken from [15, 19, 32, 42, 49, 55].

1.1 Nevanlinna theory: Basic definitions and theorems

The crucial role played by the three Nevanlinna main functions: the proximity function m, the counting function N and The characteristic function T, can be observed easily through their involvements in the two Nevanlinna fundamental theorems and in the majority of quantities measuring the growth and value distribution. Therefore, this theory has been described as a study of how the growth of these functions interrelates. We begin this section by the definitions of these three main functions.

Definition 1.1. *Let $f(z)$ be a meromorphic function. For $f \not\equiv a \in \mathbb{C}$, the proximity functions of $f(z)$ are defined by*

$$m(r,a,f) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\varphi})-a}\right| d\varphi,$$

$$m(r,\infty,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi.$$

Definition 1.2. *Let $f(z)$ be a meromorphic function. For $f \not\equiv a \in \mathbb{C}$, the counting functions of $f(z)$ are defined by*

$$N(r,a,f) = N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t,a,f)-n(0,a,f)}{t} dt + n(0,a,f)\log r,$$

$$N(r,\infty,f) = N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t} dt + n(0,f)\log r,$$

$$\overline{N}(r,a,f)=\overline{N}\left(r,\frac{1}{f-a}\right)=\int_0^r\frac{\overline{n}(t,a,f)-\overline{n}(0,a,f)}{t}dt+\overline{n}(0,a,f)\log r,$$
$$\overline{N}(r,\infty,f)=\overline{N}(r,f)=\int_0^r\frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}dt+\overline{n}(0,f)\log r,$$

where $n(t,a,f)=n(t,a)$ $n(t,\infty,f)=n(t,f)$, $\overline{n}(t,a,f)=\overline{n}(t,a)$ and $\overline{n}(t,\infty,f)=\overline{n}(t,f)$ denote respectively the number of zeroes of $f(z)-a$, the number of poles of $f(z)$ lying in $|z|\le t$, counted according to their multiplicity, the number of distinct zeroes of $f(z)-a$ and the number of distinct poles of $f(z)$ lying in $|z|\le t$.

Definition 1.3. *The characteristic function of a meromorphic function $f(z)$ is given by*

$$T(r,f)=m(r,f)+N(r,f),\ r>0.$$

Example 1.1. *Let $f(z)$ be a meromorphic function, and let $P(z)$ and $Q(z)$ be two polynomials of degree p and q respectively*

1. $T(r,\frac{P(z)}{Q(z)})=\max\{p,q\}\log r$.
2. $T(r,P(f))=pT(r,f)+O(1)$.
3. *Suppose that $a,b,c,d\in\mathbb{C}$ with $ad-bc\neq 0$, then* $T\left(\frac{af+b}{cf+d}\right)=T(r,f)+O(1)$.
4. *Suppose $Q(z)=b_qz^q+b_{q-1}z^{q-1}+...+b_0$, then* $T(r,e^{Q(z)})=m(r,e^{Q(z)})\sim\frac{|b_q|r^q}{\pi}$.
5. $T(r,e^{e^r})=m(r,e^{e^r})\sim\frac{e^r}{2\pi^3r}$.

In general, if $f(z)$ is entire, then $N(r,f)\equiv 0$, and so $T(r,f)=m(r,f)$, and this leads to a relationship between $T(r,f)$ and the maximum modulus $M(r,f)=\max_{|z|=r}|f(z)|$, expressed through two inequalities by the following theorem.

Theorem 1.1. *Suppose $f(z)$ is entire function and $0<r<R<\infty$, then*

$$T(r,f)\le\log^+M(r,f)\le\frac{R+r}{R-r}T(R,f)$$

The functions $N(r,f)$ and $T(r,f)$ are both non-decreasing of r and convex of $\log r$, unlike $m(r,f)$, which may not necessarily satisfies these properties. Besides this, these three functions satisfy the following properties

Proposition 1.1. *Let $f_1,f_2,...,f_n$ be meromorphic functions. Then*

1. $m\left(r,\sum_{j=1}^n f_j\right)\le\sum_{j=1}^n m(r,f_j)+\log n,\qquad m\left(r,\prod_{j=1}^n f_j\right)\le\prod_{j=1}^n m(r,f_j)$.
2. $N\left(r,\sum_{j=1}^n f_j\right)\le\sum_{j=1}^n N(r,f_j),\qquad N\left(r,\prod_{j=1}^n f_j\right)\le\prod_{j=1}^n N(r,f_j)$.
3. $T\left(r,\sum_{j=1}^n f_j\right)\le\sum_{j=1}^n T(r,f_j)+\log n,\qquad T\left(r,\prod_{j=1}^n f_j\right)\le\prod_{j=1}^n T(r,f_k)$.

Theorem 1.2. *(Nevanlinna's first fundamental theorem) Let $f(z)$ be a non-constant meromorphic function, and let $a \in \mathbb{C}$. Then*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1) \qquad as \quad r \to \infty. \tag{1.1}$$

Theorem 1.3. *(Nevanlinna's second fundamental theorem) Let $f(z)$ be a non-constant meromorphic function, and let $k \geq 2$. Suppose that $a_1, \ldots, a_k$ are distinct complex numbers. Then*

$$m(r, f) + \sum_{j=1}^{k} m(r, \frac{1}{f-a_j}) \leq 2T(r, f) + S(r, f), \tag{1.2}$$

where

$$S(r, f) = O\big(\log T(r, f) + \log r\big) = o(T(r, f)) \quad as \quad r \to \infty, \tag{1.3}$$

outside of a possible exceptional set $\mathcal{F} \subset [0, \infty)$ of finite linear measure.

The estimate provided by the lemma of the logarithmic derivative is the source from which the error term $S(r, f)$ arises. Perhaps this fact alone should tell us about the significance of the role played by this lemma in the Nevanlinna's second fundamental theorem in particular, and generally this lemma is considered as an indispensable tool is several other results in the value distribution theory and its applications such as, the differential equations. in the following, we only state its standard version, while some of its counterparts and their variants will be recalled and applied in the next chapters.

Theorem 1.4. *(The logarithmic derivative lemma) Let f be a meromorphic function and $k \geq 1$ be integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \tag{1.4}$$

where $S(r, f)$ satisfies (1.3).

We finish this section by recalling the definition of the central index of entire functions.

Definition 1.4. *Suppose $f(z)$ is an entire function whose Taylor expansion is $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The central index of $f(z)$ is given by*

$$V(r, f) = \max_{m \geq 0}\{m : |a_m| r^m = u(r, f)\},$$

where $u(r, f) = \max_{n \geq 0} |a_n| r^n$ is the maximum term, whose existence is always guaranteed by the convergence of the power series $\sum_{n=0}^{\infty} |a_n| r^n$ for every $r > 0$.

1.2 Growth and value distribution scales of meromorphic function

In this section, we introduce some quantities, most of which are defined in terms of the three main Nevanlinna functions, these quantities will be used to estimate the growth and value distribution of meromorphic function

Definition 1.5. *Let $f(z)$ be a meromorphic function. The order of the growth and the lower order of growth are respectively defined by*

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r},\quad \mu(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is entire, then

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\longrightarrow\infty}\frac{\log\log M(r,f)}{\log r},\quad \mu(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\longrightarrow\infty}\frac{\log\log M(r,f)}{\log r}.$$

Definition 1.6. *Let $f(z)$ be a meromorphic function with $0<\mu(f)\leq\rho(f)<\infty$. The type and the lower type are respectively given by*

$$\tau(f)=\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{r^{\rho(f)}},\quad \underline{\tau}(f)=\liminf_{r\longrightarrow\infty}\frac{T(r,f)}{r^{\mu(f)}}.$$

If $f(z)$ is entire, then

$$\tau_M(f)=\limsup_{r\longrightarrow\infty}\frac{\log M(r,f)}{r^{\rho(f)}},\quad \underline{\tau}_M(f)=\liminf_{r\longrightarrow\infty}\frac{\log M(r,f)}{r^{\mu(f)}}.$$

The order and the type are both effective indicators of the growth of meromorphic functions, but from the definition of the type, we remark that it is no longer useful for the two cases when the functions are of zero or infinite order. For that, we need to introduce other growth indicators.

Definition 1.7. *Let $f(z)$ be a meromorphic function, and let $p,q\in\mathbb{N}$ such that $p\geq q\geq 1$. The $[p,q]$-order and the lower $[p,q]$-order are respectively defined by*

$$\rho_{[p,q]}(f)=\limsup_{r\longrightarrow\infty}\frac{\log_p T(r,f)}{\log_q r},\quad \mu_{[p,q]}(f)=\liminf_{r\longrightarrow\infty}\frac{\log_p T(r,f)}{\log_q r}.$$

If $f(z)$ is entire, then

$$\rho_{[p,q]}(f)=\limsup_{r\longrightarrow\infty}\frac{\log_p T(r,f)}{\log_q r}=\limsup_{r\longrightarrow\infty}\frac{\log_{p+1} M(r,f)}{\log_q r},$$

$$\mu_{[p,q]}(f)=\liminf_{r\longrightarrow\infty}\frac{\log_p T(r,f)}{\log_q r}=\liminf_{r\longrightarrow\infty}\frac{\log\log_{p+1} M(r,f)}{\log_q r}.$$

Note that, for a meromorphic function $f(z)$, it is clear that $\mu_{[p,q]}(f)\leq\rho_{[p,q]}(f)$. If $\mu_{[p,q]}(f)=\rho_{[p,q]}(f)$, then $f(z)$ is said to be of regular $[p,q]$-growth, and if $\mu_{[p,q]}(f)<\rho_{[p,q]}(f)$, then $f(z)$ is of irregular $[p,q]$-growth. In particular, when $\mu_{[1,1]}(f)=\mu(f)=\rho(f)=\rho_{[1,1]}(f)$, $f(z)$ is of regular growth, and it is of irregular growth otherwise.

Definition 1.8. *Let $f(z)$ be a meromorphic function with $0<\mu_{[p,q]}(f)\leq\rho_{[p,q]}(f)<\infty$. The $[p,q]$-type and the lower $[p,q]$-type are respectively given by*

$$\tau_{[p,q]}(f)=\limsup_{r\longrightarrow\infty}\frac{\log_{p-1} T(r,f)}{(\log_{q-1})^{\rho_{[p,q]}(f)}},\quad \underline{\tau}_{[p,q]}(f)=\liminf_{r\longrightarrow\infty}\frac{\log_{p-1} T(r,f)}{(\log_{q-1})^{\mu_{[p,q]}(f)}}.$$

If $f(z)$ is entire, then

$$\tau_{[p,q],M}(f)=\limsup_{r\longrightarrow\infty}\frac{\log_p M(r,f)}{(\log_{q-1})^{\rho(f)}},\quad \underline{\tau}_{[p,q],M}(f)=\liminf_{r\longrightarrow\infty}\frac{\log_p M(r,f)}{(\log_{q-1})^{\mu(f)}}.$$

Definition 1.9. *Let $f(z)$ be a meromorphic function. The logarithmic order and the logarithmic lower order are respectively defined by*

$$\rho_{\log}(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log\log r},\quad \mu_{\log}(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log\log r}.$$

If $f(z)$ is entire, then

$$\rho_{\log}(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log\log r}=\limsup_{r\longrightarrow\infty}\frac{\log\log M(r,f)}{\log\log r},$$

$$\mu_{\log}(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log\log r}=\liminf_{r\longrightarrow\infty}\frac{\log\log M(r,f)}{\log\log r}.$$

Example 1.2. *In view of* Example 1.1*, we have*

1. $\rho_{[p,q]}(\frac{P(z)}{Q(z)})=0$ *for any* $p\geq q\geq 1$ *and* $\rho_{\log}(\frac{P(z)}{Q(z)})=1.$
2. $\rho(e^{Q(z)})=q$ *and* $\rho_{\log}(e^{Q(z)})=\infty.$
3. $\rho(e^{e^z})=\rho_{\log}(e^{e^z})=\infty$ *and* $\rho_{[2,1]}(e^{e^z})=1.$

Among the properties of the logarithmic order we should mention that the meromorphic functions with finite logarithmic order are of zero order. However, the reverse is not necessarily true. the logarithmic order can not take any value between zero and one. As it is shown in the above example the non-constant rational functions are of logarithmic order equals one. For further properties and examples (see [19, 20])

Definition 1.10. *Let $f(z)$ be a meromorphic function with $0<\mu_{\log}(f)\leq\rho_{\log}(f)<\infty$. The logarithmic type and the logarithmic lower type are respectively given by*

$$\tau_{\log}(f)=\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{(\log r)^{\rho_{\log}(f)}},\quad \underline{\tau}_{\log}(f)=\liminf_{r\longrightarrow\infty}\frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

If $f(z)$ is entire, then

$$\tau_{\log,M}(f)=\limsup_{r\longrightarrow+\infty}\frac{\log M(r,f)}{(\log r)^{\rho_{\log}(f)}},\quad \underline{\tau}_{\log,M}(f)=\liminf_{r\longrightarrow+\infty}\frac{\log M(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Definition 1.11. *Suppose $f(z)$ is a meromorphic function. Then, the exponent of convergence of poles of $f(z)$ is defined by*

$$\lambda\left(\frac{1}{f}\right)=\limsup_{r\longrightarrow+\infty}\frac{\log n(r,f)}{\log r}=\limsup_{r\longrightarrow+\infty}\frac{\log N(r,f)}{\log r}.$$

Definition 1.12. *Suppose $f(z)$ is a meromorphic function. Then, the logarithmic exponent of convergence of poles of $f(z)$ is defined by*

$$\lambda_{\log}\left(\frac{1}{f}\right)=\limsup_{r\longrightarrow+\infty}\frac{\log n(r,f)}{\log\log r}=\limsup_{r\longrightarrow+\infty}\frac{\log N(r,f)}{\log\log r}-1,$$

Definition 1.13. *The deficiency of $a \in \overline{\mathbb{C}}$ with respect to a meromorphic function $f(z)$ is given by*

$$\delta(a,f) = \liminf_{r\to+\infty} \frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}, \quad for \quad a \in \mathbb{C}.$$

$$\delta(\infty,f) = \liminf_{r\to+\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N(r,f)}{T(r,f)},$$

a value a is called a deficient or a defective value of $f(z)$ if the above quantity is strictly greater than zero, whereas it is obvious that $0 \leq \delta(a,f) \leq 1$. Moreover, by the second fundamental theorem, the set of the deficient values of $f(z)$ satisfies $\sum_{a\in\bar{\mathbb{C}}} \delta(a,f) \leq 2$.

1.3 Growth and value distribution scales of meromorphic function around an isolated point

In this section, we list some other quantities, which are also important in studying the growth and value distribution of meromorphic function around a singular point $z_0 \in \mathbb{C}$. For that, we first need new definitions for the main Nevanlinna functions.

Definition 1.14. *Suppose $f(z)$ is a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. The characteristic function of $f(z)$ near z_0 is defined by*

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f),$$

where

$$m_{z_0}(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi$$

is the proximity function of $f(z)$ near z_0 and

$$N_{z_0}(r,f) = -\int_\infty^r \frac{n_{z_0}(t,f) - n_{z_0}(\infty,f)}{t} dt - n_{z_0}(\infty,f)\log r$$

is its counting function near z_0. Here $n_{z_0}(t,f)$ denotes the number of poles of $f(z)$ in $\{z \in \mathbb{C} : t \leq |z-z_0|\} \cup \{\infty\}$, each pole according to its multiplicity. While the number of distinct poles of $f(z)$ in $\{z \in \mathbb{C} : t \leq |z-z_0|\} \cup \{\infty\}$ is denoted by $\overline{n}_{z_0}(t,f)$, which can be used to generate $\overline{N}_{z_0}(r,f)$ in an analogous manner to $N_{z_0}(r,f)$.

Lemma 1.1. *Let $f(z)$ be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and set $g(\omega) = f(z_0 - \frac{1}{\omega})$. Then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have*

$$T(R,g) = T_{z_0}(\frac{1}{R}, f).$$

From Lemma 1.1, it is easy to see that the properties of the characteristic function of meromorphic functions are also hold for the characteristic function of meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$.

Definition 1.15. *The maximum modulus of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$, is given by*

$$M_{z_0}(r,f) = \max\{|f(z)| : |z-z_0| = r\}$$

.

Definition 1.16. *Let $f(z)=\sum_{n=0}^{\infty}a_n\frac{1}{(z-z_0)^n}$ be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, with the maximum term $u_{z_0}(r,f)=\max_{n\geq 0}|a_n|\frac{1}{r^n}$, where $|z-z_0|=r$. The central index of $f(z)$ is defined by*

$$V_{z_0}(r,f)=\max_{m\geq 0}\left\{m:|a_m|\frac{1}{r^m}=u_{z_0}(r,f)\right\}.$$

Remark 1.1. *If $f(z)$ is non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then $g(\omega)=f(z_0-\frac{1}{\omega})$ is entire and $V_{z_0}(r,f)=V(R,g)$, where $R=\frac{1}{r}$.*

Definition 1.17. *The $[p,q]$-order and the lower $[p,q]$-order near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ are respectively defined by*

$$\rho_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q\frac{1}{r}},\quad \mu_{[p,q]}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q\frac{1}{r}}.$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$, the $[p,q]$-order and the lower $[p,q]$-order of $f(z)$ near z_0 are given by

$$\rho_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q\frac{1}{r}}=\limsup_{r\longrightarrow 0}\frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q\frac{1}{r}},$$

$$\mu_{[p,q]}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_p^+ T_{z_0}(r,f)}{\log_q\frac{1}{r}}=\liminf_{r\longrightarrow 0}\frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q\frac{1}{r}}.$$

Note that $\rho_{[1,1]}(f,z_0)=\rho(f,z_0)$ and $\mu_{[1,1]}(f,z_0)=\mu(f,z_0)$ are just the order and the lower order near z_0 of the meromorphic function $f(z)$. $\rho_{[2,1]}(f,z_0)$ and $\rho_{[p,1]}(f,z_0)$ are called, respectively, the hyper-order and the iterated p-order near z_0.

Definition 1.18. *The $[p,q]$-type and the lower $[p,q]$-type near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ with $0<\mu_{[p,q]}(f,z_0)\leq\rho_{[p,q]}(f,z_0)<\infty$, are respectively defined by*

$$\tau_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_{p-1}^+ T_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\rho_{[p,q]}(f,z_0)}},\quad \underline{\tau}_{[p,q]}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_{p-1}^+ T_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\mu_{[p,q]}(f,z_0)}}.$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$, the $[p,q]$-type and the lower $[p,q]$-type of $f(z)$ near z_0 are given by

$$\tau_{[p,q],M}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\rho_{[p,q]}(f,z_0)}},\quad \underline{\tau}_{[p,q],M}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_p^+ M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\mu_{[p,q]}(f,z_0)}}$$

Definition 1.19. *The logarithmic order and the lower logarithmic order near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ are respectively defined by*

$$\rho_{\log}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log^+ T_{z_0}(r,f)}{\log\log\frac{1}{r}},\quad \mu_{\log}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log^+ T_{z_0}(r,f)}{\log\log\frac{1}{r}}.$$

If $f(z)$ is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then

$$\rho_{\log}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log^+ T_{z_0}(r,f)}{\log\log\frac{1}{r}}=\limsup_{r\longrightarrow 0}\frac{\log^+\log^+ M_{z_0}(r,f)}{\log\log\frac{1}{r}},$$

$$\mu_{\log}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log^+ T_{z_0}(r,f)}{\log\log\frac{1}{r}}=\liminf_{r\longrightarrow 0}\frac{\log^+\log^+ M_{z_0}(r,f)}{\log\log\frac{1}{r}},$$

Notice that, from Lemma 1.1 $\rho_{\log}(f,z_0)=\rho_{\log}(g)$. Therefore, all what can be said about the properties of the logarithmic order of meromorphic functions is also valid for those which are meromorphic in $\overline{\mathbb{C}}-\{z_0\}$.

Definition 1.20. *The logarithmic type and the lower logarithmic type near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ with $0<\mu_{\log}(f,z_0)\leq\rho_{\log}(f,z_0)<\infty$, are respectively defined by*

$$\tau_{\log}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{T_{z_0}(r,f)}{(\log\frac{1}{r})^{\rho_{\log}(f,z_0)}},\quad \underline{\tau}_{\log}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{T_{z_0}(r,f)}{(\log\frac{1}{r})^{\mu_{\log}(f,z_0)}}.$$

If $f(z)$ is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$, then

$$\tau_{\log,M}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log^+M_{z_0}(r,f)}{(\log\frac{1}{r})^{\rho_{\log}(f,z_0)}},\quad \underline{\tau}_{\log,M}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log^+M_{z_0}(r,f)}{(\log\frac{1}{r})^{\mu_{\log}(f,z_0)}}.$$

Definition 1.21. *The $[p,q]$-exponent of convergence of zeros and distinct zeros near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ are respectively defined by*

$$\lambda_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+N_{z_0}(r,\frac{1}{f})}{\log_q\frac{1}{r}},\qquad \overline{\lambda}_{[p,q]}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_p^+\overline{N}_{z_0}(r,\frac{1}{f})}{\log_q\frac{1}{r}},$$

In particular, $\lambda_{[p,1]}(f,z_0)=\lambda_p(f,z_0)$ is the iterated p-exponent of convergence of zeros near z_0 of $f(z)$ and $\overline{\lambda}_{[p,1]}(f,z_0)=\overline{\lambda}_p(f,z_0)$ is the iterated p-exponent of convergence of distinct zeros.

Definition 1.22. *The logarithmic exponent of convergence of zeros and distinct zeros near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ are given by*

$$\lambda_{\log}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log^+N_{z_0}(r,\frac{1}{f})}{\log\log\frac{1}{r}}-1,\qquad \overline{\lambda}_{\log}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log^+\overline{N}_{z_0}(r,\frac{1}{f})}{\log\log\frac{1}{r}}-1.$$

Chapter 2

Linear differential equations with finite or infinite order analytic coefficients in $\overline{\mathbb{C}} - \{z_0\}$

2.1 Introduction

In this chapter, we study the growth of the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{2.1}$$

where $k \geq 2$ and the coefficients $A_0(z), ..., A_{k-1}(z)$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. It is well known that if these coefficients are entire functions, then so are all solutions of (2.1). Unfortunately that is not the case in $\overline{\mathbb{C}} - \{z_0\}$, in other words, if the coefficients $A_0(z), ..., A_{k-1}(z)$ are analytic in $\overline{\mathbb{C}} - \{z_0\}$, then the equation (2.1) may have an non-analytic function in $\overline{\mathbb{C}} - \{z_0\}$ as a solution, that can be illustrated by the following example

Example 2.1. *Consider the linear differential equation*

$$f'' + \left(\exp_2 \left\{ \frac{1}{z_0 - z} \right\} + \frac{1}{z_0 - z} \right) f' + \frac{2}{z_0 - z} \exp_2 \left\{ \frac{1}{z_0 - z} \right\} f = 0. \tag{2.2}$$

The function $f(z) = (z_0 - z)^2$ solves (2.2), and $f(z)$ is not analytic in $\overline{\mathbb{C}} - \{z_0\}$.

Fettouch and Hamouda were behind the idea of investigating the growth of solution of (2.1) in $\overline{\mathbb{C}} - \{z_0\}$, such that they discussed the relationship between the growth of solutions and that of the coefficients in term of the order and the hyper order [32], they obtained the following result

Theorem 2.1 ([32]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Assume that*

$$\max\{\rho(A_j, z_0) : j = 1, \ldots, k-1\} < \rho(A_0, z_0) < +\infty.$$

Then every solution $f(z) \not\equiv 0$ that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (2.1)*, satisfies $\rho_2(f, z_0) = \rho(A_0, z_0)$.*

The above result is a $\overline{\mathbb{C}} - \{z_0\}$ counterpart to previous theorem obtained in the complex plane by Chen and Yang [18]. The concepts of $[p,q]$-order and the $[p,q]$-type of entire functions were firstly introduced by Juneja and his coauthors [44, 45], they have been used later as more general growth indicators to estimate the growth of solutions of differential equations (see e.g. [14, 30, 31, 51, 64, 78]). In [51], J. Liu and his coauthors extended the theorem of Chen and Yang for the cases when there are some coefficients of infinite order or when there are multiple coefficients with maximal finite $[p,q]$-order. This inspired Long and Zeng to prove similar results in $\overline{\mathbb{C}}-\{z_0\}$, by which they extended Theorem 2.1, such that they proved the following theorems.

Theorem 2.2 ([55]). *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\ldots,k-1\} < \rho_{[p,q]}(A_0,z_0) < +\infty.$$

Then every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1), satisfies $\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$.

Theorem 2.3 ([55]). *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\ldots,k-1\} \le \rho_{[p,q]}(A_0,z_0) < +\infty$$

and

$$\max\{\tau_{[p,q]}(A_j,z_0) : \rho_{[p,q]}(A_j,z_0)=\rho_{[p,q]}(A_0,z_0)>0\} < \tau_{[p,q]}(A_0,z_0) < +\infty.$$

Then every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1), satisfies $\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$.

Observe that in the above theorems, the dominance of the coefficient $A_0(z)$ is assumed in term of the $[p,q]$-order or the $[p,q]$-type. So, it is natural to ask what can be said about the growth of solutions of (2.1), if the dominance of $A_0(z)$ is assumed in term of the lower $[p,q]$-order or the lower $[p,q]$-type instead ? The aim of this chapter is to answer this question by proving the following theorems, which are also considered as extensions to the results obtained in [53].

2.2 Main Results

Theorem 2.4 ([25]). *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\ldots,k-1\} < \mu_{[p,q]}(A_0,z_0) \le \rho_{[p,q]}(A_0,z_0) < +\infty.$$

Then every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1), satisfies

$$\mu_{[p,q]}(A_0,z_0)=\mu_{[p+1,q]}(f,z_0) \le \rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0).$$

Theorem 2.5 ([25]). *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\ldots,k-1\} \le \mu_{[p,q]}(A_0,z_0) \le \rho_{[p,q]}(A_0,z_0)=\rho < +\infty$$

and

$$\begin{aligned}\tau_1 &= \max\{\tau_{[p,q]}(A_j,z_0) : \rho_{[p,q]}(A_j,z_0)=\mu_{[p,q]}(A_0,z_0)>0\}\\ &< \underline{\tau}_{[p,q]}(A_0,z_0)=\underline{\tau} < +\infty.\end{aligned}$$

Then every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\mu_{[p,q]}(A_0,z_0)=\mu_{[p+1,q]}(f,z_0)\le \rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$.

Theorem 2.6 ([25]). *Let $A_0(z),\dots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that*

$$\max\{\rho_{[p,q]}(A_j,z_0): j=1,\dots,k-1\} \leq \mu_{[p,q]}(A_0,z_0) < +\infty$$

and

$$\limsup_{r\longrightarrow 0} \frac{\sum_{j=1}^{k-1} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)} < 1.$$

Then every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1), satisfies

$$\mu_{[p,q]}(A_0,z_0) = \mu_{[p+1,q]}(f,z_0) \leq \rho_{[p+1,q]}(f,z_0) = \rho_{[p,q]}(A_0,z_0).$$

2.3 Lemmas

The following lemmas are important to prove our results.

Lemma 2.1 ([32]). *Let f be non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$, let $\kappa>0$ be given real constant and $j\in\mathbb{N}$. Then there exists a set $\mathcal{F}_1\subset(0,1)$ having finite logarithmic measure and a constant $C>0$ that depends on κ and j such that for all $|z-z_0|=r\in(0,1)\setminus\mathcal{F}_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{1}{r^2}T_{z_0}(\kappa r,f)\log T_{z_0}(\kappa r,f)\right]^j,$$

Lemma 2.2 ([55]). *Let $g:(0,1)\to\mathbb{R}$, $h:(0,1)\to\mathbb{R}$ be monotone decreasing functions such that $g(r)\geq h(r)$ possibly outside an exceptional set $\mathcal{F}_2\subset(0,1)$ that has finite logarithmic measure. Then for any given $\delta>1$, there exists a constant $0<r_0<1$, such that for all $r\in(0,r_0)$, we have $g(r^\delta)\geq h(r)$.*

Lemma 2.3 ([37]). *Let f be non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$. Then, there exists a set $\mathcal{F}_3\subset(0,1)$ that has finite logarithmic measure, such that for all $j=0,1,\dots,k$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1+o(1))\left(\frac{V_{z_0}(r,f)}{z_0-z_r}\right)^j,$$

as $r\longrightarrow 0$, $r\notin\mathcal{F}_3$, where z_r is a point in the circle $|z-z_0|=r$ that satisfies $|f(z_r)|=\max\{|f(z)|: |z-z_0|=r\}$.

Lemma 2.4. *Let f be non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\mu_{[p,q]}(f,z_0)=\mu<\infty$. Then there exists a set $E_1\subset(0,1)$ having infinite logarithmic measure such that for all $|z-z_0|=r\in E_1$, we have*

$$\mu = \lim_{r\longrightarrow 0}\frac{\log_p T_{z_0}(r,f)}{\log_q \frac{1}{r}} = \lim_{r\longrightarrow 0}\frac{\log_{p+1} M_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

and for any given $\varepsilon>0$ and all $|z-z_0|=r\in E_1$

$$M_{z_0}(r,f) \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu+\varepsilon}\right\}.$$

Proof. By the definition of the lower $[p,q]$-order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and

$$\lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(r_n, f)}{\log_q \frac{1}{r_n}} = \mu.$$

Therefore, there exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$ and for any $r \in [\frac{n}{n+1} r_n, r_n]$, we get

$$\lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(r_n, f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} \leq \lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}} \leq \lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(\frac{n}{n+1} r_n, f)}{\log_q \frac{1}{r_n}}.$$

Since

$$\lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(r_n, f)}{\log_q \frac{1}{\frac{n}{n+1} r_n}} = \lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(\frac{n}{n+1} r_n, f)}{\log_q \frac{1}{r_n}} = \mu,$$

then for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we get

$$\lim_{n\longrightarrow\infty} \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}} = \mu.$$

Set $E_1 = \bigcup_{n=n_0}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then for any given $\varepsilon > 0$ and $|z - z_0| = r \in E_1$

$$M_{z_0}(r, f) \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu+\varepsilon} \right\},$$

where

$$m_{\log}(E_1) = \sum_{n=n_0}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{1}{t} dt = \sum_{n=n_0}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty.$$

Similarly, we can prove the other result. □

Lemma 2.5 ([55]). *Let f be non-constant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then*

$$\rho_{[p,q]}(f, z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r, f)}{\log_q \frac{1}{r}},$$

Lemma 2.6 ([44]). *Let f be entire function. Then*

$$\mu_{[p,q]}(f) = \liminf_{r\longrightarrow\infty} \frac{\log_p V(r, f)}{\log_q r}.$$

Lemma 2.7. *Let f be non-constant analytic function in $\overline{\mathbb{C}} - \{z_0\}$. Then*

$$\mu_{[p,q]}(f, z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r, f)}{\log_q \frac{1}{r}},$$

Proof. Set $g(\omega) = f(z_0 - \frac{1}{\omega})$. From Remark 1.1, g is entire and we have

$$V_{z_0}(r, f) = V(R, g), \quad \text{where } R = \frac{1}{r}$$

this and Lemma 2.6 lead to

$$\mu_{[p,q]}(g) = \liminf_{r\longrightarrow\infty} \frac{\log_p V(R, g)}{\log_q r} = \liminf_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r, f)}{\log_q \frac{1}{r}}.$$

On the other hand, by Lemma 1.1, we have $T(R,g) = T_{z_0}(r,f)$, which implies that $\mu_{[p,q]}(g) = \mu_{[p,q]}(f,z_0)$. Therefore, we get

$$\mu_{[p,q]}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p^+ V_{z_0}(r,f)}{\log_q \frac{1}{r}}$$

□

Lemma 2.8. *Let f be non-constant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $0 < \mu_{[p,q]}(f,z_0) = \mu < \infty$ and $0 < \underline{\tau}_{[p,q]}(f,z_0) = \underline{\tau} < \infty$. Then there exists a set $E_2 \subset (0,1)$ having infinite logarithmic measure such that for all $|z-z_0| = r \in E_2$, we have*

$$M_{z_0}(r,f) < \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu}\right\}.$$

Proof. By the definition of lower $[p,q]$-order and lower $[p,q]$-type, there exists a sequence $\{r_m\}_{m=1}^{\infty}$ tending to 0 satisfying $r_{m+1} < \frac{m}{m+1} r_m$ and

$$\lim_{m\to+\infty} \frac{\log_p^+ M_{z_0}(r_m,f)}{\left(\log_{q-1}\frac{1}{r_m}\right)^{\mu}} = \underline{\tau}.$$

For any $r \in \left[\frac{m}{m+1} r_m, r_m\right]$, we have

$$\frac{\log_p^+ M_{z_0}(r,f)}{\left(\log_{q-1}\frac{1}{r}\right)^{\mu}} \le \frac{\log_p^+ M_{z_0}(\frac{m}{m+1} r_m, f)}{\left(\log_{q-1}\frac{1}{r_m}\right)^{\mu}}$$

$$= \frac{\log_p^+ M_{z_0}(\frac{m}{m+1} r_m, f)}{\left(\log_{q-1}\frac{1}{\frac{m}{m+1} r_m}\right)^{\mu}} \cdot \frac{\left(\log_q \frac{1}{\frac{m}{m+1} r_m}\right)^{\mu}}{\left(\log_{q-1}\frac{1}{r_m}\right)^{\mu}} \underset{m\to+\infty}{\longrightarrow} \underline{\tau}.$$

Then, for any given $\varepsilon > 0$, there exists a positive integer m_0 such that for all $m \ge m_0$ and for all $r \in \left[\frac{m}{m+1} r_m, r_m\right]$, we have

$$M_{z_0}(r,f) < \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu}\right\}.$$

Set $E_2 = \bigcup_{m=m_0}^{+\infty} \left[\frac{m}{m+1} r_m, r_m\right]$. Then for any given $\varepsilon > 0$ and all $|z-z_0| = r \in E_2$

$$M_{z_0}(r,f) < \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu}\right\},$$

where

$$m_{\log}(E_2) = \sum_{m=m_0}^{+\infty} \int_{\frac{m}{m+1} r_m}^{r_m} \frac{dt}{t} = \sum_{m=m_0}^{+\infty} \log\left(1+\frac{1}{m}\right) = +\infty.$$

□

Lemma 2.9 ([55]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then the following statements hold* *(i)* $T_{z_0}(r,\frac{1}{f}) = T_{z_0}(r,f)+O(1)$;

(ii) $T_{z_0}(r,f') < O\big(T_{z_0}(r,f)+\log\frac{1}{r}\big)$, $\quad r\in(0,r_0]\backslash\mathcal{F}_4$, *where* $\mathcal{F}_4\subset(0,r_0]$ *with* $m_{\log}(\mathcal{F}_4)<\infty$.

Lemma 2.10 ([55]). *Let f be non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho_{[p,q]}(f,z_0)=\rho<\infty$. Then there exists a set $E_3\subset(0,1)$ having infinite logarithmic measure such that for all $|z-z_0|=r\in E_3$, we have*

$$\rho = \lim_{r\longrightarrow 0}\frac{\log_p T_{z_0}(r,f)}{\log_q\frac{1}{r}} = \lim_{r\longrightarrow 0}\frac{\log_{p+1} M_{z_0}(r,f)}{\log_q\frac{1}{r}},$$

and for any given $\varepsilon>0$ and all $|z-z_0|=r\in E_3$

$$T_{z_0}(r,f)\geq \exp_p\left\{(\rho-\varepsilon)\log_q\frac{1}{r}\right\}.$$

2.4 Proofs of the theorems

Proof of Theorem 2.4

Proof. We only need to prove that every solution $f\not\equiv 0$ that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0)$, because we already have from Theorem 2.2, $\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$. We rewrite (2.1) as

$$|A_0(z)|\leq\left|\frac{f^{(k)}(z)}{f(z)}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{2.3}$$

Set $\max\{\rho_{[p,q]}(A_j,z_0): j=1,\ldots,k-1\}=\rho_1<\mu_{[p,q]}(A_0,z_0)$. Then for any given ε $(0<2\varepsilon<\mu_{[p,q]}(A_0,z_0)-\rho_1)$, there exists $r_1\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_1)$, we have

$$M_{z_0}(r,A_0)\geq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\} \tag{2.4}$$

and

$$M_{z_0}(r,A_j)\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\},\ (j=1,2,\cdots,k-1). \tag{2.5}$$

By Lemma 2.1, there exists a set $\mathcal{F}_1\subset(0,1)$ that has a finite logarithmic measure and a constant $C>0$ that depends on $\kappa>0$ and $j\in\mathbb{N}$ such that for all $r=|z-z_0|$ satisfying $r\in(0,1)\setminus\mathcal{F}_1$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq C\left[\frac{1}{r^2}T_{z_0}(\kappa r,f)\log T_{z_0}(\kappa r,f)\right]^j,\ (j=1,2,\cdots,k). \tag{2.6}$$

Substituting (2.4)-(2.6) into (2.3), for the above ε and $r\in(0,r_1)\setminus\mathcal{F}_1$, we have

$$\begin{aligned}&\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\\ \leq{}& Ck\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\}\left[\frac{1}{r^2}T_{z_0}(\alpha r,f)\log T_{z_0}(\kappa r,f)\right]^k.\end{aligned} \tag{2.7}$$

By (2.7), we get

$$\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\} \leq Ck\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_1+\varepsilon}\right\}\left[\frac{1}{r}T_{z_0}(\kappa r,f)\right]^{2k}, \tag{2.8}$$

for all $|z-z_0|=r\in(0,r_1)\setminus\mathcal{F}_1$ and $|A_0(z)|=M_{z_0}(r,A_0)$. By (2.8) and Lemma 2.2, we obtain $\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0)-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get

$$\mu_{[p+1,q]}(f,z_0)\geq\mu_{[p,q]}(A_0,z_0). \tag{2.9}$$

By (2.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|. \tag{2.10}$$

By Lemma 2.3, there exists a set $\mathcal{F}_3\subset(0,1)$ that has a finite logarithmic measure, such that for all $j=0,1,...,k$ and $r\notin E_3$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|=|1+o(1)|\left(\frac{V_{z_0}(r,f)}{r}\right)^j,\ r\longrightarrow0, \tag{2.11}$$

where z is a point in the circle $|z-z_0|=r$ that satisfies $|f(z)|=M_{z_0}(r,f)$.
By Lemma 2.4, there exists a set $E_1\subset(0,1)$ having infinite logarithmic measure, such that for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_1$, we have

$$|A_0(z)|\leq M_{z_0}(r,A_0)\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}. \tag{2.12}$$

By substituting (2.5), (2.11) and (2.12) into (2.10), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_1\cap(0,r_1)\setminus\mathcal{F}_3$, we get

$$|1+o(1)|\left(V_{z_0}(r,f)\right)^k\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|\left(V_{z_0}(r,f)\right)^{k-1}, \tag{2.13}$$

then we obtain

$$V_{z_0}(r,f)\leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|,\ r\in E_1\cap(0,r_1)\setminus\mathcal{F}_3. \tag{2.14}$$

By Lemma 2.2, Lemma 2.7 and (2.14), we get $\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain

$$\mu_{[p+1,q]}(f,z_0)\leq\mu_{[p,q]}(A_0,z_0), \tag{2.15}$$

from (2.9) and (2.15), we obtain $\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0)$. □

Proof of Theorem 2.5

Proof. By Theorem 2.3, we have $\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0)$. We only need to prove that $\mu_{[p+1,q]}(f,z_0)=\mu_{[p,q]}(A_0,z_0)$.

We set $\rho_2=\max\{\rho_{[p,q]}(A_j,z_0),\rho_{[p,q]}(A_j,z_0)<\mu_{[p,q]}(A_0,z_0): j=1,\ldots,k-1\}$. If $\rho_{[p,q]}(A_j,z_0)<\mu_{[p,q]}(A_0,z_0)$, then for any given ε $(0<2\varepsilon<\min\{\mu_{[p,q]}(A_0,z_0)-\rho_2,\underline{\tau}-\tau_1\}$, there exists $r_2\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_2)$, we have

$$M_{z_0}(r,A_j)\le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}$$

$$\le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\},\ (j=1,2,...,k-1). \tag{2.16}$$

If $\rho_{[p,q]}(A_j,z_0)=\mu_{[p,q]}(A_0,z_0)$, $\tau_{[p,q]}(A_j,z_0)\le\tau_1<\underline{\tau}=\underline{\tau}_{[p,q]}(A_0,z_0)$, then there exists $r_3\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_3)$, we have

$$M_{z_0}(r,A_j)\le \exp_p\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\} \tag{2.17}$$

and

$$M_{z_0}(r,A_0)\ge \exp_p\left\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}. \tag{2.18}$$

By substituting (2.6) and (2.16) -(2.18) into (2.10), we obtain

$$\begin{aligned}&\exp_p\left\{(\underline{\tau}-\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\\ &\le kC\exp_p\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\left[\frac{1}{r}T_{z_0}(\kappa r,f)\right]^{2k},\end{aligned} \tag{2.19}$$

for all $|z-z_0|=r\in(0,r_2)\cap(0,r_3)\setminus\mathfrak{F}_1$, $r\longrightarrow 0$ and $|A_0(z)|=M_{z_0}(r,A_0)$, where $C>0$ is a constant. By Lemma 2.2 and (2.19), we have

$$\mu_{[p+1,q]}(f,z_0)\ge\mu_{[p,q]}(A_0,z_0). \tag{2.20}$$

By Lemma 2.8, there exists a set $E_2\subset(0,1)$ having infinite logarithmic measure, such that for all $|z-z_0|=r\in E_2$, we have

$$|A_0(z)|\le M_{z_0}(r,A_0)\le \exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}. \tag{2.21}$$

By combining (2.10), (2.11), (2.16), (2.17) and (2.21), for all $|z-z_0|=r\in E_2\cap(0,r_2)\cap(0,r_3)\setminus\mathfrak{F}_3$, $r\longrightarrow 0$, we have

$$\begin{aligned}&|1+o(1)|\,(V_{z_0}(r,f))^k\\ &\le kr\exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}|1+o(1)|\,(V_{z_0}(r,f))^{k-1},\end{aligned}$$

so

$$V_{z_0}(r,f) \leq kr\exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}|1+o(1)|. \tag{2.22}$$

By Lemma 2.2, Lemma 2.7 and (2.10), we obtain

$$\mu_{[p+1,q]}(f,z_0) \leq \mu_{[p,q]}(A_0,z_0). \tag{2.23}$$

Thus, from (2.20) and (2.23) we have

$$\mu_{[p+1,q]}(f,z_0) = \mu_{[p,q]}(A_0,z_0).$$

□

Proof of Theorem 2.6

Proof. By (1.1), we have

$$m_{z_0}(r,A_0) \leq \sum_{j=1}^{k-1} m_{z_0}(r,A_j) + \sum_{j=1}^{k} m_{z_0}\left(r,\frac{f^{(j)}(z)}{f(z)}\right) + O(1). \tag{2.24}$$

By Lemma 2.9, for a constant $r_0 \in (0,1)$, there exists a set $\mathcal{F}_4 \subset (0,r_0]$ with $m_l(\mathcal{F}_4) < +\infty$ such that for all $|z-z_0| = r \in (0,r_0] \setminus \mathcal{F}_4$, we have

$$\sum_{j=1}^{k} m_{z_0}\left(r,\frac{f^{(j)}(z)}{f(z)}\right) \leq O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right). \tag{2.25}$$

Setting $\limsup_{r\longrightarrow 0}\frac{\sum_{j=1}^{k-1} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)} < \sigma < 1$. Then for $r \to 0$, we have

$$\sum_{j=1}^{k-1} m_{z_0}(r,A_j) < \sigma m_{z_0}(r,A_0). \tag{2.26}$$

By substituting (2.25) and (2.26) into (2.24), we obtain for all $|z-z_0| = r \in (0,r_0] \setminus \mathcal{F}_4$, $r \to 0$

$$(1-\sigma)m_{z_0}(r,A_0) \leq O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right). \tag{2.27}$$

By the definition of lower $[p,q]-$order, for any given $\varepsilon > 0$, there exists $r_4 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_4)$, we have

$$m_{z_0}(r,A_0) = T_{z_0}(r,A_0) \geq \exp_p\left\{\left(\mu_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\}. \tag{2.28}$$

By (2.27) and (2.28), for any given $\varepsilon > 0$ and $|z-z_0| = r \in (0,r_0] \cap (0,r_4) \setminus \mathcal{F}_4$, $r \to 0$, we obtain

$$(1-\sigma)\exp_p\left\{\left(\mu_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\} \leq O\left(T_{z_0}(r,f) + \log\frac{1}{r}\right). \tag{2.29}$$

By Lemma 2.2 and (2.29), we have $\mu_{[p+1,q]}(f,z_0) \geq \mu_{[p,q]}(A_0,z_0) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mu_{[p+1,q]}(f,z_0) \geq \mu_{[p,q]}(A_0,z_0). \tag{2.30}$$

Set $\max\{\rho_{[p,q]}(A_j,z_0) : j=1,\ldots,k-1\} = \rho_3 \leq \mu_{[p,q]}(A_0,z_0) \leq \rho_{[p,q]}(A_0,z_0)$. Then for any given $\varepsilon > 0$, there exists $r_5 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_5)$, we have

$$M_{z_0}(r,A_j) \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_3+\varepsilon}\right\}$$

$$\leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\},\ (j=1,2,\cdots,k-1). \tag{2.31}$$

By substituting (2.11), (2.12) and (2.31) into (2.10), for any given $\varepsilon > 0$ and for all $|z-z_0| = r \in E_1 \cap (0,r_5) \setminus \mathcal{F}_3$, we get

$$|1+o(1)|\,(V_{z_0}(r,f))^k \leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|\,(V_{z_0}(r,f))^{k-1}, \tag{2.32}$$

By (2.32), for above ε, we get

$$V_{z_0}(r,f) \leq kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|, \tag{2.33}$$

where $|z-z_0| = r \in E_1 \cap (0,r_5) \setminus \mathcal{F}_3$ and $|f(z)| = M_{z_0}(r,f)$. By (2.33), Lemma 2.2 and Lemma 2.4, we obtain $\mu_{[p+1,q]}(f,z_0) \leq \mu_{[p,q]}(A_0,z_0)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get

$$\mu_{[p+1,q]}(f,z_0) \leq \mu_{[p,q]}(A_0,z_0). \tag{2.34}$$

Thus, from (2.30) and (2.34) we have

$$\mu_{[p+1,q]}(f,z_0) = \mu_{[p,q]}(A_0,z_0).$$

By using similar method, from (2.27) we have for $|z-z_0| = r \in (0,r_0] \setminus \mathcal{F}_4$, $r \to 0$

$$(1-\sigma)T_{z_0}(r,A_0) = (1-\sigma)m_{z_0}(r,A_0) \leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{2.35}$$

By Lemma 2.10, there exists a set $E_3 \subset (0,1)$ having infinite logarithmic measure such that for any given $\varepsilon > 0$ and all $|z-z_0| = r \in E_3$

$$T_{z_0}(r,f) \geq \exp_p\left\{\left(\rho_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\}. \tag{2.36}$$

By substituting (2.36) into (2.35), we obtain for any given $\varepsilon > 0$ and all $|z-z_0| = r \in E_3 \cap (0,r_0] \setminus \mathcal{F}_4$, $r \to 0$

$$(1-\sigma)\exp_p\left\{\left(\rho_{[p,q]}(A_0,z_0)-\varepsilon\right)\log_q\frac{1}{r}\right\}$$

$$\leq (1-\sigma)T_{z_0}(r,A_0) \leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{2.37}$$

Making use of Lemma 2.2 and Lemma 2.4, from (2.37), we get

$$\rho_{[p+1,q]}(f,z_0) \geq \rho_{[p,q]}(A_0,z_0). \tag{2.38}$$

By the definition of the $[p,q]$-order of $\rho_{[p,q]}(A_0,z_0)$ for any given $\varepsilon>0$, there exists $r_6\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_6)$, we have

$$M_{z_0}(r,A_0)\le \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_{[p,q]}(A_0,z_0)+\varepsilon}\right\}. \tag{2.39}$$

Also by substituting (2.11), (2.31) and (2.39) into (2.10), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in(0,r_5)\cap(0,r_6)\setminus\mathfrak{F}_3$, we can find that

$$V_{z_0}(r,f)\le kr\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_{[p,q]}(A_0,z_0)+\varepsilon}\right\}|1+o(1)|. \tag{2.40}$$

By using Lemma 2.5, Lemma 2.2 and (2.40), we get

$$\rho_{[p+1,q]}(f,z_0)\le\rho_{[p,q]}(A_0,z_0), \tag{2.41}$$

from (2.38) and (2.41), we conclude that

$$\rho_{[p+1,q]}(f,z_0)=\rho_{[p,q]}(A_0,z_0).$$

□

2.5 Examples

Example 2.1 $f(z)=\exp_3\left\{\frac{1}{(z_0-z)^{2n+1}}\right\}$ solves the following equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{2.42}$$

where

$$A_0(z)=-\frac{(2n+1)^2}{(z_0-z)^{4n+4}}\exp\left\{2\exp\left(\frac{1}{(z_0-z)^{2n+1}}+\frac{2}{(z_0-z)^{2n+1}}\right)\right\}$$

and

$$A_1(z)=\frac{2n+1}{(z_0-z)^{2n+2}}\exp\left\{\frac{1}{(z-z_0)^{2n+1}}\right\}+\frac{2n+1}{(z_0-z)^{2n+2}}+\frac{2n+2}{z_0-z}.$$

We have

$$\rho_{[2,1]}(A_1,z_0)=0<\mu_{[2,1]}(A_0,z_0)=\rho_{[2,1]}(A_0,z_0)=2n+1$$

Obviously, the conditions of Theorem 2.4 are satisfied and we see that

$$\mu_{[2,1]}(A_0,z_0)=\rho_{[2,1]}(A_0,z_0)=\mu_{[3,1]}(f,z_0)=\rho_{[3,1]}(f,z_0)=2n+1.$$

Example 2.2 $f(z)=\frac{1}{(z_0-z)^n}\exp_2\left\{\frac{1}{(z_0-z)^{n+1}}\right\}$ solves the following equation

$$f'''+A_2(z)f''+A_1(z)f'+A_0(z)f=0, \tag{2.43}$$

where

$$A_0(z)=\frac{n(n+1)^2}{(z_0-z)^{3n+6}}\exp\left\{\frac{3}{(z_0-z)^{n+1}}\right\}$$

$$+\left(\frac{(n+1)^2(3n+2)}{(z_0-z)^{3n+6}}+\frac{(n+1)\left(5n^2+7n+3\right)}{(z_0-z)^{2n+5}}\right)\exp\left\{\frac{2}{(z_0-z)^{n+1}}\right\}$$

$$+\left(\frac{(n+1)^3}{(z_0-z)^{3n+6}}+\frac{6(n+1)^3}{(z_0-z)^{2n+5}}+\frac{6(n+1)^3}{(z_0-z)^{n+4}}\right)\exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\}$$
$$+\frac{n(n+1)(n+2)}{(z_0-z)^3},$$

$$A_1(z)=-\frac{(n+1)^2}{(z_0-z)^{n+3}}\exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\}$$

and

$$A_2(z)=\frac{1}{(z_0-z)^{n+2}}\exp\left\{\frac{1}{(z_0-z)^{n+1}}\right\}.$$

We have

$$\max\left\{\rho_{[1,1]}(A_2,z_0),\rho_{[1,1]}(A_1,z_0)\right\}=\max\{n+1,n+1\}=n+1$$
$$=\mu_{[1,1]}(A_0,z_0)=\rho_{[1,1]}(A_0,z_0)$$

and

$$\max\left\{\tau_{[1,1]}(A_2,z_0),\tau_{[1,1]}(A_1,z_0)\right\}=1<\underline{\tau}_{[1,1]}(A_0,z_0)=3.$$

It is clear that the conditions of Theorem 2.5 are satisfied and we see that

$$\mu_{[1,1]}(A_0,z_0)=\rho_{[1,1]}(A_0,z_0)=\mu_{[2,1]}(f,z_0)=\rho_{[2,1]}(f,z_0)=n+1.$$

Example 2.3 $f(z)=\exp_2\left\{\frac{1}{2(z_0-z)}\right\}$ is a solution to equation (2.43) for the following coefficients

$$A_0(z)=\frac{1}{8(z_0-z)^6}\exp\left\{\frac{3}{2(z_0-z)}\right\},$$

$$A_1(z)=\left(-\frac{3}{(z_0-z)^3}-\frac{1}{2(z_0-z)^4}\right)\exp\left\{\frac{1}{2(z_0-z)}\right\}-\frac{2}{(z_0-z)^3}-\frac{6}{(z_0-z)^2}$$

and

$$A_2(z)=\frac{1}{2(z_0-z)^2}.$$

We have

$$\max\rho_{[1,1]}\left\{(A_2,z_0),\rho_{[1,1]}(A_1,z_0)\right\}=\max\{0,1\}=1$$
$$=\mu_{[1,1]}(A_0,z_0)=\rho_{[1,1]}(A_0,z_0),$$
$$\limsup_{r\longrightarrow 0}\frac{m_{z_0}(r,A_2)+m_{z_0}(r,A_1)}{m_{z_0}(r,A_0)}=\frac{1}{3}<1.$$

Obviously the conditions of Theorem 2.6 are verified and we see that

$$\mu_{[1,1]}(A_0,z_0)=\rho_{[1,1]}(A_0,z_0)=\mu_{[2,1]}(f,z_0)=\rho_{[2,1]}(f,z_0)=1.$$

Chapter 3

Linear differential equations with zero order analytic or meromorphic coefficients in $\overline{\mathbb{C}}-\{z_0\}$ part 1

3.1 Introduction

In this chapter, we discuss another case where the order fails to estimate the growth of solutions of equation (2.1), which requires to use different growth indicators. Furthermore, we will also consider the growth and oscillation of the following non-homogeneous differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{3.1}$$

where $A_j(z)$ $(j=0,1,\ldots,k-1)$ and $F(z)$ are analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Under different hypotheses on the coefficients of equation (2.1), Fettouch and Hamouda obtained the following results on the iterated order

Theorem 3.1 ([33]). *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite iterated order with $\max\left\{\rho_p(A_j,z_0): j\neq 0\right\}\leq\rho_p(A_0,z_0)=\rho<+\infty$, $1<p<\infty$ and $E\subset(0,1)$ be a set of infinite logarithmic measure such that for some constants $0\leq\beta<\alpha$ and any given $\varepsilon>0$, we have*

$$|A_0(z)|\geq\exp_p\left\{\frac{\alpha}{r^{\rho-\varepsilon}}\right\},$$

$$|A_j(z)|\leq\exp_p\left\{\frac{\beta}{r^{\rho-\varepsilon}}\right\},\qquad j=1,...,k-1,$$

as $r\to 0$ with $r\in E$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2.1) satisfies $\rho_{p+1}(f,z_0)=\rho_p(A_0,z_0)=\rho$.

Cherief and Hamouda in [18] considered (2.1) for the case when the coefficients are meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$, where they obtained the following theorem on the hyper order.

Theorem 3.2 ([18]). *Let* $A_0(z),...,A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}-\{z_0\}$ *satisfying* $\max\left\{\rho(A_j,z_0): j\neq 0\right\} < \rho(A_0,z_0)$ *with*

$$\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)} > 0.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}-\{z_0\}$ *of* (2.1) satisfies $\rho(A_0,z_0)\leq\rho_2(f,z_0)$.

As a generalization to Theorem 3.1 to the non-homogeneous case and also as $\overline{\mathbb{C}}-\{z_0\}$ counterpart of a theorem obtained by Jin and his coauthors in [67], Fettouch and Hamouda proved the following theorem on the iterated order and the iterated exponent of convergence of zeroes of solutions to equation (3.1)

Theorem 3.3 ([33]). *Let* $A_0(z),...,A_{k-1}(z)$ *satisfy the hypotheses of* Theorem 3.1, *and let* $F(z)\not\equiv 0$ *be an analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *with* $i(F)=q$.

i) If $q<p+1$ *or* $q=p+1$, $\rho_{p+1}(F,z_0)<\rho_p(A_0,z_0)$, *then every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) *satisfies* $\overline{\lambda}_{p+1}(f,z_0)=\lambda_{p+1}(f,z_0)=\rho_{p+1}(f,z_0)=\rho_p(A_0,z_0)$, *with at most one exceptional solution* f_0 *satisfying* $i(f_0)<p+1$ *or* $\rho_{p+1}(f,z_0)<\rho_p(A_0,z_0)$

ii) If $q>p+1$ *or* $q=p+1$, $\rho_p(A_0,z_0)<\rho_{p+1}(F,z_0)<+\infty$, *then every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (3.1) *satisfies* $i(f)=q$ *and* $\rho_q(f,z_0)=\rho_q(F,z_0)$.

It is obvious that the above results do not include the case when the coefficients are analytic or meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ of order zero, , in fact not even in the results of the precedent chapter. The main purpose of this chapter is to consider this case, such that we use the logarithmic order and the logarithmic lower order as growth indicators, our results are extensions of the above theorems and also $\overline{\mathbb{C}}\backslash\{z_0\}$ counterparts to some previous results in the complex plane $\mathbb{C}$ (see [15,29]).

3.2 Main Results

Theorem 3.4 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *such that for real constants* $\alpha,\beta,\upsilon,\theta_1$ *and* θ_2 *with* $0\leq\beta<\alpha,\upsilon\geq 1,\theta_1<\theta_2$ *satisfying*

$$|A_0(z)|\geq\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\upsilon}\right\},$$

$$|A_j(z)|\leq\exp\left\{\beta\left(\log\frac{1}{r}\right)^{\upsilon}\right\},\qquad j=1,...,k-1,$$

where $\arg(z_0-z)=\theta\in(\theta_1,\theta_2)$ *and* $|z_0-z|=r\to 0$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $\rho_{[2,2]}(f,z_0)\geq\upsilon-1$ with $\rho_{[2,2]}(f,z_0)\geq\upsilon>1$.

Theorem 3.5 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *and* $E\subset(0,1)$ *be a set of infinite logarithmic measure such that*

$$|A_0(z)|\geq\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\upsilon}\right\},$$

$$|A_j(z)| \le \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\upsilon}\right\}, \qquad j=1,...,k-1,$$

with $0 \le \beta < \alpha, \upsilon \ge 1$ *and* $|z_0 - z| = r \to 0,\ r \in E$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $\rho_{[2,2]}(f,z_0) \ge \upsilon - 1$ with $\rho_{[2,2]}(f,z_0) \ge \upsilon > 1$

Theorem 3.6 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\max\left\{\rho_{\log}(A_j,z_0) : j \ne 0\right\} \le \rho_{\log}(A_0,z_0) = \rho$ *and* $E \subset (0,1)$ *be a set of infinite logarithmic measure such that for some constants* $0 \le \beta < \alpha$ *and any given* $\varepsilon > 0$, *we have*

$$|A_0(z)| \ge \exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\rho-\varepsilon}\right\},$$

$$|A_j(z)| \le \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\rho-\varepsilon}\right\}, \qquad j=1,...,k-1,$$

as $r \to 0$ *with* $r \in E$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $\rho_{\log}(A_0,z_0) - 1 \le \rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_0,z_0) = \rho$ with $\rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) = \rho > 1$

Theorem 3.7 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\max\left\{\rho_{\log}(A_j,z_0) : j \ne 0\right\} < \rho_{\log}(A_0,z_0) = \rho$. *Then, every analytic solution* $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $\rho_{\log}(A_0,z_0) - 1 \le \rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_0,z_0)$ with $\rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) > 1$

Theorem 3.8 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with*

$$\max\left\{\rho_{\log}(A_j,z_0) : j \ne 0\right\} \le \rho_{\log}(A_0,z_0) = \rho,$$

$$\max\left\{\tau_{\log,M}(A_j,z_0) : \rho_{\log}(A_j,z_0) = \rho,\ j \ne 0\right\} < \tau_{\log,M}(A_0,z_0) = \tau \le +\infty,$$

Then, every analytic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $\rho_{\log}(A_0,z_0) - 1 \le \rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_0,z_0)$ with $\rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) > 1$

Theorem 3.9. *Let* $A_0(z),...A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\max\left\{\sigma_{\log}(A_j,z_0) : j \ne 0\right\} \le \mu_{\log}(A_0,z_0) \le \sigma_{\log}(A_0,z_0)$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),\, j\ne 0} \tau_{\log}(A_j,z_0) < \underline{\tau}_{\log}(A_0,z_0).$$

Then, every analytic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $0 \le \mu_{\log}(A_0,z_0) - 1 \le \mu_{[2,2]}(f,z_0) \le \mu_{\log}(A_0,z_0)$, with $1 < \mu_{\log}(A_0,z_0) = \mu_{[2,2]}(f,z_0) \le \sigma_{[2,2]}(f,z_0) = \sigma_{\log}(A_0,z_0) = \overline{\lambda}_{[2,2]}(f-\varphi,z_0) = \lambda_{[2,2]}(f-\varphi,z_0)$, where $\varphi(z)(\not\equiv 0)$ is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\sigma_{[2,2]}(\varphi,z_0) < \mu_{\log}(A_0,z_0)$.

Theorem 3.10. *Let* $A_0(z),...A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\max\left\{\sigma_{\log}(A_j,z_0) : j \ne s\right\} \le \mu_{\log}(A_0,z_0) \le \sigma_{\log}(A_0,z_0)$ *and*

$$\limsup_{r\longrightarrow 0} \frac{\sum_{j\ne 0} m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)} < 1.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $0 \le \mu_{\log}(A_0,z_0) - 1 \le \mu_{[2,2]}(f,z_0) \le \mu_{\log}(A_0,z_0)$, with $1 < \mu_{\log}(A_0,z_0) = \mu_{[2,2]}(f,z_0) \le \sigma_{[2,2]}(f,z_0) = \sigma_{\log}(A_0,z_0) = \overline{\lambda}_{[2,2]}(f-\varphi,z_0) = \lambda_{[2,2]}(f-\varphi,z_0)$, where $\varphi(z)(\not\equiv 0)$ is an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ satisfying $\sigma_{[2,2]}(\varphi,z_0) < \mu_{\log}(A_0,z_0)$.

Theorem 3.11. *Let* $A_0(z),...A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)}=\delta>0$, $\max\left\{\sigma_{\log}(A_j,z_0):j\neq 0\right\}\leq\mu_{\log}(A_0,z_0)\leq\sigma_{\log}(A_0,z_0)$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),j\neq 0}\tau_{\log}(A_j,z_0)<\delta\underline{\tau}_{\log}(A_0,z_0).$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $0\leq\mu_{\log}(A_0,z_0)-1\leq\mu_{[2,2]}(f,z_0)$, with $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$.

Theorem 3.12. *Let* $A_0(z),...A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)}=\delta>0$ *and*

$$\limsup_{r\longrightarrow 0}\frac{\sum_{j\neq s}m_{z_0}(r,A_j)}{m_{z_0}(r,A_0)}<1.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $0\leq\mu_{\log}(A_0,z_0)-1\leq\mu_{[2,2]}(f,z_0)$, with $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$.

Theorem 3.13. *Let* $A_0(z),...A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of finite logarithmic order with* $\lambda_{\log}(\frac{1}{A_0},z_0)+1<\mu_{\log}(A_0,z_0)$, $\max\left\{\sigma_{\log}(A_j,z_0):j\neq 0\right\}\leq\mu_{\log}(A_0,z_0)\leq\sigma_{\log}(A_0,z_0)$ *and*

$$\sum_{\sigma_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),j\neq 0}\tau_{\log}(A_j,z_0)<\underline{\tau}_{\log}(A_0,z_0).$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ *in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *of* (2.1) satisfies $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$.

Theorem 3.14 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *satisfy the hypotheses of* Theorem 3.7 and let $F(z)(\not\equiv 0)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$.

i) If $\rho_{\log}(A_0,z_0)<\rho_{[2,2]}(F,z_0)<+\infty$, then, every analytic solution $f(z)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0)=\rho_{[2,2]}(F,z_0)$.

ii) If $\rho_{\log}(A_0,z_0)>\rho_{[2,2]}(F,z_0)$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_0,z_0)$, and that $\rho_{[2,2]}(f,z_0)\geq\rho_{\log}(A_0,z_0)-1$ with at most one exceptional solution, and that $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\rho_{[2,2]}(f,z_0)$ holds for every solution f which satisfies $\rho_{[2,2]}(f,z_0)=\rho_{\log}(A_0,z_0)$.

Theorem 3.15 ([27]). *Let* $A_0(z),...,A_{k-1}(z)$ *satisfy the hypotheses of* Theorem 3.8 and let $F(z)(\not\equiv 0)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$.

i) If $\rho_{\log}(A_0,z_0)<\rho_{[2,2]}(F,z_0)<+\infty$, then, every analytic solution $f(z)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0)=\rho_{[2,2]}(F,z_0)$.

ii) If $\rho_{\log}(A_0,z_0)>\rho_{[2,2]}(F,z_0)$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_0,z_0)$, and that $\rho_{[2,2]}(f,z_0)\geq\rho_{\log}(A_0,z_0)-1$ with at most one exceptional solution, and that $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\rho_{[2,2]}(f,z_0)$ holds for every solution f which satisfies $\rho_{[2,2]}(f,z_0)=\rho_{\log}(A_0,z_0)$.

3.3 Lemmas

Lemma 3.1. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\rho_{\log}(f)=\rho$. Then there exists a subset E_1 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0|=r\in E_1$, we have*

$$\rho=\lim_{r\longrightarrow 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}$$

and for any given $\varepsilon>0$

$$M_{z_0}(r,f)>\exp\left\{\left(\log\frac{1}{r}\right)^{\rho-\varepsilon}\right\}.$$

Proof. By the definition of the logarithmic order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1}<\frac{n}{n+1}r_n$ and

$$\rho=\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{r_n}}.$$

Then, for any given $\varepsilon>0$, there exists an $n_0\in\mathbb{N}^+$ such that for all $n\geq n_0$ and for any $r\in[\frac{n}{n+1}r_n,r_n]$, we obtain

$$\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{\frac{n}{n+1}r_n}}\leq\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}\leq\frac{\log\log M_{z_0}(\frac{n}{n+1}r_n,f)}{\log\log\frac{1}{r_n}}.$$

Since

$$\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(r_n,f)}{\log\log\frac{1}{\frac{n}{n+1}r_n}}=\lim_{n\longrightarrow\infty}\frac{\log\log M_{z_0}(\frac{n}{n+1}r_n,f)}{\log\log\frac{1}{r_n}}=\rho,$$

then for any $r\in[\frac{n}{n+1}r_n,r_n]$, we get

$$\lim_{r\longrightarrow 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}=\rho. \tag{3.2}$$

Setting $E_1=\bigcup\limits_{n=n_0}^{\infty}[\frac{n}{n+1}r_n,r_n]$, then $m_{\log}(E_1)=\sum\limits_{n=n_0}^{\infty}\int_{\frac{n}{n+1}r_n}^{r_n}\frac{dt}{t}=\sum\limits_{n=n_0}^{\infty}\log\left(1+\frac{1}{n}\right)=\infty$. It follows from (3.2), for any given $\varepsilon>0$

$$M_{z_0}(r,f)>\exp\left\{\left(\log\frac{1}{r}\right)^{\rho-\varepsilon}\right\}.$$

□

Lemma 3.2. *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order with max$\{\rho_{\log}(A_j,z_0):j=0,...,k-1\}\leq\alpha_1<+\infty$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of* (2.1) satisfies $\rho_{[2,2]}(f,z_0)\leq\alpha_1$.

Proof. Let $f(z)(\not\equiv 0)$ be an analytic solution of (2.1) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By Lemma 2.3, there exists a set $\mathfrak{F}_3\subset(0,1)$ of finite logarithmic measure such that, for all $r\notin\mathfrak{F}_3$ and $r\to 0$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)}=\left(\frac{V_{z_0}(r,f)}{z_0-z_r}\right)^j\big(1+o(1)\big),\qquad j=0,...,k. \tag{3.3}$$

Setting

$$M_{z_0}(r)=\max_{|z_0-z|=r}\big\{|A_j(z)|:j=0,1,...,k-1\big\}. \tag{3.4}$$

Since $\max\left\{\rho_{\log}(A_j,z_0) : j=0,...,k-1\right\} \leq \alpha_1 < +\infty$, then for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $r_0 > r > 0$, we get

$$M_{z_0}(r) \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha_1+\varepsilon}\right\}. \tag{3.5}$$

Now, we may rewrite (2.1) as

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + \left|A_1(z)\right|\left|\frac{f'(z)}{f(z)}\right| + \left|A_0(z)\right|. \tag{3.6}$$

Then, by substituting (3.3) and (3.4) into (3.6), we obtain

$$\left(\frac{V_{z_0}(r,f)}{r}\right)^k \left|1+o(1)\right| \leq kM_{z_0}(r)\left(\frac{V_{z_0}(r,f)}{r}\right)^{k-1}\left|1+o(1)\right|. \tag{3.7}$$

It follows by (3.5) and (3.7)

$$V_{z_0}(r,f) \leq kr\exp\left\{\left(\log\frac{1}{r}\right)^{\alpha_1+\varepsilon}\right\}\left|1+o(1)\right|. \tag{3.8}$$

Therefore, by Lemma 2.5 we get $\rho_{[2,2]}(f,z_0) \leq \alpha_1$. □

Lemma 3.3. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with finite logarithmic order $1 \leq \rho_{\log}(f,z_0) = \rho < +\infty$ and finite logarithmic type $0 < \tau_{\log,M}(f,z_0) < +\infty$. Then there exists a subset E_2 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0| = r \in E_2$, we have*

$$\tau_{\log,M}(f,z_0) = \lim_{r\longrightarrow 0}\frac{\log M_{z_0}(r,f)}{(\log\frac{1}{r})^\rho}$$

and for any given $\beta < \tau_{\log,M}(f,z_0)$

$$M_{z_0}(r,f) > \exp\left\{\beta\left(\log\frac{1}{r}\right)^\rho\right\}.$$

Proof. Similarly, as in the proof of Lemma 3.1. By the definition of the logarithmic type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1}r_n$ and

$$\lim_{n\longrightarrow\infty}\frac{\log M_{z_0}(r_n,f)}{(\log\frac{1}{r_n})^\rho} = \tau_{\log,M}(f,z_0).$$

So, for any given $\varepsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$ and for any $r \in [\frac{n}{n+1}r_n, r_n]$, we have

$$\frac{\log M_{z_0}(r_n,f)}{(\log\frac{1}{\frac{n}{n+1}r_n})^\rho} \leq \frac{\log M_{z_0}(r,f)}{(\log\frac{1}{r})^\rho} \leq \frac{\log M_{z_0}(\frac{n}{n+1}r_n,f)}{(\log\frac{1}{r_n})^\rho}.$$

Since

$$\lim_{n\longrightarrow\infty}\frac{\log M_{z_0}(r_n,f)}{(\log\frac{1}{\frac{n}{n+1}r_n})^\rho} = \lim_{n\longrightarrow\infty}\frac{\log M_{z_0}(\frac{n}{n+1}r_n,f)}{(\log\frac{1}{r_n})^\rho} = \tau_{\log}(f,z_0),$$

then for any $r \in [\frac{n}{n+1}r_n, r_n]$, we obtain

$$\lim_{r\longrightarrow 0}\frac{\log M_{z_0}(r,f)}{(\log\frac{1}{r})^\rho} = \tau_{\log,M}(f,z_0). \tag{3.9}$$

By (3.9), for any given $\beta < \tau_{\log,M}(f,z_0)$, we get

$$M_{z_0}(r,f) > \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\rho}\right\}.$$

Set $E_2 = \bigcup_{n=n_0}^{\infty}[\frac{n}{n+1}r_n, r_n]$, then $m_{\log}(E_2) = \sum_{n=n_0}^{\infty}\int_{\frac{n}{n+1}r_n}^{r_n}\frac{dt}{t} = \sum_{n=n_0}^{\infty}\log\left(1+\frac{1}{n}\right) = \infty.$ □

Lemma 3.4 ([18]). *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let $k \in \mathbb{N}$. Then*

$$m_{z_0}\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right), \qquad \text{for all } r \in (0,1)\backslash\mathcal{F}_1 \quad \text{with } m_{\log}(\mathcal{F}_1) < \infty.$$

If f is of finite order, then

$$m_{z_0}\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = O\left(\log\frac{1}{r}\right), \qquad r \notin \mathcal{F}_1.$$

Lemma 3.5. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\mu = \mu_{\log}(f)$. Then there exists a set E_3 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0| = r \in E_3$, we have*

$$\lim_{r\longrightarrow 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}} = \mu.$$

Proof. The proof is similar to the proof of Lemma 2.4. Here we omit it. □

Lemma 3.6. *Let $A_0(z),...A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order . If there exists an integer $(0 \le l \le k-1)$ such that $1 \le \max\left\{\mu_{\log}(A_l,z_0), \rho_{\log}(A_j,z_0) : j \ne l\right\} \le \alpha_2$. Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of* (2.1) satisfies $\mu_{[2,2]}(f,z_0) \le \alpha_2$.

Proof. Suppose that $f(z)(\not\equiv 0)$ is an analytic solution of (2.1) in $\overline{\mathbb{C}}\backslash\{z_0\}$. By (2.1), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| = \left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + \left|A_l(z)\right|\left|\frac{f^{(l)}(z)}{f(z)}\right| + \cdots + \left|A_0(z)\right|. \tag{3.10}$$

Since $\max\left\{\rho_{\log}(A_j,z_0) : j = 0,...,k-1, j \ne l\right\} \le \alpha_2 < +\infty$. Then for any given $\varepsilon > 0$, there exists $r_1 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_1)$, we have

$$\left|A_j(z)\right| \le M_{z_0}(r,A_j) \le \exp\left\{\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)+\varepsilon}\right\} \le \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha_2+\varepsilon}\right\}, \quad j \ne l. \tag{3.11}$$

By Lemma 3.5, there exists a set $E_3 \subset (0,1)$ that has infinite logarithmic measure such that, for any given $\varepsilon > 0$ and for all $r \in E_3$ we get

$$\left|A_l(z)\right| \le M_{z_0}(r,A_l) \le \exp\left\{\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_l,z_0)+\varepsilon}\right\} \le \exp\left\{\left(\log\frac{1}{r}\right)^{\alpha_2+\varepsilon}\right\}. \tag{3.12}$$

Substituting (3.3), (3.11) and (3.12) into (3.10), for any given $\varepsilon > 0$ and for all $r \in E_3 \cap (0,r_1) \setminus \mathcal{F}_3$, we obtain

$$V_{z_0}(r,f) \le Cr\exp\left\{\left(\log\frac{1}{r}\right)^{\alpha+\varepsilon}\right\}\left(1+o(1)\right), \qquad C > 0. \tag{3.13}$$

It follows by (3.13) and Lemma 2.7 that, $\mu_{[2,2]}(f,z_0) \le \alpha_2$. □

Lemma 3.7 ([72]). *Assume $f \not\equiv 0$ is a solution of* (2.1), set $g = f - \varphi$, then g satisfies

$$g^{(k)} + A_{k-1}g^{(k-1)} + \cdots + A_1 g' + A_0 g = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi\right]. \tag{3.14}$$

Lemma 3.8. *[6] Let f be a meromorphic function in $\mathbb{C}$ with $p \geq q \geq 1$. Then*

$$\rho_{[p,q]}(f') = \rho_{[p,q]}(f).$$

Lemma 3.9. *Let f be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $p \geq q \geq 1$. Then*

$$\rho_{[p,q]}(f^{(n)}, z_0) = \rho_{[p,q]}(f, z_0), \qquad n \in \mathbb{N}.$$

Proof. It is sufficient to prove that $\rho_{[p,q]}(f', z_0) = \rho_{[p,q]}(f, z_0)$. By Lemma 1.1 , $g(\omega) = f(z_0 - \frac{1}{\omega})$ is meromorphic in $\mathbb{C}$ and $\rho_{[p,q]}(g) = \rho_{[p,q]}(f, z_0)$. By Lemma 3.8 we have $\rho_{[p,q]}(g') = \rho_{[p,q]}(g)$ where $f'(z) = \frac{1}{\omega^2}g'(\omega)$. Set $h(\omega) = \frac{1}{\omega^2}g'(\omega)$. Clearly $\rho_{[p,q]}(h) = \rho_{[p,q]}(g')$. In the other hand by Lemma 1.1, we have $\rho_{[p,q]}(h) = \rho_{[p,q]}(f', z_0)$. So, we deduce that $\rho_{[p,q]}(f, z_0) = \rho_{[p,q]}(f', z_0)$. □

Lemma 3.10. *Let $F(z) \not\equiv 0$, $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ and let f be a non-constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of* (3.1) satisfying

$$\max\left\{\rho_{[2,2]}(F, z_0), \rho_{[2,2]}(A_j, z_0) : (j = 0, ..., k-1)\right\} < \rho_{[2,2]}(f, z_0).$$

Then $\overline{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \rho_{[2,2]}(f, z_0) = \rho_{\log}(A_0, z_0)$.

Proof. We may rewrite (3.1) as

$$\frac{1}{f(z)} = \frac{1}{F(z)}\left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)} + A_0(z)\right). \tag{3.15}$$

By Lemma 2.9 and (3.15) we get

$$\begin{aligned} T_{z_0}(r, f) &= T_{z_0}\big(r, \frac{1}{f}\big) + O(1) \\ &= m_{z_0}\big(r, \frac{1}{f}\big) + N_{z_0}\big(r, \frac{1}{f}\big) + O(1) \\ &\leq \sum_{j=0}^{k-1} m_{z_0}(r, A_j) + \sum_{j=1}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f}\right) + m_{z_0}\big(r, \frac{1}{F}\big) + N_{z_0}\big(r, \frac{1}{f}\big) + O(1). \end{aligned} \tag{3.16}$$

From (3.1), it is easy to see that if f has a zero at z_1 of order $m\,(m > k)$, then F must have a zero at z_1 of order at least $m - k$. Hence

$$n\big(r, \frac{1}{f}\big) \leq k\overline{n}\big(r, \frac{1}{f}\big) + n\big(r, \frac{1}{F}\big)$$

and

$$N_{z_0}\big(r, \frac{1}{f}\big) \leq k\overline{N}_{z_0}\big(r, \frac{1}{f}\big) + N_{z_0}\big(r, \frac{1}{F}\big). \tag{3.17}$$

Again by Lemma 2.9, there exists a set $\mathcal{F}_4 \subset (0, r_0]$ that has a finite logarithmic measure such for all $|z_0 - z| = r \in (0, r_0]\backslash\, \mathcal{F}_4$, we obtain

$$\sum_{j=1}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f}\right) = O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right) \leq \frac{1}{2}T_{z_0}(r, f). \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16), we get

$$\frac{1}{2}T_{z_0}(r,f) \leq k\overline{N}_{z_0}\left(r,\frac{1}{f}\right) + T_{z_0}(r,F) + \sum_{j=0}^{k-1} T_{z_0}(r,A_j) + O(1). \tag{3.19}$$

This implies that $\rho_{[2,2]}(f,z_0) \leq \max\left\{\overline{\lambda}_{[2,2]}(f,z_0), \rho_{[2,2]}(F,z_0), \rho_{[2,2]}(A_j,z_0) : (j=0,...,k-1)\right\}$. Since

$$\max\left\{\rho_{[2,2]}(F,z_0), \rho_{[2,2]}(A_j,z_0) : (j=0,...,k-1)\right\} < \rho_{[2,2]}(f,z_0),$$

then we obtain $\rho_{[2,2]}(f,z_0) \leq \overline{\lambda}_{[2,2]}(f,z_0)$. On the other hand, by definition we have $\overline{\lambda}_{[2,2]}(f,z_0) \leq \lambda_{[2,2]}(f,z_0) \leq \rho_{[2,2]}(f,z_0)$, therefore

$$\rho_{[2,2]}(f,z_0) = \overline{\lambda}_{[2,2]}(f,z_0) = \lambda_{[2,2]}(f,z_0).$$

□

3.4 Proofs of the theorems

Proof of Theorem 3.4

Proof. We assume that f is a non constant analytic solution of (2.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. By the hypotheses of Theorem 3.4, for real constants $0 \leq \beta < \alpha, \upsilon \geq 1$ and $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ with $|z_0 - z| = r \to 0$, we have

$$|A_0(z)| \geq \exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\upsilon}\right\} \tag{3.20}$$

and

$$|A_j(z)| \leq \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\upsilon}\right\}, \qquad j = 1,...,k-1. \tag{3.21}$$

By Lemma 2.1, there exists a subset $\mathcal{F}_1 \subset (0,1)$ having finite logarithmic measure and a constant $C > 0$ that depends only on κ, such for all $r \notin \mathcal{F}_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C\left[\frac{1}{r}T_{z_0}(\kappa r, f)\right]^{2j}, \qquad j = 1,...,k. \tag{3.22}$$

Substituting (3.20) - (3.22) into (2.3), for all $r \notin \mathcal{F}_1$ and $r \to 0$, we obtain

$$\exp\left\{\alpha\left(\log\frac{1}{r}\right)^{\upsilon}\right\} \leq kC\left[\frac{1}{r}T_{z_0}(\kappa r, f)\right]^{2k} \exp\left\{\beta\left(\log\frac{1}{r}\right)^{\upsilon}\right\}. \tag{3.23}$$

From (3.23), it follows

$$\exp\left\{(\alpha-\beta)\left(\log\frac{1}{r}\right)^{\upsilon}\right\} \leq kC\left[\frac{1}{r}T_{z_0}(\kappa r, f)\right]^{2k}. \tag{3.24}$$

We conclude from (3.24) that $\rho_{[2,2]}(f,z_0) \geq \upsilon - 1$ with $\rho_{[2,2]}(f,z_0) \geq \upsilon > 1$. □

Proof of Theorem 3.5

Proof. By the hypotheses of Theorem 3.5, there exists a set $E \subset (0,1)$ of infinite logarithmic measure such that, for all $r \in E$ and $r \to 0$, (3.20) and (3.21) hold. Then, similarly as in (3.20)-(3.24) in the proof of Theorem 3.1, for all $r \in E \setminus \mathcal{F}_1$ and $r \to 0$, we get (3.24) holds which implies $\rho_{[2,2]}(f,z_0) \geq \upsilon - 1$ with $\rho_{[2,2]}(f,z_0) \geq \upsilon > 1$. □

Proof of Theorem 3.6

Proof. First, by Theorem 3.5, we can obtain $\rho_{[2,2]}(f,z_0) \geq \rho - 1 - \varepsilon$ and $\rho_{[2,2]}(f,z_0) \geq \rho - \varepsilon > 1$. Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_{[2,2]}(f,z_0) \geq \rho_{\log}(A_0,z_0) - 1 = \rho - 1 \quad \text{and} \quad \rho_{[2,2]}(f,z_0) \geq \rho_{\log}(A_0,z_0) = \rho > 1 \tag{3.25}$$

By the definition of $\rho_{\log}(A_j,z_0)$, for any given $\varepsilon > 0$ and $r \to 0$, we have

$$|A_j(z)| \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\rho+\varepsilon}\right\}, \qquad j = 0,\ldots,k-1. \tag{3.26}$$

By Lemma 2.3, there exists a set $\mathcal{F}_3 \subset (0,1)$ of finite logarithmic measure such that, for all $r \notin \mathcal{F}_3$ and $r \to 0$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j (1+o(1)), \qquad j = 0,\ldots,k, \tag{3.27}$$

where $|f(z_r)| = M_{z_0}(r,f) = \max_{|z-z_0|=r}|f(z)|$. Substituting (3.26) and (3.27) into (2.10), we get

$$\left(\frac{V_{z_0}(r)}{r}\right)^k \left|1+o(1)\right| \leq k\exp\left\{\left(\log\frac{1}{r}\right)^{\rho+\varepsilon}\right\}\left(\frac{V_{z_0}(r)}{r}\right)^{k-1}\left|1+o(1)\right|, \tag{3.28}$$

it follows

$$V_{z_0}(r) \leq kr\exp\left\{\left(\log\frac{1}{r}\right)^{\rho+\varepsilon}\right\}\left|1+o(1)\right|. \tag{3.29}$$

This implies that

$$\rho_{[2,2]}(f,z_0) \leq \rho_{\log}(A_0,z_0) = \rho. \tag{3.30}$$

From (3.25) and (3.30), we obtain

$$\rho_{\log}(A_0,z_0) - 1 \leq \rho_{[2,2]}(f,z_0) \leq \rho_{\log}(A_0,z_0) \quad \text{with} \quad \rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) > 1. \tag{3.31}$$

□

Proof of Theorem 3.7

Proof. Set $\max\left\{\rho_{\log}(A_j,z_0) : j \neq 0\right\} < \rho_0 < \rho_1 < \rho_{\log}(A_0,z_0)$. For any given $\varepsilon > 0$, there exists a $r_0 > 0$ such that for all $r_0 \geq r > 0$, we have

$$|A_j(z)| \leq \exp\left\{\left(\log\frac{1}{r}\right)^{\rho_0+\varepsilon}\right\}, \qquad j = 1,\ldots,k-1. \tag{3.32}$$

For $\rho_0+\varepsilon<\rho_1<\rho_{\log}(A_0,z_0)$, by Lemma 3.1, there exists a set $E_1\subset(0,1)$ of infinite logarithmic measure such that, for all $r\in E_1$ and $|A_0(z)|=M_{z_0}(r,A_0)$, we have

$$|A_0(z)|>\exp\left\{\left(\log\frac{1}{r}\right)^{\rho_1}\right\}. \tag{3.33}$$

Substituting (3.22), (3.31) and (3.32) into (2.3), for all $r\in E_1\backslash\mathfrak{F}_1$, we obtain

$$\exp\left\{\left(\log\frac{1}{r}\right)^{\rho_1}\right\}\le kC\left[\frac{1}{r}T_{z_0}(\lambda r,f)\right]^{2k}\exp\left\{\left(\log\frac{1}{r}\right)^{\rho_0+\varepsilon}\right\}. \tag{3.34}$$

From (3.34), we get

$$\rho_{[2,2]}(f,z_0)\ge\rho_1-1\ge 0\quad\text{with}\quad\rho_{[2,2]}(f,z_0)\ge\rho_1>1. \tag{3.35}$$

Further, by (3.35) and Lemma 3.2, we have $0\le\rho_1-1\le\rho_{[2,2]}(f,z_0)\le\rho_{\log}(A_0,z_0)$ with $1<\rho_1\le\rho_{[2,2]}(f,z_0)\le\rho_{\log}(A_0,z_0)$ which hold for each $\rho_1<\rho_{\log}(A_0,z_0)$. Thus, we obtain $\rho_{\log}(A_0,z_0)-1\le\rho_{[2,2]}(f,z_0)\le\rho_{\log}(A_0,z_0)$ and $1<\rho_{\log}(A_0,z_0)=\rho_{[2,2]}(f,z_0)$. □

Proof of Theorem 3.8

Proof. Let β_0 and β_1 be two constants such that $\max\{\tau_{\log,M}(A_j,z_0):\rho_{\log}(A_j,z_0)=\rho_{\log}(A_0,z_0)=\rho,\, j\neq 0\}<\beta_0<\beta_1<\tau_{\log,M}(A_0,z_0)$. If $\rho_{\log}(A_j,z_0)<\rho_{\log}(A_0,z_0)=\rho$, then there exists a r_0 such that for all $r_0\ge r>0$ and for any given $\varepsilon>0$, (3.32) holds. If $\rho_{\log}(A_j,z_0)=\rho_{\log}(A_0,z_0)=\rho$, then by the definition of $\tau_{\log,M}(A_j,z_0)$ for any given $\varepsilon>0$ and for sufficiently small r, we get

$$|A_j(z)|\le\exp\left\{\beta_0\left(\log\frac{1}{r}\right)^{\rho}\right\},\qquad j=1,\ldots,k-1. \tag{3.36}$$

By Lemma 3.3, there exists a set $E_2\subset(0,1)$ of infinite logarithmic measure such that, for all $r\in E_2$ and $|A_0(z)|=M_{z_0}(r,A_0)$, we obtain

$$|A_0(z)|\ge\exp\left\{\beta_1\left(\log\frac{1}{r}\right)^{\rho}\right\}. \tag{3.37}$$

Substituting (3.22), (3.32), (3.36) and (3.37) into (2.3), for all $r\in E_2\backslash\mathfrak{F}_1$, we get

$$\exp\left\{\beta_1\left(\log\frac{1}{r}\right)^{\rho}\right\}\le kC\left[\frac{1}{r}T_{z_0}(\lambda r,f)\right]^{2k}\exp\left\{\beta_0\left(\log\frac{1}{r}\right)^{\rho}\right\}. \tag{3.38}$$

This implies that

$$\rho_{[2,2]}(f,z_0)\ge\rho-1=\rho_{\log}(A_0,z_0)-1\quad\text{with}\quad\rho_{[2,2]}(f,z_0)\ge\rho=\rho_{\log}(A_0,z_0)>1. \tag{3.39}$$

Then, by (3.39) and Lemma 3.2, we have $\rho_{\log}(A_0,z_0)-1\le\rho_{[2,2]}(f,z_0)\le\rho_{\log}(A_0,z_0)$ and $1<\rho_{\log}(A_0,z_0)=\rho_{[2,2]}(f,z_0)$. □

Proof of Theorem 3.9

Proof. Suppose that $f(z)$ is an analytic solution of (2.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. By (2.24) and Lemma 3.4, there exists a set $\mathcal{F}_1 \subset (0,1)$, having finite logarithmic measure , such that for all $r \in (0,1) \setminus \mathcal{F}_1$, we have

$$m_{z_0}(r,A_0(z)) \leq O\left(\log T_{z_0}(r,f(z)) + \log\frac{1}{r}\right) + \sum_{j=1}^{k-1} m_{z_0}(r,A_j(z)), \tag{3.40}$$

which means

$$T_{z_0}(r,A_0(z)) = m_{z_0}(r,A_0(z)) \leq O\left(\log T_{z_0}(r,f(z)) + \log\frac{1}{r}\right) + \sum_{j=1}^{k-1} T_{z_0}(r,A_j(z)). \tag{3.41}$$

First, we assume $\rho = \max\left\{\rho_{\log}(A_j,z_0) : j = 1,...,k-1\right\} < \mu_{\log}(A_0,z_0) = \mu$. Then for any given $0 < 2\varepsilon < \mu-\rho$, there exists $r_1 \in (0,1)$ such that for all $|z_0 - z| = r \in (0,r_1)$, we get

$$T_{z_0}(r,A_0) \geq \left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \tag{3.42}$$

and

$$T_{z_0}(r,A_j) \leq \left(\log\frac{1}{r}\right)^{\rho+\varepsilon}, \quad j = 1,...,k-1. \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.41), for the above ε and for all $|z_0 - z| = r \in (0,r_1) \setminus \mathcal{F}_1$, we obtain

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right) + (k-1)\left(\log\frac{1}{r}\right)^{\rho+\varepsilon}. \tag{3.44}$$

Then, by $0 < 2\varepsilon < \mu-\rho$, we have

$$\big(1-o(1)\big)\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right). \tag{3.45}$$

This implies that, $\mu - 1 - \varepsilon \leq \mu_{[2,2]}(f,z_0)$. Since ε is arbitrary, we obtain $0 \leq \mu_{\log}(A_0,z_0) - 1 \leq \mu_{[2,2]}(f,z_0)$. On the other hand, by Lemma 3.6, we have $\mu_{[2,2]}(f,z_0) \leq \mu_{\log}(A_0,z_0)$. Thus, we get that every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2.1) satisfies $\mu_{\log}(A_0,z_0) - 1 \leq \mu_{[2,2]}(f,z_0) \leq \mu_{\log}(A_0,z_0)$. Furthermore, If $\mu_{\log}(A_0,z_0) > 1$ then, by (3.45) and Lemma 3.6, we obtain $\mu_{[2,2]}(f,z_0) = \mu_{\log}(A_0,z_0)$. Now we prove that $\overline{\lambda}_{[2,2]}(f-\varphi,z_0) = \lambda_{[2,2]}(f-\varphi,z_0) = \rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) > 1$. Set $g = f-\varphi$. By Theorem 3.7 and since $\varphi(z)(\not\equiv 0)$ satisfies $\rho_{[2,2]}(\varphi,z_0) < \mu_{\log}(A_0,z_0) \leq \rho_{\log}(A_0,z_0)$, then we have $\rho_{[2,2]}(g,z_0) = \rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0) \geq \mu_{\log}(A_0,z_0) > 1$. By Lemma 3.7 g satisfies (3.14), Set $G = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi$. If $G \equiv 0$, then by the first part of the proof (or by Theorem 3.7, we get $\rho_{[2,2]}(\varphi,z_0) \geq \mu_{[2,2]}(\varphi,z_0) = \mu_{\log}(A_0,z_0)$, which is a contradiction, thus $G \not\equiv 0$. Since $G \not\equiv 0$ then by Lemma 3.9, we have $\rho_{[2,2]}(G,z_0) \leq \rho_{[2,2]}(\varphi,z_0) < \mu_{\log}(A_0,z_0) \leq \rho_{\log}(A_0,z_0) = \rho_{[2,2]}(g,z_0)$. By Lemma 3.10, we obtain $\overline{\lambda}_{[2,2]}(g,z_0) = \lambda_{[2,2]}(g,z_0) = \rho_{[2,2]}(g,z_0)$. Then, we deduce that $\overline{\lambda}_{[2,2]}(f-\varphi,z_0) = \lambda_{[2,2]}(f-\varphi,z_0) = \rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0)$.

Now we assume that $\max\left\{\rho_{\log}(A_j,z_0) : j \neq 0\right\} = \mu_{\log}(A_0,z_0) = \mu$ and

$$\tau_1 = \sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),\, j\neq 0} \tau_{\log}(A_j,z_0) < \underline{\tau}_{\log}(A_0,z_0) = \underline{\tau}.$$

Then, there exists a set $J \subseteq \{j = 1,...,k\}$, such that for $j \in J$, we get $\rho_{\log}(A_j, z_0) = \mu_{\log}(A_0, z_0) = \mu$ with $\tau_1 = \sum_{j\in J} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}$, where for $j \in \{j = 1,...,k\} \setminus J$, we have $\rho_{\log}(A_j, z_0) < \mu_{\log}(A_0, z_0) = \mu$. Hence, for any given ε $\left(0 < \varepsilon < \frac{\underline{\tau}-\tau_1}{k}\right)$, there exists a constant $r_2 \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_2)$, we have

$$T_{z_0}(r, A_j) \le \left(\tau_{\log}(A_j, z_0) + \varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu}, \quad j \in J, \tag{3.46}$$

$$T_{z_0}(r, A_j) \le \left(\log\frac{1}{r}\right)^{\rho_0}, \quad j \in \{j = 1,...,k\} \setminus J, \quad 0 < \rho_0 < \mu \tag{3.47}$$

and

$$T_{z_0}(r, A_0) \ge \left(\underline{\tau} - \varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu}. \tag{3.48}$$

By substituting (3.46)-(3.48) into (3.41), for the above ε and for all $|z_0 - z| = r \in (0, r_2) \setminus \mathcal{F}_1$, we get

$$\begin{aligned}
\left(\underline{\tau} - \varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} &\le O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right) + \sum_{j\in J}\left(\tau_{\log}(A_j, z_0) + \varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\
&\quad + \sum_{j\in\{j=1,...,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\rho_0} \\
&\le O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right) + \left(\tau_1 + (k-1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \\
&\quad + (k-1)\left(\log\frac{1}{r}\right)^{\rho_0}
\end{aligned} \tag{3.49}$$

and so

$$\left(1 - o(1)\right)\left(\underline{\tau} - \tau_1 - k\varepsilon\right)\left(\log\frac{1}{r}\right)^{\mu} \le O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.50}$$

By (3.50), it follows that

$$0 \le \mu_{\log}(A_0, z_0) - 1 \le \mu_{[2,2]}(f, z_0). \tag{3.51}$$

From (3.51) and Lemma 3.6, we conclude that every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (2.1) satisfies $\mu_{\log}(A_0, z_0) - 1 \le \mu_{[2,2]}(f, z_0) \le \mu_{\log}(A_0, z_0)$. Furthermore, If $\mu_{\log}(A_0, z_0) > 1$ then, by (3.50) and Lemma 3.6, we get $\mu_{[2,2]}(f, z_0) = \mu_{\log}(A_0, z_0)$. We prove that $\overline{\lambda}_{[2,2]}(f - \varphi, z_0) = \lambda_{[2,2]}(f - \varphi, z_0) = \rho_{[2,2]}(f, z_0) = \rho_{\log}(A_0, z_0) > 1$, similarly as in the proof for the first case. □

Proof of Theorem 3.10

Proof. We assume that $\limsup_{r\longrightarrow 0} \frac{\sum_{j=1}^{k-1} m_{z_0}(r, A_j)}{m_{z_0}(r, A_0)} < \sigma < 1$. Then for $r \to 0$, we have

$$\sum_{j=0, j\neq s}^{k-1} m_{z_0}(r, A_j) < \sigma m_{z_0}(r, A_0). \tag{3.52}$$

Substituting (3.52) into (3.40), for all $r \in (0,1) \setminus \mathcal{F}_1$, we obtain

$$(1-\sigma)T_{z_0}(r, A_0) = (1-\sigma)m_{z_0}(r, A_0) \le O\left(\log T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.53}$$

By the definition of $\mu_{\log}(A_0,z_0)=\mu$, for any given $\varepsilon>0$ there exists $r_3\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_3)$, (3.42) holds. Then by Substituting (3.42) into (3.53), for any given $\varepsilon>0$ and for all $r\in(0,r_3)\setminus\mathcal{F}_1$, we get

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon}\leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right), \tag{3.54}$$

which implies that, $\mu-1-\varepsilon\leq\mu_{[2,2]}(f,z_0)$. Since ε is arbitrary, we obtain

$$0\leq\mu_{\log}(A_0,z_0)-1\leq\mu_{[2,2]}(f,z_0). \tag{3.55}$$

It follows by (3.55) and Lemma 3.6 that $\mu_{\log}(A_0,z_0)-1\leq\mu_{[2,2]}(f,z_0)\leq\mu_{\log}(A_0,z_0)$. Moreover, If $\mu_{\log}(A_0,z_0)>1$ then, by (3.54) and Lemma 3.6, we get $\mu_{[2,2]}(f,z_0)=\mu_{\log}(A_0,z_0)$. Similarly as in the proof of Theorem 3.9 we prove that $\overline{\lambda}_{[2,2]}(f-\varphi,z_0)=\lambda_{[2,2]}(f-\varphi,z_0)=\rho_{[2,2]}(f,z_0)=\rho_{\log}(A_0,z_0)>1$. □

Proof of Theorem 3.11

Proof. Suppose that $f(z)$ is a meromorphic solution of (2.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. As in the proof of Theorem 3.9, first, if $\rho=\max\left\{\rho_{\log}(A_j,z_0): j=1,...,k-1\right\}<\mu_{\log}(A_0,z_0)=\mu$. Then for any given $0<2\varepsilon<\mu-\rho$, there exists $r_4\in(0,1)$, such that for all $|z_0-z|=r\in(0,r_4)$, (3.43) holds. By the condition $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)}=\delta>0$ and the definition of $\mu_{\log}(A_0,z_0)=\mu$, for the above ε, there exists $r_5\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_5)$, we have

$$m_{z_0}(r,A_0)\geq\frac{\delta}{2}T_{z_0}(r,A_0)\geq\frac{\delta}{2}\left(\log\frac{1}{r}\right)^{\mu-\frac{\varepsilon}{2}}\geq\left(\log\frac{1}{r}\right)^{\mu-\varepsilon}. \tag{3.56}$$

By substituting (3.43) and (3.56) into (3.41), for any given ε $(0<2\varepsilon<\mu-\rho)$, and for all $|z_0-z|=r\in(0,r_4)\cap(0,r_5)\setminus\mathcal{F}_1$, we obtain

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon}\leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right)+(k-1)\left(\log\frac{1}{r}\right)^{\rho+\varepsilon}. \tag{3.57}$$

that is

$$(1-o(1))\left(\log\frac{1}{r}\right)^{\mu-\varepsilon}\leq O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{3.58}$$

It follows that, $0\leq\mu-1-\varepsilon\leq\mu_{[2,2]}(f,z_0)$ and $1<\mu-\varepsilon\leq\mu_{[2,2]}(f,z_0)$. Since ε is arbitrary, we get $0\leq\mu_{\log}(A_0,z_0)-1\leq\mu_{[2,2]}(f,z_0)$ with $1<\mu_{\log}(A_0,z_0)\leq\mu_{[2,2]}(f,z_0)$. Next if $\max\left\{\rho_{\log}(A_j,z_0): j\neq 0\right\}=\mu_{\log}(A_0,z_0)=\mu$ and $\tau_1=\sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0),j\neq0}\tau_{\log}(A_j,z_0)<\delta\underline{\tau}_{\log}(A_0,z_0)=\delta\underline{\tau}$. Then, by the condition $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_0)}{T_{z_0}(r,A_0)}=\delta>0$ and the definition of $\underline{\tau}_{\log}(A_0,z_0)=\underline{\tau}$, for any given $\varepsilon>0$, there exists a constant $r_6\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_6)$, we have

$$\begin{aligned}m_{z_0}(r,A_0)\geq(\delta-\varepsilon)T_{z_0}(r,A_0)&\geq(\delta-\varepsilon)(\underline{\tau}-\varepsilon)\left(\log\frac{1}{r}\right)^{\mu}\\&\geq(\delta\underline{\tau}-(\underline{\tau}+1)\varepsilon)\left(\log\frac{1}{r}\right)^{\mu}.\end{aligned} \tag{3.59}$$

For any given ε $\big(0 < (\underline{\tau}+k)\varepsilon < \delta\underline{\tau} - \tau_1\big)$, there exists $r_7 \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_7)$, (3.46) and (3.47) hold. Then, by substituting (3.46), (3.47) and (3.21) into (3.59), for the above ε and for all $|z_0 - z| = r \in (0, r_6) \cap (0, r_7) \setminus \mathcal{F}_1$, we get

$$\begin{aligned}\big(\delta\underline{\tau} - (\underline{\tau}+1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} &\leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right) + \sum_{j\in J}\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + \sum_{j\in\{j=1,\dots,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\rho_0} \\ &\leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right) + \big(\tau_1 + (k-1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + (k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}.\end{aligned} \tag{3.60}$$

So

$$\big(1-o(1)\big)\big(\delta\underline{\tau} - \tau_1 - (\underline{\tau}+k)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right). \tag{3.61}$$

By (3.61), we obtain $0 \leq \mu_{\log}(A_0,z_0) - 1 \leq \mu_{[2,2]}(f,z_0)$ with $1 < \mu_{\log}(A_0,z_0) \leq \mu_{[2,2]}(f,z_0)$.

□

Proof of Theorem 3.12

Proof. Let $f(z)$ be a meromorphic solution of (2.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. For any given $\varepsilon > 0$, there exists $r_8 \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_8)$, (3.52) and (3.56) hold. Then, by combining (3.40) (3.52) and (3.56),for any given $\varepsilon > 0$ and for all $|z_0 - z| = r \in (0, r_8) \setminus \mathcal{F}_1$, we have

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right), \tag{3.62}$$

It follows that $0 \leq \mu_{\log}(A_0,z_0) - 1 \leq \mu_{[2,2]}(f,z_0)$ and $1 < \mu_{\log}(A_0,z_0) \leq \mu_{[2,2]}(f,z_0)$. □

Proof of Theorem 3.13

Proof. Let $f(z)$ be a meromorphic solution of (2.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$. By (3.40), for all $r \in (0,1) \setminus \mathcal{F}_1$, we obtain

$$\begin{aligned}T_{z_0}(r,A_0(z)) &= m_{z_0}(r,A_0(z)) + N_{z_0}(r,A_0(z)) \\ &\leq O\left(\log T_{z_0}(r,f) + \log\frac{1}{r}\right) + \sum_{j=1}^{k-1} T_{z_0}(r,A_j) + N_{z_0}(r,A_0(z)).\end{aligned} \tag{3.63}$$

Also as in the proof of Theorem 3.9, first, if $\rho = \max\big\{\rho_{\log}(A_j,z_0) : j = 1,\dots,k-1\big\} < \mu_{\log}(A_0,z_0) = \mu$. Then for any given ε $\big(0 < 2\varepsilon < \mu - \rho\big)$, there exists $r_9 \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_9)$, (3.42) and (3.43) hold. By the definition of $\lambda_{\log}(\frac{1}{A_0}, z_0) = \lambda$, for any given ε $\big(0 < 2\varepsilon < \mu - \lambda - 1\big)$, there exists $r_{10} \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_{10})$, we have

$$N_{z_0}(r,A_0) \leq \left(\log\frac{1}{r}\right)^{\lambda_{\log}(\frac{1}{A_0},z_0)+1+\varepsilon}. \tag{3.64}$$

By substituting (3.42), (3.43) and (3.64) into (3.63), for sufficiently small ε satisfying $0 < 2\varepsilon < \min\{\mu-\sigma, \mu-\lambda-1\}$ and for all $|z_0 - z| = r \in (0, r_9) \cap (0, r_{10}) \setminus \mathfrak{F}_1$, we get

$$\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\Big(\log T_{z_0}(r,f) + \log\frac{1}{r}\Big) + (k-1)\left(\log\frac{1}{r}\right)^{\rho+\varepsilon} + \left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon}. \tag{3.65}$$

It follows that

$$\big(1-o(1)\big)\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} \leq O\big(\log T_{z_0}(r,f)\big), \tag{3.66}$$

that is $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. Now, if $\max\{\rho_{\log}(A_j, z_0) : j \neq 0\} = \mu_{\log}(A_0, z_0) = \mu$ and $\tau_1 = \sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_0,z_0), j\neq 0} \tau_{\log}(A_j, z_0) < \underline{\tau}_{\log}(A_0, z_0) = \underline{\tau}$. Then, for any given ε $\left(0 < \varepsilon < \frac{\underline{\tau}-\tau_1}{k}\right)$, there exists a constant $r_{11} \in (0,1)$, such that for all $|z_0 - z| = r \in (0, r_{11})$, (3.46), (3.47) and (3.48) hold. By substituting (3.46)-(3.48) and (3.64) into (3.63), for sufficiently small ε satisfying $0 < \varepsilon < \min\left\{\frac{\underline{\tau}-\tau_1}{k}, \frac{\mu-\lambda-1}{2}\right\}$ and for all $|z_0 - z| = r \in (0, r_{10}) \cup (0, r_{11}) \setminus \mathfrak{F}_1$, we obtain

$$\begin{aligned}(\underline{\tau}-\varepsilon)\left(\log\frac{1}{r}\right)^{\mu} &\leq O\Big(\log T_{z_0}(r,f) + \log\frac{1}{r}\Big) + \sum_{j\in J}\big(\tau_{\log}(A_j, z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + \sum_{j\in\{j=1,\dots,k\}\setminus J}\left(\log\frac{1}{r}\right)^{\rho_0} + \left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon} \\ &\leq O\Big(\log T_{z_0}(r,f) + \log\frac{1}{r}\Big) + \big(\tau_1 + (k-1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \\ &\quad + (k-1)\left(\log\frac{1}{r}\right)^{\rho_0} + \left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon}.\end{aligned} \tag{3.67}$$

So

$$\big(1-o(1)\big)\big(\underline{\tau}-\tau_1-k\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu} \leq O\big(\log T_{z_0}(r,f)\big). \tag{3.68}$$

From (3.68), we deduce that $1 < \mu_{\log}(A_0, z_0) \leq \mu_{[2,2]}(f, z_0)$. □

Proof of Theorem 3.14

Proof. Let f be an analytic solution in $\overline{\mathbb{C}}\setminus\{z_0\}$ of (3.1). Then f can be represented as

$$f(z) = B_1(z)f_1(z) + B_2(z)f_2(z) + \dots + B_k(z)f_k(z), \tag{3.69}$$

where $f_1, f_2, \dots, f_k$ is solution base of (2.1) (the homogeneous corresponding equation of (3.1)) and $B_1, B_2, \dots, B_k$ are suitable analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ determined by the following system of equations

$$\begin{cases} B_1'(z)f_1(z) + B_2'(z)f_2(z) + \cdots + B_k'(z)f_k(z) = 0 \\ B_1'(z)f_1'(z) + B_2'(z)f_2'(z) + \cdots + B_k'(z)f_k'(z) = 0 \\ \vdots \\ B_1'(z)f_1^{(k-1)}(z) + B_2'(z)f_2^{(k-1)}(z) + \cdots + B_k'(z)f_k^{(k-1)}(z) = F. \end{cases} \tag{3.70}$$

By (3.70), for $j=1,...,k$, we obtain

$$B'_j = F.G_j(f_1,f_2,...,f_k).W(f_1,f_2,...,f_k)^{-1}, \tag{3.71}$$

where

$$W(f_1,f_2,...,f_k) = \begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_k(z) \\ f'_1(z) & f'_2(z) & \cdots & f'_k(z) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(k-1)}(z) & f_2^{(k-1)}(z) & \cdots & f_k^{(k-1)}(z) \end{vmatrix}$$

is the Wronksian of $f_1,f_2,...,f_k$ and $G_j(f_1,f_2,...,f_k)$ is differential polynomial of $f_1,f_2,...,f_k$ and their derivatives with constant coefficients.

From (3.71) and Lemma 3.9, for $j=1,...,k$, we get

$$\rho_{[2,2]}(C_j,z_0) = \rho_{[2,2]}(C'_j,z_0) \le \max\left\{\rho_{[2,2]}(F,z_0), \rho_{[2,2]}(G_j(f_1,f_2,...,f_k),z_0), \rho_{[2,2]}(W(f_1,f_2,...,f_k),z_0)\right\}. \tag{3.72}$$

By Theorem 3.7 and the fact that $G_j(f_1,f_2,...,f_k)$ and $W(f_1,f_2,...,f_k)$ are both differential polynomial of $f_1,f_2,...,f_k$ and their derivatives with constant coefficients, we have

$$\max\left\{\rho_{[2,2]}(G_j(f_1,f_2,...,f_k),z_0), \rho_{[2,2]}(W(f_1,f_2,...,f_k),z_0)\right\} \le \rho_{[2,2]}(f_j,z_0) \le \rho_{\log}(A_0,z_0). \tag{3.73}$$

By (3.69), (3.72) and (3.73), for $j=1,...,k$, we obtain

$$\begin{aligned} \rho_{[2,2]}(f,z_0) &\le \max\left\{\rho_{[2,2]}(f_j,z_0), \rho_{[2,2]}(C_j,z_0)\right\} \\ &\le \max\left\{\rho_{[2,2]}(F,z_0), \rho_{\log}(A_0,z_0)\right\}. \end{aligned} \tag{3.74}$$

i) If $\rho_{[2,2]}(F,z_0) > \rho_{\log}(A_0,z_0)$, then from (3.1) and (3.74), we deduce that $\rho_{[2,2]}(f,z_0) = \rho_{[2,2]}(F,z_0)$.

ii) If $\rho_{[2,2]}(F,z_0) < \rho_{\log}(A_0,z_0)$, then it follows by (3.74) that $\rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_0,z_0)$. Now, we assert that all solutions f of the equation (3.1) satisfy $\rho_{[2,2]}(f,z_0) \ge \rho_{\log}(A_0,z_0) - 1$ with at most one exception. In fact, if there exist two distinct analytic solutions g_1 and g_2 of (3.1) satisfying $\rho_{[2,2]}(g_j,z_0) < \rho_{\log}(A_0,z_0) - 1$, $(j=1,2)$, then $g = g_1 - g_2$ is a nonzero analytic solution of (2.1) and satisfies $\rho_{[2,2]}(g,z_0) = \rho_{[2,2]}(g_1-g_2,z_0) < \rho_{\log}(A_0,z_0) - 1$. But by Theorem 3.7, we have $\rho_{[2,2]}(g,z_0) = \rho_{[2,2]}(g_1-g_2,z_0) \ge \rho_{\log}(A_0,z_0) - 1$. This is a contradiction. Further, if f is an analytic solution of (3.1) that satisfies $\rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0)$, then

$$\max\left\{\rho_{[2,2]}(F,z_0), \rho_{[2,2]}(A_j,z_0) : j=0,1,...,k-1\right\} < \rho_{[2,2]}(f,z_0).$$

So, the assumption of Lemma 3.10 also holds and therefore $\overline{\lambda}_{[2,2]}(f,z_0) = \lambda_{[2,2]}(f,z_0) = \rho_{[2,2]}(f,z_0) = \rho_{\log}(A_0,z_0)$.

□

Proof of Theorem 3.15

Proof. By using similar discussions as in the proof of Theorem 3.14, we obtain the assertions of Theorem 3.15. □

3.5 Examples

Here we give some examples to illustrate the sharpness of some assertions in our theorems.

Example 3.1. *For* Theorem 3.7*, we consider the analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$

$$f(z)=\frac{1}{(z-z_0)^{2n+1}},\quad n\in\mathbb{N}, \tag{3.75}$$

which is a solution to the following homogeneous complex differential equation

$$f'''(z)+A_2(z)f''(z)+A_1(z)f'(z)+A_0(z)f(z)=0, \tag{3.76}$$

where $A_0(z)=\frac{(2n+1)(2n+2)(2n+3)}{(z-z_0)^3}-\frac{(2n+1)(2n+2)(3-7i)}{(z-z_0)^2}+\frac{(2n+1)(3+7i)}{(z-z_0)}$, $A_1(z)=3+7i$ *and* $A_2(z)=3-7i$. *The coefficients* $A_j(z), j=0,1,2$ *satisfy the conditions of* Theorem 3.7*, such that*

$$\max\{\rho_{\log}(A_1),\rho_{\log}(A_2)\}=0<\rho_{\log}(A_0)=1.$$

We see that f *satisfies*

$$\rho_{\log}(A_0)-1=\rho_{[2,2]}(f)=0\leq\rho_{\log}(A_0).$$

Example 3.2. *For* Theorem 3.8*, we consider the analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$

$$f(z)=e^{\frac{1}{(z-z_0)^{2n+1}}},\quad n\in\mathbb{N}. \tag{3.77}$$

Note that f *is a solution to the homogeneous complex differential equation* (3.76)*, for*

$$A_0(z)=\frac{4(n+1)(2n+1)^2}{(z-z_0)^{4n+5}},A_1(z)=-\frac{(2n+2)(2n+3)}{(z-z_0)^2}$$

and $A_2(z)=\frac{2n+1}{(z-z_0)^{2n+2}}$. *The coefficients* $A_j(z), j=0,1,2$ *satisfy the conditions of* Theorem 3.8*, such that*

$$\max\{\rho_{\log}(A_1),\rho_{\log}(A_2)\}=\rho_{\log}(A_0)=1$$

and

$$\max\{\tau_{\log}(A_1),\tau_{\log}(A_2)\}=2n+2<\tau_{\log}(A_0)=4n+5.$$

We remark that f *satisfies*

$$\rho_{[2,2]}(f)=1=\rho_{\log}(A_0).$$

Example 3.3. *For* Theorem 3.15*, the function* f *in* (3.75) *is an analytic solution in* $\overline{\mathbb{C}}\backslash\{z_0\}$ *to the following non-homogeneous linear differential equation*

$$f'''(z)+A_2(z)f''(z)+A_1(z)f'(z)+A_0(z)f(z)=F(z), \tag{3.78}$$

where $A_0(z)=\frac{2n(2n+2)(2n+3)}{(z-z_0)^3}$, $A_1(z)=\frac{\sqrt{2}\,(2n+2)}{(z-z_0)}$, $A_2(z)=\sqrt{2}$ *and* $F(z)=\frac{2n(2n+2)(2n+3)}{(z-z_0)^3}$. *As we see* $A_j(z), i=0,1,2$ *and* $F(z)$ *satisfy the conditions of* Theorem 3.15 *in case (ii), such that*

$$\max\{\rho_{\log}(A_1),\rho_{\log}(A_2)\}=\rho_{\log}(A_0)=1,$$

$$\max\{\tau_{\log}(A_1), \tau_{\log}(A_2)\} = 1 < \tau_{\log}(A_0) = 3$$

and

$$\rho_{[2,2]}(F) = 0 < \rho_{\log}(A_0) = 1.$$

Then f satisfies

$$\rho_{[2,2]}(f) = 0 < \rho_{\log}(A_0) = 1.$$

Chapter 4

Linear differential equations with zero order analytic or meromorphic coefficients in $\overline{\mathbb{C}}-\{z_0\}$ part 2

4.1 Introduction

In this chapter, we will extend the investigations of the previous chapters to the case when arbitrary coefficient is dominating the other coefficients in the differential equations (2.1) and (3.1). This case was firstly considered by Long and Zeng in [55], such that they assumed the dominance of the arbitrary coefficient is in term of the $[p,q]$-order, and they obtained the following result

Theorem 4.1 ([55]). *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. If there exists an integer $s(0\leq s\leq k-1)$ such that $A_l(z)$ satisfies* $\max\left\{\rho_{[p,q]}(A_j,z_0):j\neq l\right\}<\rho_{[p,q]}(A_l,z_0)<+\infty$. *Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of* (2.1) satisfies $\rho_{[p+1,q]}(f,z_0)\leq\rho_{[p,q]}(A_l,z_0)\leq\rho_{[p,q]}(f,z_0)$.

As we have shown before the effectiveness of the logarithmic order and the logarithmic lower order in estimating the growth for the case when the coefficients of (2.1) and (3.1) are zero order analytic or meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$, here we also make use of them to extend the above theorem. Our results are on the logarithmic order, the logarithmic lower order and the exponent of convergence of the solutions, where the dominance of the arbitrary coefficient $A_l(z)$ is assumed in five different terms: $m_{z_0}(r,A_l),\rho_{\log}(A_l,z_0),\mu_{\log}(A_l,z_0),\tau_{\log}(A_l,z_0)$ and $\underline{\tau}_{\log}(A_l,z_0)$. These results are also generalizations to previous results obtained in the precedent chapter, and they are also $\overline{\mathbb{C}}-\{z_0\}$ counterparts to some of those obtained in (see [15,29]).

4.2 Main Results

Theorem 4.2 ([26]). *Let $A_0(z),\dots,A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0\le l\le k-1)$ such that $A_l(z)$ satisfies*

$$\limsup_{r\longrightarrow 0}\frac{\sum_{j\neq l} m_{z_0}(r,A_j)}{m_{z_0}(r,A_l)}<1 \quad \textit{and} \quad \liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\rho_{\log}(A_l,z_0)-1\le\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\le\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$.

Theorem 4.3 ([26]). *Let $A_0(z),\dots,A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0\le l\le k-1)$ such that $A_l(z)$ satisfies* $\max\{\rho_{\log}(A_j,z_0): j\neq l\}\le\rho_{\log}(A_l,z_0)$,

$$\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$$

and

$$\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0)\ge 1,\, j\neq l}\tau_{\log}(A_j,z_0)<\delta\tau_{\log}(A_l,z_0)<+\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\rho_{\log}(A_l,z_0)-1\le\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\le\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$.

Theorem 4.4 ([26]). *Let $A_0(z),\dots,A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0\le l\le k-1)$ such that $A_l(z)$ satisfies $\lambda_{\log}(\frac{1}{A_l},z_0)+1<\rho_{\log}(A_l,z_0)$,* $\max\{\rho_{\log}(A_j,z_0): j\neq l\}\le\rho_{\log}(A_l,z_0)$ *and*

$$\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0),\, j\neq l}\tau_{\log}(A_j,z_0)<\tau_{\log}(A_l,z_0)<+\infty.$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\rho_{\log}(A_l,z_0)\le\rho_{\log}(f,z_0)$.

Theorem 4.5. *Let $A_0(z),\dots A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0\le l\le k-1)$ such that $A_l(z)$ satisfies* $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$, $\max\{\rho_{\log}(A_j,z_0): j\neq l\}\le\mu_{\log}(A_l,z_0)$ *and*

$$\sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_l,z_0),\, j\neq l}\tau_{\log}(A_j,z_0)<\delta\underline{\tau}_{\log}(A_l,z_0).$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2.1) satisfies $0\le\mu_{\log}(A_l,z_0)-1\le\mu_{\log}(f,z_0)$ with $1<\mu_{\log}(A_l,z_0)\le\mu_{\log}(f,z_0)$.

Remark 4.1. *The conditions* $\max\{\rho_{\log}(A_j,z_0): j\neq l\}\le\mu_{\log}(A_l,z_0)=\mu<+\infty$ *and*

$$\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0),\, j\neq l}\tau_{\log}(A_j,z_0)<\delta\underline{\tau}_{\log}(A_l,z_0)$$

in Theorem 4.5 can be replaced by $\limsup_{r\longrightarrow 0}\frac{\sum_{j\neq l} m_{z_0}(r,A_j)}{m_{z_0}(r,A_l)}<1$ or we replace the condition

$$\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$$

by $\lambda_{\log}(\frac{1}{A_l},z_0)+1<\mu_{\log}(A_l,z_0)$, which clearly includes the assumption that $\mu_{\log}(A_l,z_0)>1$.

Theorem 4.6 ([26]). *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0 \le l \le k-1)$ such that $A_l(z)$ satisfies* $\max\left\{\rho_{\log}(A_j,z_0) : j \neq l\right\} \le \rho_{\log}(A_l,z_0) = \rho$ *and*

$$\limsup_{r\longrightarrow 0}\frac{\sum_{j\neq l} m_{z_0}(r,A_j)}{m_{z_0}(r,A_l)} < 1.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\rho_{[2,2]}(f,z_0)-1 \le \rho_{\log}(A_l,z_0)-1 \le \rho_{\log}(f,z_0)$. Furthermore, if $\rho_{\log}(A_l,z_0) > 1$, then $f(z)$ satisfies $\rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_l,z_0) \le \rho_{\log}(f,z_0)$.

Theorem 4.7 ([26]). *Let $A_0(z),...,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0 \le l \le k-1)$ such that $A_l(z)$ satisfies* $\max\left\{\rho_{\log}(A_j,z_0) : j \neq l\right\} \le \rho_{\log}(A_l,z_0) = \rho$ *and*

$$\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0),\, j\neq l} \tau_{\log}(A_j,z_0) < \tau_{\log}(A_l,z_0) < +\infty.$$

Then, every analytic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2.1) satisfies $\rho_{[2,2]}(f,z_0)-1 \le \rho_{\log}(A_l,z_0)-1 \le \rho_{\log}(f,z_0)$. Furthermore, if $\rho_{\log}(A_l,z_0) > 1$, then $f(z)$ satisfies $\rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_l,z_0) \le \rho_{\log}(f,z_0)$.

Theorem 4.8. *Let $A_0(z),...A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite logarithmic order. If there exists an integer l $(0 \le l \le k-1)$ such that $A_l(z)$ satisfies* $\max\left\{\rho_{\log}(A_j,z_0) : j \neq l\right\} \le \mu_{\log}(A_l,z_0)$ *and*

$$\sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_l,z_0),\, j\neq l} \tau_{\log}(A_j,z_0) < \underline{\tau}_{\log}(A_l,z_0).$$

Then, every meromorphic solution $f(z)(\not\equiv 0)$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2.1) satisfies $\mu_{[2,2]}(f,z_0)-1 \le \mu_{\log}(A_l,z_0)-1 \le \mu_{\log}(f,z_0)$. Further, if $\mu_{\log}(A_l,z_0) > 1$ then, $\mu_{[2,2]}(f,z_0) \le \mu_{\log}(A_l,z_0) \le \mu_{\log}(f,z_0)$.

Remark 4.2. *We can replace the condition*

$$\sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_l,z_0),\, j\neq l} \tau_{\log}(A_j,z_0) < \underline{\tau}_{\log}(A_l,z_0)$$

in Theorem 4.8 by $\limsup_{r\longrightarrow 0}\frac{\sum_{j\neq l} m_{z_0}(r,A_j)}{m_{z_0}(r,A_l)} < 1$.

Theorem 4.9 ([26]). *Let $A_0(z),...,A_{k-1}(z)$ satisfy the hypotheses of* Theorem 4.7 and let $F(z)(\not\equiv 0)$ be analytic function in $\overline{\mathbb{C}}-\{z_0\}$

i) If $\rho_{\log}(A_l,z_0) \le \rho_{[2,2]}(F,z_0) < +\infty$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0) = \rho_{[2,2]}(F,z_0)$.

ii) If $\rho_{\log}(A_l,z_0) > \rho_{[2,2]}(F,z_0)$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (3.1) satisfies $\rho_{[2,2]}(f,z_0) \le \rho_{\log}(A_l,z_0)$ and $\overline{\lambda}_{[2,2]}(f,z_0) = \lambda_{[2,2]}(f,z_0) = \rho_{[2,2]}(f,z_0)$ holds for every solution satisfies $\rho_{[2,2]}(f,z_0) = \rho_{\log}(A_l,z_0)$.

4.3 Lemmas

Lemma 4.1. *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\rho_{\log}(f) = \rho$. Then there exists a subset E_1 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0| = r \in E_1$, we have*

$$\rho = \lim_{r\longrightarrow 0} \frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}}$$

and for any given $\varepsilon > 0$

$$T_{z_0}(r,f) > \left(\log\frac{1}{r}\right)^{\rho-\varepsilon}.$$

Proof. Replacing $\log M_{z_0}(r,f)$ by $T_{z_0}(r,f)$ in the proof of Lemma 3.1, we get the proof of Lemma 4.1. □

Lemma 4.2. *Let f_1, f_2 be two meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\rho_1 = \rho_{\log}(f_1,z_0) > \rho_{\log}(f_2,z_0) = \rho_2$. Then there exists a set $E_2 \subset (0,1)$ of infinite logarithmic measure such that for all $|z-z_0| = r \in E_2$, we have*

$$\lim_{r\longrightarrow 0} \frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)} = 0.$$

Proof. By the definition of the logarithmic order, for any given $0 < \varepsilon < \frac{\rho_1-\rho_2}{2}$, there exists $r_2 \in (0,1)$ such that for all $|z-z_0| = r \in (0,r_2)$, we obtain

$$T_{z_0}(r,f_2) \leq \left(\log\frac{1}{r}\right)^{\rho_2+\varepsilon}. \tag{4.1}$$

By Lemma 4.1, there exists a set $E_1 \subset (0,1)$ of infinite logarithmic measure such that, for the above ε and for all $|z-z_0| = r \in E_1$, we have

$$T_{z_0}(r,f_1) \geq \left(\log\frac{1}{r}\right)^{\rho_1-\varepsilon}. \tag{4.2}$$

By (4.1) and (4.2), for the above ε and for all $|z-z_0| = r \in E_2 = E_1 \cap (0,r_2)$, we get

$$0 \leq \frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)} \leq \frac{\left(\log\frac{1}{r}\right)^{\rho_2+\varepsilon}}{\left(\log\frac{1}{r}\right)^{\rho_1-\varepsilon}} = \frac{1}{\left(\log\frac{1}{r}\right)^{\rho_1-\rho_2-2\varepsilon}} \longrightarrow 0, \quad \text{as} \quad r\longrightarrow 0.$$

□

Lemma 4.3. *Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with finite logarithmic order $1 \leq \rho_{\log}(f,z_0) = \rho < +\infty$ and finite logarithmic type $0 < \tau_{\log}(f,z_0) < +\infty$. Then there exists a set E_3 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0| = r \in E_3$, we have*

$$\lim_{r\longrightarrow 0} \frac{T_{z_0}(r,f)}{(\log\frac{1}{r})^{\rho}} = \tau_{\log}(f,z_0).$$

Proof. The proof can easily obtained by replacing $\log M_{z_0}(r,f)$ by $T_{z_0}(r,f)$ in the proof of Lemma 3.3. □

4.4 Proofs of the theorems

Proof of Theorem 4.2

Proof. Let $f(\not\equiv 0)$ be a meromorphic solution of (2.1) in $\overline{\mathbb{C}}-\{z_0\}$. If $\rho_{\log}(f,z_0)=\infty$, then the result is trivial . So, we suppose that $\rho_{\log}(f,z_0)<\infty$. By (2.1), we have

$$-A_l(z)=\frac{f^{(k)}(z)}{f^{(l)}(z)}+A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(l)}(z)}+\cdots+A_{l+1}(z)\frac{f^{(l+1)}(z)}{f^{(l)}(z)}+A_{l-1}(z)\frac{f^{(l-1)}(z)}{f^{(l)}(z)}+\cdots+A_0(z)\frac{f(z)}{f^{(l)}(z)}. \tag{4.3}$$

It follows that

$$m_{z_0}(r,A_l(z))\leq\sum_{j=0,j\neq l}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(l)}(z)}\right)+\sum_{j=0,j\neq l}^{k-1}m_{z_0}\big(r,A_j(z)\big)+O(1). \tag{4.4}$$

By Lemma 2.9, for a constant $r_0\in(0,1)$, there exists a set $\mathcal{F}_4\subset(0,r_0]$ of finite logarithmic measure such that for all $|z-z_0|=r\in(0,r_0]\setminus\mathcal{F}_4$ and for any given $\varepsilon>0$, we have

$$\sum_{j=0,j\neq l}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(l)}(z)}\right)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{4.5}$$

Suppose that $\limsup_{r\longrightarrow 0}\frac{\sum_{j=0,j\neq l}^{k-1}m_{z_0}(r,A_j)}{m_{z_0}(r,A_l)}<\sigma<1$. Then for $r\to 0$, we get

$$\sum_{j=0,j\neq l}^{k-1}m_{z_0}(r,A_j)<\sigma m_{z_0}(r,A_l). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), for all $|z-z_0|=r\in(0,r_0]\setminus\mathcal{F}_4$ and $r\to 0$, we obtain

$$(1-\sigma)m_{z_0}(r,A_l)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{4.7}$$

By the assumption $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$, there exists $r_1\in(0,1)$ such that, for all $|z-z_0|=r\in(0,r_1)$, we have

$$m_{z_0}(r,A_l)\geq\frac{\delta}{2}T_{z_0}(r,A_l). \tag{4.8}$$

By Lemma 4.1, there exists a set $E_1\subset(0,1)$ of infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_1$, we have

$$T_{z_0}(r,A_l)\geq\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)-\varepsilon}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_1\cap(0,r_0]\cap(0,r_1)\setminus\mathcal{F}_4$, we get

$$\frac{\delta}{2}(1-\sigma)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)-\varepsilon}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{4.10}$$

This implies that $\rho_{\log}(A_l,z_0)-1-\varepsilon\leq\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)-\varepsilon\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. Since $\varepsilon>0$ is arbitrary, we obtain $\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. □

Proof of Theorem 4.3

Proof. Let $f(\not\equiv 0)$ be a meromorphic solution of (2.1) in $\overline{\mathbb{C}}-\{z_0\}$. First, we suppose that $\max\left\{\rho_{\log}(A_j,z_0): j\neq l\right\}<\rho_{\log}(A_l,z_0)=\rho$. Then, as in the proof of Theorem 4.2, by substituting (4.5) and (4.8) into (4.4), for all $|z-z_0|=r\in(0,r_0]\cap(0,r_1)\setminus\mathcal{F}_4$, we obtain

$$\frac{\delta}{2}T_{z_0}(r,A_l)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq l}^{k-1}T_{z_0}(r,A_j). \tag{4.11}$$

By Lemma 4.2, there exists a set $E_2\subset(0,1)$ of infinite logarithmic measure such that for all $|z-z_0|=r\in E_2$, we have

$$\max\left\{\frac{T_{z_0}(r,A_j)}{T_{z_0}(r,A_l)},j\neq l\right\}\longrightarrow 0,\text{ as } r\longrightarrow 0. \tag{4.12}$$

Then, by (4.11) and (4.12) for all $r\in E_2\cap(0,r_0]\cap(0,r_1)\setminus\mathcal{F}_4$ and $r\longrightarrow 0$, we get

$$\left(\frac{\delta}{2}-o(1)\right)T_{z_0}(r,A_l)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \tag{4.13}$$

From (4.13), we deduce that $\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. Now we suppose that $\max\left\{\rho_{\log}(A_j,z_0): j\neq l\right\}=\rho_{\log}(A_l,z_0)=\rho$ and

$$\tau_1=\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0)\geq 1,j\neq l}\tau_{\log}(A_j,z_0)<\delta\tau_{\log}(A_l,z_0)=\delta\tau.$$

So, there exists a set $J_1\subseteq\{0,1,...,k-1\}\setminus\{l\}$ such that for $j\in J_1$, we have $\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0)=\rho$ with $\tau_1=\sum_{j\in J_1}\tau_{\log}(A_j,z_0)<\tau_{\log}(A_l,z_0)=\tau$ and for $j\in J_2=\{0,1,...l-1,l+1,...,k-1\}\setminus J_1$, we have $\rho_{\log}(A_j,z_0)<\rho_{\log}(A_l,z_0)=\rho$. Then, there exists $r_3\in(0,1)$, such that for all $|z-z_0|=r\in(0,r_3)$ and for any given ε $\left(0<(\tau+k)\varepsilon<\delta\tau-\tau_1\right)$, we obtain

$$T_{z_0}(r,A_j)\leq\left(\tau_{\log}(A_j,z_0)+\varepsilon\right)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)}=\left(\tau_{\log}(A_j,z_0)+\varepsilon\right)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)},\qquad j\in J_1 \tag{4.14}$$

and

$$T_{z_0}(r,A_j)\leq\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)+\varepsilon}\leq\left(\log\frac{1}{r}\right)^{\rho_0},\qquad j\in J_2, \tag{4.15}$$

where $\max\left\{\rho_{\log}(A_j,z_0): j\in J_2\right\}<\rho_0<\rho$. By the assumption $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$ and Lemma 4.3, there exists a set $E_3\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_3$, we have

$$\begin{aligned}m_{z_0}(r,A_l)&\geq(\delta-\varepsilon)T_{z_0}(r,A_l)\geq(\delta-\varepsilon)(\tau-\varepsilon)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)}\\&\geq\left(\delta\tau-(\tau+1)\varepsilon\right)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)}.\end{aligned} \tag{4.16}$$

By substituting (4.5) and (4.14)-(4.16) into (4.4), for the above ε and for all $|z-z_0|=r\in E_3\cap(0,r_0]\cap(0,r_3)\setminus\mathcal{F}_4$, we obtain

$$\begin{aligned}
&\big(\delta\tau-(\tau+1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_l,z_0)}\\
&\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq l}^{k-1}T_{z_0}(r,A_j)\\
&\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j\in J_1}\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\rho}+\sum_{j\in J_2}\left(\log\frac{1}{r}\right)^{\rho_0}\\
&\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\big(\tau_1+(k-1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\rho}+(k-1)\left(\log\frac{1}{r}\right)^{\rho_0}.
\end{aligned}\tag{4.17}$$

It follows that

$$\big(1-o(1)\big)\big(\delta\tau-\tau_1-(\tau+k)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\rho}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right),\tag{4.18}$$

which implies that, $\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $1<\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. □

Proof of Theorem 4.4

Proof. By (4.4) and (4.5), for all $r\in(0,r_0]\setminus\mathcal{F}_4$, we have

$$\begin{aligned}
T_{z_0}\big(r,A_l(z)\big)&=m_{z_0}\big(r,A_l(z)\big)+N_{z_0}\big(r,A_l(z)\big)\\
&\leq\sum_{j=0,j\neq l}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(l)}(z)}\right)+\sum_{j=0,j\neq l}^{k-1}m_{z_0}\big(r,A_j(z)\big)+N_{z_0}\big(r,A_l(z)\big)+O(1)\\
&\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq l}^{k-1}T_{z_0}\big(r,A_j(z)\big)+N_{z_0}\big(r,A_l(z)\big).
\end{aligned}\tag{4.19}$$

If $\rho_1=\max\{\rho_{\log}(A_j,z_0):j\neq l\}<\rho_{\log}(A_l,z_0)=\rho$, then there exists $r_4\in(0,1)$ such that for any given ε $(0<2\varepsilon<\rho-\rho_1)$ and for all $|z-z_0|=r\in(0,r_4)$, we obtain

$$T_{z_0}(r,A_j)\leq\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)+\varepsilon}\leq\left(\log\frac{1}{r}\right)^{\rho_1+\varepsilon},\qquad j=0,\dots,k-1,j\neq l.\tag{4.20}$$

By Lemma 4.1, there exists a set $E_1\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_1$, the assumption (4.9) holds. By the definition of $\lambda_{\log}\big(\frac{1}{A_l},z_0\big)=\lambda$, there exists $r_5\in(0,1)$ such that for any given ε $(0<2\varepsilon<\rho-\lambda-1)$ and for all $|z-z_0|=r\in(0,r_5)$, we get

$$N_{z_0}(r,A_l)\leq\left(\log\frac{1}{r}\right)^{\lambda_{\log}\left(\frac{1}{A_l},z_0\right)+1+\varepsilon}.\tag{4.21}$$

By substituting (4.9), (4.20) and (4.21) into (4.19), for sufficiently small ε satisfying $0<2\varepsilon<\min\{\rho-\rho_1,\rho-\lambda-1\}$ and for all $r\in E_1\cap(0,r_0]\cap(0,r_4)\cap(0,r_5)\setminus\mathcal{F}_4$, we have

$$\left(\log\frac{1}{r}\right)^{\rho-\varepsilon}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+(k-1)\left(\log\frac{1}{r}\right)^{\rho_1+\varepsilon}+\left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon},\tag{4.22}$$

then

$$\big(1-o(1)\big)\bigg(\log\frac{1}{r}\bigg)^{\rho-\varepsilon}\leq O\bigg(T_{z_0}(r,f)+\log\frac{1}{r}\bigg). \tag{4.23}$$

Thus, $1<\rho-\varepsilon\leq\rho_{\log}(f,z_0)$. Since $\varepsilon>0$ is arbitrary, we obtain $1<\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$. Now, if $\max\big\{\rho_{\log}(A_j,z_0):j\neq l\big\}=\rho_{\log}(A_l,z_0)=\rho$ and

$$\tau_1=\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0)\geq 1,j\neq l}\tau_{\log}(A_j,z_0)<\tau_{\log}(A_l,z_0)=\tau,$$

then as in the proof of Theorem 4.3, we assume that there exists a set $J_1\subseteq\{0,1,...,k-1\}\setminus\{l\}$ such that for $j\in J_1$, we have $\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0)=\rho$ with $\tau_1=\sum_{\rho_{\log}(A_j,z_0)=\rho_{\log}(A_l,z_0),j\neq l}\tau_{\log}(A_j,z_0)<\tau_{\log}(A_l,z_0)=\tau$ and for $j\in J_2=\{0,1,...l-1,l+1,...,k-1\}\setminus J_1$, we have $\rho_{\log}(A_j,z_0)<\rho_{\log}(A_l,z_0)=\rho$. Then, there exists a $r_3\in(0,1)$, such that for any given ε $\big(0<\varepsilon<\frac{\tau-\tau_1}{k}\big)$ and for all $|z-z_0|=r\in(0,r_3)$, the assumptions (4.14) and (4.15) hold. By Lemma 4.3, there exists a set $E_3\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_3$, we obtain

$$T_{z_0}(r,A_l)\geq\big(\tau-\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\rho}. \tag{4.24}$$

By substituting (4.14), (4.15) and (4.24) into (4.19), for sufficiently small ε satisfying $0<\varepsilon<\min\big\{\frac{\rho-\lambda-1}{2},\frac{\tau-\tau_1}{k}\big\}$ and for all $r\in E_3\cap(0,r_0]\cap(0,r_3)\cap(0,r_5)\setminus\mathcal{F}_4$, we get

$$\begin{aligned}\big(\tau-\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\rho}&\leq O\bigg(T_{z_0}(r,f)+\log\frac{1}{r}\bigg)+\sum_{j\in J_1}\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\rho}\\&\quad+\sum_{j\in J_2}\bigg(\log\frac{1}{r}\bigg)^{\rho_0}+\bigg(\log\frac{1}{r}\bigg)^{\lambda+1+\varepsilon}\\&\leq O\bigg(T_{z_0}(r,f)+\log\frac{1}{r}\bigg)+\big(\tau_1+(k-1)\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\rho}\\&\quad+(k-1)\bigg(\log\frac{1}{r}\bigg)^{\rho_0}+\bigg(\log\frac{1}{r}\bigg)^{\lambda+1+\varepsilon}.\end{aligned} \tag{4.25}$$

So

$$\big(1-o(1)\big)\big(\tau-\tau_1-k\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\rho}\leq O\bigg(T_{z_0}(r,f)+\log\frac{1}{r}\bigg), \tag{4.26}$$

which implies that $1<\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$. □

Proof of Theorem 4.5

Proof. Let f be a meromorphic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of (2.1) with $\mu_{\log}(f,z_0)<\infty$, otherwise, the result is trivial. First, if $\rho_1=\max\big\{\rho_{\log}(A_j,z_0):j=0,...,k-1,j\neq l\big\}<\mu_{\log}(A_l,z_0)=\mu$. Then for any given $0<2\varepsilon<\mu-\rho_1$, there exists $r_4\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_4)$, (4.20) holds. By $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$ and the definition of $\mu_{\log}(A_l,z_0)=\mu$, for the above ε, there exists $r_6\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_6)$, we have

$$m_{z_0}(r,A_l)\geq\frac{\delta}{2}T_{z_0}(r,A_l)\geq\frac{\delta}{2}\bigg(\log\frac{1}{r}\bigg)^{\mu_{\log}(A_l,z_0)-\frac{\varepsilon}{2}}\geq\bigg(\log\frac{1}{r}\bigg)^{\mu_{\log}(A_l,z_0)-\varepsilon}. \tag{4.27}$$

Substituting (4.5), (4.20) and (4.26) into (4.4), for the above ε and for all $|z_0-z|=r\in(0,r_0]\cap(0,r_4)\cap(0,r_6)\setminus\mathcal{F}_4$, we get

$$\begin{aligned}\left(\log\frac{1}{r}\right)^{\mu-\varepsilon} &\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq s}^{k-1}T_{z_0}(r,A_j)\\ &\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+(k-1)\left(\log\frac{1}{r}\right)^{\rho_1+\varepsilon}.\end{aligned}\tag{4.28}$$

Thus,

$$\big(1-o(1)\big)\left(\log\frac{1}{r}\right)^{\mu-\varepsilon}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right).\tag{4.29}$$

It follows that, $0\leq\mu-1-\varepsilon\leq\mu_{\log}(f,z_0)$. Since $\varepsilon>0$ is arbitrary, we obtain $0\leq\mu_{\log}(A_l,z_0)-1\leq\mu_{\log}(f,z_0)$. Now if $\max\{\rho_{\log}(A_j,z_0): j=0,...,k-1, j\neq l\}=\mu_{\log}(A_l,z_0)=\mu$ and $\tau_2=\sum_{\rho_{\log}(A_j,z_0)=\mu_{\log}(A_l,z_0),j\neq l}\tau_{\log}(A_j,z_0)<\delta\underline{\tau}_{\log}(A_l,z_0)=\delta\underline{\tau}$. Then, there exists a set $J_1\subseteq\{0,1,...,k-1\}\setminus\{l\}$ such that for $j\in J_1$, we have $\rho_{\log}(A_j,z_0)=\mu_{\log}(A_l,z_0)=\mu$ with $\tau_2=\sum_{j\in J_1}\tau_{\log}(A_j,z_0)<\underline{\tau}_{\log}(A_l,z_0)=\underline{\tau}$ and for $j\in J_2=\{0,1,...l-1,l+1,...,k-1\}\setminus J_1$, we have $\rho_{\log}(A_j,z_0)<\mu_{\log}(A_l,z_0)=\mu$. Hence, for any given ε $\big(0<(\underline{\tau}+k)\varepsilon<\delta\underline{\tau}-\tau_2\big)$, there exists $r_7\in(0,1)$, such that for all $|z_0-z|=r\in(0,r_7)$, we obtain

$$T_{z_0}(r,A_j)\leq\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)}=\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_l,z_0)},\qquad j\in J_1\tag{4.30}$$

and

$$T_{z_0}(r,A_j)\leq\left(\log\frac{1}{r}\right)^{\rho_{\log}(A_j,z_0)+\varepsilon}\leq\left(\log\frac{1}{r}\right)^{\rho_0},\qquad j\in J_2,\tag{4.31}$$

where $\max\{\rho_{\log}(A_j,z_0): j\in J_2\}<\rho_0<\mu$. By $\liminf_{r\longrightarrow0}\frac{m_{z_0}(r,A_l)}{T_{z_0}(r,A_l)}=\delta>0$ and the definition of $\underline{\tau}_{\log}(A_l,z_0)=\underline{\tau}$, for the above $\varepsilon>0$, there exists $r_8\in(0,1)$ such that for all $|z_0-z|=r\in(0,r_8)$, we have

$$\begin{aligned}m_{z_0}(r,A_l)\geq\big(\delta-\varepsilon\big)T_{z_0}(r,A_l)&\geq\big(\delta-\varepsilon\big)\big(\underline{\tau}-\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_l,z_0)}\\ &\geq\big(\delta\underline{\tau}-(\underline{\tau}+1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu_{\log}(A_l,z_0)}.\end{aligned}\tag{4.32}$$

By substituting (4.5), (4.30), (4.31) and (4.32) into (4.4), for the above ε and for all $|z_0-z|=r\in(0,r_0]\cap(0,r_7)\cap(0,r_8)\setminus\mathcal{F}_4$, we obtain

$$\begin{aligned}&\big(\delta\underline{\tau}-(\underline{\tau}+1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu}\\ &\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq l}^{k-1}T_{z_0}(r,A_j)\\ &\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j\in J_1}\big(\tau_{\log}(A_j,z_0)+\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu}+\sum_{j\in J_2}\left(\log\frac{1}{r}\right)^{\rho_0}\\ &\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\big(\tau_2+(k-1)\varepsilon\big)\left(\log\frac{1}{r}\right)^{\mu}+(k-1)\left(\log\frac{1}{r}\right)^{\rho_0},\end{aligned}\tag{4.33}$$

that is

$$\big(1-o(1)\big)\big(\delta\underline{\tau}-\tau_2-(\underline{\tau}+k)\varepsilon\big)\bigg(\log\frac{1}{r}\bigg)^{\mu}\leq O\bigg(T_{z_0}(r,f)+\log\frac{1}{r}\bigg). \tag{4.34}$$

This implies that $0\leq\mu_{\log}(A_l,z_0)-1\leq\mu_{\log}(f,z_0)$ and $1<\mu_{\log}(A_l,z_0)\leq\mu_{[\log}(f,z_0)$. □

Proof of Theorem 4.6

Proof. We assume that $f(\not\equiv 0)$ is an analytic solution of (2.1) in $\overline{\mathbb{C}}-\{z_0\}$. By Theorem 4.2, we have $0\leq\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. On the other hand, by Lemma 3.2, we have $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_l,z_0)$. Hence, $\rho_{[2,2]}(f,z_0)-1\leq\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. □

Proof of Theorem 4.7

Proof. We assume that $f(\not\equiv 0)$ is an analytic solution of (2.1) in $\overline{\mathbb{C}}-\{z_0\}$. By Theorem 4.3, we get $0\leq\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. Then, by using Lemma 3.2, we conclude that, $\rho_{[2,2]}(f,z_0)-1\leq\rho_{\log}(A_l,z_0)-1\leq\rho_{\log}(f,z_0)$ and $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_l,z_0)\leq\rho_{\log}(f,z_0)$ if $\rho_{\log}(A_l,z_0)>1$. □

Proof of Theorem 4.8

Proof. By Theorem 4.5 and Lemma 3.6, we get the assertions of Theorem 4.8. □

Proof of Theorem 4.9

Proof. Here we use a similar discussion as in the proof of Theorem 3.14. We suppose $f(z)$ is an analytic solution in $\overline{\mathbb{C}}-\{z_0\}$ of (3.1), then f can be represented in the form (3.69). Hence, the assumptions (3.70)-(3.72) hold. By Theorem 4.7 and the fact that $G_j(f_1,f_2,...,f_k)$ and $W(f_1,f_2,...,f_k)$ are both differential polynomial of $f_1,f_2,...,f_k$ and their derivatives with constant coefficients, we have

$$\max\big\{\rho_{[2,2]}(G_j(f_1,f_2,...,f_k),z_0),\rho_{[2,2]}(W(f_1,f_2,...,f_k),z_0)\big\}\leq\rho_{[2,2]}(f_j,z_0)\leq\rho_{\log}(A_l,z_0). \tag{4.35}$$

By (3.69), (3.72) and (4.35), for $j=1,...,k$, we get

$$\begin{aligned}\rho_{[2,2]}(f,z_0)&\leq\max\big\{\rho_{[2,2]}(f_j,z_0),\rho_{[2,2]}(B_j,z_0)\big\}\\&\leq\max\big\{\rho_{[2,2]}(F,z_0),\rho_{\log}(A_l,z_0)\big\}.\end{aligned} \tag{4.36}$$

i) If $\rho_{[2,2]}(F,z_0)\geq\rho_{\log}(A_l,z_0)$, then by (3.1) and (4.36), we deduce that $\rho_{[2,2]}(f,z_0)=\rho_{[2,2]}(F,z_0)$.

ii) If $\rho_{[2,2]}(F,z_0)<\rho_{\log}(A_l,z_0)$, then by (4.36) , we obtain $\rho_{[2,2]}(f,z_0)\leq\rho_{\log}(A_l,z_0)$. Further, assume that a solution f of (3.1) satisfies $\rho_{[2,2]}(f,z_0)=\rho_{\log}(A_l,z_0)$. Then, there holds

$$\max\big\{\rho_{[2,2]}(F,z_0),\rho_{[2,2]}(A_j,z_0):(j=0,...,k-1)\big\}<\rho_{[2,2]}(f,z_0).$$

By Lemma 3.10, we conclude that $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\rho_{[2,2]}(f,z_0)=\rho_{\log}(A_l,z_0)$. □

Chapter 5

Linear difference equations with zero order meromorphic coefficients

5.1 Introduction

This chapter is devoted to study the growth of the meromorphic solutions of homogeneous and non-homogeneous linear difference equations

$$A_k(z)f(z+c_k)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{5.1}$$

$$A_k(z)f(z+c_k)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=F(z), \tag{5.2}$$

where $A_0(z),\ldots,A_k(z)$ and $F(z)$ are meromorphic functions of finite logarithmic order, $c_i(i=1,\ldots,k,k\in\mathbb{N})$ are distinct non-zero complex constants. In [81], Zhou and Zheng considered the growth of the difference equation (5.2) , and proved the following result.

Theorem 5.1 ([81]). *Let $A_j(z)(j=0,1,...,k)$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\le l\le k)$ such that $A_l(z)$ satisfies*

$$\lambda(\frac{1}{A_l})<\rho(A_l)<\infty,$$

$$\max\{\rho(A_j):\ j=0,1...k,j\neq l\}\le\rho(A_l),$$

$$\sum_{\rho(A_j)=\rho(A_l),j\neq l}\tau(A_j)<\tau(A_l)<\infty.$$

1. *If $\rho(F)<\rho(A_l)$, or $\rho(F)=\rho(A_l)$ and $\sum_{\rho(A_j)=\rho(A_l),j\neq l}\tau(A_j)+\tau(F)<\tau(A_l)$, or $\rho(F)=\rho(A_l)$ and $\sum_{\rho(A_j)=\rho(A_l),}\tau(A_j)<\tau(F)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (5.2) satisfies $\rho(f)\ge\rho(A_l)$.
2. If $\rho(F)>\rho(A_l)$, then every meromorphic solution $f(z)$ of (5.2) satisfies $\rho(f)\ge\rho(F)$.

There are many other results have been obtained by many different mathematicians on studying the growth of solutions of the linear difference equations, where their coefficients are entire or meromorphic functions (see e.g. [5,7,10,48,54,56,80]). However, a few of them concentrated on the case when the coefficients are of zero order entire or meromorphic functions. In [4], Belaïdi considered this case for the special homogeneous case of (5.1)

$$A_k(z)f(z+k)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \tag{5.3}$$

where $A_k(z),\ldots,A_0(z)$ are entire or meromorphic functions of finite logarithmic order, and obtained the following results on the logarithmic order and the logarithmic lower order of solutions.

Theorem 5.2 ([4]). *Let $A_j(z)(j=0,1,\ldots,k)$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_l(z)$ satisfies*

$$\lambda_{\log}(\frac{1}{A_l})<\rho_{\log}(A_l)<\infty,$$

$$\max\{\rho_{\log}(A_j):\ j=0,1,\ldots,k,j\neq l\}\leq\rho_{\log}(A_l),$$

$$\sum_{\rho_{\log}(A_j)=\rho_{\log}(A_l),j\neq l}\tau_{\log}(A_j)<\tau_{\log}(A_l)<\infty.$$

If f is a meromorphic solution of (5.3), then $\rho_{\log}(f)\geq\rho_{\log}(A_l)+1$.

Theorem 5.3 ([4]). *Let $A_j(z)(j=0,1,\ldots,k)$ be entire functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_l(z)$ satisfies*

$$\max\{\rho_{\log}(A_j):\ j=0,1,\ldots,k,j\neq l\}\leq\mu_{\log}(A_l),$$

$$\max\{\tau_{\log}(A_j):\ \rho_{\log}(A_j)=\mu_{\log}(A_l):\ j=0,1,\ldots,k,j\neq l\}<\underline{\tau}_{\log}(A_l)<\infty.$$

Then every meromorphic solution $f(z)\not\equiv 0$ of (5.3) satisfies $\mu_{\log}(f)\geq\mu_{\log}(A_l)+1$.

The main aim of this chapter, by using the logarithmic lower order, we extend Theorem 5.1 to the case when the coefficients are zero order meromorphic functions and thus we generalize Theorem 5.2 and Theorem 5.3 to the non-homogeneous case.

5.2 Main Results

Theorem 5.4 ([24]). *Let $A_j(z)(j=0,1,\ldots,k)$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_l(z)$ satisfies $\delta(\infty,A_l)>0$ and*

$$\limsup_{r\longrightarrow+\infty}\frac{\sum_{j=0,j\neq l}^{k}m(r,A_j)}{m(r,A_l)}<1.$$

Then every meromorphic solution $f(z)(\not\equiv 0)$ of (5.1) satisfies $\mu_{\log}(f)\geq\mu_{\log}(A_l)+1$.

Theorem 5.5 ([24]). *Let $A_j(z)(j=0,1,\ldots,k)$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_l(z)$ satisfies $\delta(\infty,A_l)>0$ and* $\max\{\rho_{\log}(A_j):\ j=0,1,\ldots,k,j\neq l\}<\mu_{\log}(A_l)\leq\rho_{\log}(A_l)<\infty$. *Then every meromorphic solution $f(z)(\not\equiv 0)$ of* (5.1) satisfies $\mu_{\log}(f)\geq\mu_{\log}(A_l)+1$.

Theorem 5.6 ([24]). *Let $A_j(z)(j=0,1,\dots,k)$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\le l\le k)$ such that $A_l(z)$ satisfies $\delta(\infty,A_l)>0$ and $\max\{\rho_{\log}(A_j):\ j=0,1,\dots,k, j\neq l\}<\mu_{\log}(A_l)\le\rho_{\log}(A_l)<\infty$.*

1. *If $\mu_{\log}(F)<\mu_{\log}(A_l)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (5.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(A_l)$. Further, if $F(z)\equiv 0$, then $\mu_{\log}(f)\ge\mu_{\log}(A_l)+1$.

2. If $\mu_{\log}(F)>\mu_{\log}(A_l)$, then every meromorphic solution $f(z)$ of (5.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(F)$.

5.3 Lemmas

For the proof of our results we need the following lemmas.

Lemma 5.1 ([8,9]). *Let f be a meromorphic function with finite logarithmic lower order $1\le\mu_{\log}(f)<+\infty$. Then there exists a subset E_1 of $[1,+\infty)$ that has infinite logarithmic measure such that for all $r\in E_1$, we have*

$$T(r,f)<(\log r)^{\mu_{\log}(f)+\varepsilon}.$$

Lemma 5.2 ([22]). *Let α,R,R' be real numbers such that $0<\alpha<1, R>0$ and let η non-zero complex number. Then, there is a positive constant C_α depending only α such that for a given meromorphic function f we have, when $|z|=r$, $\max\{1,r+|\eta|\}<R<R'$, the estimate*

$$\begin{aligned} m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)\le&\frac{2|\eta|R}{(R-r-|\eta|)^2}\left(m(R,f)+m(R,\frac{1}{f})\right)\\ &+\frac{2R'}{R'-R}\left(\frac{|\eta|}{R-r-|\eta|}+\frac{C_\alpha|\eta|^\alpha}{(1-\alpha)r^\alpha}\right)\\ &\times\left(N(R',f)+N(R',\frac{1}{f})\right).\end{aligned}$$

Lemma 5.3. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1\neq\eta_2$ and let f be finite logarithmic lower order meromorphic function. Let μ be the logarithmic lower order of f. Then for each $\varepsilon>0$, there exists a subset $E_2\subset(1,+\infty)$ of infinite logarithmic measure such that for all $r\in E_2$, we have*

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=O\big((\log r)^{\mu-1+\varepsilon}\big).$$

Proof. We have

$$\begin{aligned} m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)\le& m\left(r,\frac{f(z+\eta_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta_2)}\right)\\ \le& m\left(r,\frac{f(z+\eta_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta_1)}\right)\\ &+m\left(r,\frac{f(z+\eta_2)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta_2)}\right).\end{aligned}\tag{5.4}$$

Since f has finite logarithmic lower order $\mu_{\log}(f) = \mu < \infty$, so by Lemma 5.1, for any given $\varepsilon\big(0 < \varepsilon < 2\big)$, there exists a subset E_2 of infinite logarithmic measure such that for all $r \in E_2$, we have

$$T(r,f) \leq (\log r)^{\mu+\frac{\varepsilon}{2}}. \tag{5.5}$$

By Lemma 5.2, we obtain from (5.4)

$$\begin{aligned} m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right) \leq & \frac{2|\eta_1|R}{(R-r-|\eta_1|)^2}\left(m\big(R,f\big)+m\big(R,\frac{1}{f}\big)\right) \\ &+\frac{2R'}{R'-R}\left(\frac{|\eta_1|}{R-r-|\eta_1|}+\frac{C_\alpha|\eta_1|^\alpha}{(1-\alpha)r^\alpha}\right)\left(N\big(R',f\big)+N\big(R',\frac{1}{f}\big)\right) \\ &+\frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\left(m\big(R,f\big)+m\big(R,\frac{1}{f}\big)\right) \\ &+\frac{2R'}{R'-R}\left(\frac{|\eta_2|}{R-r-|\eta_2|}+\frac{C_\alpha|\eta_2|^\alpha}{(1-\alpha)r^\alpha}\right)\left(N\big(R',f\big)+N\big(R',\frac{1}{f}\big)\right) \\ =&\left(\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2}+\frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\right)\left(m\big(R,f\big)+m\big(R,\frac{1}{f}\big)\right) \\ &+\frac{2R'}{R'-R}\left(\frac{|\eta_1|}{R-r-|\eta_1|}+\frac{C_\alpha|\eta_1|^\alpha}{(1-\alpha)r^\alpha}+\frac{|\eta_2|}{R-r-|\eta_2|}\right. \\ &\left.+\frac{C_\alpha|\eta_2|^\alpha}{(1-\alpha)r^\alpha}\right)\left(N\big(R',f\big)+N\big(R',\frac{1}{f}\big)\right). \end{aligned} \tag{5.6}$$

We choose $\alpha = 1-\frac{\varepsilon}{2}$, $R = 2r$, $R' = 3r$ and $r > \max\{|\eta_1|,|\eta_2|,\frac{1}{2}\}$ in (5.6), we obtain

$$\begin{aligned} m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right) \leq & \left(\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right)\left(m\big(2r,f\big)+m\big(2r,\frac{1}{f}\big)\right) \\ &+6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{2C_\alpha|\eta_1|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}+\frac{|\eta_2|}{r-|\eta_2|}+\frac{2C_\alpha|\eta_2|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right) \\ &\times\left(N\big(3r,f\big)+N\big(3r,\frac{1}{f}\big)\right) \\ \leq & 4\left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}+6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{|\eta_2|}{r-|\eta_2|}\right.\right. \\ &\left.\left.+\frac{2C_\alpha\big(|\eta_1|^{1-\frac{\varepsilon}{2}}+|\eta_2|^{1-\frac{\varepsilon}{2}}\big)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right]T\big(3r,f\big). \end{aligned} \tag{5.7}$$

Using the estimate (5.5), we get

$$\begin{aligned} m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right) \leq & 4K\left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2}+\frac{4|\eta_2|r}{(r-|\eta_2|)^2}+6\left(\frac{|\eta_1|}{r-|\eta_1|}+\frac{|\eta_2|}{r-|\eta_2|}\right.\right. \\ &\left.\left.+\frac{2C_\alpha\big(|\eta_1|^{1-\frac{\varepsilon}{2}}+|\eta_2|^{1-\frac{\varepsilon}{2}}\big)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right]\big(\log 3r\big)^{\mu+\frac{\varepsilon}{2}} \\ \leq & M\big(\log r\big)^{\mu+\varepsilon-1}, \end{aligned}$$

where $K > 0$, $M > 0$ are some constants. The proof is completed. □

Lemma 5.4 ([4]). *Let c_1, c_2 be two arbitrary complex numbers such that $c_1 \neq c_2$ and let f be finite logarithmic order meromorphic function. Let ρ be the logarithmic order of f. Then for each $\varepsilon > 0$, we have*

$$m\left(r,\frac{f(z+c_1)}{f(z+c_2)}\right) = O\big((\log r)^{\rho-1+\varepsilon}\big).$$

Lemma 5.5 ([34]). *Let f be a meromorphic function, c be a non-zero complex constant. Then we have that for $r \longrightarrow \infty$*

$$\big(1+o(1)\big)T(r-|c|,f) \leq T(r,f(z+c)) \leq \big(1+o(1)\big)T(r+|c|,f).$$

It follows that $\rho_{\log}(f(z+c)) = \rho_{\log}(f)$ and $\mu_{\log}(f(z+c)) = \mu_{\log}(f)$.

5.4 Proofs of the theorems

In our proofs, we suppose always that f is of finite logarithmic order $\big(\rho_{\log}(f) < \infty\big)$, otherwise the results are trivial.

Proof of Theorem 5.4

Proof. Let $f(z)(\not\equiv 0)$ be a meromorphic solution of (5.1). We divide (5.1) by $f(z+c_l)$ to get

$$-A_l(z) = \sum_{j=1, j\neq l}^{k} A_j(z)\frac{f(z+c_j)}{f(z+c_l)} + A_0(z)\frac{f(z)}{f(z+c_l)}, \tag{5.8}$$

it follows

$$m(r,A_l(z)) \leq \sum_{j=0, j\neq l}^{k} m(r,A_j(z)) + \sum_{j=1, j\neq l}^{k} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1). \tag{5.9}$$

By (5.9) and Lemma 5.3, for any given $\varepsilon > 0$, there exists a subset $E_2 \subset (1,+\infty)$ of infinite logarithmic measure such that for all $r \in E_2$, we have

$$\begin{aligned} m(r,A_l(z)) &\leq \sum_{j=0, j\neq l}^{k} m(r,A_j(z)) + \sum_{j=1, j\neq l}^{k} O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big) + O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big) + O(1) \\ &\leq \sum_{j=0, j\neq l}^{k} m(r,A_j(z)) + O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \end{aligned} \tag{5.10}$$

Assume that $\operatorname{mlim\,sup}_{r\longrightarrow+\infty} \frac{\sum_{j=0, j\neq l}^{k} m(r,A_j)}{m(r,A_l)} < \beta < 1$, then for sufficiently large r, we have

$$\sum_{j=0, j\neq l}^{k} m(r,A_j) < \beta m(r,A_l). \tag{5.11}$$

Combining (5.10) and (5.11), for any given $\varepsilon > 0$ and for all $r \in E_2$, we obtain

$$(1-\beta)m(r,A_l(z)) \leq O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \tag{5.12}$$

Setting

$$\liminf_{r\longrightarrow+\infty} \frac{m(r,A_l)}{T(r,A_l)} = \delta(\infty,A_l) = \delta > 0. \tag{5.13}$$

By (5.12) and the definition of $\mu_{\log}(A_l)$, for any given ε and sufficiently large r, we have

$$m(r,A_l) \geq \frac{\delta}{2}T(r,A_l) \geq \frac{\delta}{2}(\log r)^{\mu_{\log}(A_l)-\frac{\varepsilon}{2}} \geq (\log r)^{\mu_{\log}(A_l)-\varepsilon}. \tag{5.14}$$

Substituting (5.14) into (5.12), for any given $\varepsilon > 0$ and for all $r \in E_2$, we get

$$(1-\beta)(\log r)^{\mu_{\log}(A_l)-\varepsilon} \leq O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big), \tag{5.15}$$

which implies that $\mu_{\log}(A_l)+1-2\varepsilon \leq \mu_{\log}(f)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\mu_{\log}(A_l)+1 \leq \mu_{\log}(f)$. □

Proof of Theorem 5.5

Proof. By (5.9), for any given $\varepsilon > 0$ and for all $r \in E_2$, we have

$$m(r,A_l(z)) \leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \tag{5.16}$$

Suppose that $\max\{\rho_{\log}(A_j): j=0,1,\ldots,k, j\neq l\} = \rho < \mu_{\log}(A_l)$. Then, by the definition of $\rho_{\log}(A_j)$, $j=0,1,\ldots,k, j\neq l$, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_l)-\rho}{2}\right)$ and sufficiently large r, we have

$$T(r,A_j) \leq (\log r)^{\rho+\varepsilon}, \qquad j=0,1,\ldots,k, j\neq l. \tag{5.17}$$

Substituting (5.12) and (5.17) into (5.16), for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_l)-\rho}{2}\right)$ and for all $r \in E_2$, we obtain

$$(\log r)^{\mu_{\log}(A_l)-\varepsilon} \leq k(\log r)^{\rho+\varepsilon} + O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \tag{5.18}$$

Then

$$(1-o(1))(\log r)^{\mu_{\log}(A_l)-\varepsilon} \leq O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \tag{5.19}$$

It follows that $\mu_{\log}(A_l)+1-2\varepsilon \leq \mu_{\log}(f)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\mu_{\log}(A_l)+1 \leq \mu_{\log}(f)$. □

Proof of Theorem 5.6

Proof. Let $f(z)(\not\equiv 0)$ be a meromorphic solution of (5.2). We divide (5.2) by $f(z+c_l)$ to get

$$-A_l(z) = \sum_{j=1,j\neq l}^{k} A_j(z)\frac{f(z+c_j)}{f(z+c_l)} + A_0(z)\frac{f(z)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}, \tag{5.20}$$

it follows

$$\begin{aligned} m(r,A_l(z)) \leq & \sum_{j=0,j\neq l}^{k} m(r,A_j(z)) + \sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r,\frac{f(z)}{f(z+c_l)}\right) \\ & + m(r,F(z)) + m\left(r,\frac{1}{f(z+c_l)}\right) + O(1). \end{aligned} \tag{5.21}$$

By (5.21), Lemma 5.4 and Lemma 5.5, for any given $\varepsilon > 0$, we have

$$\begin{aligned} m(r,A_l(z)) \leq & \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) \\ & + T(r,F(z)) + T(r,f(z+c_l)) + O(1) \\ \leq & \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) \\ & + T(r,F(z)) + \big(1+o(1)\big)T(r+|c_l|,f(z)) \\ \leq & \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + T(r,F(z)) + 2T(2r,f(z)). \end{aligned} \tag{5.22}$$

Setting $\liminf_{r\longrightarrow+\infty} \frac{m(r,A_l)}{T(r,A_l)} = \delta(\infty,A_l) = \delta > 0$ and $\max\{\rho_{\log}(A_j)\colon j=0,1,\ldots,k, j\neq l\} = \rho < \mu_{\log}(A_l)$. Then, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_l)-\rho}{2}\right)$ and sufficiently large r, the assumptions (5.14) and (5.17) hold.

(1) If $\mu_{\log}(F) < \mu_{\log}(A_l)$, then by Lemma 5.1, there exists a subset E_1 with infinite logarithmic measure such that for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_l)-\mu_{\log}(F)}{2}\right)$ and for all $r\in E_1$, we have

$$T(r,F) \leq (\log r)^{\mu_{\log}(F)+\varepsilon}. \tag{5.23}$$

By substituting (5.14), (5.17) and (5.23) into (5.22), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_l)-\rho}{2}, \frac{\mu_{\log}(A_l)-\mu_{\log}(F)}{2}\right\}$$

and for all $r\in E_1$, we obtain

$$\begin{aligned} (\log r)^{\mu_{\log}(A_l)-\varepsilon} \leq & k(\log r)^{\rho+\varepsilon} + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) \\ & + (\log r)^{\mu_{\log}(F)+\varepsilon} + O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big), \end{aligned} \tag{5.24}$$

which implies that

$$(1-o(1))(\log r)^{\mu_{\log}(A_l)-\varepsilon} \leq O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \tag{5.25}$$

By (5.25), we get $\mu_{\log}(A_l)-2\varepsilon \leq \rho_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we deduce that $\mu_{\log}(A_l) \leq \rho_{\log}(f)$.

(2) Let f be a meromorphic solution of (5.2). If $\mu_{\log}(F) > \mu_{\log}(A_l)$, then for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(F)-\mu_{\log}(A_l)}{2}\right)$ and sufficiently large r, we have

$$T(r,F) \geq (\log r)^{\mu_{\log}(F)-\varepsilon}. \tag{5.26}$$

By Lemma 5.1, there exists a subset E_1 with infinite logarithmic measure such that for the above ε and for all $r\in E_1$, we obtain

$$T(r,A_l) \leq (\log r)^{\mu_{\log}(A_l)+\varepsilon}. \tag{5.27}$$

By (5.2) and Lemma 5.5, we have

$$\begin{aligned} T(r,F(z)) &\leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + T(r,A_l(z)) + \sum_{j=1}^{k} T(r,f(z+c_j)) \\ &\quad + T(r,f(z)) + O(1) \qquad (5.28) \\ &\leq \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + T(r,A_l(z)) + (2k+1)T(2r,f(z)) + O(1). \end{aligned}$$

Substituting (5.17), (5.26) and (5.27) into (5.28), for the above ε and for all $r \in E_1$, we get

$$(\log r)^{\mu_{\log}(F)-\varepsilon} \leq k(\log r)^{\rho+\varepsilon} + (\log r)^{\mu_{\log}(A_l)+\varepsilon} + (2k+1)T(2r,f(z)) + O(1). \qquad (5.29)$$

So

$$(1-o(1))(\log r)^{\mu_{\log}(F)-\varepsilon} \leq O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \qquad (5.30)$$

It follows that $\mu_{\log}(F) - 2\varepsilon \leq \rho_{\log}(f)$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_{\log}(F) \leq \rho_{\log}(f)$.

□

Chapter 6

Linear delay-differential equations with zero order meromorphic coefficients

6.1 Introduction

This chapter is devoted to consider the homogeneous and non-homogeneous linear delay-differential equations

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i)=0, \tag{6.1}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i)=F(z), \tag{6.2}$$

where $A_{ij}(z)$ $(i=0,1,\ldots,n,\, j=0,1,\ldots,m, n,m\in\mathbb{N})$ and $F(z)$ are meromorphic functions, $c_i(i=0,\ldots,n)$ are distinct non-zero complex constants. Recently, the research on the growth properties of meromorphic solutions of the complex delay-differential equations has gathered increasing attention (see e.g. [11,12,17,52,71,81]. In [12], Bellaama and Belaïdi considered the growth of equations (6.1) and (6.2) for the case where one arbitrary coefficient dominates the rest of the coefficients either by its lower order or by its lower type, and obtained the following theorem.

Theorem 6.1 ([12]). *Consider the delay differential equation with meromorphic coefficients. Suppose that one of the coefficients, say A_{l0} with $\mu(A_{l0})>0$, is dominate in the sens that:*

(i) $\max\{\mu(A_{ab}),\rho(\mathcal{S})\}\le\mu(A_{l0})<\infty$;

(ii) $\underline{\tau}(A_{l0})>\underline{\tau}(A_{ab})$, *whenever* $\mu(A_{l0})=\mu(A_{ab})$;

(iii) $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau(A_{ij})+\tau(F)<\underline{\tau}(A_{l0})<\infty$, *whenever* $\mu(A_{l0})=\rho(\mathcal{S})$;

(iv) $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau(A_{ij})+\underline{\tau}(A_{ab})<\underline{\tau}(A_{l0})<\infty$, *whenever* $\mu(A_{l0})=\mu(A_{ab})=\rho(\mathcal{S})$;

(v) $\lambda\left(\frac{1}{A_{l0}}\right)<\mu(A_{l0})<\infty$, *where* $\mathcal{S}:=\{F,A_{ij}\colon (i,j)\neq(a,b),(l,0)\}$ *and* $\rho(\mathcal{S}):=\max\{\rho(g)\colon g\in\mathcal{S}\}$.

Then any meromorphic solution f of (6.2) satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (6.2) satisfies $\rho(f) \geq \mu(A_{l0})+1$.

They also in [11], proved the following result in which the assumption on the dominance of the coefficient $A_{l0}(z)$ by the lower order or the lower type, was made in different way

Theorem 6.2 ([11]). *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m\in\mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_{l0}(z)$ satisfies*

$$\lambda(\frac{1}{A_{l0}}) < \mu(A_{l0}) < \infty,$$

$$\max\{\rho(A_{ij})\colon (i,j)\neq(l,0)\} \leq \mu(A_{l0}),$$

$$\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0)} \tau(A_{ij}) < \underline{\tau}(A_{l0}) < \infty.$$

1. *If $\rho(F) < \mu(A_{l0})$, or $\rho(F) = \mu(A_{l0})$ and $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0)} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{l0})$, or $\mu(F) = \mu(A_{l0})$ and $\sum_{\rho(A_{ij})=\mu(A_{l0}),(i,j)\neq(l,0)} \tau(A_{ij}) + \underline{\tau}(A_{l0}) < \underline{\tau}(F)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (6.2) satisfies $\rho(f) \geq \mu(A_{l0})$. Further, if $F(z)\equiv 0$, then $\rho(f) \geq \mu(A_{l0})+1$.
2. If $\mu(F) > \mu(A_{l0})$, then every meromorphic solution $f(z)$ of (6.2) satisfies $\rho(f) \geq \mu(F)$.

As an answer to the question how to express the growth of solutions of (6.1), for the case when its coefficients are meromorphic functions of order zero. In [9], Belaïdi used the logarithmic order and obtained the following theorem.

Theorem 6.3 ([9]). *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m\in\mathbb{N})$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_{l0}(z)$ satisfies*

$$\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\} < \rho_{\log}(A_{l0}),$$

$$\delta(\infty, A_{l0}) > 0.$$

Then every meromorphic solution $f(z)(\not\equiv 0)$ of (6.1) satisfies $\rho_{\log}(f) \geq \rho_{\log}(A_{l0})+1$.

In [13], Biswas generalized the above result to non-homogeneous equation (6.2), by proving the following theorem

Theorem 6.4 ([13]). *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m\in\mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\leq l\leq k)$ such that $A_{l0}(z)$ satisfies*

$$\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\} < \rho_{\log}(A_{l0}),$$

$$\delta(\infty, A_{l0}) > 0.$$

1. *If $\rho_{\log}(F) < \rho_{\log}(A_{l0})$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (6.2) satisfies $\rho_{\log}(f) \geq \rho_{\log}(A_{l0})$.

2. If $\rho_{\log}(F) > \rho_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of (6.1) satisfies $\rho_{\log}(f) \geq \rho_{\log}(F)$.

The main purpose of this chapter is to continue investigating the logarithmic order of meromorphic solutions of equations (6.1) and (6.2) to extend and improve the above theorems. In our results, alongside the other additional conditions, the dominance of the arbitrary coefficient A_{l0} is assumed in term of the logarithmic lower order and the logarithmic lower type in two different way as in Theorem 6.1 and Theorem 6.2.

6.2 Main Results

Theorem 6.5 ([23]). *Let $A_{ij}(z)$ $(i = 0,1,\ldots,n, j = 0,1,\ldots,m, n,m \in \mathbb{N})$ be meromorphic functions, and $a,l \in \{0,1,...,n\}$, $b \in \{0,1,...,m\}$ such that $(a,b) \neq (l,0)$. Suppose that one of the coefficients, say A_{l0} with $\lambda_{\log}\left(\frac{1}{A_{l0}}\right) + 1 < \mu_{\log}(A_{l0}) < \infty$ is dominate in the sens that:*

(i) $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(\mathcal{S})\} \leq \mu_{\log}(A_{l0}) < \infty$;

(ii) $\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$;

(iii) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \rho_{\log}(\mathcal{S})$;

(iv) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab}) < \underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(\mathcal{S})$, *where* $\mathcal{S} := \{F, A_{ij} \colon (i,j) \neq (a,b),(l,0)\}$ *and* $\rho_{\log}(\mathcal{S}) := \max\{\rho_{\log}(g) \colon g \in \mathcal{S}\}$.

Then any meromorphic solution f of (6.2) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (6.1) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Theorem 6.6 ([23]). *Let $A_{ij}(z)$ $(i = 0,1,\ldots,n, j = 0,1,\ldots,m, n,m \in \mathbb{N})$ be meromorphic functions, and $a,l \in \{0,1,...,n\}$, $b \in \{0,1,...,m\}$ such that $(a,b) \neq (l,0)$. Suppose that one of the coefficients, say A_{l0} with $\mu(A_{l0}) > 0$ and $\delta(\infty, A_{l0}) > 0$ is dominate in the sens that:*

(i) $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(\mathcal{S})\} \leq \mu_{\log}(A_{l0}) < \infty$;

(ii) $\delta\underline{\tau}_{\log}(A_{l0}) > \underline{\tau}_{\log}(A_{ab})$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab})$;

(iii) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) < \delta\underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \rho_{\log}(\mathcal{S})$;

(iv) $\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab}) < \delta\underline{\tau}_{\log}(A_{l0}) < \infty$, *whenever* $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(\mathcal{S})$, *where* $\mathcal{S} := \{F, A_{ij} \colon (i,j) \neq (a,b),(l,0)\}$ *and* $\rho_{\log}(\mathcal{S}) := \max\{\rho_{\log}(g) \colon g \in \mathcal{S}\}$.

Then any meromorphic solution f of (6.2) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$ if $F(z)(\not\equiv 0)$. Further if $F(z)(\equiv 0)$, then any meromorphic solution $f(z)(\not\equiv 0)$ of (6.1) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Theorem 6.7 ([24]). *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m\in\mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\le l\le k)$ such $A_{l0}(z)$ satisfies $\delta(\infty,A_{l0})>0$ and* $\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})\le\rho_{\log}(A_{l0})<\infty$.

1. *If $\mu_{\log}(F)<\mu_{\log}(A_{l0})$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (6.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(A_{l0})$. Further, if $F(z)\equiv 0$, then $\mu_{\log}(f)\ge\mu_{\log}(A_{l0})+1$.

2. If $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of (6.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(F)$.

Remark 6.1. *We can also replace the condition* $\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})\le\rho_{\log}(A_{l0})$ *in* Theorem 6.7 by

$$\limsup_{r\longrightarrow+\infty}\frac{\sum_{(i,j)\neq(l,0)}m(r,A_{ij})}{m(r,A_{l0})}<1$$

for the homogeneous case $F(z)\equiv 0$.

Theorem 6.8 ([23]). *Let $A_{ij}(z)$ $(i=0,1,\ldots,n, j=0,1,\ldots,m, n,m\in\mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer $l(0\le l\le k)$ such that $A_{l0}(z)$ satisfies*

$$\lambda_{\log}(\frac{1}{A_{l0}})+1<\mu_{\log}(A_{l0})<\infty,$$

$$\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}\le\mu_{\log}(A_{l0}),$$

$$\tau=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0})<\infty.$$

1. *If $\rho_{\log}(F)<\mu_{\log}(A_{l0})$, or $\rho_{\log}(F)=\mu_{\log}(A_{l0})$ and $\tau+\tau_{\log}(F)<\underline{\tau}_{\log}(A_{l0})$, or $\mu_{\log}(F)=\mu_{\log}(A_{l0})$ and $\tau+\underline{\tau}_{\log}(A_{l0})<\underline{\tau}_{\log}(F)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of* (6.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(A_{l0})$. Further, if $F(z)\equiv 0$, then $\mu_{\log}(f)\ge\mu_{\log}(A_{l0})+1$.

2. If $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of (6.2) satisfies $\rho_{\log}(f)\ge\mu_{\log}(F)$.

Remark 6.2. *The condition $\lambda_{\log}(\frac{1}{A_{l0}})+1<\mu_{\log}(A_{l0})$ in* Theorem 6.8 can be replaced by $\delta(\infty,A_{l0})>0$ with $\delta\underline{\tau}_{\log}(A_{l0})$ instead of $\underline{\tau}_{\log}(A_{l0})$, the only difference between the two conditions that by the condition $\delta(\infty,A_{l0})>0$ the case when $\mu_{\log}(A_{l0})=1$ is also included.

6.3 Lemmas

For the proof of our results we need the following lemmas.

Lemma 6.1 ([39]). *Let k and j be integers such that $k>j\ge 0$. Let f be a meromorphic function in the plane $\mathbb{C}$ such that $f^{(j)}$ does not vanish identically. Then, there exists an $r_0>1$ such that*

$$m\left(r,\frac{f^{(k)}}{f^{(j)}}\right)\le(k-j)\log^{+}\frac{\rho(T(\rho,f))}{r(\rho-r)}+\log\frac{k!}{j!}+5.3078(k-j),$$

for all $r_0 < r < \rho < +\infty$. If f is of finite order s, then

$$\limsup_{r\to+\infty} \frac{m\left(r, \frac{f^{(k)}}{f^{(j)}}\right)}{\log r} \leq \max\{0, (k-j)(s-1)\}.$$

Lemma 6.2. *Let f be a meromorphic function with finite logarithmic lower order $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a subset E_2 of $[1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E_2$, we have*

$$\underline{\tau}_{\log}(f) = \lim_{r\longrightarrow+\infty} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Consequently, for any given $\varepsilon > 0$ and for all $r \in E_2$, we have

$$T(r,f) < (\underline{\tau}_{\log}(f) + \varepsilon)(\log r)^{\mu_{\log}(f)}.$$

Proof. By the definition of the logarithmic lower type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$, and

$$\underline{\tau}_{\log}(f) = \lim_{r_n\to+\infty} \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we have

$$\frac{T(\frac{n}{n+1} r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} \leq \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}} \leq \frac{T(r_n,f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}}.$$

It follows that

$$\left(\frac{\log \frac{n}{n+1} r_n}{\log r_n}\right)^{\mu_{\log}(f)} \frac{T(\frac{n}{n+1} r_n, f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}} \leq \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}$$
$$\leq \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}} \left(\frac{\log r_n}{\log \frac{n}{n+1} r_n}\right)^{\mu_{\log}(f)}. \tag{6.3}$$

Set

$$E_2 = \bigcup_{n=n_1}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right].$$

Then from (6.3, we obtain

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}} = \lim_{r_n\to+\infty} \frac{T(r_n,f)}{(\log r_n)^{\mu_{\log}(f)}} = \underline{\tau}_{\log}(f),$$

so for any given $\varepsilon > 0$ and all sufficiently large $r \in E_2$, we get

$$T(r,f) < \left(\underline{\tau}_{\log}(f) + \varepsilon\right) (\log r)^{\mu_{\log}(f)},$$

where $m_{\log}(E_2) = \int_{E_2} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\left(1+\frac{1}{n}\right) = +\infty.$ □

Lemma 6.3 ([38]). *Let f be a meromorphic function and $k \geq 1$ be an integer. Then we have*

$$T(r, f^{(k)}) \leq (k+1) T(r,f) + S(r,f),$$

where $S(r,f)$ satisfies the condition (1.3). In particular, if f is of finite order, then (1.3) holds without the excluded set.

6.4 Proof of the theorems

Proof of Theorem 6.5

Proof. Let $f(z)$ be a meromorphic solution of (6.2). If $f(z)$ has infinite logarithmic order, then the result holds. Now, we suppose that $\rho_{\log}(f) < \infty$. We divide (6.2) by $f(z+c_l)$ to get

$$-A_{l0}(z) = \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} A_{ij}\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)}$$

$$+\sum_{j=0,j\neq b}^{m} A_{aj}\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\frac{f(z+c_a)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj}\frac{f^{(j)}(z+c_l)}{f(z+c_l)}$$

$$+A_{ab}\frac{f^{(b)}(z+c_a)}{f(z+c_a)}\frac{f(z+c_a)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}. \tag{6.4}$$

By (6.4) and Lemma 5.5, for sufficiently large r, we have

$$m(r,A_{l0}(z)) \leq \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} m(r,A_{ij}(z)) + m(r,A_{ab}(z))$$

$$+\sum_{j=1}^{m} m(r,A_{lj}(z)) + \sum_{j=0,j\neq b}^{m} m(r,A_{aj}(z)) + \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)$$

$$+\sum_{i=0,i\neq l,a}^{n} m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right) + \sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right) + 2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)$$

$$+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right) + m(r,F(z)) + m\left(r,\frac{1}{f(z+c_l)}\right) + O(1)$$

$$\leq \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + T(r,A_{ab}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z))$$

$$+\sum_{j=0,j\neq b}^{m} T(r,A_{aj}(z)) + \sum_{i=0,i\neq l,a}^{n}\sum_{j=0}^{m} m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right)$$

$$+\sum_{i=0,i\neq l,a}^{n} m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right) + \sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right) + 2m\left(r,\frac{f(z+c_a)}{f(z+c_l)}\right)$$

$$+\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_a)}{f(z+c_a)}\right) + T(r,F(z)) + 2T(2r,f) + O(1). \tag{6.5}$$

From Lemma 6.1, for sufficiently large r, we obtain

$$m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) \leq 2j\log^{+} T(2r,f),\ (i=0,1,\ldots,n, j=1,\ldots,m). \tag{6.6}$$

By Lemma 5.4, for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right) = O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),\ (i=0,1,\ldots,n, i\neq l). \tag{6.7}$$

By substituting (6.6) and (6.7) into (6.5), for any given $\varepsilon > 0$ and all sufficiently large r, we obtain

$$m(r,A_{l0}(z)) \leq \sum_{i=0,i\neq l,a}^{n} \sum_{j=0}^{m} T(r,A_{ij}(z)) + T(r,A_{ab}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + \sum_{j=0,j\neq b}^{m} T(r,A_{aj}(z))$$

$$+O\left(\log^{+} T(2r,f)\right) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + T(r,F(z)) + 2T(2r,f). \tag{6.8}$$

Let us set

$$\delta = \delta(\infty, A_{l0}) > 0. \tag{6.9}$$

Now, we divide this proof into four cases:

Case (i): If $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(\mathcal{S})\} < \mu_{\log}(A_{l0})$, then by the definition of $\mu_{\log}(A_{l0})$ and (6.9), for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$m(r,A_{l0}) \geq \frac{\delta}{2} T(r,A_{l0}) \geq \frac{\delta}{2} (\log r)^{\mu_{\log}(A_{l0})-\frac{\varepsilon}{2}} \geq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}. \tag{6.10}$$

By the definition of $\rho_{\log}(\mathcal{S})$ for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r,g) \leq (\log r)^{\rho_{\log}(\mathcal{S})+\varepsilon}, \quad g \in \mathcal{S}. \tag{6.11}$$

By the definition of $\mu_{\log}(A_{ab})$ and Lemma 5.1, there exists a subset $E_1 \subset (1, +\infty)$ of infinite logarithmic measure such that for any given $\varepsilon > 0$ and for all sufficiently large $r \in E_1$, we have

$$T(r,A_{ab}) \leq (\log r)^{\mu_{\log}(A_{ab})+\varepsilon}. \tag{6.12}$$

Set $\rho = \max\{\mu_{\log}(A_{ab}), \rho_{\log}(\mathcal{S})\}$, then from (6.11) and (6.12), for any given $\varepsilon > 0$ and for all $r \in E_1$, it follows

$$\max\{T(r,A_{ab}), T(r,g)\} \leq (\log r)^{\rho+\varepsilon}. \tag{6.13}$$

Also, from the definition of $\rho_{\log}(f)$ for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r,f) \leq (\log r)^{\rho_{\log}(f)+\varepsilon}. \tag{6.14}$$

By substituting (6.10), (6.13) and (6.14) into (6.8), for all $r \in E_1$, we get

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq ((n-1)(m+1)+2m+1)(\log r)^{\rho+\varepsilon} + O(\log(\log r))$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + (\log r)^{\rho+\varepsilon} + O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \tag{6.15}$$

Now, we choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_{\log}(A_{l0}) - \rho$, for all $r \in E_1$, it follows from (6.15) that

$$(1-o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

this means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

Similarly, for the homogeneous case, by (6.1), (6.6) and (6.7), we obtain

$$m(r,A_{l0}(z)) \leq \sum_{i=0,i\neq l,a}^{n} \sum_{j=0}^{m} T(r,A_{ij}(z)) + T(r,A_{ab}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + \sum_{j=0,j\neq b}^{m} T(r,A_{aj}(z))$$

$$+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{6.16}$$

Then, by substituting (6.10) and (6.13) into (6.16), for all sufficiently large $r \in E_1$, we have

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq ((n-1)(m+1)+2m+1)(\log r)^{\rho+\varepsilon}+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{6.17}$$

For the above ε and for all $r \in E_1$, we deduce from (6.17) that

$$(1-o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-2\varepsilon \leq \rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})+1$.

Case (ii): If $\beta=\rho_{\log}(\mathcal{S})<\mu_{\log}(A_{l0})=\mu_{\log}(A_{ab})$ and $\delta\underline{\tau}_{\log}(A_{l0})>\underline{\tau}_{\log}(A_{ab})$, then by the definition of $\underline{\tau}_{\log}(A_{l0})$ and (6.9), for any given $\varepsilon>0$ and all sufficiently large r, we have

$$\begin{aligned} m(r,A_{l0}) &\geq (\delta-\varepsilon)T(r,A_{l0}) \geq (\delta-\varepsilon)(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &\geq \left(\delta\underline{\tau}_{\log}(A_{l0})-(\underline{\tau}_{\log}(A_{l0})+\delta)\varepsilon+\varepsilon^2\right)(\log r)^{\mu_{\log}(A_{l0})} \\ &\geq \left(\delta\underline{\tau}_{\log}(A_{l0})-(\underline{\tau}_{\log}(A_{l0})+\delta)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}. \end{aligned} \tag{6.18}$$

By the definition of $\underline{\tau}_{\log}(A_{ab})$ and Lemma 6.2, there exists a subset $E_1 \subset (1,+\infty)$ of infinite logarithmic measure such that for any given $\varepsilon>0$ and for all sufficiently large $r \in E_1$, we obtain

$$T(r,A_{ab}) \leq (\underline{\tau}_{\log}(A_{ab})+\varepsilon)(\log r)^{\mu_{\log}(A_{ab})}=(\underline{\tau}_{\log}(A_{ab})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.19}$$

By substituting (6.11), (6.14), (6.18) and (6.19) into (6.8), for all sufficiently large $r \in E_1$, we get

$$\begin{aligned} \left(\delta\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-(\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} &\leq ((n-1)(m+1)+2m)(\log r)^{\beta+\varepsilon} \\ +O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\beta+\varepsilon}+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \end{aligned} \tag{6.20}$$

Now, we choose sufficiently small ε satisfying $0<\varepsilon<\min\{\frac{\mu_{\log}(A_{l0})-\beta}{2},\frac{\delta\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})}{\underline{\tau}_{\log}(A_{l0})+\delta+1}\}$, for all sufficiently large $r \in E_1$, by (3.20), we obtain

$$\begin{aligned} (1-o(1))\left(\delta\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-(\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} \\ \leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right), \end{aligned}$$

this means, $\mu_{\log}(A_{l0})-\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

Next, for the homogeneous case, by substituting (6.11), (6.18) and (6.19) into (6.16), for all sufficiently large $r \in E_1$, we have

$$\begin{aligned} \left(\delta\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-(\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})} &\leq ((n-1)(m+1)+2m)(\log r)^{\beta+\varepsilon} \\ +O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \end{aligned} \tag{6.21}$$

For the above ε and all sufficiently large $r \in E_1$, from (6.16), we obtain

$$(1-o(1))\left(\delta\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-(\underline{\tau}_{\log}(A_{l0})+\delta+1)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-\varepsilon \leq \rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})+1$.

Case (iii): When $\mu_{\log}(A_{ab}) < \mu_{\log}(A_{l0}) = \rho_{\log}(\mathcal{S})$ and

$$\tau_1 = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F)$$

$$= \tau + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0}),\ \tau = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)} \tau_{\log}(A_{ij}).$$

Then, there exists a subset $J \subseteq \{0,1,\ldots,n\}\times\{0,1,\ldots,m\}\setminus\{(l,0),(a,b)\}$ such that for all $(i,j)\in J$, when $\rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0})$, we have $\sum_{(i,j)\in J} \tau_{\log}\left(A_{ij}\right) < \delta\underline{\tau}_{\log}(A_{l0}) - \tau_{\log}(F)$, and for $(i,j)\in\Pi = \{0,1,\ldots,n\}\times\{0,1,\ldots,m\}\setminus(J\cup\{(l,0),(a,b)\})$ we have $\rho_{\log}\left(A_{ij}\right) < \mu_{\log}(A_{l0})$. Hence, for any given $\varepsilon>0$ and all sufficiently large r, we get

$$T\left(r,A_{ij}\right) \leq \begin{cases} \left(\tau_{\log}(A_{ij})+\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}, & \text{if } (i,j)\in J, \\ (\log r)^{\rho_{\log}(A_{ij})+\varepsilon} \leq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, & \text{if } (i,j)\in\Pi \end{cases} \tag{6.22}$$

and

$$T(r,F) \leq \begin{cases} \left(\tau_{\log}(F)+\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}, & \text{if } \rho_{\log}(F)=\mu_{\log}(A_{l0}), \\ (\log r)^{\rho_{\log}(F)+\varepsilon} \leq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, & \text{if } \rho_{\log}(F)<\mu_{\log}(A_{l0}). \end{cases} \tag{6.23}$$

By substituting (6.12), (6.14), (6.18), (6.22)and (6.23) into (6.8), for all sufficiently large $r\in E_1$, we get

$$\begin{aligned}(\delta\underline{\tau}_{\log}(A_{l0}) - \tau_1 - &\left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right) + (\log r)^{\mu_{\log}(A_{ab})+\varepsilon} + O(\log(\log r)) \\ &+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right).\end{aligned} \tag{6.24}$$

We may choose sufficiently small ε satisfying $0<\varepsilon<\min\{\frac{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab})}{2}, \frac{\underline{\tau}_{\log}(A_{l0})-\tau_1}{\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1}\}$, for all sufficiently large $r\in E_1$, by (6.24), we obtain

$$(1-o(1))(\delta\underline{\tau}_{\log}(A_{l0}) - \tau_1 - \left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n+1\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

this means, $\mu_{\log}(A_{l0})-\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.

Further, for the homogeneous case, by substituting (6.12), (6.18) and (6.22) into (6.16), for all sufficiently large $r\in E_1$, we get

$$(\delta\underline{\tau}_{\log}(A_{l0}) - \tau - \left(\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n\right)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right) + (\log r)^{\mu_{\log}(A_{ab})+\varepsilon} + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{6.25}$$

For ε sufficiently small satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab})}{2}, \frac{\underline{\tau}_{\log}(A_{l0})-\tau}{\underline{\tau}_{\log}(A_{l0})+\delta+mn+m+n}\right\},$$

and for all sufficiently large $r \in E_1$, from (6.25) we conclude

$$(1-o(1))(\delta \underline{\tau}_{\log}(A_{l0}) - \tau - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n\right) \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0}) - \varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$.

Case (iv): When $\mu_{\log}(A_{l0}) = \mu_{\log}(A_{ab}) = \rho_{\log}(\mathcal{S})$ and

$$\tau_3 = \sum_{\rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0}), (i,j) \neq (l,0),(a,b)} \tau_{\log}(A_{ij}) + \tau_{\log}(F) + \underline{\tau}_{\log}(A_{ab})$$

$$= \tau_2 + \tau_{\log}(F) < \delta \underline{\tau}_{\log}(A_{l0}),$$

$$\tau_2 = \sum_{\rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0}), (i,j) \neq (l,0),(a,b)} \tau_{\log}(A_{ij}) + \underline{\tau}_{\log}(A_{ab}).$$

Then, by substituting (6.14), (6.18), (6.19), (6.22) and (6.23) into (6.8), for all sufficiently large $r \in E_1$, we get

$$(\delta \underline{\tau}_{\log}(A_{l0}) - \tau_3 - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 2\right) \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right) + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right). \quad (6.26)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\delta \underline{\tau}_{\log}(A_{l0}) - \tau_3}{\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 2}$, for all sufficiently large $r \in E_1$, we deduce from (6.26) that

$$(1-o(1))(\delta \underline{\tau}_{\log}(A_{l0}) - \tau_3 - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 2\right) \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

this means, $\mu_{\log}(A_{l0}) - \varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

Also for the homogeneous case, by substituting (6.18), (6.19) and (6.22) into (6.16), for all sufficiently large $r \in E_1$, we have

$$(\delta \underline{\tau}_{\log}(A_{l0}) - \tau_2 - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 1)\right) \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right) + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \quad (6.27)$$

Thus, for sufficiently small ε satisfying $0 < \varepsilon < \frac{\delta \underline{\tau}_{\log}(A_{l0}) - \tau_2}{\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 1}$, for all sufficiently large $r \in E_1$, from (6.26) we obtain

$$(1-o(1))(\delta \underline{\tau}_{\log}(A_{l0}) - \tau_2 - \left(\underline{\tau}_{\log}(A_{l0}) + \delta + mn + m + n + 1\right) \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0}) - \varepsilon \leq \rho_{\log}(f) - 1$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) + 1$. $\square$

Proof of Theorem 6.6

Proof. Let $f(z)$ be a meromorphic solution of (6.2). If $f(z)$ has infinite logarithmic order, then the result holds. Now, we suppose that $\rho_{\log}(f) < \infty$. By the definition of $\lambda_{\log}\left(\frac{1}{A_{l0}}\right)$, for any given $\varepsilon > 0$ with sufficiently large r, we have

$$N(r,A_{l0}) \leq (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{6.28}$$

By (6.8) and (6.28), for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$\begin{aligned} T(r,A_{l0}(z)) &= m(r,A_{l0}(z)) + N(r,A_{l0}(z)) \\ &\leq \sum_{i=0,i\neq l,a}^{n} \sum_{j=0}^{m} T(r,A_{ij}(z)) + T(r,A_{ab}(z)) \\ &+ \sum_{j=1}^{m} T(r,A_{lj}(z)) + \sum_{j=0,j\neq b}^{m} T(r,A_{aj}(z)) + O\left(\log^{+} T(2r,f)\right) \\ &+ O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) + T(r,F(z)) + 2T(2r,f) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \end{aligned} \tag{6.29}$$

This proof is also divided into four cases:

Case (i): If $\max\{\mu_{\log}(A_{ab}), \rho_{\log}(\mathcal{S})\} < \mu_{\log}(A_{l0})$, then by the definition of $\mu_{\log}(A_{l0})$ for any given $\varepsilon > 0$ and all sufficiently large r, we have

$$T(r,A_{l0}) \geq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}. \tag{6.30}$$

By substituting (6.13), (6.14) and (6.30) into (6.29), for any given $\varepsilon > 0$ and all sufficiently large $r \in E_1$, we get

$$\begin{aligned} (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} &\leq ((n-1)(m+1)+2m+2)(\log r)^{\rho+\varepsilon} + O(\log(\log r)) + O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right) \\ &+ O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right) + (\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \end{aligned} \tag{6.31}$$

Now, we choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\left\{\mu_{\log}(A_{l0}) - \rho, \mu_{\log}(A_{l0}) - \lambda_{\log}\left(\frac{1}{A_{l0}}\right) - 1\right\},$$

for all sufficiently large $r \in E_1$, it follows from (6.31) that

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

that means, $\mu_{\log}(A_{l0}) - 2\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon > 0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Similarly, for the homogeneous case, by (6.16) and (6.28), we have

$$\begin{aligned} T(r,A_{l0}(z)) &= m(r,A_{l0}(z)) + N(r,A_{l0}(z)) \\ &\leq \sum_{i=0,i\neq l,a}^{n} \sum_{j=0}^{m} T(r,A_{ij}(z)) + T(r,A_{ab}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + \sum_{j=0,j\neq b}^{m} T(r,A_{aj}(z)) \end{aligned}$$

$$+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{6.32}$$

Then, by substituting (6.13) and (6.30) into (6.32), for all sufficiently large $r \in E_1$, we have

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq ((n-1)(m+1)+2m+1)(\log r)^{\rho+\varepsilon}+O(\log(\log r))$$

$$+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{6.33}$$

For sufficiently small ε satisfying

$$0<2\varepsilon<\min\left\{\mu_{\log}(A_{l0})-\rho, \mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1\right\},$$

and all sufficiently large $r \in E_1$, we deduce from (6.33) that

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-2\varepsilon \leq \rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})+1$.

Case (ii): If $\beta=\rho_{\log}(\mathcal{S})<\mu_{\log}(A_{l0})=\mu_{\log}(A_{ab})$ and $\underline{\tau}_{\log}(A_{l0})>\underline{\tau}_{\log}(A_{ab})$, then by the definition of $\underline{\tau}_{\log}(A_{l0})$, for any given $\varepsilon>0$ and all sufficiently large r, we have

$$T(r,A_{l0}) \geq (\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.34}$$

By substituting (6.11), (6.14), (6.19) and (6.34) into (6.29), for all sufficiently large $r \in E_1$, we get

$$(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq ((n-1)(m+1)+2m+1)(\log r)^{\beta+\varepsilon}$$

$$+(\underline{\tau}_{\log}(A_{ab})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{6.35}$$

Now, we choose sufficiently small ε satisfying $0<2\varepsilon<\min\{\mu_{\log}(A_{l0})-\beta, \mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1, \underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})\}$, for all sufficiently large $r \in E_1$, it follows from (6.35) that

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-2\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq (\log r)^{\rho_{\log}(f)+\varepsilon},$$

this means, $\mu_{\log}(A_{l0})-\varepsilon \leq \rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

Next, for the homogeneous case, by substituting (6.11), and (6.19) into (6.32), for all sufficiently large $r \in E_1$, we have

$$(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq ((n-1)(m+1)+2m)(\log r)^{\beta+\varepsilon}+(\underline{\tau}_{\log}(A_{ab})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}$$

$$+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \tag{6.36}$$

Now, we choose sufficiently small ε satisfying $0<2\varepsilon<\min\{\mu_{\log}(A_{l0})-\beta, \mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1, \underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})\}$, for all sufficiently large $r \in E_1$, we deduce from (6.36) that

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\underline{\tau}_{\log}(A_{ab})-2\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-\varepsilon\leq\rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})+1$.

Case (iii): When $\mu_{\log}(A_{ab})<\mu_{\log}(A_{l0})=\rho_{\log}(\mathcal{S})$ and

$$\tau_1=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau_{\log}(A_{ij})+\tau_{\log}(F)$$

$$=\tau+\tau_{\log}(F)<\underline{\tau}_{\log}(A_{l0}),\ \tau=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau_{\log}(A_{ij}).$$

Then, for any given $\varepsilon>0$ and all sufficiently large r, (6.22) and (6.23) hold. By substituting (6.12), (6.14), (6.22), (6.23) and (6.34) into (6.29), for all sufficiently large $r\in E_1$, we get

$$\left(\underline{\tau}_{\log}(A_{l0})-\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}\leq\sum_{(i,j)\in J}\left(\tau_{\log}\left(A_{ij}\right)+\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}$$

$$+\sum_{(i,j)\in\Pi}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}+(\log r)^{\mu_{\log}(A_{ab})+\varepsilon}+O\left(\log\left(\log r\right)\right)+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+\left(\tau_{\log}(F)+\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}$$

$$\leq\left(\tau_1+(mn+m+n)\varepsilon\right)(\log r)^{\mu_{\log}(A_{l0})}+O\left(\log r\right)^{\mu_{\log}(A_{l0})-\varepsilon}$$

$$+(\log r)^{\mu_{\log}(A_{ab})+\varepsilon}+O\left(\log\left(\log r\right)\right)+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)$$

$$+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \qquad (6.37)$$

We may choose sufficiently small ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab})}{2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1}{2},\frac{\underline{\tau}_{\log}(A_{l0})-\tau_1}{mn+m+n+1}\right\},$$

for all sufficiently large $r\in E_1$, by (6.37) we have

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau_1-(mn+m+n+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

this means, $\mu_{\log}(A_{l0})-\varepsilon\leq\rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.

Further, for the homogeneous case, by substituting (6.12), (6.22) and (6.34) into (6.32), for all sufficiently large $r\in E_1$, we get

$$(\underline{\tau}_{\log}(A_{l0})-\tau-(mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)$$

$$+(\log r)^{\mu_{\log}(A_{ab})+\varepsilon}+O\left(\log\left(\log r\right)\right)+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}. \qquad (6.38)$$

We may choose sufficiently small ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\mu_{\log}(A_{ab})}{2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1}{2},\frac{\underline{\tau}_{\log}(A_{l0})-\tau}{mn+m+n}\right\},$$

for all sufficiently large $r\in E_1$, by (6.38) we have

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau-(mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-\varepsilon\leq\rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})+1$.
Case (iv): When $\mu_{\log}(A_{l0})=\mu_{\log}(A_{ab})=\rho_{\log}(\mathcal{S})$ and

$$\tau_3=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau_{\log}(A_{ij})+\tau_{\log}(F)+\underline{\tau}_{\log}(A_{ab})$$

$$=\tau_2+\tau_{\log}(F)<\underline{\tau}_{\log}(A_{l0}),$$

$$\tau_2=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0),(a,b)}\tau_{\log}(A_{ij})+\underline{\tau}_{\log}(A_{ab}).$$

Then, by substituting (6.14), (6.19), (6.22), (6.23) and (6.34) into (6.29), for all sufficiently large $r\in E_1$, we have

$$\begin{aligned}(\underline{\tau}_{\log}(A_{l0})-\tau_3-(mn+m+n+2)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}&\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)\\&+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)\\&+O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}.\end{aligned}\tag{6.39}$$

Now, we may choose sufficiently small ε satisfying $0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1}{2},\frac{\underline{\tau}_{\log}(A_{l0})-\tau_3}{mn+m+n+2}\right\}$, for all sufficiently large $r\in E_1$, we deduce from (6.39) that

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau_3-(mn+m+n+2)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq O\left((\log r)^{\rho_{\log}(f)+\varepsilon}\right),$$

this means, $\mu_{\log}(A_{l0})-\varepsilon\leq\rho_{\log}(f)$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.
Further, for the homogeneous case, by substituting (6.19), (6.22) and (6.34) into (6.32), for all sufficiently large $r\in E_1$, we get

$$\begin{aligned}(\underline{\tau}_{\log}(A_{l0})-\tau_2-(mn+m+n+1)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}&\leq O\left((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\right)\\&+O(\log(\log r))+O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right)+(\log r)^{\lambda_{\log}\left(\frac{1}{A_{l0}}\right)+1+\varepsilon}.\end{aligned}\tag{6.40}$$

Therefore, for ε satisfying $0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\lambda_{\log}\left(\frac{1}{A_{l0}}\right)-1}{2},\frac{\underline{\tau}_{\log}(A_{l0})-\tau_2}{mn+m+n+1}\right\}$ and for all sufficiently large $r\in E_1$, by (6.40) we have

$$(1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau_2-(mn+m+n+1)\,\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right),$$

that is, $\mu_{\log}(A_{l0})-\varepsilon\leq\rho_{\log}(f)-1$ and since $\varepsilon>0$ is arbitrary, then $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})+1$. □

Proof of Theorem 6.7

Proof. Let $f(z)(\not\equiv0)$ be a meromorphic solution of (6.2). We divide (6.2) by $f(z+c_l)$ to get

$$-A_{l0}(z)=\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}A_{ij}(z)\frac{f^{(j)}(z+c_i)f(z+c_i)}{f(z+c_i)f(z+c_l)}+\sum_{j=1}^{m}A_{ij}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}.\tag{6.41}$$

By (6.41), it follows

$$\begin{aligned} m(r,A_{l0}(z)) \leq & \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} m(r,A_{ij}(z)) + \sum_{j=1}^{m} m(r,A_{lj}(z)) \\ & + \sum_{i=0}^{n}\sum_{j=1}^{m} m\left(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\right) + \sum_{i=0,i\neq l}^{n} m\left(r,\frac{f(z+c_i)}{f(z+c_l)}\right) \\ & + m(r,F(z)) + m\left(r,\frac{1}{f(z+c_l)}\right) + O(1). \end{aligned} \tag{6.42}$$

Combining (6.6) and (6.7) with (6.42), then by using Lemma 5.5, for any given $\varepsilon > 0$, we get

$$\begin{aligned} m(r,A_{l0}(z)) \leq & \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + O\left(\log^{+} T(2r,f)\right) \\ & + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + T(r,F(z)) \\ & + \big(1+o(1)\big)T(r+|c_l|,f(z)) + O(1) \\ \leq & \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + O(\log(\log r)) \\ & + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + T(r,F(z)) + 2T(2r,f(z)) \\ \leq & \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + O(\log(\log r)) \\ & + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + T(r,F(z)) + O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \end{aligned} \tag{6.43}$$

We suppose that

$$\delta(\infty,A_{l0}) = \delta > 0 \tag{6.44}$$

and $\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\} = \rho < \mu_{\log}(A_{l0})$. Then by (6.44) and the definitions of $\mu_{\log}(A_{l0})$ and $\rho_{\log}(A_{ij})$, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\rho}{2}\right)$ and sufficiently large r, we have

$$m(r,A_{l0}) \geq \frac{\delta}{2}T(r,A_{l0}) \geq \frac{\delta}{2}(\log r)^{\mu_{\log}(A_{l0})-\frac{\varepsilon}{2}} \geq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}. \tag{6.45}$$

and

$$T(r,A_{ij}) \leq (\log r)^{\rho_{\log}(A_{ij})+\varepsilon} \leq (\log r)^{\rho+\varepsilon}, \qquad (i,j)\neq(l,0). \tag{6.46}$$

(1) If $\mu_{\log}(F) < \mu_{\log}(A_{l0})$, then by Lemma 5.1, there exists a subset E_1 with infinite logarithmic measure such that for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\mu_{\log}(F)}{2}\right)$ and for all $r\in E_1$, we have

$$T(r,F) \leq (\log r)^{\mu_{\log}(F)+\varepsilon}. \tag{6.47}$$

By substituting (6.45)-(6.47) into (6.43), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\rho}{2},\frac{\mu_{\log}(A_{l0})-\mu_{\log}(F)}{2}\right\}$$

and for all $r\in E_1$, we get

$$\begin{aligned} (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq & (n(m+1)+n)(\log r)^{\rho+\varepsilon} + O(\log(\log r)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) \\ & + (\log r)^{\mu_{\log}(F)+\varepsilon} + O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big), \end{aligned} \tag{6.48}$$

which implies that

$$(1-o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \le O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big), \tag{6.49}$$

that is, $\mu_{\log}(A_{l0})-2\varepsilon \le \rho_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we obtain $\mu_{\log}(A_{l0}) \le \rho_{\log}(f)$. Further, if $F(z)\equiv 0$, then by (6.6), (6.42) and Lemma 5.3, there exists a subset E_2 with infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $r\in E_2$, we have

$$\begin{aligned} m(r,A_{l0}(z)) \le & \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + O(\log(\log r)) \\ & + O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big). \end{aligned} \tag{6.50}$$

Substituting (6.45) and (6.46) into (6.50), for any given ε satisfying $0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\rho}{2}$ and for all $r\in E_2$, we obtain

$$(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \le (n(m+1)+m)(\log r)^{\rho+\varepsilon} + O(\log(\log r)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big). \tag{6.51}$$

It follows that

$$(1-o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \le O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big). \tag{6.52}$$

So, $\mu_{\log}(A_{l0})+1-2\varepsilon \le \rho_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we conclude that $\mu_{\log}(A_{l0})+1 \le \rho_{\log}(f)$.

(2) Let f be a meromorphic solution of (6.2). If $\mu_{\log}(F) > \mu_{\log}(A_{l0})$, then by (6.2), Lemma 5.5 and Lemma 6.3, we have

$$\begin{aligned} T(r,F(z)) \le & \sum_{(i,j)\neq(l,0)} T(r,A_{ij}(z)) + T(r,A_{l0}(z)) + \sum_{i=0}^{n}\sum_{j=0}^{m} T\big(r,f^{(j)}(z+c_i)\big) + O(1) \\ \le & \sum_{(i,j)\neq(l,0)} T(r,A_{ij}(z)) + T(r,A_{l0}(z)) + \sum_{i=0}^{n}\sum_{j=0}^{m} \big((j+1)T\big(r,f(z+c_i)\big) \\ & + S(r,f)\big) + O(1) \\ \le & \sum_{(i,j)\neq(l,0)} T(r,A_{ij}(z)) + T(r,A_{l0}(z)) + O\big(T(2r,f(z))\big) + o(T(r,f)). \end{aligned} \tag{6.53}$$

By the definition of $\mu_{\log}(F)$, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(F)-\mu_{\log}(A_{l0})}{2}\right)$ and sufficiently large r, we have

$$T(r,F) \ge (\log r)^{\mu_{\log}(F)-\varepsilon}. \tag{6.54}$$

By Lemma 5.1, there exists a subset E_1 of infinite logarithmic measure, such that for any given $\varepsilon>0$ and for all $r\in E_1$, we have

$$T(r,A_{l0}) \le (\log r)^{\mu_{\log}(A_{l0})+\varepsilon}. \tag{6.55}$$

Substituting the assumptions (6.46), (6.54) and (6.55) into (6.53), for any given ε satisfying $0<\varepsilon<\frac{\mu_{\log}(F)-\mu_{\log}(A_{l0})}{2}$ and for all $r\in E_1$, we get

$$\begin{aligned}(\log r)^{\mu_{\log}(F)-\varepsilon} &\leq \sum_{(i,j)\neq(l,0)}(\log r)^{\rho+\varepsilon}+(\log r)^{\mu_{\log}(A_{l0})+\varepsilon}+O\big(T(2r,f(z))\big)+o(T(r,f))\\ &=(n(m+1)+n)(\log r)^{\rho+\varepsilon}+(\log r)^{\mu_{\log}(A_{l0})+\varepsilon}+O\big(T(2r,f(z))\big)+o(T(r,f)).\end{aligned} \tag{6.56}$$

Then

$$\big(1-o(1)\big)(\log r)^{\mu_{\log}(F)-\varepsilon}\leq O\big(T(2r,f(z))\big)+o(T(r,f)). \tag{6.57}$$

It follows by (6.57) that $\rho_{\log}(f)\geq\mu_{\log}(F)-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we deduce that $\rho_{\log}(f)\geq\mu_{\log}(F)$.

□

Proof of Theorem 6.8

Proof. Let $f(z)(\not\equiv 0)$ be a meromorphic solution of (6.2). By (6.43), for any given $\varepsilon>0$, we have

$$\begin{aligned}T(r,A_{l0}(z)) &= m(r,A_{l0}(z))+N(r,A_{l0}(z))\\ &\leq \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij}(z))+\sum_{j=1}^{m}T(r,A_{lj}(z))+O(\log(\log r))\\ &\quad +O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big)+T(r,F(z))+O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big)\\ &\quad +N(r,A_{l0}(z)),\end{aligned} \tag{6.58}$$

(1) If $\rho_{\log}(F)<\mu_{\log}(A_{l0})$, then for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\rho_{\log}(F)}{2}\right)$ and sufficiently large r, we have

$$T(r,F)\leq(\log r)^{\rho_{\log}(F)+\varepsilon}. \tag{6.59}$$

Suppose that $\rho=\max\{\rho_{\log}(A_{ij}):\ (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})$. Then by the definitions of $\mu_{\log}(A_{l0})$ and $\rho_{\log}(A_{ij})$, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\rho}{2}\right)$ and sufficiently large r, we get

$$T(r,A_{l0})\geq(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \tag{6.60}$$

and

$$T(r,A_{ij})\leq(\log r)^{\rho_{\log}(A_{ij})+\varepsilon}\leq(\log r)^{\rho+\varepsilon},\qquad (i,j)\neq(l,0). \tag{6.61}$$

By the definition of $\lambda_{\log}(\frac{1}{A_{l0}})$, for any given ε $\left(0<\varepsilon<\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2}\right)$ and sufficiently large r, we have

$$N(r,A_{l0})\leq(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}. \tag{6.62}$$

By substituting the assumptions (6.59) - (6.62) into (6.58), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\rho}{2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2},\frac{\mu_{\log}(A_{l0})-\rho_{\log}(F)}{2}\right\}$$

and sufficiently large r, we obtain

$$\begin{aligned}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq& \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij}(z)) + \sum_{j=1}^{m} T(r,A_{lj}(z)) + O(\log(\log r)) \\ &+ O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + (\log r)^{\rho_{\log}(F)+\varepsilon} \\ &+ O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big) + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}.\end{aligned} \tag{6.63}$$

Then

$$\big(1-o(1)\big)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leq O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \tag{6.64}$$

which implies that $\rho_{\log}(f) \geq \mu_{\log}(A_{l0}) - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$. Now we suppose that $\max\{\rho_{\log}(A_{ij})\colon (i,j) \neq (l,0)\} = \mu_{\log}(A_{l0})$ and

$$\tau = \sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0}).$$

Then there exist two sets $\Gamma_1 \subseteq \{(i,j) : i = 0,1,...,n, j = 0,1,,...,m,(i,j) \neq (l,0)\}$ and $\Gamma_2 = \{(i,j) : i = 0,1,...,n, j = 0,1,,...,m,(i,j) \neq (l,0)\} \setminus \Gamma_1$, such that for $(i,j) \in \Gamma_1$, we have $\rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0})$ with $\tau = \sum_{(i,j)\in\Gamma_1} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0})$ and for $(i,j) \in \Gamma_2$, we have $\rho_{\log}(A_{ij}) < \mu_{\log}(A_{l0})$. Hence, for any given ε $\left(0 < \varepsilon < \frac{\underline{\tau}_{\log}(A_{l0})-\tau}{mn+m+n+1}\right)$ and sufficiently large r, we get

$$T(r,A_{ij}) \leq (\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}, \qquad (i,j) \in \Gamma_1 \tag{6.65}$$

and

$$T(r,A_{ij}) \leq (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, \qquad (i,j) \in \Gamma_2. \tag{6.66}$$

By the definition of $\underline{\tau}_{\log}(A_{l0})$, for the above ε and sufficiently large r, we have

$$T(r,A_{l0}) \geq (\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.67}$$

By substituting the assumptions (6.59), (6.62), (6.65), (6.66) and (6.67) into (6.58), for any given ε satisfying

$$0 < \varepsilon < \min\left\{\frac{\underline{\tau}_{\log}(A_{l0})-\tau}{mn+m+n+1}, \frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2}, \frac{\mu_{\log}(A_{l0})-\rho_{\log}(F)}{2}\right\}$$

and for sufficiently large r, we obtain

$$\begin{aligned}&(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &\leq \sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + \sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\ &\quad + O(\log(\log r)) + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + (\log r)^{\rho_{\log}(F)+\varepsilon} \\ &\quad + O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big) + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon} \\ &\leq (\tau+(mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O\big((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\big) + O(\log(\log r)) \\ &\quad + O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big) + (\log r)^{\rho_{\log}(F)+\varepsilon} \\ &\quad + O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big) + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}.\end{aligned} \tag{6.68}$$

Thus,

$$\big(1-o(1)\big)\big(\underline{\tau}_{\log}(A_{l0})-\tau-(mn+m+n+1)\varepsilon\big)(\log r)^{\mu_{\log}(A_{l0})}\leq O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \tag{6.69}$$

It follows by (6.69) that $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$.

If $\rho_{\log}(F)=\mu_{\log}(A_{l0})$ and $\tau+\tau_{\log}(F)<\underline{\tau}_{\log}(A_{l0})$, then for any given $\varepsilon>0$ and for sufficiently large r, we have

$$T(r,F)\leq(\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.70}$$

By substituting the assumptions (6.62), (6.65)-(6.67) and (6.70) into (6.58), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\underline{\tau}_{\log}(A_{l0})-\tau-\tau_{\log}(F)}{mn+m+n+2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2}\right\}$$

and for sufficiently large r, we get

$$\begin{aligned}
&(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\
&\leq\sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\
&\quad+O(\log(\log r))+O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big)+(\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\
&\quad+O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big)+(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}\\
&\leq(\tau+(mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O\big((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\big)+O(\log(\log r))\\
&\quad+O\big((\log r)^{\rho_{\log}(f)-1+\varepsilon}\big)+(\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\
&\quad+O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big)+(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}.
\end{aligned} \tag{6.71}$$

It follows

$$\big(1-o(1)\big)\big(\underline{\tau}_{\log}(A_{l0})-\tau-\tau_{\log}(F)-(mn+m+n+2)\varepsilon\big)(\log r)^{\mu_{\log}(A_{l0})}\leq O\big((\log r)^{\rho_{\log}(f)+\varepsilon}\big). \tag{6.72}$$

This implies that $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\rho_{\log}(f)\geq\mu_{\log}(A_{l0})$. If $\mu_{\log}(F)=\mu_{\log}(A_{l0})$ and $\tau+\underline{\tau}_{\log}(A_{l0})<\underline{\tau}_{\log}(F)$, then for any sufficiently small $\varepsilon>0$ and for sufficiently large r, we have

$$T(r,F)>(\underline{\tau}_{\log}(F)-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.73}$$

By Lemma 6.2, there exists a subset E_1 of infinite logarithmic measure, such that for any given $\varepsilon>0$ and for all $r\in E_1$, we have

$$T(r,A_{l0})\leq(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{6.74}$$

Substituting the assumptions (6.65), (6.66), (6.73) and (6.74), into (6.53), for every sufficiently small ε satisfying $0 < \varepsilon < \frac{\underline{\tau}_{\log}(F)-\tau-\underline{\tau}_{\log}(A_{l0})}{mn+m+n+2}$ and for all $r \in E_3$, we obtain

$$\begin{aligned}
&(\underline{\tau}_{\log}(F)-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\
&\quad \leq \sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\
&\qquad +(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O\big(T(2r,f(z))\big)+o(T(r,f)) \\
&\quad \leq (\tau+\underline{\tau}_{\log}(A_{l0})+(mn+m+n+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O\big((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\big) \\
&\qquad +O\big(T(2r,f(z))\big)+o(T(r,f)).
\end{aligned} \tag{6.75}$$

So

$$\begin{aligned}
&\big(1-o(1)\big)\big(\underline{\tau}_{\log}(F)-\tau-\underline{\tau}_{\log}(A_{l0})-(mn+m+n+2)\varepsilon\big)(\log r)^{\mu_{\log}(A_{l0})} \\
&\qquad \leq O\big(T(2r,f(z))\big)+o(T(r,f)),
\end{aligned} \tag{6.76}$$

which implies that $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.
Further for the homogeneous case $F(z)\equiv 0$, by (6.50), for any given $\varepsilon > 0$ and for all $r \in E_2$, we have

$$\begin{aligned}
T(r,A_{l0}(z)) &= m(r,A_{l0}(z))+N(r,A_{l0}(z)) \\
&\leq \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij})+\sum_{j=1}^{m}T(r,A_{lj})+O(\log(\log r)) \\
&\quad +O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big)+N(r,A_{l0}(z)).
\end{aligned} \tag{6.77}$$

If $\rho=\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})$, then by substituting the assumptions (6.60) - (6.62) into (6.77), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\mu_{\log}(A_{l0})-\rho}{2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2}\right\}$$

and for all $r \in E_2$, we obtain

$$\begin{aligned}
(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} &\leq \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}(\log r)^{\rho+\varepsilon}+\sum_{j=1}^{m}(\log r)^{\rho+\varepsilon}+O(\log(\log r)) \\
&\quad +O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big)+(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon} \\
&\leq (n(m+1)+m)(\log r)^{\rho+\varepsilon}+O(\log(\log r))+O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big) \\
&\quad +(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}.
\end{aligned} \tag{6.78}$$

Then

$$\big(1-o(1)\big)(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\leq O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big), \tag{6.79}$$

which implies that $\mu_{\log}(f)\geq\mu_{\log}(A_{l0})+1-2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we deduce that $\mu_{\log}(f)\geq\mu_{\log}(A_{l0})+1$. If $\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}=\mu_{\log}(A_{l0})$ and

$$\tau=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0}),$$

then by substituting the assumptions (6.62), (6.65), (6.66) and (6.67) into (6.77), for any given ε satisfying

$$0<\varepsilon<\min\left\{\frac{\underline{\tau}_{\log}(A_{l0})-\tau}{mn+m+n+1},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(\frac{1}{A_{l0}})-1}{2}\right\}$$

and for all $r\in E_2$, we get

$$\begin{aligned}(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}&\\ \leq \sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}&+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\ +O(\log(\log r))+O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big)&+(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}\\ \leq(\tau+(mn+m+n)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}&+O\big((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\big)\\ +O(\log(\log r))+O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big)&+(\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}.\end{aligned} \tag{6.80}$$

It follows that

$$\big(1-o(1)\big)\big(\underline{\tau}_{\log}(A_{l0})-\tau-(mn+m+n+1)\varepsilon\big)(\log r)^{\mu_{\log}(A_{l0})}\leq O\big((\log r)^{\mu_{\log}(f)-1+\varepsilon}\big), \tag{6.81}$$

that is $\mu_{\log}(f)\geq\mu_{\log}(A_{l0})+1-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\mu_{\log}(f)\geq\mu_{\log}(A_{l0})+1$.

(2) Let f be a meromorphic solution of (6.2). If $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, then we suppose that $\rho=\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})$. By using a similar reasoning method as in (6.53)-(6.57) from the proof of Theorem 6.7, we get $\mu_{\log}(F)\leq\rho_{\log}(f)$.
Now, we suppose that $\max\{\rho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}=\mu_{\log}(A_{l0})$ and

$$\tau=\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0}).$$

Then by substituting the assumptions (6.54), (6.65), (6.66) and (6.67) into (6.53), for any given ε satisfying $0<\varepsilon<\frac{\mu_{\log}(F)-\mu_{\log}(A_{l0})}{2}$ and for all $r\in E_3$, we have

$$\begin{aligned}(\log r)^{\mu_{\log}(F)-\varepsilon}&\leq\sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\ &\quad+(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O\big(T(2r,f(z))\big)+o(T(r,f))\\ &\leq(\tau+\underline{\tau}_{\log}(A_{l0})+(mn+m+n+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O\big((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\big)\\ &\quad+O\big(T(2r,f(z))\big)+o(T(r,f)).\end{aligned} \tag{6.82}$$

It follows that

$$\big(1-o(1)\big)(\log r)^{\mu_{\log}(F)-\varepsilon}\leq O\big(T(2r,f(z))\big)+o(T(r,f)). \tag{6.83}$$

By (6.83), we conclude that $\rho_{\log}(f)\geq\mu_{\log}(F)$.

□

6.5 Examples

The following example is for illustrating the sharpness of some assertions in Theorem 6.6.

Example 6.1. *For* Theorem 6.6*, we consider the meromorphic function*

$$f(z)=\frac{1}{z^5} \tag{6.84}$$

which is a solution to the delay-differential equation

$$A_{20}(z)f(z-2i)+A_{11}(z)f'(z+i)+A_{10}(z)f(z+i)$$

$$+A_{01}(z)f'(z)+A_{00}(z)f(z)=F(z), \tag{6.85}$$

where $A_{20}(z)=\frac{2}{3}(z-2i)^4$, $A_{11}(z)=2e$, $A_{10}(z)=\frac{10e}{z+i}$, $A_{01}(z)=\frac{i}{2}$, $A_{00}(z)=\frac{5i}{2z}$ *and* $F(z)=\frac{2}{3(z-2i)}$. *Obviously,* $A_{ij}(z)$ $(i=0,1,2,j=0,1)$ *and* $F(z)$ *satisfy the conditions in Case* (iii) *of* Theorem 6.6*, such that*

$$\delta(\infty,A_{20})=1>0,$$

$$\mu_{\log}(A_{11})=0<\max\{\rho_{\log}(F),\rho_{\log}(A_{ij}),(i,j)\neq(1,1),(2,0)\}=\mu_{\log}(A_{20})=1$$

and

$$\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{20}),(i,j)\neq(1,1),(2,0)}\tau_{\log}(A_{ij})+\tau_{\log}(F)=3<\delta\underline{\tau}_{\log}(A_{20})=4.$$

We see that f *satisfies*

$$\mu_{\log}(f)=1=\rho_{\log}(A_{20}).$$

The meromorphic function $f(z)=\frac{1}{z^5}$ *is a solution of equation* (6.85) *for the coefficients* $A_{20}(z)=3(z-2i)^7$, $A_{11}(z)=\frac{1}{z-i}$, $A_{10}(z)=\frac{5}{z^2+1}$, $A_{01}(z)=\frac{i}{2}$, $A_{00}(z)=\frac{5i}{2z}$ *and* $F(z)=3(z-2i)^2$. *Clearly,* $A_{ij}(z)$ $(i=0,1,2,j=0,1)$ *and* $F(z)$ *satisfy the conditions in Case* (iv) *of* Theorem 6.6 *such that*

$$\delta(\infty,A_{20})=1>0,$$

$$\mu_{\log}(A_{11})=\max\{\rho_{\log}(F),\rho_{\log}(A_{ij}),(i,j)\neq(1,1),(2,0)\}=\mu_{\log}(A_{20})=1$$

and

$$\sum_{\rho_{\log}(A_{ij})=\mu_{\log}(A_{20}),(i,j)\neq(1,1),(2,0)}\tau_{\log}(A_{ij})+\tau_{\log}(F)+\underline{\tau}_{\log}(A_{11})=6<\delta\underline{\tau}_{\log}(A_{20})=7.$$

We see that f *satisfies* $\rho_{\log}(f)=1=\mu_{\log}(A_{20})$.

Conclusion

In this thesis, Nevanlinna theory of value distribution has been applied to study the behavior of solutions through two important aspects: the growth and the oscillation. This study included linear differential equations with analytic or meromorphic coefficients in the extended complex plane except at a finite singular point, linear difference equations with meromorphic coefficients and linear delay-differential equations also with meromorphic coefficients. This thesis may address several questions related to the growth and value distribution of solutions to these three types of problems, but it also opens the door to many other intriguing questions, such as:

Q1: Can we obtain similar results to those in the third and fourth chapters for the case when the coefficients are analytic or meromorphic functions in the unit disk?

Q2: What can be said about the growth and value distribution of the linear difference polynomials generated by meromorphic solutions of the higher order complex linear difference equations of Chapter 5 ?

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