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THESIS

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By :

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Quantitative Study of Some Fractional Differential Equations With q-Difference

On July 02, 2024 in front of the committee composed of

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THESIS

Dedication

I dedicate this modest work to:

My dear parents, may Allah bless them.

My sisters and My brothers.

All members of my large family.

My dear close friends.

Nadia ALLOUCH, 2024

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Contents

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Abstract

The main goal of this thesis is to present a set results on the existence, uniqueness and stability of certain classes of the initial value problems and boundary value problems for fractional q-difference equations and impulsive fractional q-difference equations involving Caputo's fractional q-derivative. The results have been proven analytically, where the existence results are based on some classical fixed point theorems (Banach, Schaefer, Krasnoselskii, Non-linear alternative of Leray-Schauder) as well as Mönch's fixed point theorem combined with the technique of Kuratowski's measure of noncompactness, while the stability results depend on the techniques of Ulam-Hyers stability and Ulam-Hyers-Rassias stability. To support our results, we provide different illustrative examples in each chapter.

Key-words and phrases: Fractional q-calculus; Quantum calculus; Caputo fractional q-derivative; Fractional q-difference equations; Impulsive fractional q-difference equations; Initial value problem; Boundary value problem; Banach space; Existence; Uniqueness; Fixed point theorems; Kuratowski measure of noncompactness; Ulam-Hyers stability; Ulam-Hyers-Rassias stability.

AMS Subject Classification: 05A30, 26A33, 34A08, 34A12, 34A37, 34Bxx, 39A13, 47H10.

Résumé

L'objectif principal de cette thèse est de présenter un ensemble des résultats sur l'existence, l'unicité et la stabilité de certaines classes de problèmes à valeurs initiales et problèmes aux limites pour les équations q-différence fractionnaires et les équations q-différence fractionnaires impulsives impliquant la q-dérivée fractionnaire de Caputo. Les résultats ont été prouvés analytiquement, où les résultats d'existence sont basés sur certaines théorèmes classiques du point fixe (Banach, Schaefer, Krasnoselskii, alternative non linéaire de Leray-Schauder) et ainsi que sur le théorème du point fixe de Mönch combiné avec la technique de la mesure de non-compacité de Kuratowski, alors que les résultats de stabilité sont basés sur des techniques de la stabilité d'Ulam-Hyers et la stabilité d'Ulam-Hyers-Rassias. Pour étayer nos résultats, on offre différents exemples illustratifs dans chaque chapitre.

Mots-clés et phrases: q-Calcul fractionnaire; Calcul quantique; La q-Dérivé fractionnaire au sens de Caputo; Équations q-différence fractionnaires; Équations q-différence fractionnaires impulsives; Problème à valeur initiale; Problème aux limites; Espace de Banach; Existence; Unicité; Théorèmes de point fixe; Mesure de Kuratowski de non-compacité; Stabilité d'Ulam-Hyers; Stabilité d'Ulam-Hyers-Rassias.

Classification AMS: 05A30, 26A33, 34A08, 34A12, 34A37, 34Bxx, 39A13, 47H10.

الملخص

الهدف الرئيسي من هذه الأطروحة هو تقديم مجموعة من النتائج حول وجود، وحدانية واستقرار حلول لفئات معينة من مسائل القيمة الأولية والحدية للمعادلات $\boldsymbol{q}$-التفاضلية الكسرية والمعادلات $\boldsymbol{q}$-التفاضلية الكسرية الاندفاعية التي تتضمن مشتق كسري ل q-Caput. وقد تم إثبات النتائج تحليلياً، حيث تعتمد نتائج الوجود على بعض نظريات النقطة الثابتة (Banach، Schaefer، Krasnoselskii، البديل غير الخطي Leray-Schauder) وبالإضافة إلى نظرية Mönch للنقطة الثابتة جنبًا إلى جنب مع تقنية مقياس عدم التوافق لKuratowski، و في حين تعتمد نتائج الاستقرار على تقنيات استقرار Ulam-Hyers و استقرار Ulam-Hyers-Rassias. لدعم نتائجنا، نقدم أمثلة توضيحية مختلفة في كل فصل.

Notations

$\mathbb{N}$	:	Set of natural numbers.
$\mathbb{N}^*$	:	Set of natural numbers without zero.
$\mathbb{R}$	:	Set of real numbers.
$\mathbb{R}^*$	:	Set of real numbers without zero.
$\mathbb{R}_+$	:	Set of positive real numbers.
$\mathbb{C}$	:	Set of complex numbers.
$conv\mathscr{A}$	:	Convex hull of bounded set $\mathscr{A}$.
$\overline{conv}\mathscr{A}$	:	Convex hull and closed of bounded set $\mathscr{A}$.
sup	:	Supremum.
inf	:	Infimum.
$[.]$	:	Integer part of a real number.
$\mathbb{E}$	:	Banach space.
$\\|.\\|$	:	Norm of Banach space $\mathbb{E}$.
$\Gamma_q(.)$	:	q-Gamma function.
$\mathrm{B}_q(.,.)$	:	q-Beta function.
$\mathfrak{D}_q$	:	q-Derivative.
$\mathfrak{I}_q$	:	q-Integral.
$\mathfrak{I}^{\beta}_{q,\mathfrak{a}}$	:	Riemann-Liouville's fractional q-integral of order $\beta \geq 0$; $q \in (0,1)$.
${}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}$	:	Riemann-Liouville's fractional q-derivative of order $\beta \geq 0$; $q \in (0,1)$.
${}^{C}\mathfrak{D}^{\alpha}_{q,\mathfrak{a}}$	:	Caputo's fractional q-derivative of order $\beta \geq 0$; $q \in (0,1)$.

Introduction

Fractional calculus is one of the fundamental areas of mathematical analysis, which is a generalization of ordinary differentiation and integration to an arbitrary real or complex order. The idea of fractional calculus has a long history, it began with a conversation between two mathematicians *Leibnitz* and *L'Hopital* at the end of the 17^{th} century; *Leibnitz* introduced the derivative symbol $\frac{d^n y}{dx^n}$ of order $n \in \mathbb{N}$ and *L'Hopital* posed a question to *Leibnitz* in 1695 : "*What if n be* $\frac{1}{2}$", *Leibnitz* replied: "*It will lead to a paradox*" and added: "*From this apparent paradox, one day useful consequences will be drawn.*" (see [81]). Out of this conversation fractional calculus was born.

Through time, this question has attracted the interest and investigation by many mathematicians, in particular: *Euler (1730), Lagrange (1772), Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Hadamard (1892), Riesz (1922)*, and others, which contributed to the development of fractional calculus. However, this theory can be considered a new topic because it was the talk of researchers at scientific conferences and seminars. The first conference on fractional calculus and its applications is attributed to *Ross*, who organized it in June 1974 at the University of New Haven, edited the conference procedures in [81]. The first monograph is attributed to *Oldham* and *Spanier* who published a book dedicated to fractional calculus in 1974 [76].

Recently, there has been a great interest on the theory of the integrals, derivatives of the arbitrary order, and fractional differential equations by scholars, with a lot of works that appeared on it, including books of *Samko et al.* [85], *Miller et al.* [70], *Podlubny* [77], *Hilfer* [55], *Kilbas et al.* [64] and *Tarasov* [92]. Moreover, fractional differential equations provide a comprehensive scheme for analysing regular and complex systems in many fields, such as physics, engineering, biology, economics, material sciences and social sciences. These are some applications:

- **In Physics:** Fractional differential equations are essential for modelling anomalous diffusion and explaining how the system changes over time [48]. Furthermore, many non-linear physical phenomena that include gas bubbles in liquids can be explained by using the nonlinear fractional-stochastic wave equation [72].

- **In Engineering:** Fractional differential equations are employed to simulate damping systems and viscoelastic materials [47], with giving more precise explanations of mechanical behaviour. Fractional-order controllers can also be used to progress the performance and stability of control systems [41].

- **In Biology:** Fractional order models can be used to study how the disease is spread in societies. Specifically, an effective and acceptable explanation of the COVID-19 pandemic and its spreading variant can be obtained using non-integer order of COVID-19 models [19, 97].

- **In Economics:** A fractional-order growth model with time delay can be used to effectively represent economic growth by adding a time lag to the capital stock [69].

As a consequence, several authors paid attention of fractional differential equations and investigated the existence and stability of solutions to initial and boundary value problems for fractional differential equations; the reader can see the books of *Abbas et al* [4, 3] and *Benchohra et al* [36, 37], the papers [9, 10, 31, 32, 33, 34, 35, 39, 40, 45, 46, 53] and the references therein.

Q-Difference calculus or quantum calculus is also a significant branch in mathematical analysis, and is considered a link between mathematics and physics. The history of q-difference calculus dates back to 1910, thanks to the works of *Jackson* [58, 59], the first researcher who created q-calculus in a systematic manner and presented the idea of the q-derivative, q-integral and certain classical concepts. The essential definitions and properties of q-calculus can be read in the book of *Kac and Cheung* [63]. The q-difference calculus has played a very important role in physical phenomena; for example, the physicist *Fock* studied the symmetry of hydrogen atoms using the q-difference equations.

By integrating q-difference calculus with fractional calculus you get fractional q-difference calculus, which is a generalization of q-difference calculus to an arbitrary real or complex order. Moreover, the fractional q-calculus was first developed at the end of the 1960s thanks to the contributions of *Al-Salam* [27] who proposed the theory of the fractional q-calculus, beginning from the q-analogue of Cauchy's formula and *Agarwal* [11] who addressed some fractional q-integral and fractional q-derivatives operators. In addition, *Rajković et al.* in [78, 79, 86, 88] expanded the concepts of the fractional q-calculus and discussed its properties.

Fractional q-calculus, especially fractional q-difference equations are essential for modeling a large number of phenomena in various fields of science and engineering, such as:

- **In Physics:** Fractional q-difference equations can be used to simulate anomalous diffusion processes, like as subdiffusion or superdiffusion. It is also possible to model complex behaviors in statistical physics [8].

- **In Mechanics:** Fractional q-calculus provides a mathematical framework for modeling complex dynamics and phenomena. It can be used to model non-linear dynamics in mechanical systems as well as frictional dynamics in mechanical systems, including viscoelastic materials and contact mechanics.

- **In Chemistry:** Fractional q-difference equations can be applied to models of chemical networks and chemical reaction kinetics, which provide a framework for describing the speeds of chemical reactions involving complex molecular interactions.

- **In Medicine:** Fractional q-difference equations can facilitate modeling of diseases with complex dynamics, such as cancer, neurodegenerative disorders and Corona virus (COVID-19). In addition, fractional q-difference equations can be helpful in analysing biomedical signals, including electrocardiogram (ECG) and electroencephalogram (EEG).

- **In Economics:** Fractional q-calculus techniques can be applied to solve economics optimisation problems such as production scheduling and resource allocation. The dynamics of financial time series data, such as currency rates and stock prices, can also be represented employing fractional q-difference equations.

Further, fractional q-difference equations have attracted the attention of mathematicians and engineers in recent times, due to their application in many areas. So that they discussed and investigated the existence and stability of its solutions; for details, see the books of *Annaby* and *Mansour* [28], *Abbas et al.* [1] and the papers of *Ahmed et al.* [13, 15, 17] and *Abbas et al.* [2, 5, 6]. As a result, initial and boundary value problems for fractional q-difference equations involving Caputo's fractional q-derivative have become of significance among researchers; for more information, see the works [5, 7, 8, 13, 14, 15, 42, 56, 67, 68, 84, 98] and the references therein.

On the other hand, the research into the theory of impulsive differential equations began in the 1960s by *Milman* and *Myshkis* [71] and achieved great progress over time with contributions from mathematicians due to its importance and applications in various fields, including in physics, chemistry, biology, control theory and population dynamics. This makes it a vital and active field of research in modern mathematics and applied sciences. Recently, impulsive fractional differential equations and impulsive fractional q-difference equations have drawn the attention of several scholars, who have examined the existence and stability of their solutions; for example, refer to references [2, 13, 16, 26, 20, 38, 52, 61, 75, 93].

Fixed point theories offers basic tools for examining existence and uniqueness of solutions to various non-linear problems. They often depend on some specific properties (e.g. contraction, complete continuity, ...). Lately, fixed point theory has proven to be a highly useful and significant instrument in studying a variety of phenomena in a wide range of scientific and engineering domains. Fixed point theory plays an essential role in solving fractional differential equations and their applications such as initial and boundary value problems. Among the most famous fixed point theories are Banach's theorem, Schaefer's and Krasnoselskii's theorems, nonlinear alternative of Leary-Schauder's theorem. In addition to fixed point theorem of Mönch [73, 74] combined with Kuratowski's measure of non-compactness.

The concept of the measure of non-compactness was first introduced in the mid-20th century by the Polish mathematician *Kuratowski* [66], after whom the concept is named. The first study of the Kuratowski's measure of non-compactness was attributed to *Banas* and *Goebe* [29], and then it was developed and applied to many works; see the papers of *Szufla* [89], *Akhmerov et al* [18], *Guo et al* [51] and *Banas et al* [30]. the technique of Kuratowski's measures of non-compactness is a useful and important tool in mathematical analysis, particularly in functional analysis, differential equations and dynamic systems.

Ulam [95] was the first to raise the topic of the functional equations stability in a 1940 speech at Wisconsin University . In 1941, *Hyers* introduced and proved the stability theory of functional equations (for more information see [57]). Later, this type of stability was called Ulam-Hyers stability. *Rassias* [80] generalised the Hyers theorem in 1978, and established the Ulam-Hyers stability of linear mappings in Banach space. Following this finding, several works were published in order to expand the results of Ulam-Hyers stability and apply them to ordinary differential equations and fractional differential equations; refer to the papers of *Rus* [82, 83], *Wang et al.* [96], *Dahmani et al.* [46] and *Taieb et al.* [90, 91], and the monograph of *Jung* [62] and *Abbas et al.* [4]. However, many academics have focused on Ulam-Hyers stability and Ulam-Hyers-Rassias stability of fractional q-difference equations; see the references [1, 6, 42, 56, 61, 67, 68].

The main goal of this thesis is to study of the existence, uniqueness and Ulam stability of the solutions of some initial and boundary value problems for fractional q-difference equations involving Caputo's fractional q-derivative. This thesis consists of an introduction, four chapters and a conclusion with some perspectives, which is organized as follows:

- **Chapter 1:** This chapter contains preliminary concepts, main definitions and notations necessary to understanding the content of this thesis. In the first section, we explain the terms and definitions of functional analysis tools. The quantum calculus is reviewed in the second section. The third section focuses on the basic definitions and properties of the fractional q-calculus. In the fourth section, we give a summary of Kuratowski's measures of non-compactness. Then, we present some classical fixed point theorems in the last section.

- **Chapter 2:** In this chapter, we create some existence and uniqueness results for the boundary value problem for fractional q-difference equations involving the Caputo's fractional q-derivative of the following form:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 0 < \beta \leq 1, \; t \in \mathrm{J} = [0, \mathrm{T}], \\ \mathfrak{a} z(0) + \mathfrak{b} z(\mathrm{T}) = \mathfrak{c}, \end{cases} \tag{1}$$

where $q \in (0,1)$, $\mathrm{T} > 0$ and ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (0,1]$, $\phi : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a given function with $\mathbb{E}$ is Banach space and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are real constants such that $\mathfrak{a} + \mathfrak{b} \neq 0$.
Firstly, we begin by presenting the integrable solution to the boundary value problem (1). Following that, we provide the main results of existence, the first results are based on Banach's contraction principle, Schaefer's fixed point theorem and Leray-Schauder non-linear alternative. The second result depends on Mönch's fixed point theorem combined with the Kuratowski's measure of non-compactness. Finally, we give an illustrative example at the end each section.

- **Chapter 3:** This chapter is concerned with determining the results of the existence and stability of the boundary value problem for fractional q-difference equations with integral conditions, which are given by:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 1 < \beta \leq 2, \; t \in \mathrm{J} = [0, \mathrm{T}], \\ z(0) - z'(0) = \int_0^{\mathrm{T}} \varphi(s, z(s))\, ds, \\ z(\mathrm{T}) + z'(\mathrm{T}) = \int_0^{\mathrm{T}} \psi(s, z(s))\, ds, \end{cases} \tag{2}$$

where $q \in (0,1)$, $\mathrm{T} > 0$ and ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (1,2]$, $\phi : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a given function and $\varphi, \psi : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ are continuous functions with $\mathbb{E}$ is Banach space.

First of all, we start by giving the integrable solution to the problem (2). Then, in the second part, we prove the existence, uniqueness and Ulam stability of solutions to the boundary value problem for fractional q-difference equations with integral conditions (2) with $\mathbb{E} = \mathbb{R}$, by applying some fixed point theorems (Banach, Schaefer) and Ulam-Hyers and Ulam-Hyers-Rassias stabilities techniques. In the third part, we discuss another result of the existence of solutions to the boundary value problem for fractional q-difference equations with integral conditions (2) in Banach spaces, using Mönch's fixed point theorem and the Kuratowski's measure of noncompactness. At the conclusion of each section, we provide examples to illustrate the main results.

- **Chapter 4:** In this chapter, we establish the existence, uniqueness and stability results for the initial value problem for impulsive fractional q-difference equations involving Caputo's fractional q-derivative, given as follows:

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 1 < \beta \leq 2,\ t \in \mathrm{J} = [0, \mathrm{T}],\ t \neq t_i,\ i = 1, \cdots, n, \\ \Delta z\,|_{t=t_i} = \mathscr{I}_i(z(t_i^-)),\ i = 1, \cdots, n, \\ \Delta z'\,|_{t=t_i} = \overline{\mathscr{I}}_i(z(t_i^-)),\ i = 1, \cdots, n, \\ z(0) = z_0, \quad z'(0) = z_0^*, \end{cases} \tag{3}$$

where $q \in (0,1)$, $\mathrm{T} > 0$, ${}^{\mathrm{C}}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (1,2]$, and $\phi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathscr{I}_i, \overline{\mathscr{I}}_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$ are given functions, and $z_0, z_0^* \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \mathrm{T} < +\infty$, $\Delta z\,|_{t=t_i} = z(t_i^+) - z(t_i^-)$ and $\Delta z'\,|_{t=t_i} = z'(t_i^+) - z'(t_i^-)$, $z(t_i^+) = \lim_{\varepsilon \to 0^+} z(t_i + \varepsilon)$ and $z(t_i^-) = \lim_{\varepsilon \to 0^-} z(t_i + \varepsilon)$ represent the right and left limits of z at $t = t_i$, $i = 1, \cdots, n$.
In first section, we present the integrable solution to the initial value problem (3). The second section focuses on the main results of the existence and Ulam stability for solutions of the initial value problem for impulsive fractional q-difference equations (3), such that the first results of existence depend on Banach's contraction principle and Krasnoselskii's fixed point theorem. The second results of stabilities are based on Ulam-Hyers and Ulam-Hyers-Rassias stability. In the last, we offer an example that illustrates our main results.

Finally, we close our work with a conclusion and some perspectives.

Chapter 1

Materials and Preliminaries

This chapter constitutes a preliminary part in which we recall the fundamental notions and results of the functional analysis theory and fractional q-calculus, which represent essential tools in our study. Fractional q-calculus is a generalization of fractional calculus and thus retains many basic properties.

The chapter is made in several sections. In the first section, we introduce the concepts and definitions of functional analysis tools. The second section contains an overview of quantum calculus (q-difference). Then, in the third section, we recall the elementary definitions and basic concepts related to the theory of fractional q-calculus and its properties. The last two sections are devoted to presenting the basic properties of Kuratowski's measure of non-compactness and some classical fixed point theorems that play an essential role in our results concerning fractional q-difference equations.

1 Notations and Essential Concepts

This section contains the notations, definitions and essential concepts of the functional analysis theory and operator; we suggest the reader to return to the following original sources [43, 44, 49, 60, 64, 65, 85].

Definition 1.1 *[43](**Cauchy Sequence**)*
Let $(\mathbb{E}, \|.\|)$ be a normed vector space and let $(z_n)_n$ be a sequence of elements of $\mathbb{E}$, we say that $(z_n)_n$ is a Cauchy sequence if

$$\forall \epsilon > 0, \exists \, N_\epsilon \in \mathbb{N}, \forall p, q \geq N_\epsilon \Rightarrow \|z_p - z_q\| < \epsilon.$$

Definition 1.2 *[65](**Complete Space**)*
A normed vector space $(\mathbb{E}, \|.\|)$ is said to be complete, if any Cauchy sequence $(z_n)_n$ of elements of $\mathbb{E}$ is convergent.

Definition 1.3 *[65](**Banach Space)***
Any complete normed vector space $(\mathbb{E}, \|.\|)$ *is known a Banach space.*

Definition 1.4 *[60](**Lipschitz Mapping)***
Let $(\mathbb{E}, \|.\|)$ *be a normed vector space. A map* $\phi : \mathbb{E} \to \mathbb{E}$ *is said to be Lipschitzian if*

$$\exists \mathfrak{L} > 0, \forall y, z \in \mathbb{E}, \text{ such as}: \|\phi(y) - \phi(z)\| \leq \mathfrak{L}\|y - z\|.$$

Definition 1.5 *[60](**Contraction Mapping)***
A map $\phi : \mathbb{E} \to \mathbb{E}$ *is said to be contraction, if it's Lipschitzian with* $\mathfrak{L} \in (0, 1)$.

Definition 1.6 *[44](**Carathéodory Mapping)***
A map $\phi : [\mathfrak{a}, \mathfrak{b}] \times \mathbb{E} \to \mathbb{E}$ *is said to be Carathéodory, if*

1. *The map* $t \to \phi(t, z)$ *is measurable for all* $z \in \mathbb{E}$, *and*
2. *The map* $z \to \phi(t, z)$ *is continuous for almost each* $t \in [\mathfrak{a}, \mathfrak{b}]$.

Definition 1.7 *[60] (**Compact Operator)***
Let $\mathbb{E}, \mathbb{F}$ *be two Banach spaces. A linear operator* $\mathscr{A} : \mathbb{E} \to \mathbb{F}$ *is compact if it transforms all bounded set of* $\mathbb{E}$ *into a relatively compact set of* $\mathbb{F}$.

Definition 1.8 *[60] (**Completely Continuous Operator)***
Let $\mathbb{E}, \mathbb{F}$ *be two Banach spaces. The operator* $\mathscr{A} : \mathbb{E} \to \mathbb{F}$ *is said to be completely continuous, if it's continuous and compact.*

Now, let $(\mathrm{E}, \|.\|)$ be a Banach space and $\mathrm{J} = [\mathfrak{a}, \mathfrak{b}]$ be an interval of $\mathbb{R}$, then we give some functional spaces [44, 49, 64, 85]:

- Consider $\mathrm{C}(\mathrm{J}, \mathbb{E})$ the Banach space of continuous functions $z : \mathrm{J} \to \mathbb{E}$, equipped with the norm
$$\|z\|_\infty = \sup_{t \in \mathrm{J}} |z(t)|.$$
- Let $\mathrm{C}^2(\mathrm{J}, \mathbb{E})$ the Banach space of differentiable functions from J into $\mathbb{E}$ whose first and second derivatives are continuous.
- Let $\mathrm{L}^1(\mathrm{J}, \mathbb{R})$ the Banach space of measurable functions from J into $\mathbb{R}$ which are Lebesgue integrable, with the norm
$$\|z\|_{\mathrm{L}^1} = \int_{\mathrm{J}} |z(t)| dt.$$
- Let $\mathrm{L}^\infty(\mathrm{J}, \mathbb{R})$ the Banach space of measurable functions $z : \mathrm{J} \to \mathbb{R}$ which are essentially bounded, with the norm
$$\|z\|_{\mathrm{L}^\infty} = ess\sup_{t \in \mathrm{J}} |z(t)| = \inf\{c > 0 \; : \; |z(t)| \leq c \text{ a.e } t \in \mathrm{J}\}.$$

Next, given a set $\mathcal{V}$ of functions $v : \mathrm{J} \to \mathbb{E}$, let

$$\begin{aligned} \mathcal{V}(t) &= \{v(t) : v \in \mathcal{V}\};\ t \in \mathrm{J}, \\ \mathcal{V}(\mathrm{J}) &= \{v(t) : v \in \mathcal{V};\ t \in \mathrm{J}\}. \end{aligned}$$

Finally, we present the Arzela-Ascoli's theorem which plays an important role in our work.

Theorem 1.9 *[64]* ***(Arzela-Ascoli)***
Let X *be a subset of* $\mathrm{C}(\mathrm{J}, \mathbb{E})$ *with* $\mathbb{E}$ *is finite space. Then,* X *is relatively compact in* $\mathrm{C}(\mathrm{J}, \mathbb{E})$ *if and only if*

(i) X *is uniformly bounded, i.e.:*

$$\exists \mathcal{M} > 0 : \|\phi(z)\|_\infty \leq \mathcal{M}; \quad \forall z \in \mathrm{J} \text{ and } \phi \in \mathrm{X}.$$

(ii) X *is equi-continuous, i.e.:*

$$\forall \epsilon > 0, \exists \delta > 0 : \|y - z\|_\infty < \delta \Rightarrow \|\phi(y) - \phi(z)\|_\infty < \epsilon \quad \forall y, z \in \mathrm{J} \text{ and } \phi \in \mathrm{X}.$$

2 Quantum Calculus (q-Difference)

In 1910, *Jackson* [58, 59], was the first scientist to developed quantum calculus in a systematic way, and introduced the notion of the q-derivative, the q-integral and some classical concepts.

This section reviews the fundamental definitions and some notations of the q-difference, as well as its properties and some examples. For details, see references [28, 63, 78, 88].

In this thesis, we assume that $q \in (0,1)$. For all $\alpha \in \mathbb{R}$, we set:

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}. \tag{1.1}$$

Definition 2.1 *[63] The* q*-factorial of a positive integer* n *is defined by:*

$$[0]_q = 1, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q; \quad n \in \mathbb{N}. \tag{1.2}$$

Definition 2.2 *[63] The q-analogue of the power $(\mathfrak{a}-\mathfrak{b})^{(n)}$ is expressed by:*

$$(\mathfrak{a}-\mathfrak{b})^{(0)}=1, \quad (\mathfrak{a}-\mathfrak{b})^{(n)}=(\mathfrak{a}-\mathfrak{b})_q^n=\prod_{i=0}^{n-1}\left(\mathfrak{a}-\mathfrak{b}q^i\right); \quad n\in\mathbb{N},\ \mathfrak{a},\mathfrak{b}\in\mathbb{R}. \tag{1.3}$$

In general, if $\beta\in\mathbb{R}$, we have:

$$(\mathfrak{a}-\mathfrak{b})^{(\beta)}=\mathfrak{a}^{\beta}\prod_{i=0}^{\infty}\left(\frac{\mathfrak{a}-\mathfrak{b}q^i}{\mathfrak{a}-\mathfrak{b}q^{i+\beta}}\right); \quad \mathfrak{a},\mathfrak{b}\in\mathbb{R}. \tag{1.4}$$

Notice that, if $\mathfrak{b}=0$, then $\mathfrak{a}^{(\beta)}=\mathfrak{a}^{\beta}$.

Properties 2.3 *[78] For all $\mathfrak{a},\mathfrak{b},\beta\in\mathbb{R}_+$ and $n,m\in\mathbb{N}$. The following formulas are correct:*

$$\left(\mathfrak{a}-\mathfrak{b}q^m\right)^{(\beta)}=\mathfrak{a}^{\beta}\left(1-q^m\frac{\mathfrak{b}}{\mathfrak{a}}\right)^{(\beta)}. \tag{1.5}$$

$$\frac{\left(\mathfrak{a}-\mathfrak{b}q^m\right)^{(\beta)}}{(\mathfrak{a}-\mathfrak{b})^{(\beta)}}=\frac{\left(\mathfrak{a}-\mathfrak{b}q^{\beta}\right)^{(m)}}{(\mathfrak{a}-\mathfrak{b})^{(m)}}. \tag{1.6}$$

$$\left(q^n-q^m\right)^{(\beta)}=0; \quad m\le n. \tag{1.7}$$

2.1 q-Derivative

In this part, we review a few definitions and properties of the q-derivative.

Definition 2.4 *[63] Let ϕ be an arbitrary function. The q-differential is defined by:*

$$d_q\phi(x)=\phi(x)-\phi(qx).$$

In particular,

$$d_qx=x(1-q).$$

Definition 2.5 *[63] The q-derivative of a function ϕ is defined by:*

$$\left(\mathfrak{D}_q\phi\right)(x)=\frac{d_q\phi(x)}{d_qx}=\frac{\phi(x)-\phi(qx)}{(1-q)x}; \quad x\neq 0. \tag{1.8}$$

$$\left(\mathfrak{D}_q\phi\right)(0)=\lim_{x\to 0}\left(\mathfrak{D}_q\phi\right)(x).$$

Note that,

$$\lim_{q\to 1}\left(\mathfrak{D}_q\phi\right)(x)=\frac{d\phi(x)}{dx}=\phi'(x).$$

Example 2.6 *[63] Let $\phi(x) = x^n$ where $n \in \mathbb{N}$. Then, we have:*

$$\mathfrak{D}_q x^n = [n]_q x^{n-1}. \tag{1.9}$$

In effect,

$$\begin{aligned} \mathfrak{D}_q x^n &= \frac{x^n - (qx)^n}{(1-q)x}, \\ &= \frac{\left(1-q^n\right)x^n}{(1-q)x}, \\ &= \frac{1-q^n}{1-q} x^{n-1}, \\ &= [n]_q x^{n-1}. \end{aligned}$$

Properties 2.7 *[63] Let ϕ, ψ be two functions and for γ, $\lambda \in \mathbb{R}$. The properties of the q-derivative are as follows:*

1. *The q-derivative $\mathfrak{D}_q$ is a linear operator, such that:*

$$\mathfrak{D}_q\left(\gamma\phi(x) + \lambda\psi(x)\right) = \gamma\left(\mathfrak{D}_q\phi\right)(x) + \lambda\left(\mathfrak{D}_q\psi\right)(x).$$

2. *The q-derivative of the product of the functions ϕ and ψ is given by:*

$$\mathfrak{D}_q\left(\phi(x).\psi(x)\right) = \phi(qx)\left(\mathfrak{D}_q\psi\right)(x) + \psi(x)\left(\mathfrak{D}_q\phi\right)(x). \tag{1.10}$$

3. *The q-derivative of the quotient of the functions ϕ and ψ is given by:*

$$\mathfrak{D}_q\left(\frac{\phi(x)}{\psi(x)}\right) = \frac{\left(\mathfrak{D}_q\phi\right)(x)\psi(qx) - \left(\mathfrak{D}_q\psi\right)(x)\phi(qx)}{\psi(x)\psi(qx)}; \quad \text{with } \psi(x) \neq 0,\ \psi(qx) \neq 0. \tag{1.11}$$

Proposition 2.8 *[63] For $n \in \mathbb{N}^*$, we have:*

(i)

$$\mathfrak{D}_q (x-\mathfrak{a})^{(n)} = [n]_q (x-\mathfrak{a})^{(n-1)}. \tag{1.12}$$

(ii)

$$\mathfrak{D}_q (\mathfrak{a}-x)^{(n)} = -[n]_q \left(\mathfrak{a}-qx\right)^{(n-1)}. \tag{1.13}$$

Proof.

(i) By using mathematical induction. Clearly, the proposition (1.12) is correct when $n = 1$, $\left(\text{because } \mathfrak{D}_q (x-\mathfrak{a}) = [1]_q = 1\right)$.

Let $n \in \mathbb{N}^*$. Assume that the proposition (1.12) is correct for some order i, i.e.:

$$\mathfrak{D}_q (x-\mathfrak{a})^{(i)} = [i]_q (x-\mathfrak{a})^{(i-1)},$$

and we'll prove that the proposition is correct for $i+1$, i.e.:

$$\mathfrak{D}_q (x-\mathfrak{a})^{(i+1)} = [i+1]_q (x-\mathfrak{a})^{(i)}.$$

According to the formula (1.3) $\left(\text{we have } (x-\mathfrak{a})^{(i+1)} = (x-\mathfrak{a})^{(i)}\left(x-q^i\mathfrak{a}\right)\right)$ and using the rule of the product (1.10), we get:

$$\begin{aligned}
\mathfrak{D}_q (x-\mathfrak{a})^{(i+1)} &= \mathfrak{D}_q (x-\mathfrak{a})^{(i)} \left(x-q^i\mathfrak{a}\right), \\
&= \left(qx-q^i\mathfrak{a}\right)\mathfrak{D}_q (x-\mathfrak{a})^{(i)} + (x-\mathfrak{a})^{(i)}\mathfrak{D}_q\left(x-q^i\mathfrak{a}\right), \\
&= q[i]_q\left(x-q^{i-1}\mathfrak{a}\right)(x-\mathfrak{a})^{(i-1)} + (x-\mathfrak{a})^{(i)}, \\
&= q[i]_q (x-\mathfrak{a})^{(i)} + (x-\mathfrak{a})^{(i)}, \\
&= \left(q[i]_q+1\right)(x-\mathfrak{a})^{(i)}, \\
&= [i+1]_q (x-\mathfrak{a})^{(i)}.
\end{aligned}$$

So, the proposition is correct for $i+1$, hence, the proposition (1.12) holds.

(ii) According to the formula (1.3) and for $n \in \mathbb{N}^*$, we have:

$$\begin{aligned}
(\mathfrak{a}-x)^{(n)} &= (\mathfrak{a}-x)\left(\mathfrak{a}-qx\right)\left(\mathfrak{a}-q^2x\right)\cdots\left(\mathfrak{a}-q^{n-1}x\right), \\
&= (\mathfrak{a}-x)\, q\left(q^{-1}\mathfrak{a}-x\right) q^2\left(q^{-2}\mathfrak{a}-x\right)\cdots q^{n-1}\left(q^{1-n}\mathfrak{a}-x\right), \\
&= q^{\frac{n(n-1)}{2}}(-1)(x-\mathfrak{a})(-1)\left(x-q^{-1}\mathfrak{a}\right)(-1)\left(x-q^{-2}\mathfrak{a}\right)\cdots(-1)\left(x-q^{1-n}\mathfrak{a}\right), \\
&= (-1)^n q^{\frac{n(n-1)}{2}}(x-\mathfrak{a})\left(x-q^{-1}\mathfrak{a}\right)\left(x-q^{-2}\mathfrak{a}\right)\cdots\left(x-q^{1-n}\mathfrak{a}\right).
\end{aligned}$$

Then,

$$(\mathfrak{a}-x)^{(n)} = (-1)^n q^{\frac{n(n-1)}{2}}\left(x-q^{1-n}\mathfrak{a}\right)^{(n)}. \tag{1.14}$$

Using the formula (1.14) and proposition (1.12), we find:

$$\begin{aligned}
\mathfrak{D}_q (\mathfrak{a}-x)^{(n)} &= (-1)^n q^{\frac{n(n-1)}{2}}\mathfrak{D}_q\left(x-q^{1-n}\mathfrak{a}\right)^{(n)}, \\
&= (-1)^n q^{\frac{n(n-1)}{2}}[n]_q\left(x-q^{1-n}\mathfrak{a}\right)^{(n-1)}, \\
&= -[n]_q q^{n-1}(-1)^{n-1} q^{\frac{(n-1)(n-2)}{2}}\left(x-q^{2-n}\left(q^{-1}\mathfrak{a}\right)\right)^{(n-1)}, \\
&= -[n]_q q^{n-1}\left(q^{-1}\mathfrak{a}-x\right)^{(n-1)}, \\
&= -[n]_q\left(\mathfrak{a}-qx\right)^{(n-1)}.
\end{aligned}$$

∎

Definition 2.9 *[63] Let $\phi : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$. The q-derivative of order $n \in \mathbb{N}$ is defined by:*

$$\left(\mathfrak{D}_q^n \phi\right)(x) = \left(\mathfrak{D}_q \mathfrak{D}_q^{n-1} \phi\right)(x); \quad x \in [\mathfrak{a}, \mathfrak{b}], \; n \in \{1, 2, \ldots\}, \tag{1.15}$$

and

$$\left(\mathfrak{D}_q^0 \phi\right)(x) = \phi(x).$$

2.2 q-Integral

This part contains the definitions, notations and some properties of the q-integral (Jackson's Integral).

Definition 2.10 *[63] The function Φ is a q-antiderivative of the function ϕ if $\mathfrak{D}_q \Phi(x) = \phi(x)$. It's rated by:*

$$\Phi(x) = \int \phi(x) d_q x.$$

Jackson's Integral

[63] Let ϕ be an arbitrary function, for constructing its q-antiderivative Φ such that:

$$\Phi(x) = \int \phi(x) d_q x. \tag{1.16}$$

Then, we introduce the linear operator $\widehat{\mathrm{M}}_q$, which is defined by [63]:

$$\widehat{\mathrm{M}}_q (\Phi(x)) = \Phi(qx). \tag{1.17}$$

By applying the q-derivative $\mathfrak{D}_q$ on the formula (1.16), we obtain:

$$\left(\mathfrak{D}_q \Phi\right)(x) = \mathfrak{D}_q \left(\int \phi(x) d_q x\right) \Rightarrow \frac{\Phi(x) - \Phi(qx)}{(1-q)x} = \phi(x).$$

According to the formula (1.17), we get:

$$\frac{\left(1 - \widehat{\mathrm{M}}_q\right) \Phi(x)}{(1-q)x} = \phi(x).$$

Thus,

$$\Phi(x) = \frac{1}{\left(1 - \widehat{\mathrm{M}}_q\right)} (1-q) x \phi(x).$$

Using expansion of the geometric series, we find:

$$\Phi(x) = (1-q) \sum_{n=0}^{\infty} \widehat{\mathrm{M}}_q^n \left(x \phi(x)\right).$$

Since $\widehat{\mathrm{M}}_q^n\left(x\phi(x)\right) = q^n x\phi\left(q^n x\right)$, we can write:

$$\Phi(x) = (1-q)\sum_{n=0}^{\infty} q^n x\phi\left(q^n x\right).$$

Hence,

$$\int \phi(x)d_q x = (1-q)x\sum_{n=0}^{\infty} q^n \phi\left(q^n x\right),$$

this last expression is called the **Jackson's integral**.

Set $\mathrm{J}_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.11 *[63] Let $\phi : \mathrm{J}_t \to \mathbb{R}$ be a function. The q-integral is given by:*

$$(\mathfrak{I}_q\phi)(x) = \int_0^x \phi(t)d_q t = (1-q)x\sum_{n=0}^{\infty} q^n\phi\left(q^n x\right), \tag{1.18}$$

provided that the series converges.

Definition 2.12 *[63] The q-integral of a function $\phi : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$, is defined by:*

$$(\mathfrak{I}_{q,\mathfrak{a}}\phi)(x) = \int_{\mathfrak{a}}^{\mathfrak{b}} \phi(t)d_q t = \int_0^{\mathfrak{b}} \phi(t)d_q t - \int_0^{\mathfrak{a}} \phi(t)d_q t. \tag{1.19}$$

Remark 2.13 *In the case of putting the lower limit of integration is $\mathfrak{a} = q^n\mathfrak{b}$, where $n \in \mathbb{N}$. The q-integral which depends on q, n and $\mathfrak{b}$ is defined as follows:*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \phi(t)d_q t = \int_{q^n\mathfrak{b}}^{\mathfrak{b}} \phi(t)d_q t = (1-q)\mathfrak{b}\sum_{i=0}^{n-1} q^i\phi\left(q^i\mathfrak{b}\right). \tag{1.20}$$

Example 2.14 *[63] Let $\phi(x) = x^n$, $n \in \mathbb{N}$. Then, we have:*

$$\mathfrak{I}_q x^n = \frac{x^{n+1}}{[n+1]_q}. \tag{1.21}$$

In effect,

$$\begin{aligned} \mathfrak{I}_q x^n &= \int_0^x t^n d_q t, \\ &= (1-q)x^{n+1}\sum_{i=0}^{\infty} q^{i(n+1)}, \\ &= \frac{1-q}{1-q^{n+1}}x^{n+1}, \\ &= \frac{x^{n+1}}{[n+1]_q}. \end{aligned}$$

Definition 2.15 *[63] Let ϕ, ψ be two q-differentiable functions on $[\mathfrak{a},\mathfrak{b}]$. The q-integration by parts is defined as:*

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \phi(x)\left(\mathfrak{D}_q\psi\right)(x)d_qx = \phi(\mathfrak{b})\psi(\mathfrak{b}) - \phi(\mathfrak{a})\psi(\mathfrak{a}) - \int_{\mathfrak{a}}^{\mathfrak{b}} \psi(qx)\left(\mathfrak{D}_q\phi\right)(x)d_qx. \tag{1.22}$$

Proposition 2.16 *[63, 88] The properties of the q-integral and q-derivative are as follows:*

1.
$$\left(\mathfrak{D}_q\mathfrak{I}_q\phi\right)(x) = \phi(x). \tag{1.23}$$

2. *If ϕ is continuous at 0, then:*
$$\left(\mathfrak{I}_q\mathfrak{D}_q\phi\right)(x) = \phi(x) - \phi(0). \tag{1.24}$$

In general [88], for all $n \in \mathbb{N}$, we have:

1.
$$\left(\mathfrak{D}_q^n\mathfrak{I}_{q,\mathfrak{a}}^n\phi\right)(x) = \phi(x). \tag{1.25}$$

2.
$$\left(\mathfrak{I}_{q,\mathfrak{a}}^n\mathfrak{D}_q^n\phi\right)(x) = \phi(x) - \sum_{i=0}^{n-1}\frac{\left(\mathfrak{D}_q^i\phi\right)(\mathfrak{a})}{[i]_q!}(x-\mathfrak{a})^{(i)}. \tag{1.26}$$

2.3 q-Exponential Functions

In this part, we shall define two q-analogues of the exponential functions.

Definition 2.17 *[63] The two q-exponential functions are as follows:*

$$e_q^x = \sum_{i=0}^{\infty}\frac{x^i}{[i]_q!} = \frac{1}{\left(1-(1-q)x\right)^{(\infty)}}, \tag{1.27}$$

and

$$\mathrm{E}_q^x = \sum_{i=0}^{\infty} q^{\frac{i(i-1)}{2}}\frac{x^i}{[i]_q!} = \left(1+(1-q)x\right)^{(\infty)}. \tag{1.28}$$

Remark 2.18 *[63] The two q-exponential functions are closely related. From (1.27) and (1.28), we see that:*

$$e_q^x\mathrm{E}_q^{-x} = 1.$$

Proposition 2.19 *[63] The q-derivative of the q-exponential functions are given by:*

(i) $\mathfrak{D}_q e_q^x = e_q^x$.

(ii) $\mathfrak{D}_q \mathrm{E}_q^x = \mathrm{E}_q^{qx}$.

2.4 q-Special Functions

This part includes the definitions and properties of the q-Gamma and q-Beta functions.

q-Gamma Function

Definition 2.20 *[63] The q-Gamma function is given by:*

$$\Gamma_q(\beta) = \frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}}; \quad \beta > 0. \tag{1.29}$$

The q-Gamma function admits a q-integral representation, which is defined by:

$$\Gamma_q(\beta) = \int_0^{\infty} x^{\beta-1} \mathrm{E}_q^{-qx} d_q x; \quad \beta > 0. \tag{1.30}$$

Properties 2.21 *[63] The q-Gamma function has the following properties:*

1. *For any $\beta > 0$, we have:*
$$\Gamma_q(\beta+1) = [\beta]_q \Gamma_q(\beta). \tag{1.31}$$

2. *For $n \in \mathbb{N}$, we have:*
$$\Gamma_q(n+1) = [n]_q! \quad with \quad \Gamma_q(1) = 1. \tag{1.32}$$

Proof.

1. For $\beta > 0$, we have:

$$\begin{aligned} \Gamma_q(\beta+1) &= \frac{(1-q)^{(\beta)}}{(1-q)^{\beta}}, \\ &= \frac{(1-q^{\beta})(1-q)^{(\beta-1)}}{(1-q)(1-q)^{\beta-1}}, \\ &= [\beta]_q \Gamma_q(\beta). \end{aligned}$$

2. Applying the property (1.31), for any $n \in \mathbb{N}$ with $\Gamma_q(1) = 1$, we have:

$$\begin{aligned} \Gamma_q(n+1) &= [n]_q \Gamma_q(n), \\ &= [n]_q [n-1]_q \Gamma_q(n-1), \\ &\vdots \\ &= [n]_q [n-1]_q \cdots [2]_q [1]_q \Gamma_q(1), \\ &= [n]_q!. \end{aligned}$$

■

q-Beta Function

Definition 2.22 *[63] The q-Beta function is given by:*

$$B_q(\alpha,\beta) = \int_0^1 x^{\alpha-1}\left(1-qx\right)^{(\beta-1)} d_q x; \quad \alpha,\beta > 0. \tag{1.33}$$

Proposition 2.23 *[63] The q-Beta and q-Gamma functions have the following relationship:*

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}; \quad \alpha,\beta > 0. \tag{1.34}$$

3 Fractional q-Calculus

At the end of the sixties, *Al-Salam* [27] and *Agarwal* [11] suggested the fractional q-difference calculus, so that they provided some types of the fractional q-integral and q-derivative operators.

This section concentrates on the fundamental definitions and some properties of the fractional q-calculus, can be located in [11, 27, 78, 79, 86, 88] and references therein.

3.1 Riemann-Liouville's Fractional q-Integral

In this part, we will review the essential definitions and properties of the Riemann-Liouville's fractional q-integral.

The fractional q-integral of the Riemann-Liouville type is based on the q-analogue of Cauchy's formula (see [27, 78]), which is the calculation of the q-integral repeated n times which is obtained by:

$$\begin{aligned} \left(\mathfrak{I}^n_{q,\mathfrak{a}}\phi\right)(x) &= \int_{\mathfrak{a}}^x d_q t \int_{\mathfrak{a}}^t d_q t_{n-1} \int_{\mathfrak{a}}^{t_{n-1}} d_q t_{n-2} \cdots \int_{\mathfrak{a}}^{t_2} \phi(t_1) d_q t_1, \\ &= \frac{1}{[n-1]_q!} \int_{\mathfrak{a}}^x (x-qt)^{(n-1)} \phi(t) d_q t. \end{aligned} \tag{1.35}$$

By generalizing the formula (1.35) to the real positive order β and replacing the q-factorial function with the q-Gamma function, we will have the following definition:

Definition 3.1 *[11, 78] Let ϕ be a function defined on $[\mathfrak{a},\mathfrak{b}]$ and for $\beta \in \mathbb{R}_+$. The Riemann-Liouville's fractional q-integral of order β is given by:*

$$\begin{cases} \left(\mathfrak{I}^0_{q,\mathfrak{a}}\phi\right)(x) = \phi(x); & \text{if } \beta = 0, \\ \left(\mathfrak{I}^\beta_{q,\mathfrak{a}}\phi\right)(x) = \frac{1}{\Gamma_q(\beta)} \int_a^x (x-qt)^{(\beta-1)} \phi(t) d_q t; & \text{if } \beta > 0. \end{cases} \tag{1.36}$$

Notice that, if $\beta = 1$, then $\left(\mathfrak{I}^1_{q,\mathfrak{a}}\phi\right)(x) = \left(\mathfrak{I}_{q,\mathfrak{a}}\phi\right)(x)$.

Example 3.2 *Let $\phi(x)=x$, $x>0$ and for $\beta\geq 0$. Then, we have:*

$$\mathfrak{I}_q^{\beta}x=\frac{x^{(\beta+1)}}{\Gamma_q(\beta+2)}. \tag{1.37}$$

In effect, by applying the property (1.13) and q-integration by parts (1.22), we get:

$$\begin{aligned}
\mathfrak{I}_q^{\beta}x &= \frac{1}{\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}t\,d_qt,\\
&= -\frac{1}{[\beta]_q\Gamma_q(\beta)}\int_0^x t\mathfrak{D}_q(x-t)^{(\beta)}d_qt,\\
&= -\frac{1}{\Gamma_q(\beta+1)}\left(\left[t(x-t)^{(\beta)}\right]_0^x-\int_0^x (x-qt)^{(\beta)}d_qt\right),\\
&= \frac{1}{\Gamma_q(\beta+1)}\int_0^x (x-qt)^{(\beta)}d_qt,\\
&= -\frac{1}{[\beta+1]_q\Gamma_q(\beta+1)}\int_0^x \mathfrak{D}_q(x-t)^{(\beta+1)}d_qt,\\
&= -\frac{1}{\Gamma_q(\beta+2)}\left[(x-t)^{(\beta+1)}\right]_0^x,\\
&= \frac{x^{(\beta+1)}}{\Gamma_q(\beta+2)}.
\end{aligned}$$

Remark 3.3 *[86] For every $\phi(x)$ defined on $(0,\mathfrak{b})$ and $\beta\in\mathbb{R}_+$, the following fact is true:*

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(\mathfrak{a})=\frac{1}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{\mathfrak{a}}(\mathfrak{a}-qt)^{(\beta-1)}\phi(t)d_qt=0. \tag{1.38}$$

The next result is essential in clarifying the properties and lemmas of fractional q-calculus.

Lemma 3.4 *[78] The following identification is correct for every $\nu,\alpha,\beta\in\mathbb{R}_+$:*

$$\sum_{n=0}^{\infty}q^{\alpha n}\frac{(1-q^{1-n}\nu)^{(\alpha-1)}(1-q^{1+n})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}=\frac{(1-q\nu)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}. \tag{1.39}$$

Theorem 3.5 *[86] For each $\beta\in\mathbb{R}_+$ and $x\in(\mathfrak{a},\mathfrak{b})$. Then, the following relation is valid:*

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)=\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta+1}\mathfrak{D}_q\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(\beta+1)}(x-\mathfrak{a})^{(\beta)}. \tag{1.40}$$

Proof. Using the property (1.13) and q-integration by parts (1.22), we get:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \frac{1}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta-1)}\phi(t)d_qt,\\ &= -\frac{1}{[\beta]_q\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}\mathfrak{D}_q(x-t)^{(\beta)}\phi(t)d_qt,\\ &= -\frac{1}{\Gamma_q(\beta+1)}\left(\left[(x-t)^{(\beta)}\phi(t)\right]_{\mathfrak{a}}^{x}-\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta)}\left(\mathfrak{D}_q\phi\right)(t)d_qt\right),\\ &= \frac{\phi(\mathfrak{a})}{\Gamma_q(\beta+1)}(x-\mathfrak{a})^{(\beta)}+\frac{1}{\Gamma_q(\beta+1)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta)}\left(\mathfrak{D}_q\phi\right)(t)d_qt,\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta+1}\mathfrak{D}_q\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(\beta+1)}(x-\mathfrak{a})^{(\beta)}.\end{aligned}$$

■

Lemma 3.6 *[78] For $\alpha,\beta\in\mathbb{R}_+$ and let $x<\mathfrak{a}$, the relationship shown below is true:*

$$\int_0^{\mathfrak{a}}(x-qt)^{(\alpha-1)}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(t)d_qt=0. \tag{1.41}$$

Lemma 3.7 *[78] For each $\beta\in\mathbb{R}_+$, $\lambda\in(-1,+\infty)$. Then, we have:*

$$\mathfrak{I}_{q,\mathfrak{a}}^{\beta}(x-\mathfrak{a})^{(\lambda)}=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}(x-\mathfrak{a})^{(\beta+\lambda)};\quad 0<\mathfrak{a}<x<\mathfrak{b}. \tag{1.42}$$

Particularly, for $\lambda=0$, we have:

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}1\right)(x)=\frac{(x-\mathfrak{a})^{(\beta)}}{\Gamma_q(\beta+1)}. \tag{1.43}$$

Proof. For $\beta\in\mathbb{R}_+$ and $\lambda\neq 0$, then by Definition 3.1, we have:

$$\begin{aligned}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}(x-\mathfrak{a})^{(\lambda)} &= \frac{1}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta-1)}(t-\mathfrak{a})^{(\lambda)}d_qt,\\ &= \frac{1}{\Gamma_q(\beta)}\left(\int_0^{x}(x-qt)^{(\beta-1)}(t-\mathfrak{a})^{(\lambda)}d_qt-\int_0^{\mathfrak{a}}(x-qt)^{(\beta-1)}(t-\mathfrak{a})^{(\lambda)}d_qt\right).\end{aligned}$$

On the one hand, according to Definition 2.11 and formula (1.7), we get:

$$\begin{aligned}\int_0^{\mathfrak{a}}(x-qt)^{(\beta-1)}(t-\mathfrak{a})^{(\lambda)}d_qt &= (1-q)\mathfrak{a}\sum_{n=0}^{\infty}q^n\left(x-q^{n+1}\mathfrak{a}\right)^{(\beta-1)}\left(q^n\mathfrak{a}-\mathfrak{a}\right)^{(\lambda)},\\ &= \mathfrak{a}^{(\lambda+1)}(1-q)\sum_{n=0}^{\infty}q^n\left(x-q^{n+1}\mathfrak{a}\right)^{(\beta-1)}\left(q^n-1\right)^{(\lambda)},\\ &= 0.\end{aligned}$$

On the other hand, applying Definition 2.11 and using the formula (1.39), we find:

$$\begin{aligned}
\int_0^x (x-qt)^{(\beta-1)}(t-\mathfrak{a})^{(\lambda)} d_q t &= (1-q)x\sum_{n=0}^{\infty} q^n \left(x-q^{n+1}x\right)^{(\beta-1)}\left(q^n x-\mathfrak{a}\right)^{(\lambda)}, \\
&= x^{(\beta+\lambda)}(1-q)\sum_{n=0}^{\infty} q^n \left(1-q^{n+1}\right)^{(\beta-1)}\left(q^n-\frac{\mathfrak{a}}{x}\right)^{(\lambda)}, \\
&= x^{(\beta+\lambda)}(1-q)\sum_{n=0}^{\infty} q^{n(1+\lambda)} \left(1-q^{n+1}\right)^{(\beta-1)}\left(1-\frac{\mathfrak{a}}{q^n x}\right)^{(\lambda)}, \\
&= x^{(\beta+\lambda)}(1-q)\sum_{n=0}^{\infty} q^{n(1+\lambda)} \left(1-q^{n+1}\right)^{(\beta-1)}\left(1-\frac{\mathfrak{a}}{qx}q^{1-n}\right)^{(\lambda)}, \\
&= x^{(\beta+\lambda)}(1-q)\frac{\left(1-\frac{\mathfrak{a}}{x}\right)^{(\beta+\lambda)}(1-q)^{(\beta-1)}(1-q)^{(\lambda)}}{(1-q)^{(\beta+\lambda)}}, \\
&= (x-\mathfrak{a})^{(\beta+\lambda)}(1-q)\frac{(1-q)^{(\beta-1)}(1-q)^{(\lambda)}}{(1-q)^{(\beta+\lambda)}}, \\
&= (x-\mathfrak{a})^{(\beta+\lambda)}(1-q)\frac{(1-q)^{(\beta-1)}(1-q)^{(\lambda)}(1-q)^{\beta-1}(1-q)^{\lambda}}{(1-q)^{(\beta+\lambda)}(1-q)^{\beta-1}(1-q)^{\lambda}}, \\
&= (x-\mathfrak{a})^{(\beta+\lambda)}\frac{(1-q)^{\beta+\lambda}(1-q)^{(\beta-1)}(1-q)^{(\lambda)}}{(1-q)^{(\beta+\lambda)}(1-q)^{\beta-1}(1-q)^{\lambda}}, \\
&= (x-\mathfrak{a})^{(\beta+\lambda)}\frac{\Gamma_q(\beta)\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathfrak{I}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} &= \frac{1}{\Gamma_q(\beta)}\left((x-\mathfrak{a})^{(\beta+\lambda)}\frac{\Gamma_q(\beta)\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}-0\right), \\
&= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}(x-\mathfrak{a})^{(\beta+\lambda)}.
\end{aligned}$$

Particularly, for $\lambda = 0$, we have:

$$\begin{aligned}
\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}1\right)(x) &= \frac{1}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^x (x-qt)^{(\beta-1)} d_q t, \\
&= -\frac{1}{[\beta]_q\Gamma_q(\beta)}\int_{\mathfrak{a}}^x \mathfrak{D}_q(x-t)^{(\beta)} d_q t, \\
&= -\frac{1}{\Gamma_q(\beta+1)}\left[(x-t)^{(\beta)}\right]_{\mathfrak{a}}^x, \\
&= \frac{(x-\mathfrak{a})^{(\beta)}}{\Gamma_q(\beta+1)}.
\end{aligned}$$

■

Properties 3.8 *[78, 86] The fractional q-integral of the Riemann-Liouville has the following properties:*

1. ***Linearity:*** *Let $\phi,\psi : [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}$, for $\gamma,\lambda \in \mathbb{R}$ and for any $\beta \in \mathbb{R}_+$. Then, we have:*

$$\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\left(\gamma\phi(x)+\lambda\psi(x)\right) = \gamma\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi\right)(x)+\lambda\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\psi\right)(x).$$

2. ***Semi-group and commutativity:*** *Let $\phi : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$ and for any $\alpha, \beta \in \mathbb{R}_+$. Then, we have:*

$$\begin{aligned}\left(\mathfrak{I}^{\alpha}_{q,\mathfrak{a}}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{I}^{\alpha+\beta}_{q,\mathfrak{a}}\phi\right)(x),\\ &= \left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\mathfrak{I}^{\alpha}_{q,\mathfrak{a}}\phi\right)(x).\end{aligned} \tag{1.44}$$

Proof.

1. Let $\phi, \psi : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$, for $\gamma, \lambda \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$, we have:

$$\begin{aligned}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\left(\gamma\phi(x)+\lambda\psi(x)\right) &= \frac{1}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta-1)}\left(\gamma\phi(t)+\lambda\psi(t)\right)d_qt,\\ &= \frac{\gamma}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta-1)}\phi(t)d_qt+\frac{\lambda}{\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\beta-1)}\psi(t)d_qt,\\ &= \gamma\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi\right)(x)+\lambda\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\psi\right)(x).\end{aligned}$$

2. Let $\phi : [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}$ and for $\alpha, \beta \in \mathbb{R}_+$, then by Definition 3.1 and formula (1.19), we have:

$$\begin{aligned}\left(\mathfrak{I}^{\alpha}_{q,\mathfrak{a}}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \frac{1}{\Gamma_q(\alpha)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\alpha-1)}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi(t)d_qt,\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\mathfrak{a}}^{x}(x-qt)^{(\alpha-1)}\int_{\mathfrak{a}}^{t}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt,\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\left(\int_0^x(x-qt)^{(\alpha-1)}-\int_0^{\mathfrak{a}}(x-qt)^{(\alpha-1)}\right)\\ &\quad\times\left(\int_0^t(t-qs)^{(\beta-1)}\phi(s)d_qs-\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qs\right)d_qt,\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x(x-qt)^{(\alpha-1)}\int_0^t(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt\\ &\quad-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x(x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt\\ &\quad-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^{\mathfrak{a}}(x-qt)^{(\alpha-1)}\int_0^t(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt\\ &\quad+\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^{\mathfrak{a}}(x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt,\\ &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x(x-qt)^{(\alpha-1)}\int_0^t(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt\\ &\quad-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x(x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt.\end{aligned}$$

R.P Agarwal in the article [11] proved that the following equality is correct:

$$\left(\mathfrak{I}^{\alpha}_{q,0}\mathfrak{I}^{\beta}_{q,0}\phi\right)(x) = \left(\mathfrak{I}^{\alpha+\beta}_{q,0}\phi\right)(x). \tag{1.45}$$

As a result, we remark:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left(\mathfrak{I}_{q,0}^{\alpha}\mathfrak{I}_{q,0}^{\beta}\phi\right)(x)-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt,\\ &= \left(\mathfrak{I}_{q,0}^{\alpha+\beta}\phi\right)(x)-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt.\end{aligned}$$

Moreover, we have:

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha+\beta}\phi\right)(x) = \left(\mathfrak{I}_{q,0}^{\alpha+\beta}\phi\right)(x)-\left(\mathfrak{I}_{q,0}^{\alpha+\beta}\phi\right)(\mathfrak{a}). \tag{1.46}$$

According to the relation (1.46), we obtain:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha+\beta}\phi\right)(x)+\left(\mathfrak{I}_{q,0}^{\alpha+\beta}\phi\right)(\mathfrak{a})-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\alpha-1)}\\ &\quad\times\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt,\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha+\beta}\phi\right)(x)+\frac{1}{\Gamma_q(\alpha+\beta)}\int_0^{\mathfrak{a}}(x-qt)^{(\alpha+\beta-1)}\phi(t)d_qt\\ &\quad-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\alpha-1)}\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt.\end{aligned}$$

Then,

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) = \left(\mathfrak{I}_{q,\mathfrak{a}}^{\alpha+\beta}\phi\right)(x)+\mathfrak{C},$$

with

$$\begin{aligned}\mathfrak{C} &= \frac{1}{\Gamma_q(\alpha+\beta)}\int_0^{\mathfrak{a}}(x-qt)^{(\alpha+\beta-1)}\phi(t)d_qt-\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\alpha-1)}\\ &\quad\times\int_0^{\mathfrak{a}}(t-qs)^{(\beta-1)}\phi(s)d_qsd_qt.\end{aligned}$$

By applying Definition 2.11, we find:

$$\begin{aligned}\mathfrak{C} &= \frac{\mathfrak{a}(1-q)}{\Gamma_q(\alpha+\beta)}\sum_{i=0}^{\infty}q^i\left(x-q^{i+1}\mathfrak{a}\right)^{(\alpha+\beta-1)}\phi\left(q^i\mathfrak{a}\right)-\frac{\mathfrak{a}x(1-q)^2}{\Gamma_q(\alpha)\Gamma_q(\beta)}\\ &\quad\times\sum_{n=0}^{\infty}q^n\left(x-q^{n+1}x\right)^{(\alpha-1)}\sum_{i=0}^{\infty}q^i\left(q^nx-q^{i+1}\mathfrak{a}\right)^{(\beta-1)}\phi\left(q^i\mathfrak{a}\right),\\ &= \mathfrak{a}(1-q)\sum_{i=0}^{\infty}q^i\left[\frac{\left(x-q^{i+1}\mathfrak{a}\right)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}-\frac{x(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)}\right.\\ &\quad\left.\times\sum_{n=0}^{\infty}q^n\left(x-q^{n+1}x\right)^{(\alpha-1)}\left(q^nx-q^{i+1}\mathfrak{a}\right)^{(\beta-1)}\right]\phi\left(q^i\mathfrak{a}\right).\end{aligned}$$

So,

$$\mathfrak{C} = \mathfrak{a}(1-q)\sum_{i=0}^{\infty}q^ic_i\phi\left(q^i\mathfrak{a}\right),$$

with

$$c_i = \frac{\left(x - q^{i+1}\mathfrak{a}\right)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{x(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{n=0}^{\infty} q^n \left(x - q^{n+1}x\right)^{(\alpha-1)} \left(q^n x - q^{i+1}\mathfrak{a}\right)^{(\beta-1)}.$$

According to the definition of q-Gamma (1.29), we have:

$$\Gamma_q(\alpha+\beta) = \frac{(1-q)^{(\alpha+\beta-1)}}{(1-q)^{\alpha+\beta-1}}, \tag{1.47}$$

$$\Gamma_q(\alpha)\Gamma_q(\beta) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}} \frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}}. \tag{1.48}$$

Using the formulas (1.47)-(1.48), we get:

$$\begin{aligned} c_i &= \frac{\left(x - q^{i+1}\mathfrak{a}\right)^{(\alpha+\beta-1)} (1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha+\beta-1)}} - \frac{x(1-q)(1-q)^{\alpha-1}(1-q)^{\beta-1}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} \\ &\quad \times \sum_{n=0}^{\infty} q^n \left(x - q^{n+1}x\right)^{(\alpha-1)} \left(q^n x - q^{i+1}\mathfrak{a}\right)^{(\beta-1)}, \\ &= \frac{x^{(\alpha+\beta-1)} \left(1 - q^{i+1}\frac{\mathfrak{a}}{x}\right)^{(\alpha+\beta-1)} (1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha+\beta-1)}} - \frac{x^{(\alpha+\beta-1)}(1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} \\ &\quad \times \sum_{n=0}^{\infty} q^{\beta n} \left(1 - q^{n+1}\right)^{(\alpha-1)} \left(1 - q^{i+1-n}\frac{\mathfrak{a}}{x}\right)^{(\beta-1)}. \end{aligned}$$

If we take $\nu = \frac{\mathfrak{a}}{x} q^i$, we can write:

$$\begin{aligned} c_i &= \frac{x^{(\alpha+\beta-1)} \left(1 - \nu q\right)^{(\alpha+\beta-1)} (1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha+\beta-1)}} - \frac{x^{(\alpha+\beta-1)}(1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} \\ &\quad \times \sum_{n=0}^{\infty} q^{\beta n} \left(1 - q^{n+1}\right)^{(\alpha-1)} \left(1 - \nu q^{1-n}\right)^{(\beta-1)}. \end{aligned}$$

By formula (1.39), we find:

$$c_i = \frac{x^{(\alpha+\beta-1)} \left(1 - \nu q\right)^{(\alpha+\beta-1)} (1-q)^{\alpha+\beta-1}}{(1-q)^{(\alpha+\beta-1)}} - \frac{x^{(\alpha+\beta-1)}(1-q)^{\alpha+\beta-1} \left(1 - \nu q\right)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} = 0.$$

Hence,

$$\mathfrak{C} = \mathfrak{a}(1-q) \sum_{i=0}^{\infty} q^i c_i \phi\left(q^i \mathfrak{a}\right) = 0.$$

Consequently,

$$\left(\mathfrak{I}^{\alpha}_{q,\mathfrak{a}} \mathfrak{I}^{\beta}_{q,\mathfrak{a}} \phi\right)(x) = \left(\mathfrak{I}^{\alpha+\beta}_{q,\mathfrak{a}} \phi\right)(x).$$

In the same way, we show that:

$$\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}} \mathfrak{I}^{\alpha}_{q,\mathfrak{a}} \phi\right)(x) = \left(\mathfrak{I}^{\alpha+\beta}_{q,\mathfrak{a}} \phi\right)(x).$$

■

3.2 Riemann-Liouville's Fractional q-Derivative

This part focuses on the basic concepts and properties of the Riemann-Liouville's fractional q-derivative.

Definition 3.9 *[79, 88] Let ϕ be a function defined on $[\mathfrak{a},\mathfrak{b}]$ and for $\beta\in\mathbb{R}$. The Riemann-Liouville's fractional q-derivative of order β is defined by:*

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x)=\begin{cases}\left(\mathfrak{I}^{-\beta}_{q,\mathfrak{a}}\phi\right)(x); & \text{if } \beta<0,\\ \phi(x); & \text{if } \beta=0,\\ \left(\mathfrak{D}^{[\beta]}_{q}\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\phi\right)(x); & \text{if } \beta>0,\end{cases}\tag{1.49}$$

where $[\beta]$ is the integer part of β.

Example 3.10 *[86] Let $\phi(x)=(x-\mathfrak{a})^{(\lambda)}$, $0<\mathfrak{a}<x<\mathfrak{b}$. Then, for all $\beta,\lambda\in\mathbb{R}_+\setminus\mathbb{N}$ and for $(\beta-\lambda)\notin\mathbb{N}$, we have:*

$${}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)}=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\beta+1)}(x-\mathfrak{a})^{(\lambda-\beta)}.\tag{1.50}$$

In effect,

$${}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)}=\mathfrak{D}^{[\beta]}_{q}\left(\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)}\right).$$

Using the formula (1.42), we get:

$$\begin{aligned}{}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} &= \mathfrak{D}^{[\beta]}_{q}\left(\frac{\Gamma_q(\lambda+1)}{\Gamma_q([\beta]-\beta+\lambda+1)}(x-\mathfrak{a})^{([\beta]-\beta+\lambda)}\right),\\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q([\beta]-\beta+\lambda+1)}\mathfrak{D}^{[\beta]}_{q}(x-\mathfrak{a})^{([\beta]-\beta+\lambda)},\\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q([\beta]-\beta+\lambda+1)}\frac{\Gamma_q([\beta]-\beta+\lambda+1)}{\Gamma_q([\beta]-\beta+\lambda+1-[\beta])}(x-\mathfrak{a})^{([\beta]-\beta+\lambda-[\beta])},\\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\beta+1)}(x-\mathfrak{a})^{(\lambda-\beta)}.\end{aligned}$$

In particular, for $\lambda=0$, we have:

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}1\right)(x)=\frac{(x-\mathfrak{a})^{(-\beta)}}{\Gamma_q(1-\beta)}.\tag{1.51}$$

If $(\beta-\lambda)\in\mathbb{N}$, then we have:

$${}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)}=0.\tag{1.52}$$

Remark 3.11 *[86] For every $\phi(x)$ defined on $(0,\mathfrak{b})$ and $\beta\in\mathbb{R}_+$, the following fact is true:*

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(\mathfrak{a})=\left(\mathfrak{D}^{[\beta]}_{q}\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\phi\right)(\mathfrak{a})=0.\tag{1.53}$$

Proposition 3.12 *[86] For every $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$ and $0 < \mathfrak{a} < x < \mathfrak{b}$, the following properties is correct:*

1.

$$\left(\mathfrak{D}_q\, {}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \left({}^{RL}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x). \tag{1.54}$$

2.

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) = \left({}^{RL}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x) - \frac{\phi(\mathfrak{a})}{\Gamma_q(-\beta)}(x-\mathfrak{a})^{(-\beta-1)}. \tag{1.55}$$

Proof.

1. According to Definitions 3.9 and 2.9, we get:

$$\begin{aligned}
\left(\mathfrak{D}_q\, {}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{D}_q\mathfrak{D}^{[\beta]}_q\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \left(\mathfrak{D}^{[\beta]+1}_q\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \left({}^{RL}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x).
\end{aligned}$$

2. They exist $n \in \mathbb{N}$ such as $n < \beta < n+1$, then $[\beta] = n+1$. Using Proposition 2.16, the formulas (1.43)-(1.12) and property (1.44), according to the equation (1.54), we find:

$$\begin{aligned}
\left({}^{RL}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{D}_q\, {}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \left(\mathfrak{D}_q\mathfrak{D}^{[\beta]}_q\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \left(\mathfrak{D}_q\mathfrak{D}^{n+1}_q\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \mathfrak{D}^{n+2}_q\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}\left[\left(\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \phi(\mathfrak{a})\right],\\
&= \left(\mathfrak{D}^{n+2}_q\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \phi(\mathfrak{a})\left(\mathfrak{D}^{n+2}_q\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}1\right)(x),\\
&= \left(\mathfrak{D}^{n+1}_q\mathfrak{D}_q\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(n+2-\beta)}\mathfrak{D}^{n+2}_q(x-\mathfrak{a})^{(n+1-\beta)},\\
&= \left(\mathfrak{D}^{n+1}_q\mathfrak{I}^{n+1-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(n+2-\beta)}\frac{\Gamma_q(n+2-\beta)}{\Gamma_q(-\beta)}(x-\mathfrak{a})^{(-\beta-1)},\\
&= \left(\mathfrak{D}^{[\beta]}_q\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(-\beta)}(x-\mathfrak{a})^{(-\beta-1)},\\
&= \left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(-\beta)}(x-\mathfrak{a})^{(-\beta-1)}.
\end{aligned}$$

■

Corollary 3.13 *[88] The semigroup property of the Riemann–Liouville's fractional q-derivative is not valid, i.e., for any $\alpha, \beta \in \mathbb{R}_+$, we have:*

$$\left({}^{RL}\mathfrak{D}^{\alpha}_{q,\mathfrak{a}}\, {}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) \neq \left({}^{RL}\mathfrak{D}^{\alpha+\beta}_{q,\mathfrak{a}}\phi\right)(x).$$

3.3 Caputo's Fractional q-Derivative

In this part, we review the definition and some properties of the fractional q-derivative of the Caputo type.

Definition 3.14 *[79, 88] Let ϕ be a function defined on $[\mathfrak{a},\mathfrak{b}]$ and for $\beta \in \mathbb{R}$. The Caputo's fractional q-derivative of order β is given by:*

$$\left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \begin{cases} \left(\mathfrak{I}^{-\beta}_{q,\mathfrak{a}}\phi\right)(x); & \text{if } \beta < 0, \\ \phi(x); & \text{if } \beta = 0, \\ \left(\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta]}_{q}\phi\right)(x); & \text{if } \beta > 0, \end{cases} \tag{1.56}$$

where $[\beta]$ is the integer part of β.

Example 3.15 *[86] Let $\phi(x) = (x-\mathfrak{a})^{(\lambda)}$, $0 < \mathfrak{a} < x < \mathfrak{b}$. Then, for $\beta \in \mathbb{R}_{+} \setminus \mathbb{N}$ and $\lambda > [\beta]-1$, we have:*

$${}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\beta+1)}(x-\mathfrak{a})^{(\lambda-\beta)}. \tag{1.57}$$

In effect,

$$\begin{aligned} {}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} &= \mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\left(\mathfrak{D}^{[\beta]}_{q}(x-\mathfrak{a})^{(\lambda)}\right), \\ &= \mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\left(\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+1-[\beta])}(x-\mathfrak{a})^{(\lambda-[\beta])}\right), \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+1-[\beta])}\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda-[\beta])}. \end{aligned}$$

Applying the formula (1.42), we find:

$$\begin{aligned} {}^{C}\mathfrak{D}^{\alpha}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+1-[\beta])}\frac{\Gamma_q(\lambda+1-[\beta])}{\Gamma_q([\beta]-\beta+\lambda-[\beta]+1)}(x-\mathfrak{a})^{([\beta]-\beta+\lambda-[\beta])}, \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\beta+1)}(x-\mathfrak{a})^{(\lambda-\beta)}. \end{aligned}$$

In particular, for $\lambda \in \mathbb{N}$ and $\beta > \lambda$, we have:

$${}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(\lambda)} = \mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\left(\mathfrak{D}^{[\beta]}_{q}(x-\mathfrak{a})^{(\lambda)}\right) = 0. \tag{1.58}$$

Remark 3.16 *The Caputo's q-derivative of a constant function $\mathfrak{c}$ is zero, i.e.:*

$$\left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{c}\right)(x) = \left(\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta]}_{q}\mathfrak{c}\right)(x) = 0. \tag{1.59}$$

Remark 3.17 *[86] For every $\phi(x)$ defined on $(0,\mathfrak{b})$ and $\beta \in \mathbb{R}_{+}$, the following fact is true:*

$$\left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(\mathfrak{a}) = \left(\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta]}_{q}\phi\right)(\mathfrak{a}) = 0. \tag{1.60}$$

Proposition 3.18 *[86, 88] For any* $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$ *and* $0 < \mathfrak{a} < x < \mathfrak{b}$, *the following properties is valid:*

1.

$$\left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) = \left({}^{C}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x). \tag{1.61}$$

2.

$$\left(\mathfrak{D}_q\,{}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \left({}^{C}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x) + \frac{\left(\mathfrak{D}^{[\beta]}_q\phi\right)(\mathfrak{a})}{\Gamma_q([\beta]-\beta)}(x-\mathfrak{a})^{([\beta]-\beta-1)}. \tag{1.62}$$

Proof.

1. If $\beta = n+\epsilon$, $n \in \mathbb{N}$, $0 < \epsilon < 1$, then $[\beta] = n+1$ and $[\beta+1] = n+2$. According to Definition 3.14, we obtain:

$$\begin{aligned}
\left({}^{C}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{I}^{[\beta+1]-(\beta+1)}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta+1]}_q\phi\right)(x),\\
&= \left(\mathfrak{I}^{n+2-(n+\epsilon+1)}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x),\\
&= \left(\mathfrak{I}^{1-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x),\\
&= \left(\mathfrak{I}^{1-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+1}_q\mathfrak{D}_q\phi\right)(x),\\
&= \left(\mathfrak{I}^{n+1-(n+\epsilon)}_{q,\mathfrak{a}}\mathfrak{D}^{n+1}_q\mathfrak{D}_q\phi\right)(x),\\
&= \left(\mathfrak{I}^{[\beta]-\beta}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta]}_q\mathfrak{D}_q\phi\right)(x),\\
&= \left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x).
\end{aligned}$$

2. If $\beta = n+\epsilon$, $n \in \mathbb{N}$, $0 < \epsilon < 1$, then $[\beta] = n+1$ and $[\beta+1] = n+2$. Using Theorem 3.5, we find:

$$\begin{aligned}
\left(\mathfrak{D}_q\,{}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{D}_q\mathfrak{I}^{[\beta]-(\beta)}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta]}_q\phi\right)(x),\\
&= \left(\mathfrak{D}_q\mathfrak{I}^{n+1-(n+\epsilon)}_{q,\mathfrak{a}}\mathfrak{D}^{n+1}_q\phi\right)(x),\\
&= \left(\mathfrak{D}_q\mathfrak{I}^{1-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+1}_q\phi\right)(x),\\
&= \mathfrak{D}_q\left[\left(\mathfrak{I}^{1-\epsilon+1}_{q,\mathfrak{a}}\mathfrak{D}_q\mathfrak{D}^{n+1}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(2-\epsilon)}(x-\mathfrak{a})^{(1-\epsilon)}\right],\\
&= \left(\mathfrak{D}_q\mathfrak{I}^{2-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(2-\epsilon)}\mathfrak{D}_q(x-\mathfrak{a})^{(1-\epsilon)}.
\end{aligned}$$

By property (1.44) and relations (1.12),(1.23), we get:

$$\begin{aligned}
\left(\mathfrak{D}_q\,{}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{D}_q\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{I}^{1-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(2-\epsilon)}[1-\epsilon]_q(x-\mathfrak{a})^{(-\epsilon)},\\
&= \left(\mathfrak{I}^{1-\epsilon}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(1-\epsilon)}(x-\mathfrak{a})^{(-\epsilon)},\\
&= \left(\mathfrak{I}^{1+n-\beta}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(1+n-\beta)}(x-\mathfrak{a})^{(n-\beta)},\\
&= \left(\mathfrak{I}^{n+2-(\beta+1)}_{q,\mathfrak{a}}\mathfrak{D}^{n+2}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{n+1}_q\phi\right)(\mathfrak{a})}{\Gamma_q(1+n-\beta)}(x-\mathfrak{a})^{(n-\beta)},\\
&= \left(\mathfrak{I}^{[\beta+1]-(\beta+1)}_{q,\mathfrak{a}}\mathfrak{D}^{[\beta+1]}_q\phi\right)(x) + \frac{\left(\mathfrak{D}^{[\beta]}_q\phi\right)(\mathfrak{a})}{\Gamma_q([\beta]-\beta)}(x-\mathfrak{a})^{([\beta]-\beta-1)},\\
&= \left({}^{C}\mathfrak{D}^{\beta+1}_{q,\mathfrak{a}}\phi\right)(x) + \frac{\left(\mathfrak{D}^{[\beta]}_q\phi\right)(\mathfrak{a})}{\Gamma_q([\beta]-\beta)}(x-\mathfrak{a})^{([\beta]-\beta-1)}.
\end{aligned}$$

■

Corollary 3.19 *[86] The semigroup property of the Caputo's fractional q-derivative is not valid, i.e., for any* $\alpha,\beta\in\mathbb{R}_+$, *we have:*

$$\left({}^{C}\mathfrak{D}^{\alpha}_{q,\mathfrak{a}}\,{}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) \neq \left({}^{C}\mathfrak{D}^{\alpha+\beta}_{q,\mathfrak{a}}\phi\right)(x). \tag{1.63}$$

3.4 Relationships Between Fractional q-Operators

This part will explain the link between the two types of fractional q-derivatives, as well as the relationships between the fractional q-integral and fractional q-derivatives.

Theorem 3.20 *[86] Let* $\beta\in\mathbb{R}_+\setminus\mathbb{N}$ *and* $0<\mathfrak{a}<x<\mathfrak{b}$. *The relation among the Riemann-Liouville's and Caputo's fractional q-derivatives is given as follows:*

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \left({}^{C}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) + \sum_{i=0}^{[\beta]-1}\frac{\left(\mathfrak{D}^{i}_q\phi\right)(\mathfrak{a})}{\Gamma_q(i-\beta+1)}(x-\mathfrak{a})^{(i-\beta)}. \tag{1.64}$$

Proof. For any $\beta\in\mathbb{R}_+\setminus\mathbb{N}$ with $\beta=n+\epsilon$, where $n\in\mathbb{N}$, $\epsilon\in(0,1)$. By using mathematical induction, we will prove that Theorem 3.20.

First, for $n=0$, i.e.: $0<\beta<1$. By Theorem 3.5 and property (1.44), we have:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{1-\beta}\phi\right)(x) &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{2-\beta}\mathfrak{D}_q\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}(x-\mathfrak{a})^{(1-\beta)},\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{I}_{q,\mathfrak{a}}^{1-\beta}\mathfrak{D}_q\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}(x-\mathfrak{a})^{(1-\beta)},\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}(x-\mathfrak{a})^{(1-\beta)}.\end{aligned}$$

Applying $\mathfrak{D}_q$ to the previous equation, we obtain:

$$\left(\mathfrak{D}_q\mathfrak{I}_{q,\mathfrak{a}}^{1-\beta}\phi\right)(x) = \left(\mathfrak{D}_q\mathfrak{I}_{q,\mathfrak{a}}\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}\mathfrak{D}_q(x-\mathfrak{a})^{(1-\beta)}.$$

Then, according to the propositions (1.23) and (1.12), we get:

$$\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) = \left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)+\frac{\phi(\mathfrak{a})}{\Gamma_q(1-\beta)}(x-\mathfrak{a})^{(-\beta)}.$$

Next, we assume that the relation (1.64) holds for $\beta=n+\epsilon$ where $\epsilon\in(0,1)$, for $n\in\mathbb{N}$, and we will show that it's holds for $\beta=n+\epsilon+1$. In fact, by formula (1.54), we have:

$$\begin{aligned}\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon+1}\phi\right)(x),\\ &= \left(\mathfrak{D}_q\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x).\end{aligned}\tag{1.65}$$

The following hypothesis is satisfied,

$$\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x)=\left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x)+\sum_{i=0}^{n}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon+1)}(x-\mathfrak{a})^{(i-n-\epsilon)}.$$

Therefrom, according to the equality (1.65), we find:

$$\begin{aligned}\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \mathfrak{D}_q\left[\left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x)+\sum_{i=0}^{n}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon+1)}(x-\mathfrak{a})^{(i-n-\epsilon)}\right],\\ &= \left(\mathfrak{D}_q\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x)+\sum_{i=0}^{n}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon+1)}\mathfrak{D}_q(x-\mathfrak{a})^{(i-n-\epsilon)},\\ &= \left(\mathfrak{D}_q\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon}\phi\right)(x)+\sum_{i=0}^{n}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon)}(x-\mathfrak{a})^{(i-n-\epsilon-1)}.\end{aligned}$$

Using the property (1.62), we get:

$$\begin{aligned}\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{n+\epsilon+1}\phi\right)(x)+\frac{\left(\mathfrak{D}_q^{n+1}\phi\right)(\mathfrak{a})}{\Gamma_q(1-\epsilon)}(x-\mathfrak{a})^{(-\epsilon)}+\sum_{i=0}^{n}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon)}(x-\mathfrak{a})^{(i-n-\epsilon-1)},\\ &= \left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)+\sum_{i=0}^{n+1}\frac{\left(\mathfrak{D}_q^{i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-n-\epsilon)}(x-\mathfrak{a})^{(i-n-\epsilon-1)}.\end{aligned}$$

■

Lemma 3.21 *[86] For $\beta \in \mathbb{R}_+$ and let $0 < \mathfrak{a} < x < \mathfrak{b}$. The connections among fractional q-integral and fractional q-derivative of the Riemann Liouville type are given as follows:*

1.

$$\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\,{}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \phi(x). \tag{1.66}$$

2.

$$\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) = \phi(x). \tag{1.67}$$

Proof.

1. For $0 < \beta < 1$, according to the property (1.24), we have:

$$\phi(x) = \left(\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \phi(\mathfrak{a}). \tag{1.68}$$

Applying $\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}$ on equality (1.68) and using the properties (1.43)-(1.44), we get:

$$\begin{aligned}
\left(\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \phi(\mathfrak{a})\left(\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}1\right)(x),\\
&= \left(\mathfrak{I}^{2-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}(x-\mathfrak{a})^{(1-\beta)}.
\end{aligned}$$

Then, we apply $\mathfrak{D}_q$ to the previous equation, we find:

$$\begin{aligned}
\left({}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{D}_q\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\phi\right)(x),\\
&= \left(\mathfrak{D}_q\mathfrak{I}^{2-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(2-\beta)}\mathfrak{D}_q(x-\mathfrak{a})^{(1-\beta)},\\
&= \left(\mathfrak{D}_q\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(1-\beta)}(x-\mathfrak{a})^{(-\beta)},\\
&= \left(\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(1-\beta)}(x-\mathfrak{a})^{(-\beta)}.
\end{aligned}$$

By relations (1.42), (1.44) and equality (1.68), we obtain:

$$\begin{aligned}
\left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\,{}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\mathfrak{I}^{1-\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(1-\beta)}\mathfrak{I}^{\beta}_{q,\mathfrak{a}}(x-\mathfrak{a})^{(-\beta)},\\
&= \left(\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \frac{\phi(\mathfrak{a})}{\Gamma_q(1-\beta)}\frac{\Gamma_q(1-\beta)}{\Gamma_q(\beta+1-\beta)}(x-\mathfrak{a})^{(\beta-\beta)},\\
&= \left(\mathfrak{I}_{q,\mathfrak{a}}\mathfrak{D}_q\phi\right)(x) + \phi(\mathfrak{a}),\\
&= \phi(x).
\end{aligned}$$

If $\beta = n + \epsilon$ where $\epsilon \in (0,1)$, $n \in \mathbb{N}$. Placing $\beta = \beta - 1$ and $\phi \rightarrow {}^{RL}\mathfrak{D}^{\beta-1}_{q,\mathfrak{a}}\phi$, from Theorem 3.5 and using property (1.54), we find:

$$\begin{aligned}
\left(\mathfrak{I}^{\beta-1}_{q,\mathfrak{a}}\,{}^{RL}\mathfrak{D}^{\beta-1}_{q,\mathfrak{a}}\phi\right)(x) &= \left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\mathfrak{D}_q\,{}^{RL}\mathfrak{D}^{\beta-1}_{q,\mathfrak{a}}\phi\right)(x) + \frac{\left({}^{RL}\mathfrak{D}^{\beta-1}_{q,\mathfrak{a}}\phi\right)(\mathfrak{a})}{\Gamma_q(\beta)}(x-\mathfrak{a})^{(\beta-1)},\\
&= \left(\mathfrak{I}^{\beta}_{q,\mathfrak{a}}\,{}^{RL}\mathfrak{D}^{\beta}_{q,\mathfrak{a}}\phi\right)(x) + \frac{\left({}^{RL}\mathfrak{D}^{\beta-1}_{q,\mathfrak{a}}\phi\right)(\mathfrak{a})}{\Gamma_q(\beta)}(x-\mathfrak{a})^{(\beta-1)}.
\end{aligned}$$

So, by formula (1.53), we have $\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta-1}\phi\right)(\mathfrak{a})=0$, and we can write:

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)=\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta-1}\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta-1}\phi\right)(x).$$

Repeating this relationship n times, we obtain:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta-n}\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta-n}\phi\right)(x),\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\epsilon}\,{}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\epsilon}\phi\right)(x),\\ &= \phi(x).\end{aligned}$$

2. Using the properties (1.44) and (1.25), for $\beta\in\mathbb{R}_+$, then by Definition 3.9, we get:

$$\begin{aligned}\left({}^{\mathrm{RL}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left(\mathfrak{D}_{q}^{[\beta]}\mathfrak{I}_{q,\mathfrak{a}}^{[\beta]-\beta}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x),\\ &= \left(\mathfrak{D}_{q}^{[\beta]}\mathfrak{I}_{q,\mathfrak{a}}^{[\beta]-\beta+\beta}\phi\right)(x),\\ &= \left(\mathfrak{D}_{q}^{[\beta]}\mathfrak{I}_{q,\mathfrak{a}}^{[\beta]}\phi\right)(x),\\ &= \phi(x).\end{aligned}$$

■

Lemma 3.22 *[86] Let $\beta\in\mathbb{R}_+$ and $0<\mathfrak{a}<x<\mathfrak{b}$. The connections among Riemann-Liouville's fractional q-integral and Caputo's fractional q-derivative are given as follows:*

1.

$$\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)=\phi(x)-\sum_{i=0}^{[\beta]-1}\frac{\left(\mathfrak{D}_{q}^{i}\phi\right)(\mathfrak{a})}{[i]_q!}(x-\mathfrak{a})^{(i)}. \qquad (1.69)$$

In particular, for $\beta\in(0,1)$ and $\mathfrak{a}=0$, we have:

$$\left(\mathfrak{I}_{q}^{\beta}\,{}^{\mathrm{C}}\mathfrak{D}_{q}^{\beta}\phi\right)(x)=\phi(x)-\phi(0). \qquad (1.70)$$

2.

$$\left({}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x)=\phi(x). \qquad (1.71)$$

Proof.

1. For $\beta\in\mathbb{R}_+$, by applying the properties (1.44) and (1.26), we get:

$$\begin{aligned}\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\,{}^{\mathrm{C}}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\mathfrak{I}_{q,\mathfrak{a}}^{[\beta]-\beta}\mathfrak{D}_{q}^{[\beta]}\phi\right)(x),\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta+[\beta]-\beta}\mathfrak{D}_{q}^{[\beta]}\phi\right)(x),\\ &= \left(\mathfrak{I}_{q,\mathfrak{a}}^{[\beta]}\mathfrak{D}_{q}^{[\beta]}\phi\right)(x),\\ &= \phi(x)-\sum_{i=0}^{[\beta]-1}\frac{\left(\mathfrak{D}_{q}^{i}\phi\right)(\mathfrak{a})}{[i]_q!}(x-\mathfrak{a})^{(i)}.\end{aligned}$$

Particularly, for $\beta \in (0,1)$ and $\mathfrak{a} = 0$, we have:

$$\left(\mathfrak{I}_q^{\beta}\, {}^{C}\mathfrak{D}_q^{\beta}\phi\right)(x) = \phi(x) - \phi(0).$$

2. For $\beta \in \mathbb{R}_+$, by placing $\phi \to \mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi$ in Theorem 3.20, from the relation (1.67) and formula (1.38), we find:

$$\begin{aligned}
\left({}^{C}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) &= \left({}^{RL}\mathfrak{D}_{q,\mathfrak{a}}^{\beta}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(x) - \sum_{i=0}^{[\beta]-1} \frac{\left(\mathfrak{D}_q^{i}\mathfrak{I}_{q,\mathfrak{a}}^{\beta}\phi\right)(\mathfrak{a})}{\Gamma_q(i-\beta+1)}(x-\mathfrak{a})^{(i-\beta)}, \\
&= \phi(x) - \sum_{i=0}^{[\beta]-1} \frac{\left(\mathfrak{I}_{q,\mathfrak{a}}^{\beta-i}\phi\right)(\mathfrak{a})}{\Gamma_q(i-\beta+1)}(x-\mathfrak{a})^{(i-\beta)}, \\
&= \phi(x).
\end{aligned}$$

■

4 Kuratowski's Measure of Non-Compactness

This section includes the basic concepts and certain properties of the Kuratowski's measure of non-compactness. For more details; see references [18, 29, 94].

Definition 4.1 *[29, 94] Let $\mathbb{E}$ be a Banach space and $\Omega_{\mathbb{E}}$ be the family of bounded subsets of $\mathbb{E}$. The Kuratowski's measure of non-compactness is the map $\mu : \Omega_{\mathbb{E}} \to \mathbb{R}_+$ defined as:*

$$\mu(\mathscr{A}) = \inf\{\varepsilon > 0 : \mathscr{A} \subset \cup_{i=1}^{m}\mathscr{A}_i \text{ and } diam(\mathscr{A}_i) \le \varepsilon\}; \quad where\, \mathscr{A} \in \Omega_{\mathbb{E}}.$$

Properties 4.2 *[29, 94] The Kuratowski's measure of non-compactness has the following properties:*

(1) $\mu(\mathscr{A}) = 0 \Leftrightarrow \overline{\mathscr{A}}$ *is compact ($\mathscr{A}$ is relatively compact).*

(2) $\mu(\mathscr{A}) = \mu(\overline{\mathscr{A}})$.

(3) $\mathscr{A} \subseteq \mathscr{B} \Rightarrow \mu(\mathscr{A}) \le \mu(\mathscr{B})$.

(4) $\mu(\mathscr{A} + \mathscr{B}) \le \mu(\mathscr{A}) + \mu(\mathscr{B})$.

(5) $\mu(\gamma\mathscr{A}) = |\gamma|\mu(\mathscr{A}),\ \gamma \in \mathbb{R}$.

(6) $\mu(conv\mathscr{A}) = \mu(\mathscr{A})$.

(7) $\mu(\mathscr{A} + x_0) = \mu(\mathscr{A}),$ *for every* $x_0 \in \mathbb{E}$.

Where $conv\mathscr{A}$ and $\overline{\mathscr{A}}$ denote the convex hull and the closure of the bounded set $\mathscr{A}$, respectively.

5 Fixed Point Theorems

Fixed point theorems are incredibly helpful tools in mathematics, especially for solving differential equations. In fact, these theorems provide sufficient conditions for which a certain function admits a fixed point, guaranteeing the existence of solution to a given problem.

In this section, we will present some of fixed point theorems that we will need for this thesis. For more information; see references [12, 50, 51, 65, 74, 87, 89].

Definition 5.1 *[65](**Fixed Point**)*
Let ϕ a continuous function on an interval $\mathrm{J}=[\mathfrak{a},\mathfrak{b}]$, we say that $z^*\in \mathrm{J}$ is a fixed point of ϕ such that:

$$\phi(z^*)=z^*.$$

Theorem 5.2 *[50] (**Banach Contraction Principle**)*
Let X be a non-empty closed subset of a Banach space $\mathbb{E}$ and $\mathscr{H}:\mathrm{X}\to\mathrm{X}$ a contraction mapping, then $\mathscr{H}$ has a unique fixed point.

Theorem 5.3 *[87](**Schaefer**)*
Let $\mathbb{E}$ be a Banach space and $\mathscr{H}:\mathbb{E}\to\mathbb{E}$ be a completely continuous operator. If the set

$$\Omega(\mathscr{H}):=\left\{z\in\mathbb{E}: z=\gamma\mathscr{H}(z),\ for\ \gamma\in(0,1)\right\}$$

is bounded, then $\mathscr{H}$ has at least one fixed point.

Theorem 5.4 *[12, 50] (**Nonlinear alternative of Leray-Schauder**)*
Let $\mathbb{E}$ be a Banach space and X a closed, convex subset of $\mathbb{E}$. Let $\mathscr{U}$ be an open subset of X with $0\in\mathscr{U}$ and $\mathscr{H}:\overline{\mathscr{U}}\to\mathrm{X}$ a continuous and compact operator. Then either

(i) $\mathscr{H}$ has fixed points on $\overline{\mathscr{U}}$, or

(ii) There exist $z\in\partial\mathscr{U}$ and $\gamma\in(0,1)$ with $z=\gamma\mathscr{H}(z)$.

Theorem 5.5 *[87] (**Krasnoselskii**)*
Let X be a closed, convex non-empty subset of a Banach space $\mathbb{E}$, suppose that $\mathscr{A},\mathscr{B}:\mathrm{X}\to\mathbb{E}$ are two maps satisfying the following three conditions:

(i) $\mathscr{A}y+\mathscr{B}z\in\mathrm{X}\ (\forall y,z\in\mathrm{X})$

(ii) $\mathscr{A}$ is a continuous and a compact mapping.

(iii) $\mathscr{B}$ is a contraction mapping.

Then, there exists z in X such that $\mathscr{A}z+\mathscr{B}z=z$.

Next, we review the fixed point theorem of Mönch and an essential lemma.

Theorem 5.6 *[74, 89] (**Mönch**)*
Let X *be a bounded, closed and convex subset of a Banach space* $\mathbb{E}$ *such that* $0 \in \mathrm{X}$, *and let* $\mathcal{H}$ *a continuous mapping of* X *into* X. *If the implication*

$$\mathcal{V} = \overline{conv}\mathcal{H}(\mathcal{V}) \ \text{ or } \ \mathcal{V} = \mathcal{H}(\mathcal{V}) \cup \{0\} \Rightarrow \mu(\mathcal{V}) = 0 \tag{1.72}$$

holds for every subset $\mathcal{V}$ *of* X, *then* $\mathcal{H}$ *has a fixed point.*

Lemma 5.7 *[51] If* $\mathcal{V} \subset \mathrm{C}(\mathrm{J} = [\mathfrak{a}, \mathfrak{b}], \mathbb{E})$ *is a bounded and equi-continuous set, then*

1. *The function* $t \to \mu(\mathcal{V}(t))$ *is continuous on* J.
2. $\mu\left(\left\{\int_{\mathrm{J}} z(t)dt \ : \ z \in \mathcal{V}\right\}\right) \leq \int_{\mathrm{J}} \mu(\mathcal{V}(t))dt.$

Chapter 2

Boundary Value Problem for Fractional q-Difference Equations of Order $\beta \in (0,1]$

1 Introduction and Motivation

Recently, scientists have been interested in developing and expanding as several types of fractional differential equations as possible, because of its application and modeling in many phenomena. Among types of equations that have attracted attention are the fractional q-difference equations. Therefore, researchers discussed and studied the existence of solutions to the initial and boundary value problems for fractional q-difference equations that involves the Caputo's fractional q-derivative. For more information; you can see these references [5, 6, 15, 28, 98].

Bonchohra *et al.* in [35] established the existence of solutions to the first-order boundary value problem for the following fractional differential equations involving the Caputo's fractional derivative:

$$^{c}D^{\alpha} y(t) = f(t, y(t)); \quad t \in [0, T],\ 0 < \alpha < 1,$$

$$\mathfrak{a} y(0) + \mathfrak{b} y(T) = \mathfrak{c},$$

where $^{c}D^{\alpha}$ is the Caputo's fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are real constants with $\mathfrak{a} + \mathfrak{b} \neq 0$. They provided two existence results: one based on the fixed point theorem of Banach and the other on the fixed point theorem of Schaefer.

In [39], Benhamida *et al.* studied the existence of solutions to the boundary value problem of the Caputo-Hadamard's fractional differential equations of the following form:

$$^{c}_{H}D^{r} y(t) = f(t, y(t)); \quad \text{for a.e } t \in [1, T],\ 0 < r \leq 1,$$

$$\mathfrak{a} y(1) + \mathfrak{b} y(T) = \mathfrak{c},$$

where ${}^{c}_{\mathrm{H}}\mathrm{D}^{r}$ is the Caputo-Hadamard's fractional derivatives, $f : [1,\mathrm{T}] \times \mathbb{E} \to \mathbb{E}$ is a given function and $\mathbb{E}$ is a Banach space, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{R}$ such that $\mathfrak{a} + \mathfrak{b} \neq 0$. The authors used the fixed point theorem of Mönch and Kuratowski's measure of non-compactness to study the existence of solutions.

Motivated by the previously stated works, in this chapter, we are interested in studying the existence and uniqueness of solutions to the boundary value problem for fractional q-difference equations that involves the Caputo's fractional q-derivative, which is given as follows:

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_{q}^{\beta} z\right)(t) = \phi(t, z(t)); \quad 0 < \beta \leq 1, \ t \in \mathrm{J} = [0, \mathrm{T}], \\ \\ \mathfrak{a} z(0) + \mathfrak{b} z(\mathrm{T}) = \mathfrak{c}, \end{cases} \tag{2.1}$$

where $q \in (0,1)$, $\mathrm{T} > 0$ and ${}^{\mathrm{C}}\mathfrak{D}_{q}^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (0,1]$, $\phi : \mathrm{J} \times \mathbb{E} \to \mathbb{E}$ is a given function with $\mathbb{E}$ is Banach space and $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are real constants such that $\mathfrak{a} + \mathfrak{b} \neq 0$.

The chapter's remaining are organised in the following way: In Sect.2, we present the integrable solution to the boundary value problem (2.1). Next, in Sect.3, we prove the existence and uniqueness results for solutions of the boundary value problem for fractional q-difference equations (2.1) with $\mathbb{E} = \mathbb{R}$, by applying some fixed point theorems (Banach contraction principal, Schaefer and Leray-Schauder non-linear alternative). In Sect.4, we investigate another existence result for solutions of the boundary value problem for fractional q-difference equations (2.1) in Banach space, which is based on the fixed point theorem of Mönch combined with the technique of the Kuratowski's measure of non-compactness. In order to support our existence theorems, we conclude each section with illustrative examples.

2 Representation of the Integrable Solution

This section contains the definition and lemma of the integral solution to the boundary value problem (2.1), which is essential for the rest of the chapter.

To begin with, let's define what is meant by the integral solution to the boundary value problem (2.1).

Definition 2.1 *A function $z \in \mathrm{C}(\mathrm{J}, \mathbb{E})$ is said to be a solution of the boundary value problem (2.1), if z satisfies the fractional q-difference equation $\left({}^{\mathrm{C}}\mathfrak{D}_{q}^{\beta} z\right)(t) = \phi(t, z(t))$ on J where $\beta \in (0,1]$, and satisfies the boundary condition $\mathfrak{a} z(0) + \mathfrak{b} z(\mathrm{T}) = \mathfrak{c}$.*

The following lemma is necessary for the existence of solutions to the boundary value problem (2.1).

Lemma 2.2 *Let* $\theta : \mathrm{J} \to \mathbb{E}$ *be a continuous function. The integral solution of the following fractional boundary value problem:*

$$\begin{cases} \left({}^{C}\mathfrak{D}_{q}^{\beta} z\right)(t) = \theta(t); \quad 0 < \beta \leq 1, \ t \in \mathrm{J} = [0, \mathrm{T}], \\ \mathfrak{a} z(0) + \mathfrak{b} z(\mathrm{T}) = \mathfrak{c}. \end{cases} \tag{2.2}$$

Given by:

$$z(t) = \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}} \int_0^{\mathrm{T}} \frac{(\mathrm{T} - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}. \tag{2.3}$$

Proof. By applying the Riemann-Liouville's fractional q-integral of order $\beta \in (0, 1]$ on both sides of the equation for the problem (2.2), and according to Lemma 3.22, we have:

$$z(t) = \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + c_0. \tag{2.4}$$

Next, we will determine the constant c_0 by utilising the boundary condition of the problem (2.2), we get:

$$\mathfrak{a} z(0) + \mathfrak{b} z(\mathrm{T}) = \mathfrak{c} \quad \Rightarrow \quad \mathfrak{a} c_0 + \mathfrak{b}\left(\mathfrak{I}_q^{\beta} \theta(\mathrm{T}) + c_0\right) = \mathfrak{c}.$$

Since $\mathfrak{a} + \mathfrak{b} \neq 0$, hence,

$$\begin{aligned} c_0 &= \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}} - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}} \mathfrak{I}_q^{\beta} \theta(\mathrm{T}), \\ &= \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}} - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}} \int_0^{\mathrm{T}} \frac{(\mathrm{T} - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s. \end{aligned}$$

By changing c_0 in equation (2.4), we find:

$$z(t) = \int_0^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}} \int_0^{\mathrm{T}} \frac{(\mathrm{T} - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}.$$

The proof is finished. ■

3 Boundary Value Problem for Fractional q-Difference Equations

[1] This section focuses on proving the results of the existence and uniqueness of solutions to the boundary value problem (2.1) with $\mathbb{E}=\mathbb{R}$, through the use of several fixed point theorems. This means that we will deal with the following boundary value problem:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 0 < \beta \leq 1, \ t \in J = [0, T], \\ \\ \mathfrak{a} z(0) + \mathfrak{b} z(T) = \mathfrak{c}, \end{cases} \tag{2.5}$$

where $q \in (0,1)$, $T > 0$ and ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (0,1]$, $\phi : J \times \mathbb{R} \to \mathbb{R}$ is a given function and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are real constants such that $\mathfrak{a} + \mathfrak{b} \neq 0$.

Now, we present the following hypotheses which will be used in the remaining parts:

(A_1) The function $\phi : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A_2) The function ϕ satisfies the Lipschitz condition, i.e.: There exists a constant $\mathfrak{L} > 0$, such that for every $t \in J$ and every $y, z \in \mathbb{R}$, we have:

$$|\phi(t, y) - \phi(t, z)| \leq \mathfrak{L}|y - z|.$$

(A_3) There exists a constant $\mathscr{M} > 0$, such that for every $t \in J$ and every $z \in \mathbb{R}$, we have:

$$|\phi(t, z)| \leq \mathscr{M}.$$

3.1 Existence and Uniqueness Result

This part shows the uniqueness of solutions to the boundary value problem (2.5), which depends on the theorem of Banach contraction principle (Theorem 5.2).

Theorem 3.1 *Suppose that the hypotheses* (A_1) *and* (A_2) *hold. If*

$$0 < \left(1 + \frac{|\mathfrak{b}|}{|\mathfrak{a} + \mathfrak{b}|}\right) \frac{\mathfrak{L} T^{(\beta)}}{\Gamma_q(\beta + 1)} < 1. \tag{2.6}$$

Then, the boundary value problem (2.5) has a unique solution on J.

[1]**N. Allouch**, S. Hamani and J. Henderson, *Boundary Value Problem for Fractional q-Difference Equations*, Nonlinear Dynamics and Systems Theory, **24(2)**, (2024), 111-122.

Proof. Firstly, we convert the problem (2.5) into a fixed point problem and define the operator

$$\mathscr{H} : \mathrm{C}(\mathrm{J},\mathbb{R}) \longrightarrow \mathrm{C}(\mathrm{J},\mathbb{R})$$

By:

$$(\mathscr{H}z)(t) = \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}. \quad (2.7)$$

According to Lemma 2.2, it is evident that the fixed points of the operator $\mathscr{H}$ are the solutions of the boundary value problem (2.5).

Next, we will demonstrate that the operator $\mathscr{H}$ is a contraction mapping on $\mathrm{C}(\mathrm{J},\mathbb{R})$. Let $y,z \in \mathrm{C}(\mathrm{J},\mathbb{R})$ and for each $t \in \mathrm{J}$, then we have:

$$\begin{aligned}|(\mathscr{H}y)(t)-(\mathscr{H}z)(t)| &= \Bigg|\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big(\phi(s,y(s))-\phi(s,z(s))\big)\,d_qs \\ &\quad -\frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big(\phi(s,y(s))-\phi(s,z(s))\big)\,d_qs\Bigg|.\end{aligned}$$

Thus,

$$\begin{aligned}|(\mathscr{H}y)(t)-(\mathscr{H}z)(t)| &\le \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|\phi(s,y(s))-\phi(s,z(s))\big|\,d_qs \\ &\quad +\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|\phi(s,y(s))-\phi(s,z(s))\big|\,d_qs.\end{aligned}$$

Using the hypothesis (A_2), we get:

$$\begin{aligned}|(\mathscr{H}y)(t)-(\mathscr{H}z)(t)| &\le \mathfrak{L}\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|y(s)-z(s)\big|\,d_qs \\ &\quad +\frac{\mathfrak{L}|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|y(s)-z(s)\big|\,d_qs.\end{aligned}$$

Thanks to the formula (1.43) and for each $t \in \mathrm{J}$, we obtain:

$$\begin{aligned}\|\mathscr{H}(y)-\mathscr{H}(z)\|_\infty &\le \sup_{t\in\mathrm{J}}\left(\mathfrak{L}\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|y(s)-z(s)\big|\,d_qs\right) \\ &\quad +\sup_{t\in\mathrm{J}}\left(\frac{\mathfrak{L}|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\big|y(s)-z(s)\big|\,d_qs\right), \\ &\le \mathfrak{L}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\sup_{t\in\mathrm{J}}|y-z|d_qs \\ &\quad +\frac{\mathfrak{L}|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\sup_{t\in\mathrm{J}}|y-z|d_qs, \\ &\le \frac{\mathfrak{L}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|y-z\|_\infty + \frac{\mathfrak{L}|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|y-z\|_\infty.\end{aligned}$$

Consequently,

$$\|\mathscr{H}(y)-\mathscr{H}(z)\|_\infty \leq \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{\mathfrak{L}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|y-z\|_\infty.$$

As a result, by condition (2.6), $\mathscr{H}$ is a contraction operator, and according to the theorem of Banach contraction principle, we deduce that the operator $\mathscr{H}$ has a unique fixed point, which is the unique solution to the boundary value problem (2.5). ∎

3.2 Existence Results

This part contains the results of the existence of solutions to the boundary value problem (2.5), so that we given two results of the existence depend on various fixed point theorems.

The first outcome depends on fixed point theorem of Schaefer (Theorem 5.3).

Theorem 3.2 *Assume that the hypotheses* (A_1) *and* (A_3) *are satisfied. Then, the boundary value problem (2.5) has at least one solution on* J.

Proof. To demonstrate this result, we will use the fixed point theorem of Schaefer, i.e. we'll prove that the operator $\mathscr{H}$ defined by (2.7) has a fixed point. So, the proof will be presented in four steps.
Step 1: *$\mathscr{H}$ is a continuous operator on* $C(\mathrm{J},\mathbb{R})$.
Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $z_n \to z$ in $C(\mathrm{J},\mathbb{R})$. Then, for every $t\in\mathrm{J}$, we have:

$$\begin{aligned}|(\mathscr{H}z_n)(t)-(\mathscr{H}z)(t)| \leq& \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z_n(s))-\phi(s,z(s))\right| d_qs\\ &+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z_n(s))-\phi(s,z(s))\right| d_qs.\end{aligned}$$

Therefore, for every $t\in\mathrm{J}$, we get:

$$\|\mathscr{H}(z_n)-\mathscr{H}(z)\|_\infty \leq \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|\phi(.,z_n(.))-\phi(.,z(.))\|_\infty.$$

Since ϕ is a continuous function, i.e.:

$$\|\phi(.,z_n(.))-\phi(.,z(.))\|_\infty \to 0 \quad as \quad n\to\infty.$$

Then,

$$\|\mathscr{H}(z_n)-\mathscr{H}(z)\|_\infty \to 0 \quad as \quad n\to\infty.$$

Hence, $\mathscr{H}$ is a continuous operator on $C(\mathrm{J},\mathbb{R})$.

Step 2: *$\mathcal{H}$ maps bounded sets into bounded sets in* $C(J,\mathbb{R})$.

In fact, it suffices to prove that for all $r > 0$ there exists a constant $\mathfrak{R} > 0$, such that for every $z \in \mathfrak{B}_r = \{z \in C(J,\mathbb{R}) : \|z\|_\infty \leq r\}$, we have $\|\mathcal{H}(z)\|_\infty \leq \mathfrak{R}$.

Let $z \in \mathfrak{B}_r$. Then, for every $t \in J$, we have:

$$\begin{aligned}|(\mathcal{H}z)(t)| &= \left|\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}\right|,\\ &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}.\end{aligned}$$

According to the hypothesis (A_3), we find:

$$|(\mathcal{H}z)(t)| \leq \mathcal{M}\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs + \mathcal{M}\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}.$$

Using the formula (1.43) and for each $t \in J$, we get:

$$|(\mathcal{H}z)(t)| \leq \frac{\mathcal{M}T^{(\beta)}}{\Gamma_q(\beta+1)} + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\frac{\mathcal{M}T^{(\beta)}}{\Gamma_q(\beta+1)} + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}.$$

So,

$$\|\mathcal{H}(z)\|_\infty \leq \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{\mathcal{M}T^{(\beta)}}{\Gamma_q(\beta+1)} + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|} := \mathfrak{R}.$$

Hence, $\mathcal{H}$ is an uniformly bounded operator on $\mathfrak{B}_r$.

Step 3: *$\mathcal{H}$ maps bounded sets into equi-continuous sets of* $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and let $\mathfrak{B}_r$ be a bounded set of $C(J,\mathbb{R})$ as in Step 2. For $z \in \mathfrak{B}_r$, then we have:

$$|(\mathcal{H}z)(t_2)-(\mathcal{H}z)(t_1)| = \left|\int_0^{t_2} \frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \int_0^{t_1} \frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\right|.$$

Therefore,

$$\begin{aligned}|(\mathcal{H}z)(t_2)-(\mathcal{H}z)(t_1)| &\leq \int_0^{t_1} \frac{\left((t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right)}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs\\ &\quad+\int_{t_1}^{t_2} \frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs.\end{aligned}$$

Applying the hypothesis (A_3), we obtain:

$$\begin{aligned}|(\mathcal{H}z)(t_2)-(\mathcal{H}z)(t_1)| &\leq \frac{\mathcal{M}}{\Gamma_q(\beta)}\int_0^{t_1}\left((t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right)d_qs\\ &\quad+\frac{\mathcal{M}}{\Gamma_q(\beta)}\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}d_qs.\end{aligned}$$

After calculating the integrals, we find:

$$|(\mathscr{H}z)(t_2)-(\mathscr{H}z)(t_1)| \leq \frac{\mathscr{M}}{\Gamma_q(\beta+1)}\left(t_2^{(\beta)}-t_1^{(\beta)}\right).$$

As $t_1 \to t_2$, the inequality above's right-hand side tends to zero, i.e.:

$$|(\mathscr{H}z)(t_2)-(\mathscr{H}z)(t_1)| \to 0 \quad as \quad t_1 \to t_2.$$

Consequently, $\mathscr{H}$ is an equi-continuous operator. Due to the results obtained in Steps 1, 2 and 3, and according to the theorem of Arzela-Ascoli (Theorem 1.9), we can deduce that $\mathscr{H}$ is a completely continuous operator.

Step 4: *A priori bound.*

Now, we'll show that the set $\Omega=\left\{z\in C(J,\mathbb{R}) : z=\gamma\mathscr{H}(z), 0<\gamma<1\right\}$ is bounded.

Let $z\in\Omega$. Then, for every $t\in J$, we have:

$$\begin{aligned} z(t) &= \gamma(\mathscr{H}z)(t), \\ &= \gamma\left(\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}\right). \end{aligned}$$

Therefore, from the hypothesis (A_3) and by estimation in Step 2, it follows that for $\gamma\in(0,1)$ and for each $t\in J$, we obtain:

$$\begin{aligned} |z(t)| &\leq \gamma|(\mathscr{H}z)(t)|, \\ &\leq \|\mathscr{H}(z)\|_\infty, \\ &\leq \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{\mathscr{M}T^{(\beta)}}{\Gamma_q(\beta+1)} + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|} := \mathfrak{R}. \end{aligned}$$

So,

$$\|z\|_\infty \leq \mathfrak{R} < +\infty.$$

Consequently, the set Ω is bounded.

As a result of Steps 1 to 4 and according to the fixed point theorem of Shaefer, we conclude that the operator $\mathscr{H}$ has at least one fixed point which is the solution to the boundary value problem (2.5). ■

The second outcome depends on non-linear alternative of Leray-Schauder theorem (Theorem 5.4).

Theorem 3.3 *Suppose that the hypothesis* (A_1) *is satisfied and the following hypotheses hold:*

(A_4) *There exist* $\Phi_\phi \in L^1(J,\mathbb{R}_+)$ *and* $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous and non-decreasing, such that for each* $t \in J$ *and each* $z \in \mathbb{R}$*, we have:*

$$|\phi(t,z)| \leq \Phi_\phi(t)\Psi(|z|).$$

(A_5) *There exists a constant positive* $\nu > 0$*, such that:*

$$\frac{\nu}{\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\Psi(\nu)(\mathfrak{I}_q^\beta \Phi_\phi)(T)+\frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}} > 1.$$

Then, the boundary value problem (2.5) has at least one solution on J.

Proof. To establish that the operator $\mathscr{H}$ defined by (2.7) has a fixed point, we will apply the non-linear alternative of Leray-Schauder theorem. The operator $\mathscr{H}$ is continuous and completely continuous as shown in Theorem 3.2.
Let z be such that for every $t \in J$, we use the equation $z(t) = \gamma(\mathscr{H}z)(t)$ for $\gamma \in (0,1)$. Then, thanks to the hypothesis (A_4) and for each $t \in J$, we have:

$$\begin{aligned} |z(t)| &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_q s + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}, \\ &\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\Phi_\phi(s)\Psi(|z|)d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\Phi_\phi(s)\Psi(|z|)d_q s + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}. \end{aligned}$$

Therefore, for every $t \in J$, we get:

$$\begin{aligned} \|z\|_\infty &\leq (\mathfrak{I}_q^\beta \Phi_\phi)(T)\Psi(\|z\|_\infty) + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}(\mathfrak{I}_q^\beta \Phi_\phi)(T)\Psi(\|z\|_\infty) + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}, \\ &\leq \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\Psi(\|z\|_\infty)(\mathfrak{I}_q^\beta \Phi_\phi)(T) + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}. \end{aligned}$$

Thus,

$$\frac{\|z\|_\infty}{\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\Psi(\|y\|_\infty)(\mathfrak{I}_q^\beta \Phi_\phi)(T)+\frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}} \leq 1.$$

So, by hypothesis (A_5), there exists ν such that $\|z\|_\infty \neq \nu$. Let's set

$$\mathscr{U} = \{z \in C(J,\mathbb{R}) : \|z\|_\infty < \nu\}.$$

Clearly, the operator $\mathscr{H} : \overline{\mathscr{U}} \to C(J,\mathbb{R})$ is completely continuous. From the choice of $\mathscr{U}$, there is no $z \in \partial\mathscr{U}$ such that $z = \gamma\mathscr{H}(z)$, for some $\gamma \in (0,1)$. Therefore, according to the theorem of Leray-Schauder non-linear alternative, we conclude that the operator $\mathscr{H}$ has at least one fixed point $z \in \overline{\mathscr{U}}$, which is the solution to the boundary value problem (2.5).

■

3.3 An Example

Consider the following boundary value problem for fractional q-difference equation:

$$\begin{cases} \left({}^{C}\mathfrak{D}^{1/2}_{1/3} z\right)(t) = \frac{e^{-t^2} z(t)}{(6+t)(1+z(t))}; \quad 0 < \beta \leq 1,\ t \in J = [0,1], \\ \\ z(0) + z(1) = 0, \end{cases} \tag{2.8}$$

where $q = \frac{1}{3}$, $\beta = \frac{1}{2}$, $\mathfrak{a} = \mathfrak{b} = 1$, $\mathfrak{c} = 0$, $T = 1$, and

$$\phi(t,z) = \frac{e^{-t^2} z}{(6+t)(1+z)};\ (t,z) \in J \times \mathbb{R}_+.$$

Let $y, z \in \mathbb{R}_+$ and for each $t \in J = [0,1]$. Then, we have:

$$\begin{aligned} |\phi(t,y) - \phi(t,z)| &= \left| \frac{e^{-t^2}}{(6+t)} \left(\frac{y}{1+y} - \frac{z}{1+z} \right) \right|, \\ &\leq \frac{e^{-t^2}}{(6+t)} |y - z|, \\ &\leq \frac{1}{6} |y - z|. \end{aligned}$$

Since $\mathfrak{L} = \frac{1}{6}$, then the hypothesis (A_2) holds. Next, we will check that the condition (2.6) is satisfied with $T = 1$. In fact,

$$\begin{aligned} \left(1 + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right) \frac{\mathfrak{L} T^{(\beta)}}{\Gamma_q(\beta+1)} &= \left(1 + \frac{1}{2}\right) \frac{1}{6\Gamma_{1/3}(\frac{3}{2})}, \\ &\simeq 0.2666 < 1. \end{aligned}$$

Consequently, according to Theorem 3.1, the boundary value problem (2.8) has a unique solution on $[0,1]$.

4 Boundary Value Problem for Fractional q-Difference Equations in Banach Space

[2] This section discusses the study of the existence of solutions to the boundary value problem (2.1) in Banach space $\mathbb{E}$ with the norm $\|.\|$, by using the fixed point theorem of Mönch and the Kuratowski's measure of non-compactness. which is an essential technique for finding differential equations solutions.

Below, we present the following hypotheses that will be employed in the remaining part:

(A_6) The function $\phi : J \times \mathbb{E} \to \mathbb{E}$ satisfy the Carathéodory conditions.

(A_7) There exists $\mathfrak{p} \in L^\infty(J, \mathbb{R}_+)$, such that for every $t \in J$ and every $z \in \mathbb{E}$, we have:

$$\|\phi(t,z)\| \le \mathfrak{p}(t)\|z\|.$$

(A_8) For every $t \in J$ and every bounded set $\mathscr{B} \subset \mathbb{E}$, we have:

$$\mu(\phi(t,\mathscr{B})) \le \mathfrak{p}(t)\mu(\mathscr{B}).$$

4.1 Existence Result

This part focuses on proving the result of the existence of solutions to the boundary value problem (2.1), which is based on Mönch's fixed point theorem (Theorem 5.6).

Theorem 4.1 *Assume that the hypotheses* (A_6),(A_7) *and* (A_8) *are satisfied. If*

$$\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{L^\infty} < 1. \tag{2.9}$$

Then, the boundary value problem (2.1) has at least one solution on J.

Proof. To begin with, we convert the problem (2.1) into a fixed point problem and define the operator

$$\mathscr{H} : C(J,\mathbb{E}) \longrightarrow C(J,\mathbb{E})$$

Given by:

$$(\mathscr{H}z)(t) \;=\; \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs + \frac{\mathfrak{c}}{\mathfrak{a}+\mathfrak{b}}.$$

[2] **N. Allouch** and S. Hamani, *Boundary Value Problem for Fractional q-Difference Equations in Banach Space*, Rocky Mountain J. Math., **53(4)**, (2023), 1001-1010.

From Lemma 2.2, it is evident that the fixed points of the operator $\mathscr{H}$ are the solutions of the boundary value problem (2.1).

Let $\omega > 0$, we consider the set:

$$\mathscr{D}_{\omega} = \{z \in C(J,\mathbb{E}) \,:\, \|z\|_{\infty} \leq \omega\}. \tag{2.10}$$

Evidently, the set $\mathscr{D}_{\omega}$ is closed, bounded and convex of $C(J,\mathbb{E})$.

Next, we are going to prove that the operator $\mathscr{H}$ satisfies the conditions of Mönch's fixed point theorem. Thus, we offer the proof in three steps.

Step 1: *$\mathscr{H}$ is a continuous operator on* $C(J,\mathbb{E})$.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $z_n \to z$ in $C(J,\mathbb{E})$. Then, for every $t \in J$, we have:

$$\begin{aligned} |(\mathscr{H}z_n)(t)-(\mathscr{H}z)(t)| \;\leq\; & \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \left|\phi(s,z_n(s))-\phi(s,z(s))\right| d_q s \\ & + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \left|\phi(s,z_n(s))-\phi(s,z(s))\right| d_q s. \end{aligned}$$

Therefore, for each $t \in J$, we give:

$$\|\mathscr{H}(z_n)-\mathscr{H}(z)\| \;\leq\; \left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\phi(s,z_n(s))-\phi(s,z(s))\|.$$

Let $\rho > 0$, such that:

$$\|z_n\|_{\infty} \leq \rho \quad \text{and} \quad \|z\|_{\infty} \leq \rho.$$

From the hypothesis (A_7), we get:

$$\|\phi(s,z_n(s))-\phi(s,z(s))\| \leq 2\rho\mathfrak{p}(s) := \delta(s); \quad \delta(s) \in L^{\infty}(J,\mathbb{R}_+).$$

Since ϕ is a Carathéodory's function, and thanks to the Lebesgue's dominated convergence theorem, we find:

$$\|\mathscr{H}(z_n)-\mathscr{H}(z)\|_{\infty} \to 0 \quad as \quad n \to \infty.$$

Thus, $\mathscr{H}$ is a continuous operator on $C(J,\mathbb{E})$.

Step 2: *$\mathscr{H}$ maps $\mathscr{D}_{\omega}$ into $\mathscr{D}_{\omega}$.*

Let $z \in \mathscr{D}_{\omega}$ and using hypothesis (A_7), for every $t \in J$, we have:

$$\begin{aligned} |(\mathscr{H}z)(t)| \;\leq\; & \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_q s + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}, \\ \leq\; & \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\mathfrak{p}(s)\|z\|d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\mathfrak{p}(s)\|z\| + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|}. \end{aligned}$$

By the set (2.10) and for all $t \in \mathrm{J}$, we find:

$$\begin{aligned}
\|\mathcal{H}(z)\| &\leq \frac{\omega \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{\mathrm{L}^\infty} + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\frac{\omega \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{\mathrm{L}^\infty} + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|},\\
&\leq \omega\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{\mathrm{L}^\infty} + \frac{|\mathfrak{c}|}{|\mathfrak{a}+\mathfrak{b}|},\\
&\leq \omega.
\end{aligned}$$

Consequently,

$$\|\mathcal{H}(z)\|_\infty \leq \omega.$$

Step 3: *$\mathcal{H}(\mathcal{D}_\omega)$ is bounded and equi-continuous.*
According to Step 2, it's clear that $\mathcal{H}(\mathcal{D}_\omega) \subset \mathrm{C}(\mathrm{J}, \mathbb{E})$ is bounded.
Next, we prove that the equi-continuity of $\mathcal{H}(\mathcal{D}_\omega)$. Let $z \in \mathcal{D}_\omega$ and let $t_1,\ t_2 \in \mathrm{J}$ such that $t_1 < t_2$, we have:

$$\begin{aligned}
|(\mathcal{H}z)(t_2) - (\mathcal{H}z)(t_1)| &= \left|\int_0^{t_2}\frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs - \int_0^{t_1}\frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\right|,\\
&\leq \int_0^{t_1}\frac{\left((t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right)}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs\\
&\quad + \int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}|\phi(s,z(s))|d_qs.
\end{aligned}$$

Applying the hypothesis (A_7) and by the set (2.10), we get:

$$\begin{aligned}
|(\mathcal{H}z)(t_2) - (\mathcal{H}z)(t_1)| &\leq \frac{1}{\Gamma_q(\beta)}\int_0^{t_1}\left((t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right)\mathfrak{p}(s)\|z\|d_qs\\
&\quad + \frac{1}{\Gamma_q(\beta)}\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}\mathfrak{p}(s)\|z\|d_qs,\\
&\leq \frac{\omega\|\mathfrak{p}\|_{\mathrm{L}^\infty}}{\Gamma_q(\beta)}\int_0^{t_1}\left((t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right)d_qs\\
&\quad + \frac{\omega\|\mathfrak{p}\|_{\mathrm{L}^\infty}}{\Gamma_q(\beta)}\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}d_qs.
\end{aligned}$$

After calculating the integrals, we obtain:

$$|(\mathcal{H}z)(t_2) - (\mathcal{H}z)(t_1)| \leq \frac{\omega\|\mathfrak{p}\|_{\mathrm{L}^\infty}}{\Gamma_q(\beta+1)}\left(t_2^{(\beta)} - t_1^{(\beta)}\right).$$

As $t_1 \to t_2$, the inequality above's right-hand side tends to zero, i.e.:

$$|(\mathcal{H}z)(t_2) - (\mathcal{H}z)(t_1)| \to 0 \quad as \quad t_1 \to t_2.$$

Consequently, the equi-continuity of $\mathcal{H}(\mathcal{D}_\omega)$. So, $\mathcal{H}(\mathcal{D}_\omega) \subset \mathcal{D}_\omega$.

Finally, we prove that the implication (1.72) holds.
Let $\mathcal{V}$ be a subset of $\mathcal{D}_\omega$ such that $\mathcal{V} \subset \overline{conv}(\mathcal{H}(\mathcal{V} \cup \{0\})$. Since $\mathcal{V}$ is bounded and equicontinuous, and thus the function $t \to v(t) = \mu(\mathcal{V}(t))$ is continuous on J. Thank to the hypothesis (A_8), Lemma 5.7, and using the properties of the measure μ, then for every $t \in J$, we have:

$$\begin{aligned}
v(t) &\leq \mu(\mathcal{H}(\mathcal{V})(t) \cup \{0\}),\\
&\leq \mu(\mathcal{H}(\mathcal{V})(t)),\\
&\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \mathfrak{p}(s)\mu(\mathcal{V}(s))d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \mathfrak{p}(s)\mu(\mathcal{V}(s))d_q s,\\
&\leq \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \|\mathfrak{p}\|_{L^\infty}\|v\|_\infty d_q s + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\int_0^T \frac{(T-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \|\mathfrak{p}\|_{L^\infty}\|v\|_\infty d_q s.
\end{aligned}$$

Thus, for each $t \in J$, we find:

$$\begin{aligned}
v(t) &\leq \frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{L^\infty}\|v\|_\infty + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{L^\infty}\|v\|_\infty,\\
&\leq \|v\|_\infty\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{L^\infty}.
\end{aligned}$$

This implies that,

$$\|v\|_\infty\left[1-\left(1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right)\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\|\mathfrak{p}\|_{L^\infty}\right] \leq 0.$$

According to the condition (2.9), we obtain $\|v\|_\infty = 0$, i.e.: $v(t) = 0$ for every $t \in J$. So, $\mathcal{V}(t)$ is relatively compact in $\mathbb{E}$. In light of the theorem of Ascoli-Arzela (Theorem 1.9), $\mathcal{V}$ is relatively compact in $\mathcal{D}_\omega$. Thanks to Theorem 5.6, we deduce that the operator $\mathcal{H}$ has a fixed point which represents a solution to the boundary value problem (2.1). ■

4.2 An Example

Let

$$\mathbb{E} = l^1 = \left\{(z_1, z_2, \cdots, z_n, \cdots) \ : \ \sum_{n=1}^{\infty}|z_n| < \infty\right\},$$

be our Banach space with the norm

$$\|z\| = \sum_{n=1}^{\infty}|z_n|.$$

Following that, we examine the boundary value problem for fractional q-difference equation of the from:

$$\begin{cases} \left({}^{C}\mathfrak{D}_{1/2}^{1/4} z_n\right)(t) = \frac{\Gamma_{1/2}(\frac{3}{4}) t^2 \cos(t)|z_n(t)|}{16(|z_n(t)|+1)}; \quad 0 < \beta \le 1, \ t \in \mathrm{J} = [0,1], \\ z_n(0) + z_n(1) = 1, \end{cases} \tag{2.11}$$

where $q = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\mathrm{T} = 1$, $\mathfrak{a} = \mathfrak{b} = \mathfrak{c} = 1$, and

$$z = (z_1, z_2, \ldots, z_n, \ldots),$$

$$\phi = (\phi_1, \phi_2, \ldots, \phi_n, \ldots).$$

And

$$\phi_n(t,z) = \frac{\Gamma_{1/2}(\frac{3}{4}) t^2 \cos(t)|z_n|}{16(|z_n|+1)}; \quad (t,z) \in \mathrm{J} \times \mathbb{E}.$$

It is evident that the hypotheses (A_6) and (A_7) are satisfied, where:

$$\mathfrak{p}(t) = \frac{\Gamma_{1/2}(\frac{3}{4}) t^2 \cos(t)}{16}.$$

Next, we will verify that the condition (2.9) is satisfied. In fact,

$$\begin{aligned} \left(1 + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right) \frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)} \|\mathfrak{p}\|_{L^\infty} &= \left(1 + \frac{1}{2}\right) \frac{\Gamma_{1/2}(\frac{3}{4})}{16\Gamma_{1/2}(\frac{5}{4})} \\ &\simeq 0.1172 < 1. \end{aligned}$$

Consequently, according to Theorem 4.1, the boundary value problem (2.11) has at least one solution on $[0,1]$.

5 Conclusion

In this chapter, we have provided sufficient conditions for the existence of solutions to the boundary value problem for fractional q-difference equations involving the Caputo's fractional q-derivative. Consequently, we obtained the results of the existence and uniqueness of solutions to the boundary value problem for fractional q-difference equations, through the use of different fixed point theorems (Banach contraction principle, Schaefer and non-linear alternative of Leray-Schauder). Additionally, we studied another existence result for solutions of the boundary value problem for fractional q-difference equation in Banach space. This result is based on Kuratowski's measure technique of non-compactness and Mönch's fixed point theorem. To illustrate our findings, we presented numerical examples.

Chapter 3

Boundary Value Problem for Fractional q-Difference Equations of Order $\beta \in (1,2]$ with Integral Conditions

1 Introduction and Motivation

Fractional q-difference equations play an essential role in modeling many phenomena in different areas, and are currently studied by many academics in various fields of science and engineering. In recent years, several scholars have investigated the existence and Ulam stability of solutions to the boundary value problems for fractional q-difference equations involving the Caputo's fractional q-derivative. For more details; see the works [1, 6, 56, 68, 98].

In [33], Benchohra et *al.* established the existence and uniqueness of solutions to the boundary value problem (BVP for short) with fractional order differential equations and non-linear integral conditions involving the Caputo's fractional derivative of the following form:

$$ {}^{c}D^{\alpha} y(t) = f(t, y(t)); \quad t \in [0, T],\ 1 < \alpha \leq 2, $$

$$ y(0) - y'(0) = \int_0^T g(s, y(s))\,ds, $$

$$ y(T) + y'(T) = \int_0^T h(s, y(s))\,ds, $$

where ${}^{c}D^{\alpha}$ is the Caputo's fractional derivative, and $f, g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. The researchers used some fixed point theorems (Banach, Schaefer, Leray-Schuder, Burton-Kirk) to verify the existence of solutions.

Primarily motivated by the work mentioned above, our main goal in this chapter is to examine the existence, uniqueness and Ulam stability of solutions to the boundary value problem for fractional q-difference equations involving Caputo's fractional q-derivative with non-linear integral conditions of the following type:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 1 < \beta \leq 2, \ t \in J = [0, T], \\ \\ z(0) - z'(0) = \int_0^T \varphi(s, z(s)) ds, \\ \\ z(T) + z'(T) = \int_0^T \psi(s, z(s)) ds, \end{cases} \tag{3.1}$$

where $q \in (0,1)$, $T > 0$ and ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (1,2]$, $\phi : J \times \mathbb{E} \to \mathbb{E}$ is a given function and $\varphi, \psi : J \times \mathbb{E} \to \mathbb{E}$ are continuous functions with $\mathbb{E}$ is Banach space.

The rest of the chapter is structured as follows: In Section 2, we offer the integrable solution of the boundary value problem (3.1). The Section 3 focuses on studying the existence and Ulam stability of fractional q-difference equations with integral boundary conditions (3.1) with $\mathbb{E} = \mathbb{R}$, so that we provide two results for the existence: one depends on Banach contraction principal theorem and the other on Schaefer's fixed point theorem. In addition to presenting the stabilities results, which are based on Ulam-Hyers and Ulam-Hyers-Rassias stabilities techniques. In Section 4, we establish another existence result for solutions to the boundary value problem for fractional q-difference equations with integral conditions (3.1) in Banach spaces that depends on Mönch's fixed point theorem and Kuratowski's measure of non-compactness. To illustrate our results, we give examples at the end of each section.

2 Representation of the Integrable Solution

The definition and lemma of the integral solution to the problem (3.1) are presented in this section and are important for understanding the remainder of the chapter.

Firstly, let us define what is meant by the integral solution to the problem (3.1).

Definition 2.1 *A function $z \in C(J, \mathbb{E})$ is said to be a solution of the problem (3.1), if z satisfies the fractional q-difference equation $\left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t))$ on J where $\beta \in (1,2]$, and satisfies the following integral boundary conditions:*

$$z(0) - z'(0) = \int_0^T \varphi(s, z(s)) ds \quad \textit{and} \quad z(T) + z'(T) = \int_0^T \psi(s, z(s)) ds.$$

The following lemma is essential for the existence of solutions to the problem (3.1).

Lemma 2.2 *Let $w, u, v \in \mathrm{C}(\mathrm{J}, \mathbb{E})$. The integral solution of the following fractional problem:*

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_q^{\beta} z\right)(t) = w(t); \quad 1 < \beta \leq 2, \ t \in \mathrm{J} = [0, \mathrm{T}], \\ \\ z(0) - z'(0) = \int_0^{\mathrm{T}} u(s)ds, \\ \\ z(\mathrm{T}) + z'(\mathrm{T}) = \int_0^{\mathrm{T}} v(s)ds. \end{cases} \tag{3.2}$$

Given by:

$$z(t) = \mathcal{K}(t) + \int_0^{\mathrm{T}} \mathcal{G}_q(t,s) w(s) d_q s, \tag{3.3}$$

where

$$\mathcal{K}(t) = \frac{(1+\mathrm{T}-t)}{(2+\mathrm{T})} \int_0^{\mathrm{T}} u(s)ds + \frac{(1+t)}{(2+\mathrm{T})} \int_0^{\mathrm{T}} v(s)ds, \tag{3.4}$$

and

$$\mathcal{G}_q(t,s) = \begin{cases} \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-1)}}{(2+\mathrm{T})\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-2)}}{(2+\mathrm{T})\Gamma_q(\beta-1)}; & 0 \leq s < t, \\ \\ -\frac{(1+t)(\mathrm{T}-qs)^{(\beta-1)}}{(2+\mathrm{T})\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-2)}}{(2+\mathrm{T})\Gamma_q(\beta-1)}; & t \leq s \leq \mathrm{T}. \end{cases} \tag{3.5}$$

Proof. Let's start by applying the Riemann-Liouville's fractional q-integral of order $\beta \in (1,2]$ to both sides of the equation for the problem (3.2), and thanks to Lemma 3.22, we obtain:

$$z(t) = \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s) d_q s + c_0 + c_1 t. \tag{3.6}$$

Next, we use the integral conditions of the problem (3.2) to find the constants c_0 and c_1. This gives us:

$$c_0 - c_1 = \int_0^{\mathrm{T}} u(s)ds, \tag{3.7}$$

and

$$c_0 + (1+\mathrm{T})c_1 + \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s) d_q s + \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s) d_q s = \int_0^{\mathrm{T}} v(s)ds. \tag{3.8}$$

From the equations (3.7) and (3.8), we find:

$$c_1 = \frac{1}{(2+\mathrm{T})}\left(\int_0^{\mathrm{T}} v(s)ds - \int_0^{\mathrm{T}} u(s)ds - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s)d_qs\right),$$

and

$$c_0 = \frac{(1+\mathrm{T})}{(2+\mathrm{T})}\int_0^{\mathrm{T}} u(s)ds + \frac{1}{(2+\mathrm{T})}\left(\int_0^{\mathrm{T}} v(s)ds - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s)d_qs\right).$$

By substituting c_0 and c_1 into equation (3.6), we get:

$$\begin{aligned} z(t) &= \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs + \frac{(1+\mathrm{T})}{(2+\mathrm{T})}\int_0^{\mathrm{T}} u(s)ds + \frac{1}{(2+\mathrm{T})}\Big(\int_0^{\mathrm{T}} v(s)ds - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs \\ &\quad - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s)d_qs\Big) + \frac{t}{(2+\mathrm{T})}\Big(\int_0^{\mathrm{T}} v(s)ds - \int_0^{\mathrm{T}} u(s)ds - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs \\ &\quad - \int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s)d_qs\Big), \\ &= \frac{(1+\mathrm{T}-t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} u(s)ds + \frac{(1+t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} v(s)ds + \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs \\ &\quad - \frac{(1+t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-1)}}{\Gamma_q(\beta)} w(s)d_qs - \frac{(1+t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} \frac{(\mathrm{T}-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} w(s)d_qs. \end{aligned}$$

According to the fact that $\int_0^{\mathrm{T}} = \int_0^t + \int_t^{\mathrm{T}}$, we obtain:

$$z(t) = \mathscr{K}(t) + \int_0^{\mathrm{T}} \mathscr{G}_q(t,s) w(s)d_qs,$$

with

$$\mathscr{K}(t) = \frac{(1+\mathrm{T}-t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} u(s)ds + \frac{(1+t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} v(s)ds,$$

and

$$\mathscr{G}_q(t,s) = \begin{cases} \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-1)}}{(2+\mathrm{T})\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-2)}}{(2+\mathrm{T})\Gamma_q(\beta-1)}; & 0 \le s < t, \\ -\frac{(1+t)(\mathrm{T}-qs)^{(\beta-1)}}{(2+\mathrm{T})\Gamma_q(\beta)} - \frac{(1+t)(\mathrm{T}-qs)^{(\beta-2)}}{(2+\mathrm{T})\Gamma_q(\beta-1)}; & t \le s \le \mathrm{T}. \end{cases}$$

Lastly, we confirm that z is a solution to the problem (3.2). The proof is completed. ■

3 Existence and Ulam Stability for Fractional q-Difference Equations with Integral Boundary Conditions

[1] In the following section, we will establish the results for the existence, uniqueness and stability of solutions to the problem (3.1) with $\mathbb{E} = \mathbb{R}$, by employing certain fixed point theorems and Ulam stability techniques. This implies that the following problem will be addressed:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 1 < \beta \leq 2, \ t \in \mathrm{J} = [0, \mathrm{T}], \\ z(0) - z'(0) = \int_0^{\mathrm{T}} \varphi(s, z(s)) ds, \\ z(\mathrm{T}) + z'(\mathrm{T}) = \int_0^{\mathrm{T}} \psi(s, z(s)) ds, \end{cases} \tag{3.9}$$

where $q \in (0,1)$, $\mathrm{T} > 0$ and ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (1,2]$, $\phi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\varphi, \psi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

3.1 Existence and Uniqueness Result

In this part, we discusses the uniqueness of solutions to the problem (3.9) by applying the Banach contraction principle theorem (Theorem 5.2).

Theorem 3.1 *Suppose that the following hypotheses are satisfied:*

(H_1) *The function* $\phi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(H_2) *There exist positive constants* $\mathfrak{L}_\phi, \mathfrak{L}_\varphi$ *and* $\mathfrak{L}_\psi$*, such that for every* $t \in \mathrm{J}$ *and every* $y, z \in \mathbb{R}$*, we have:*

$$|\phi(t, y) - \phi(t, z)| \leq \mathfrak{L}_\phi |y - z|,$$

$$|\varphi(t, y) - \varphi(t, z)| \leq \mathfrak{L}_\varphi |y - z|,$$

$$|\psi(t, y) - \psi(t, z)| \leq \mathfrak{L}_\psi |y - z|.$$

If

$$0 < (\mathfrak{L}_\varphi + \mathfrak{L}_\psi) \frac{\mathrm{T}(1 + \mathrm{T})}{(2 + \mathrm{T})} + \mathfrak{L}_\phi \mathcal{G}_q^* \mathrm{T} < 1, \tag{3.10}$$

where

$$\mathcal{G}_q^* = \sup_{(t,s) \in \mathrm{J} \times \mathrm{J}} |\mathcal{G}_q(t, s)|.$$

Then, the problem (3.9) has a unique solution on J.

[1]**N. Allouch** and S. Hamani, *Existence and Ulam Stability for Fractional q-Difference Equation with Integral Boundary Conditions.* (Submitted)

Proof. To prove this result, we first transform problem (3.9) into a fixed point problem and consider the operator:

$$\mathcal{N} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$$

Given by:

$$(\mathcal{N} z)(t) \;=\; \mathcal{K}(t) + \int_0^T \mathcal{G}_q(t,s)\phi(s,z(s))\,d_q s, \tag{3.11}$$

where

$$\mathcal{K}(t) = \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))\,ds + \frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))\,ds, \tag{3.12}$$

and

$$\mathcal{G}_q(t,s) = \begin{cases} \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-1)}}{(2+T)\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-2)}}{(2+T)\Gamma_q(\beta-1)}; & 0 \le s < t, \\ -\frac{(1+t)(T-qs)^{(\beta-1)}}{(2+T)\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-2)}}{(2+T)\Gamma_q(\beta-1)}; & t \le s \le T. \end{cases} \tag{3.13}$$

It is clear from Lemma 2.2 that the solutions of the problem (3.9) are the fixed points of the operator $\mathcal{N}$.

Following that, we will show that the operator $\mathcal{N}$ is a contraction mapping on $C(J,\mathbb{R})$. Let $y,z \in C(J,\mathbb{R})$ and for every $t \in J$, then we have:

$$\begin{aligned} |(\mathcal{N} y)(t) - (\mathcal{N} z)(t)| \;=\; & \Bigg| \frac{(1+T-t)}{(2+T)}\int_0^T \big(\varphi(s,y(s)) - \varphi(s,z(s))\big)\,ds \\ & + \frac{(1+t)}{(2+T)}\int_0^T \big(\psi(s,y(s)) - \psi(s,z(s))\big)\,ds \\ & + \int_0^T \mathcal{G}_q(t,s)\big(\phi(s,y(s)) - \phi(s,z(s))\big)\,d_q s \Bigg|. \end{aligned}$$

This means that,

$$\begin{aligned} |(\mathcal{N} y)(t) - (\mathcal{N} z)(t)| \;\le\; & \frac{(1+T-t)}{(2+T)}\int_0^T \big|\varphi(s,y(s)) - \varphi(s,z(s))\big|\,ds \\ & + \frac{(1+t)}{(2+T)}\int_0^T \big|\psi(s,y(s)) - \psi(s,z(s))\big|\,ds \\ & + \int_0^T |\mathcal{G}_q(t,s)|\,\big|\phi(s,y(s)) - \phi(s,z(s))\big|\,d_q s. \end{aligned}$$

Applying the hypothesis (H_2), we obtain:

$$\begin{aligned} |(\mathcal{N} y)(t) - (\mathcal{N} z)(t)| \;\le\; & \mathfrak{L}_\varphi \frac{(1+T-t)}{(2+T)}\int_0^T \big|y(s) - z(s)\big|\,ds + \mathfrak{L}_\psi \frac{(1+t)}{(2+T)}\int_0^T \big|y(s) - z(s)\big|\,ds \\ & + \mathfrak{L}_\phi \int_0^T |\mathcal{G}_q(t,s)|\,\big|y(s) - z(s)\big|\,d_q s. \end{aligned}$$

Therefore, for each $t \in \mathrm{J}$, we get:

$$
\begin{aligned}
\|\mathcal{N}(y)-\mathcal{N}(z)\|_\infty &\leq \sup_{t\in \mathrm{J}}\left(\mathfrak{L}_\varphi \frac{(1+\mathrm{T}-t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} \left|y(s)-z(s)\right| ds\right) + \sup_{t\in \mathrm{J}}\left(\mathfrak{L}_\psi \frac{(1+t)}{(2+\mathrm{T})}\int_0^{\mathrm{T}} \left|y(s)-z(s)\right| ds\right) \\
&\quad + \sup_{t\in \mathrm{J}}\left(\mathfrak{L}_\phi \int_0^{\mathrm{T}} |\mathcal{G}_q(t,s)|\left|y(s)-z(s)\right| d_q s\right), \\
&\leq \mathfrak{L}_\varphi \frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\left\|y-z\right\|_\infty + \mathfrak{L}_\psi \frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\left\|y-z\right\|_\infty + \mathfrak{L}_\phi \mathcal{G}_q^* \mathrm{T}\left\|y-z\right\|_\infty .
\end{aligned}
$$

Thus,

$$
\|\mathcal{N}(y)-\mathcal{N}(z)\|_\infty \leq \left((\mathfrak{L}_\varphi+\mathfrak{L}_\psi)\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})} + \mathfrak{L}_\phi \mathcal{G}_q^* \mathrm{T}\right)\|y-z\|_\infty .
$$

Consequently, from condition (3.10), the operator $\mathcal{N}$ is a contraction, and according to the theorem of Banach contraction principle, we conclude that the operator $\mathcal{N}$ has a unique fixed point, which is the unique solution to the problem (3.9). ■

3.2 Existence Result

In the next part, we investigate the existence of solutions to the problem (3.9) through the use of Schaefer's fixed point theorem (Theorem 5.3).

Theorem 3.2 *Suppose that the hypothesis* (H_1) *holds and the following hypothesis is satisfied:*

- (H_3) *There exist positive constants* $\mathcal{M}_\phi, \mathcal{M}_\varphi$ *and* $\mathcal{M}_\psi$*, such that for every* $t \in \mathrm{J}$ *and every* $z \in \mathbb{R}$*, we have:*

$$|\phi(t,z)| \leq \mathcal{M}_\phi,$$

$$|\varphi(t,z)| \leq \mathcal{M}_\varphi,$$

$$|\psi(t,z)| \leq \mathcal{M}_\psi.$$

Then, the problem (3.9) has at least one solution on J.

Proof. In order to demonstrate that the operator $\mathcal{N}$ defined by (3.11) has a fixed point, we will apply the fixed point theorem of Schaefer. There will be four steps to proof.
Let us consider:

$$\mathcal{G}_q^* = \sup_{(t,s)\in \mathrm{J}\times \mathrm{J}} |\mathcal{G}_q(t,s)|.$$

Step 1: $\mathscr{N}$ *is a continuous operator on* $C(J,\mathbb{R})$.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $z_n \to z$ in $C(J,\mathbb{R})$. Then, for every $t \in J$, we have:

$$\begin{aligned}|(\mathscr{N}z_n)(t)-(\mathscr{N}z)(t)| \leq\ & \frac{(1+T-t)}{(2+T)}\int_0^T \left|\varphi(s,z_n(s))-\varphi(s,z(s))\right| ds \\ & +\frac{(1+t)}{(2+T)}\int_0^T \left|\psi(s,z_n(s))-\psi(s,z(s))\right| ds \\ & +\int_0^T |\mathscr{G}_q(t,s)|\left|\phi(s,z_n(s))-\phi(s,z(s))\right| d_q s.\end{aligned}$$

Thus, for each $t \in J$, we find:

$$\begin{aligned}\|\mathscr{N}(z_n)-\mathscr{N}(z)\|_\infty \leq\ & \frac{T(1+T)}{(2+T)}\left\|\varphi(.,z_n(.))-\varphi(.,z(.))\right\|_\infty+\frac{T(1+T)}{(2+T)}\left\|\psi(.,z_n(.))-\psi(.,z(.))\right\|_\infty \\ & +\mathscr{G}_q^* T\left\|\phi(.,z_n(.))-\phi(.,z(.))\right\|_\infty.\end{aligned}$$

Since ϕ, φ and ψ are continuous functions, i.e.:

$$\begin{aligned}\|\phi(.,z_n(.))-\phi(.,z(.))\|_\infty \to 0 \quad & as \quad n\to\infty, \\ \|\varphi(.,z_n(.))-\varphi(.,z(.))\|_\infty \to 0 \quad & as \quad n\to\infty, \\ \|\psi(.,z_n(.))-\psi(.,z(.))\|_\infty \to 0 \quad & as \quad n\to\infty.\end{aligned}$$

So,

$$\|\mathscr{N}(z_n)-\mathscr{N}(z)\|_\infty \to 0 \quad as \quad n\to\infty.$$

As a consequence, the operator $\mathscr{N}$ is continuous on $C(J,\mathbb{R})$.

Step 2: $\mathscr{N}$ *maps bounded sets into bounded sets in* $C(J,\mathbb{R})$.

In actually, It is sufficient to show that for all $r^*>0$ there exists a positive constant $\mathfrak{R}^*>0$, such that for every $z \in \mathfrak{B}^*_{r^*}=\{z\in C(J,\mathbb{R}) : \|z\|_\infty \leq r^*\}$, we have $\|\mathscr{N}(z)\|_\infty \leq \mathfrak{R}^*$.

Let $z \in \mathfrak{B}^*_{r^*}$. Then, for every $t \in J$, we have:

$$\begin{aligned}|(\mathscr{N}z)(t)| =\ & \left|\frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))ds+\frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))ds+\int_0^T \mathscr{G}_q(t,s)\phi(s,z(s))d_q s\right|, \\ \leq\ & \frac{(1+T-t)}{(2+T)}\int_0^T |\varphi(s,z(s))|ds+\frac{(1+t)}{(2+T)}\int_0^T |\psi(s,z(s))|ds+\int_0^T |\mathscr{G}_q(t,s)||\phi(s,z(s))|d_q s.\end{aligned}$$

Applying the hypothesis (H_3) and for each $t \in J$, we obtain:

$$|(\mathscr{N}z)(t)| \leq \mathscr{M}_\varphi\frac{T(1+T)}{(2+T)}+\mathscr{M}_\psi\frac{T(1+T)}{(2+T)}+\mathscr{M}_\phi\mathscr{G}_q^* T.$$

Consequently,

$$\|\mathscr{N}(z)\|_\infty \leq (\mathscr{M}_\varphi+\mathscr{M}_\psi)\frac{T(1+T)}{(2+T)}+\mathscr{M}_\phi\mathscr{G}_q^* T := \mathfrak{R}^*.$$

Thus, the operator $\mathscr{N}$ is uniformly bounded on $\mathfrak{B}^*_{r^*}$.

Step 3:$\mathcal{N}$ *maps bounded sets into equi-continuous sets of* $C(J,\mathbb{R})$.

Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and let $\mathfrak{B}^*_{r^*}$ be a bounded set of $C(J,\mathbb{R})$ as in Step 2. For $z \in \mathfrak{B}^*_{r^*}$, then we have:

$$\begin{aligned}|(\mathcal{N}z)(t_2)-(\mathcal{N}z)(t_1)| &= \left|\frac{(t_1-t_2)}{(2+T)}\int_0^T \varphi(s,z(s))ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T \psi(s,z(s))ds \right.\\ &\quad \left. + \int_0^T \left(\mathcal{G}_q(t_2,s)-\mathcal{G}_q(t_1,s)\right)\phi(s,z(s))d_qs\right|,\\ &\le \frac{|t_1-t_2|}{(2+T)}\int_0^T |\varphi(s,z(s))|ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T |\psi(s,z(s))|ds\\ &\quad + \int_0^T \left|\mathcal{G}_q(t_2,s)-\mathcal{G}_q(t_1,s)\right| |\phi(s,z(s))|d_qs.\end{aligned}$$

From hypothesis (H_3), we find:

$$|(\mathcal{N}z)(t_2)-(\mathcal{N}z)(t_1)| \le \mathcal{M}_\varphi \frac{T|t_1-t_2|}{(2+T)} + \mathcal{M}_\psi \frac{T(t_2-t_1)}{(2+T)} + \mathcal{M}_\phi T \sup_{s\in J}\left|\mathcal{G}_q(t_2,s)-\mathcal{G}_q(t_1,s)\right|.$$

As $t_1 \to t_2$, the inequality above's right-hand side tends to zero, i.e.:

$$|(\mathcal{N}z)(t_2)-(\mathcal{N}z)(t_1)| \to 0 \quad as \quad t_1 \to t_2.$$

Thus, the operator $\mathcal{N}$ is equi-continuous.

As a result of Steps 1 to 3 and thanks to the Arzela-Ascoli's theorem (Theorem 1.9), we deduce that the operator $\mathcal{N}$ is completely continuous.

Step 4: *A priori bound.*

Let's now show that the set $\Omega^* = \left\{z \in C(J,\mathbb{R}) : z = \gamma\mathcal{N}(z), \gamma \in (0,1)\right\}$ is bounded.

Let $z \in \Omega^*$ and for each $t \in J$, then we have:

$$\begin{aligned}z(t) &= \gamma(\mathcal{N}z)(t),\\ &= \gamma\left(\frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))ds + \frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))ds + \int_0^T \mathcal{G}_q(t,s)\phi(s,z(s))d_qs\right).\end{aligned}$$

Hence, by hypothesis (H_3) and using the estimation in Step 2, it follows that for $\gamma \in (0,1)$ and for every $t \in J$, we find:

$$\begin{aligned}|z(t)| &\le \gamma|(\mathcal{N}z)(t)|,\\ &\le \|\mathcal{N}(z)\|_\infty,\\ &\le (\mathcal{M}_\varphi + \mathcal{M}_\psi)\frac{T(1+T)}{(2+T)} + \mathcal{M}_\phi \mathcal{G}^*_q T := \mathfrak{R}^*.\end{aligned}$$

Then,

$$\|z\|_\infty \le \mathfrak{R}^* < +\infty.$$

Thus, the set Ω^* is bounded.

As a consequence of Steps 1 to 4 and according to Schaefer's fixed point theorem, we deduce that the operator $\mathcal{N}$ has at least one fixed point which is the solution to the problem (3.9). ∎

3.3 Ulam Stability Results

This section focuses on defining and studying different types of Ulam stability to the problem (3.9), by applying Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Relying on the references [1, 6, 22, 46, 56, 82, 83, 91], we give the following definitions:

Definition 3.3 *The problem (3.9) is Ulam-Hyers stable if there exists a real number* $\eta > 0$, *such that for every* $\epsilon > 0$ *and for every solution* $y \in C(J, \mathbb{R})$ *of the following inequality:*

$$|({}^{C}\mathfrak{D}_q^{\beta} y)(t) - \phi(t, y(t))| \leq \epsilon, \quad 1 < \beta \leq 2, \ t \in J, \tag{3.14}$$

there exists a solution $z \in C(J, \mathbb{R})$ *of the problem (3.9) with the norm*

$$\|y - z\|_{\infty} \leq \eta\epsilon.$$

Definition 3.4 *The problem (3.9) is generalized Ulam-Hyers stable if there exists* $\kappa \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\kappa(0) = 0$, *such that for every* $\epsilon > 0$ *and for every solution* $y \in C(J, \mathbb{R})$ *of the inequality (3.14), there exists a solution* $z \in C(J, \mathbb{R})$ *of the problem (3.9) with the norm*

$$\|y - z\|_{\infty} \leq \kappa(\epsilon).$$

Definition 3.5 *The problem (3.9) is Ulam-Hyers-Rassias stable with respect to* σ *if there exists* $\eta_\sigma > 0$, *such that for every* $\epsilon > 0$ *and for every solution* $y \in C(J, \mathbb{R})$ *of the following inequality:*

$$|({}^{C}\mathfrak{D}_q^{\beta} y)(t) - \phi(t, y(t))| \leq \epsilon\sigma(t), \quad 1 < \beta \leq 2, \ t \in J, \tag{3.15}$$

there exists a solution $z \in C(J, \mathbb{R})$ *of the problem (3.9) with the norm*

$$\|y - z\|_{\infty} \leq \eta_\sigma \epsilon\sigma(t), \ t \in J.$$

Remark 3.6 *Clearly : Definition 3.3* $\Rightarrow$ *Definition 3.4.*

Now, we present main results related to Ulam stabilities of the problem (3.9).

Theorem 3.7 *Suppose that the hypotheses* (H_1)-(H_2) *and condition (3.10) are satisfied. Then, the problem (3.9) is Ulam-Hyers stable.*

Proof. Let $y \in C(J,\mathbb{R})$ be a solution of the inequality (3.14) and let $z \in C(J,\mathbb{R})$ be the unique solution of the problem (3.9). Then, from Lemma 2.2, we have:

$$z(t) = \mathscr{K}(t) + \int_0^T \mathscr{G}_q(t,s)\phi(s,z(s))d_qs,$$

where the functions $\mathscr{K}(t)$ and $\mathscr{G}_q(t,s)$ are given by the equations (3.12) and (3.13), respectively. By integration of the inequality (3.14) and for every $t \in J$, we get:

$$\begin{aligned}\left|\begin{array}{c} y(t) - \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,y(s))ds \\ -\frac{(1+t)}{(2+T)}\int_0^T \psi(s,y(s))ds - \int_0^T \mathscr{G}_q(t,s)\phi(s,y(s))d_qs \end{array}\right| &\leq \mathfrak{I}_q^{\beta}\epsilon, \\ &\leq \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}\epsilon.\end{aligned}$$

Therefore, we can write:

$$\begin{aligned}
|y(t)-z(t)| &\leq \left|y(t) - \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))ds - \frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))ds \right. \\
&\quad \left. - \int_0^T \mathscr{G}_q(t,s)\phi(s,z(s))d_qs\right|, \\
&\leq \left|y(t) - \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,y(s))ds - \frac{(1+t)}{(2+T)}\int_0^T \psi(s,y(s))ds \right. \\
&\quad - \int_0^T \mathscr{G}_q(t,s)\phi(s,y(s))d_qs + \frac{(1+T-t)}{(2+T)}\int_0^T \big(\varphi(s,y(s))-\varphi(s,z(s))\big)ds \\
&\quad \left. + \frac{(1+t)}{(2+T)}\int_0^T \big(\psi(s,y(s))-\psi(s,z(s))\big)ds + \int_0^T \mathscr{G}_q(t,s)\big(\phi(s,y(s))-\phi(s,z(s))\big)d_qs\right|, \\
&\leq \left|y(t) - \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,y(s))ds - \frac{(1+t)}{(2+T)}\int_0^T \psi(s,y(s))ds \right. \\
&\quad \left. - \int_0^T \mathscr{G}_q(t,s)\phi(s,y(s))d_qs\right| + \left|\frac{(1+T-t)}{(2+T)}\int_0^T \big(\varphi(s,y(s))-\varphi(s,z(s))\big)ds \right. \\
&\quad \left. + \frac{(1+t)}{(2+T)}\int_0^T \big(\psi(s,y(s))-\psi(s,z(s))\big)ds + \int_0^T \mathscr{G}_q(t,s)\big(\phi(s,y(s))-\phi(s,z(s))\big)d_qs\right|.
\end{aligned}$$

Then, for each $t \in J$, we obtain:

$$\begin{aligned}
|y(t)-z(t)| &\leq \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}\epsilon + \frac{(1+T-t)}{(2+T)}\int_0^T \big|\varphi(s,y(s))-\varphi(s,z(s))\big|ds + \frac{(1+t)}{(2+T)} \\
&\quad \times \int_0^T \big|\psi(s,y(s))-\psi(s,z(s))\big|ds + \int_0^T |\mathscr{G}_q(t,s)|\big|\phi(s,y(s))-\phi(s,z(s))\big|d_qs.
\end{aligned}$$

Applying the hypothesis (H_2) and for every $t \in J$, we find:

$$\|y - z\|_\infty \leq \frac{T^{(\beta)}}{\Gamma_q(\beta+1)}\epsilon + \left((\mathfrak{L}_\varphi + \mathfrak{L}_\psi)\frac{T(1+T)}{(2+T)} + \mathfrak{L}_\phi \mathcal{G}_q^* T\right)\|y - z\|_\infty.$$

By condition (3.10), we get:

$$\begin{aligned}\|y - z\|_\infty &\leq \frac{\frac{T^{(\beta)}}{\Gamma_q(\beta+1)}}{1 - \left((\mathfrak{L}_\varphi + \mathfrak{L}_\psi)\frac{T(1+T)}{(2+T)} + \mathfrak{L}_\phi \mathcal{G}_q^* T\right)}\epsilon,\\ &:= \eta\epsilon.\end{aligned}$$

Thus, the problem (3.9) is Ulam-Hyers stable. ■

Corollary 3.8 *If we take $\kappa(\epsilon) = \eta\epsilon$; $\kappa(0) = 0$, we conclude that the problem (3.9) is generalized Ulam-Hyers stable.*

Theorem 3.9 *Suppose that the hypotheses (H_1)-(H_2) and condition (3.10) are satisfied, and the following hypothesis holds:*

(H_4) *Let $\sigma \in C(J, \mathbb{R}_+)$ be an increasing function. There exists $\lambda_\sigma > 0$, such that for every $t \in J$, we have:*

$$\mathfrak{I}_q^\beta \sigma(t) \leq \lambda_\sigma \sigma(t).$$

Then, the problem (3.9) is Ulam-Hyers-Rassias stable.

Proof. Let $y \in C(J, \mathbb{R})$ be a solution of the inequality (3.15) and let $z \in C(J, \mathbb{R})$ be the unique solution of the problem (3.9). Then, according to Lemma 2.2, we have:

$$z(t) = \mathcal{K}(t) + \int_0^T \mathcal{G}_q(t,s)\phi(s, z(s))d_q s,$$

where the functions $\mathcal{K}(t)$ and $\mathcal{G}_q(t,s)$ are given by the equations (3.12) and (3.13), respectively. Through integration of the inequality (3.15) and for every $t \in J$, we find:

$$\begin{aligned}\left|\begin{array}{c} y(t) - \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s, y(s))ds \\ -\frac{(1+t)}{(2+T)}\int_0^T \psi(s, y(s))ds - \int_0^T \mathcal{G}_q(t,s)\phi(s, y(s))d_q s \end{array}\right| &\leq \mathfrak{I}_q^\beta \epsilon\sigma(t),\\ &\leq \epsilon\lambda_\sigma\sigma(t).\end{aligned}$$

Then, we can write:

$$\begin{aligned}
|y(t)-z(t)| &\leq \Bigg| y(t)-\frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))ds-\frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))ds \\
&\quad -\int_0^T \mathcal{G}_q(t,s)\phi(s,z(s))d_qs\Bigg|, \\
&\leq \Bigg| y(t)-\frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,y(s))ds-\frac{(1+t)}{(2+T)}\int_0^T \psi(s,y(s))ds \\
&\quad -\int_0^T \mathcal{G}_q(t,s)\phi(s,y(s))d_qs+\frac{(1+T-t)}{(2+T)}\int_0^T \big(\varphi(s,y(s))-\varphi(s,z(s))\big)ds \\
&\quad +\frac{(1+t)}{(2+T)}\int_0^T \big(\psi(s,y(s))-\psi(s,z(s))\big)ds+\int_0^T \mathcal{G}_q(t,s)\big(\phi(s,y(s))-\phi(s,z(s))\big)d_qs\Bigg|, \\
&\leq \Bigg| y(t)-\frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,y(s))ds-\frac{(1+t)}{(2+T)}\int_0^T \psi(s,y(s))ds \\
&\quad -\int_0^T \mathcal{G}_q(t,s)\phi(s,y(s))d_qs\Bigg|+\Bigg|\frac{(1+T-t)}{(2+T)}\int_0^T \big(\varphi(s,y(s))-\varphi(s,z(s))\big)ds \\
&\quad +\frac{(1+t)}{(2+T)}\int_0^T \big(\psi(s,y(s))-\psi(s,z(s))\big)ds+\int_0^T \mathcal{G}_q(t,s)\big(\phi(s,y(s))-\phi(s,z(s))\big)d_qs\Bigg|,
\end{aligned}$$

Therefore, for each $t\in J$, we give:

$$\begin{aligned}
|y(t)-z(t)| &\leq \epsilon\lambda_\sigma\sigma(t)+\frac{(1+T-t)}{(2+T)}\int_0^T \big|\varphi(s,y(s))-\varphi(s,z(s))\big|ds+\frac{(1+t)}{(2+T)} \\
&\quad \times\int_0^T \big|\psi(s,y(s))-\psi(s,z(s))\big|ds+\int_0^T |\mathcal{G}_q(t,s)|\big|\phi(s,y(s))-\phi(s,z(s))\big|d_qs.
\end{aligned}$$

Using the hypothesis (H_2) and for every $t\in J$, we get:

$$\|y-z\|_\infty \leq \epsilon\lambda_\sigma\sigma(t)+\left((\mathfrak{L}_\varphi+\mathfrak{L}_\psi)\frac{T(1+T)}{(2+T)}+\mathfrak{L}_\phi\mathcal{G}_q^*T\right)\|y-z\|_\infty.$$

From condition (3.10), we obtain:

$$\begin{aligned}
\|y-z\|_\infty &\leq \frac{\epsilon\lambda_\sigma\sigma(t)}{1-\left((\mathfrak{L}_\varphi+\mathfrak{L}_\psi)\frac{T(1+T)}{(2+T)}+\mathfrak{L}_\phi\mathcal{G}_q^*T\right)}, \\
&:= \eta_\sigma\epsilon\sigma(t).
\end{aligned}$$

Consequently, the problem (3.9) is Ulam-Hyers-Rassias stable. ■

3.4 Examples

This part includes two examples that illustrate our main results.

Example 3.10 *Consider the following boundary value problem for fractional q-difference equation:*

$$\begin{cases} \left({}^{C}\mathfrak{D}_{1/7}^{8/7}z\right)(t) = \frac{te^{-t}}{t+6}\cos(z(t)); \quad 1<\beta\leq 2,\ t\in \mathrm{J}=[0,1], \\ \\ z(0)-z'(0) = \int_0^1 \frac{s}{2}\cos(z(s))ds, \\ \\ z(1)+z'(1) = \int_0^1 \frac{s}{2}\sin(z(s))ds, \end{cases} \tag{3.16}$$

where $\beta = \frac{8}{7},\ q=\frac{1}{7}, \mathrm{T}=1,$ *and*

$$\begin{aligned} \phi(t,z) &= \frac{te^{-t}}{t+6}\cos(z), \qquad (t,z)\in \mathrm{J}\times\mathbb{R}, \\ \varphi(t,z) &= \frac{t}{2}\cos(z), \qquad (t,z)\in \mathrm{J}\times\mathbb{R}, \\ \psi(t,z) &= \frac{t}{2}\sin(z), \qquad (t,z)\in \mathrm{J}\times\mathbb{R}. \end{aligned}$$

Obviously, the functions ϕ, φ *and* ψ *are continuous.*

Let $y,z\in\mathbb{R}$ *and for every* $t\in \mathrm{J}=[0,1]$. *Then, we have:*

$$\begin{aligned} |\phi(t,y)-\phi(t,z)| &= \left|\frac{te^{-t}}{t+6}\left(\cos(y)-\cos(z)\right)\right|, \\ &\leq \frac{te^{-t}}{t+6}|\cos(y)-\cos(z)|, \\ &\leq \frac{1}{6}|y-z|, \end{aligned}$$

and

$$\begin{aligned} |\varphi(t,y)-\varphi(t,z)| &\leq \frac{1}{2}|y-z|, \\ |\psi(t,y)-\psi(t,z)| &\leq \frac{1}{2}|y-z|. \end{aligned}$$

Then, the hypothesis (H_2) *is satisfied with* $\mathfrak{L}_\phi = \frac{1}{6}$ *and* $\mathfrak{L}_\varphi = \mathfrak{L}_\psi = \frac{1}{2}$. *Hence, from the equation (3.13) and for each* $\mathrm{J}=[0,1]$, *we give:*

$$\mathcal{G}_q^* = \sup_{(t,s)\in \mathrm{J}\times \mathrm{J}} |\mathcal{G}_q(t,s)| = \frac{5}{3\Gamma_q(\beta)} + \frac{2}{3\Gamma_q(\beta-1)}.$$

Now, we will verify that the condition (3.17) is satisfied with $\mathrm{T}=1$. *In effect,*

$$\begin{aligned}(\mathfrak{L}_{\varphi}+\mathfrak{L}_{\psi})\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}+\mathfrak{L}_{\phi}\mathcal{G}_q^{*}\mathrm{T} &= \frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{6}\left(\frac{5}{3\Gamma_{1/7}(\frac{8}{7})}+\frac{2}{3\Gamma_{1/7}(\frac{1}{7})}\right),\\ &\simeq 0.9833<1.\end{aligned}$$

Hence, thanks to Theorem 3.1, the problem (3.16) has a unique solution on $[0,1]$ *and all the conditions of Theorem 3.7 are satisfied. Thus, the problem (3.16) is Ulam-Hyers stable.*
Next, let $\sigma(t)=t^2$ *for each* $t\in\mathrm{J}=[0,1]$, *then we have:*

$$\mathfrak{I}_{1/7}^{8/7}\sigma(t)=\frac{\Gamma_{1/7}(3)}{\Gamma_{1/7}(\frac{29}{7})}t^{2+\frac{8}{7}}\leq\frac{\Gamma_{1/7}(3)}{\Gamma_{1/7}(\frac{29}{7})}t^{2}=\lambda_{\sigma}\sigma(t).$$

Then, the hypothesis (H_4) *holds with* $\sigma(t)=t^2$ *and* $\lambda_\sigma=\frac{\Gamma_{1/7}(3)}{\Gamma_{1/7}(\frac{29}{7})}$. *Hence, all the hypotheses of Theorem 3.9 are satisfied. So, the problem (3.16) is Ulam-Hyers-Rassias stable.*
On the other hand, let $z\in\mathbb{R}$ *and for each* $t\in\mathrm{J}$, *we have:*

$$\begin{aligned}|\phi(t,z)| &\leq \frac{1}{6},\\ |\varphi(t,z)| &\leq \frac{1}{2},\\ |\psi(t,z)| &\leq \frac{1}{2}.\end{aligned}$$

Hence, the hypothesis (H_3) *holds with* $\mathcal{M}_\phi=\frac{1}{6}$ *and* $\mathcal{M}_\varphi=\mathcal{M}_\psi=\frac{1}{2}$.
Consequently, according to Theorem 3.2, the problem (3.16) has at least one solution on $[0,1]$.

Example 3.11 *Consider the following boundary value problem for fractional* q*-difference equation:*

$$\begin{cases}\left({}^{\mathrm{C}}\mathfrak{D}_{1/2}^{4/3}z\right)(t)=\frac{t^2|z(t)|+1}{6e^t(1+|z(t)|)}; \quad 1<\beta\leq 2,\ t\in\mathrm{J}=[0,1],\\ z(0)-z'(0)=\int_0^1\frac{s^2+1}{3}|z(s)|ds,\\ z(1)+z'(1)=\int_0^1\frac{s^2-1}{5}|z(s)|ds,\end{cases}\tag{3.17}$$

where $\beta=\frac{4}{3}$, $q=\frac{1}{2}$, $\mathrm{T}=1$, *and*

$$\begin{aligned}\phi(t,z)&=\frac{t^2z+1}{6e^t(1+z)}, &&(t,z)\in\mathrm{J}\times\mathbb{R}_+,\\ \varphi(t,z)&=\frac{t^2+1}{3}z, &&(t,z)\in\mathrm{J}\times\mathbb{R}_+,\\ \psi(t,z)&=\frac{t^2-1}{5}z, &&(t,z)\in\mathrm{J}\times\mathbb{R}_+.\end{aligned}$$

Evidently, the functions ϕ,φ *and* ψ *are continuous.*

Let $y, z \in \mathbb{R}_+$ *and for each* $t \in \mathrm{J} = [0,1]$. *Then, we have:*

$$\begin{aligned} |\phi(t,y)-\phi(t,z)| &\leq \frac{t^2}{6e^t}|y-z|, \\ &\leq \frac{1}{6}|y-z|, \end{aligned}$$

and

$$|\varphi(t,y)-\varphi(t,z)| \leq \frac{1}{3}|y-z|,$$

$$|\psi(t,y)-\psi(t,z)| \leq \frac{1}{5}|y-z|.$$

So, the hypothesis (H_2) *holds with* $\mathfrak{L}_\phi = \frac{1}{6}$, $\mathfrak{L}_\varphi = \frac{1}{3}$ *and* $\mathfrak{L}_\psi = \frac{1}{5}$. *Therefore, according to the equation (3.13) and for every* $\mathrm{J} = [0,1]$, *we give:*

$$\mathscr{G}_q^* = \sup_{(t,s)\in \mathrm{J}\times \mathrm{J}} |\mathscr{G}_q(t,s)| = \frac{5}{3\Gamma_q(\beta)} + \frac{2}{3\Gamma_q(\beta-1)}.$$

Now, we will verify that the condition (3.17) is satisfied with $\mathrm{T}=1$. *In effect,*

$$\begin{aligned} (\mathfrak{L}_\varphi+\mathfrak{L}_\psi)\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})} + \mathfrak{L}_\phi \mathscr{G}_q^* \mathrm{T} &= \frac{2}{3}\left(\frac{1}{3}+\frac{1}{5}\right) + \frac{1}{6}\left(\frac{5}{3\Gamma_{1/2}(\frac{4}{3})} + \frac{2}{3\Gamma_{1/2}(\frac{1}{3})}\right), \\ &\simeq 0.7057 < 1. \end{aligned}$$

Thus, according to Theorem 3.1, the problem (3.17) has a unique solution on $[0,1]$ *and all the conditions of Theorem 3.7 are satisfied. So, the problem (3.17) is Ulam-Hyers stable. Next, let* $\sigma(t) = t^2$ *for every* $t \in \mathrm{J} = [0,1]$, *then we have:*

$$\mathfrak{I}_{1/2}^{4/3}\sigma(t) = \frac{\Gamma_{1/2}(3)}{\Gamma_{1/2}(\frac{13}{3})} t^{2+\frac{3}{4}} \leq \frac{3}{2\Gamma_{1/2}(\frac{13}{3})} t^2 = \lambda_\sigma \sigma(t).$$

Hence, the hypothesis (H_4) *holds with* $\sigma(t) = t^2$ *and* $\lambda_\sigma = \frac{3}{2\Gamma_{1/2}(\frac{13}{3})}$. *Therefore, all the conditions of Theorem 3.9 are satisfied. Thus, the problem (3.17) is Ulam-Hyers-Rassias stable. On the other hand, let* $z \in \mathbb{R}_+$ *and for every* $t \in \mathrm{J}$, *we have:*

$$\begin{aligned} |\phi(t,z)| &\leq \frac{1}{6}, \\ |\varphi(t,z)| &\leq \frac{1}{3}, \\ |\psi(t,z)| &\leq \frac{1}{5}. \end{aligned}$$

Then, the hypothesis (H_3) *is satisfied with* $\mathscr{M}_\phi = \frac{1}{6}$, $\mathscr{M}_\varphi = \frac{1}{3}$ *and* $\mathscr{M}_\psi = \frac{1}{5}$.
Consequently, thanks to Theorem 3.2, the problem (3.17) has at least one solution on $[0,1]$.

4 Boundary Value Problem for Fractional q-Difference Equations with Integral Conditions in Banach Spaces

[2] The main goal of this section is to show the result of the existence of solutions to the boundary value problem (3.1) in Banach space $\mathbb{E}$ with the norm $\|.\|$, through the use of the Kuratowski's measure of non-compactness and Mönch's fixed point theorem.

4.1 Existence Result

In this part, we prove the existence of solutions to the problem (3.1), which depends on the fixed point theorem of Mönch (Theorem 5.6).

Theorem 4.1 *Suppose that the following hypotheses are satisfied:*

(H_5) *The functions* $\phi, \varphi, \psi : J \times \mathbb{E} \to \mathbb{E}$ *satisfy the Carathéodory conditions.*

(H_6) *There exist* $p_\phi, p_\varphi, p_\psi \in L^\infty(J, \mathbb{R}_+)$, *such that for every* $t \in J$ *and every* $z \in \mathbb{E}$, *we have:*

$$\begin{aligned} \|\phi(t,z)\| &\le p_\phi(t)\|z\|, \\ \|\varphi(t,z)\| &\le p_\varphi(t)\|z\|, \\ \|\psi(t,z)\| &\le p_\psi(t)\|z\|. \end{aligned}$$

(H_7) *For all* $t \in J$ *and every bounded set* $\mathscr{B} \subset \mathbb{E}$, *we have:*

$$\begin{aligned} \mu(\phi(t,\mathscr{B})) &\le p_\phi(t)\mu(\mathscr{B}), \\ \mu(\varphi(t,\mathscr{B})) &\le p_\varphi(t)\mu(\mathscr{B}), \\ \mu(\psi(t,\mathscr{B})) &\le p_\psi(t)\mu(\mathscr{B}). \end{aligned}$$

If

$$\frac{T(1+T)}{(2+T)}\left(\|p_\varphi\|_{L^\infty} + \|p_\psi\|_{L^\infty}\right) + \mathscr{G}_q^* T\|p_\phi\|_{L^\infty} < 1. \tag{3.18}$$

where

$$\mathscr{G}_q^* = \sup_{(t,s)\in J\times J} |\mathscr{G}_q(t,s)|.$$

Then, the problem (3.1) has at least one solution J.

[2]**N. Allouch**, J.R. Graef and S.Hamani, *Boundary Value Problem for Fractional q-Difference Equations with Integral Conditions in Banach space*, Fractal Fract., **6 (5)**, (2022), 11 page.

Proof. In order to illustrate this result, we first convert the problem (3.1) into a fixed point problem and consider the operator:

$$\mathcal{N} : C(J, \mathbb{E}) \longrightarrow C(J, \mathbb{E})$$

By:

$$(\mathcal{N} z)(t) = \mathcal{K}(t) + \int_0^T \mathcal{G}_q(t,s)\phi(s,z(s))d_q s, \tag{3.19}$$

where

$$\mathcal{K}(t) = \frac{(1+T-t)}{(2+T)}\int_0^T \varphi(s,z(s))ds + \frac{(1+t)}{(2+T)}\int_0^T \psi(s,z(s))ds,$$

and

$$\mathcal{G}_q(t,s) = \begin{cases} \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-1)}}{(2+T)\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-2)}}{(2+T)\Gamma_q(\beta-1)}; & 0 \le s < t, \\ -\frac{(1+t)(T-qs)^{(\beta-1)}}{(2+T)\Gamma_q(\beta)} - \frac{(1+t)(T-qs)^{(\beta-2)}}{(2+T)\Gamma_q(\beta-1)}; & t \le s \le T. \end{cases}$$

Obviously, according to Lemma 2.2, the fixed points of the operator $\mathcal{N}$ are solutions of the problem (3.1).
Let $\omega^* > 0$, we consider the set:

$$\mathcal{D}^*_{\omega^*} = \{z \in C(J,\mathbb{E}) \ : \ \|z\|_\infty \le \omega^*\}. \tag{3.20}$$

It is clear that $\mathcal{D}^*_{\omega^*}$ is a bounded, closed and convex set of $C(J,\mathbb{E})$.

Next, we shall show that the operator $\mathcal{N}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be provided in three steps.
Step 1: *$\mathcal{N}$ is a continuous operator on* $C(J,\mathbb{E})$.
Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence with $z_n \to z$ in $C(J,\mathbb{E})$. Then, for every $t \in J$, we have:

$$\begin{aligned} |(\mathcal{N} z_n)(t) - (\mathcal{N} z)(t)| \le & \frac{(1+T-t)}{(2+T)}\int_0^T \left|\varphi(s,z_n(s)) - \varphi(s,z(s))\right| ds \\ & + \frac{(1+t)}{(2+T)}\int_0^T \left|\psi(s,z_n(s)) - \psi(s,z(s))\right| ds \\ & + \int_0^T |\mathcal{G}_q(t,s)| \left|\phi(s,z_n(s)) - \phi(s,z(s))\right| d_q s. \end{aligned}$$

Therefore, for each $t \in J$, we obtain:

$$\begin{aligned}\|\mathcal{N}(z_n)-\mathcal{N}(z)\| \leq\ & \frac{T(1+T)}{(2+T)}\left\|\varphi(s,z_n(s))-\varphi(s,z(s))\right\|+\frac{T(1+T)}{(2+T)}\left\|\psi(s,z_n(s))-\psi(s,z(s))\right\| \\ & +\mathcal{G}_q^*T\left\|\phi(s,z_n(s))-\phi(s,z(s))\right\|.\end{aligned}$$

Let $\rho > 0$ be such that:

$$\|z_n\|_\infty \leq \rho \quad \text{and} \quad \|z\|_\infty \leq \rho.$$

Using the hypothesis (H_6), we give:

$$\begin{aligned}\|\phi(s,z_n(s))-\phi(s,z(s))\| &\leq 2\rho\mathfrak{p}_\phi(s) := \delta_\phi(s); && \delta_\phi(s) \in L^\infty(J,\mathbb{R}_+).\\ \|\varphi(s,z_n(s))-\varphi(s,z(s))\| &\leq 2\rho\mathfrak{p}_\varphi(s) := \delta_\varphi(s); && \delta_\varphi(s) \in L^\infty(J,\mathbb{R}_+).\\ \|\psi(s,z_n(s))-\psi(s,z(s))\| &\leq 2\rho\mathfrak{p}_\psi(s) := \delta_\psi(s); && \delta_\psi(s) \in L^\infty(J,\mathbb{R}_+).\end{aligned}$$

Since ϕ, φ, and ψ are Carathéodory's functions, and thanks to the Lebesgue dominated convergence theorem, we get:

$$\|\mathcal{N}(z_n)-\mathcal{N}(z)\|_\infty \to 0 \quad as \quad n\to\infty.$$

Thus, $\mathcal{N}$ is a continuous operator on $C(J,\mathbb{E})$.

Step 2: $\mathcal{N}$ *maps* $\mathcal{D}^*_{\omega^*}$ *into* $\mathcal{D}^*_{\omega^*}$.

Let $z \in \mathcal{D}^*_{\omega^*}$ and applying hypothesis (H_6), for every $t \in J$, we have:

$$\begin{aligned}|(\mathcal{N}z)(t)| &\leq \frac{(1+T-t)}{(2+T)}\int_0^T|\varphi(s,z(s))|ds+\frac{(1+t)}{(2+T)}\int_0^T|\psi(s,z(s))|ds+\int_0^T|\mathcal{G}_q(t,s)||\phi(s,z(s))|d_qs,\\ &\leq \frac{(1+T-t)}{(2+T)}\int_0^T\mathfrak{p}_\varphi(s)\|z\|ds+\frac{(1+t)}{(2+T)}\int_0^T\mathfrak{p}_\psi(s)\|z\|ds+\int_0^T|\mathcal{G}_q(t,s)|\mathfrak{p}_\phi(s)\|z\|d_qs.\end{aligned}$$

From the set (3.20) and for each $t \in J$, we obtain:

$$\begin{aligned}|(\mathcal{N}z)(t)| &\leq \frac{\omega^*T(1+T)}{(2+T)}\|\mathfrak{p}_\varphi\|_{L^\infty}+\frac{\omega^*T(1+T)}{(2+T)}\|\mathfrak{p}_\psi\|_{L^\infty}+\omega^*\mathcal{G}_q^*T\|\mathfrak{p}_\phi\|_{L^\infty},\\ &\leq \omega^*\left(\left(\|\mathfrak{p}_\varphi\|_{L^\infty}+\|\mathfrak{p}_\psi\|_{L^\infty}\right)\frac{T(1+T)}{(2+T)}+\mathcal{G}_q^*T\|\mathfrak{p}_\phi\|_{L^\infty}\right),\\ &\leq \omega^*.\end{aligned}$$

Thus,

$$\|\mathcal{N}(z)\|_\infty \leq \omega^*.$$

Step 3: *$\mathcal{N}(\mathcal{D}^*_{\omega^*})$ is bounded and equi-continuous.*
In light of Step 2, It is evident that $\mathcal{N}(\mathcal{D}^*_{\omega^*}) \subset C(J,\mathbb{E})$ is bounded.
Next, we demonstrate that the equi-continuity of $\mathcal{N}(\mathcal{D}^*_{\omega^*})$. Let $z \in \mathcal{D}^*_{\omega^*}$ and let $t_1,\ t_2 \in J$ such that $t_1 < t_2$, then we have:

$$\begin{aligned}
|(\mathcal{N}z)(t_2) - (\mathcal{N}z)(t_1)| &= \Bigg|\frac{(t_1-t_2)}{(2+T)}\int_0^T \varphi(s,z(s))ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T \psi(s,z(s))ds \\
&\quad + \int_0^T \left(\mathcal{G}_q(t_2,s) - \mathcal{G}_q(t_1,s)\right)\phi(s,z(s))d_qs\Bigg|, \\
&\leq \frac{|t_1-t_2|}{(2+T)}\int_0^T \|\varphi(s,z(s))\|ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T |\psi(s,z(s))|ds \\
&\quad + \int_0^T \left|\mathcal{G}_q(t_2,s) - \mathcal{G}_q(t_1,s)\right| |\phi(s,z(s))|d_qs.
\end{aligned}$$

Using the hypothesis (H_6), we find:

$$\begin{aligned}
|(\mathcal{N}z)(t_2) - (\mathcal{N}z)(t_1)| &\leq \frac{|t_1-t_2|}{(2+T)}\int_0^T \mathfrak{p}_\varphi(s)\|z\|ds + \frac{(t_2-t_1)}{(2+T)}\int_0^T \mathfrak{p}_\psi(s)\|z\|ds \\
&\quad + \int_0^T |\mathcal{G}(t_2,s) - \mathcal{G}(t_1,s)|\mathfrak{p}_\phi(s)\|z\|d_qs.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(\mathcal{N}z)(t_2) - (\mathcal{N}z)(t_1)| &\leq \omega^* T\frac{|t_1-t_2|}{(2+T)}\|\mathfrak{p}_\varphi\|_{L^\infty} + \omega^* T\frac{(t_2-t_1)}{(2+T)}\|\mathfrak{p}_\psi\|_{L^\infty} \\
&\quad + \omega^* T\|\mathfrak{p}_\phi\|_{L^\infty}\sup_{s\in J}\left|\mathcal{G}_q(t_2,s) - \mathcal{G}_q(t_1,s)\right|.
\end{aligned}$$

As $t_1 \to t_2$, the inequality above's right-hand side tends to zero, i.e.:

$$|(\mathcal{N}z)(t_2) - (\mathcal{N}z)(t_1)| \to 0 \quad as \quad t_1 \to t_2.$$

Thus, the equi-continuity of $\mathcal{N}(\mathcal{D}^*_{\omega^*})$. So, $\mathcal{N}(\mathcal{D}^*_{\omega^*}) \subset \mathcal{D}^*_{\omega^*}$.

Lastly, we show the validity of the implication (1.72).
Let $\mathcal{V}$ be a subset of $\mathcal{D}^*_{\omega^*}$ such that $\mathcal{V} \subset \overline{conv}(\mathcal{N}(\mathcal{V} \cup \{0\})$. Since $\mathcal{V}$ is bounded and equi-continuous, the function $t \to v(t) = \mu(\mathcal{V}(t))$ is continuous on J. Then, from hypothesis (H_7), Lemma 5.7, and applying the properties of the measure μ, for every $t \in J$, we have:

$$\begin{aligned}
v(t) &\leq \mu(\mathcal{N}(\mathcal{V})(t) \cup \{0\}), \\
&\leq \mu(\mathcal{N}(\mathcal{V})(t)), \\
&\leq \frac{(1+T-t)}{(2+T)}\int_0^T \mathfrak{p}_\varphi(s)\mu(\mathcal{V}(s))ds + \frac{(1+t)}{(2+T)}\int_0^T \mathfrak{p}_\psi(s)\mu(\mathcal{V}(s))ds + \int_0^T |\mathcal{G}_q(t,s)|\mathfrak{p}_\phi(s)\mu(\mathcal{V}(s))d_qs.
\end{aligned}$$

So, for each $t \in \mathrm{J}$, we get:

$$\begin{aligned} v(t) &\leq \frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\|\mathfrak{p}_{\varphi}\|_{\mathrm{L}^{\infty}}\|v\|_{\infty} + \frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\|\mathfrak{p}_{\psi}\|_{\mathrm{L}^{\infty}}\|v\|_{\infty} + \mathscr{G}_q^*\mathrm{T}\|\mathfrak{p}_{\phi}\|_{\mathrm{L}^{\infty}}\|v\|_{\infty}, \\ &\leq \|v\|_{\infty}\left(\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\left(\|\mathfrak{p}_{\varphi}\|_{\mathrm{L}^{\infty}} + \|\mathfrak{p}_{\psi}\|_{\mathrm{L}^{\infty}}\right) + \mathscr{G}_q^*\mathrm{T}\|\mathfrak{p}_{\phi}\|_{\mathrm{L}^{\infty}}\right). \end{aligned}$$

This implies that,

$$\|v\|_{\infty}\left[1 - \left(\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\left(\|\mathfrak{p}_{\varphi}\|_{\mathrm{L}^{\infty}} + \|\mathfrak{p}_{\psi}\|_{\mathrm{L}^{\infty}}\right) + \mathscr{G}_q^*\mathrm{T}\|\mathfrak{p}_{\phi}\|_{\mathrm{L}^{\infty}}\right)\right] \leq 0.$$

By condition (3.18), we observe that $\|v\|_{\infty} = 0$, i.e.: $v(t) = 0$ for all $t \in \mathrm{J}$. Thus, $\mathcal{V}(t)$ is relatively compact in $\mathbb{E}$. Thanks to Ascoli-Arzelà theorem (Theorem 1.9), $\mathcal{V}$ is relatively compact in $\mathscr{D}^*_{\omega^*}$. According to Mönch's fixed point theorem, we conclude that the operator $\mathscr{N}$ has a fixed point that represents the solution to the problem (3.1). ■

4.2 An Example

Let

$$\mathbb{E} = l^1 = \left\{(z_1, z_2, \cdots, z_n, \cdots) \ : \ \sum_{n=1}^{\infty}|z_n| < \infty\right\},$$

be our Banach space with the norm

$$\|z\| = \sum_{n=1}^{\infty}|z_n|.$$

Now, we consider the following boundary value problem for fractional q-difference equation:

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_{1/4}^{3/2} z_n\right)(t) = \frac{z_n(t)}{(e^t+5)(z_n(t)+1)}; \quad 1 < \beta \leq 2, \ t \in \mathrm{J} = [0,1], \\ z_n(0) - z_n'(0) = \int_0^1 \frac{s^3-1}{9} z_n(s)ds, \\ z_n(1) + z_n'(1) = \int_0^1 \frac{s^3+1}{6} z_n(s)ds, \end{cases} \tag{3.21}$$

where $\beta = \frac{3}{2}$, $q = \frac{1}{4}$, $\mathrm{T} = 1$, and

$$\begin{aligned} \phi_n(t,z) &= \frac{z_n(t)}{(e^t+5)(z_n(t)+1)}; & (t,z) \in \mathrm{J}\times\mathbb{E}, \\ \varphi_n(t,z) &= \frac{t^3-1}{9} z_n; & (t,z) \in \mathrm{J}\times\mathbb{E}, \\ \psi_n(t,z) &= \frac{t^3+1}{6} z_n; & (t,z) \in \mathrm{J}\times\mathbb{E}. \end{aligned}$$

with

$$z = (z_1, z_2, \ldots, z_n, \ldots),$$

$$\phi = (\phi_1, \phi_2, \ldots, \phi_n, \ldots),$$

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots),$$

$$\psi = (\psi_1, \psi_2, \ldots, \psi_n, \ldots).$$

Evidently, the hypotheses (H_5) and (H_6) are satisfied, with

$$\mathfrak{p}_\phi(t) = \frac{1}{e^t + 5}, \quad \mathfrak{p}_\varphi(t) = \frac{t^3}{9} \quad \text{and} \quad \mathfrak{p}_\psi(t) = \frac{t^3}{6}.$$

By equation (3.13), we give:

$$\mathcal{G}_q^* = \sup_{(t,s)\in \mathrm{J}\times\mathrm{J}} |\mathcal{G}_q(t,s)| = \frac{5}{3\Gamma_q(\beta)} + \frac{2}{3\Gamma_q(\beta-1)}.$$

Next, we confirm that the condition (3.18) holds with $\mathrm{T} = 1$. In fact,

$$\begin{aligned}\frac{\mathrm{T}(1+\mathrm{T})}{(2+\mathrm{T})}\left(\|\mathfrak{p}_\varphi\|_{\mathrm{L}^\infty} + \|\mathfrak{p}_\psi\|_{\mathrm{L}^\infty}\right) + \mathcal{G}_q^*\mathrm{T}\|\mathfrak{p}_\phi\|_{\mathrm{L}^\infty} &= \frac{2}{3}\left(\frac{1}{9}+\frac{1}{6}\right) + \frac{1}{6}\left(\frac{5}{3\Gamma_{1/4}(\frac{3}{2})} + \frac{2}{3\Gamma_{1/4}(\frac{1}{2})}\right),\\ &\simeq 0.5564 < 1.\end{aligned}$$

Thus, thanks to Theorem 4.1, the problem (3.21) has a solution on $[0,1]$.

5 Conclusion

In this study, we have presented the results of the existence, uniqueness and stability of solutions to the boundary value problem for fractional q-difference equations involving the Caputo's fractional q-derivative with non-linear integral conditions, by using some fixed point theorems and Ulam stability techniques. Moreover, we have also given an additional result for the existence of solutions to the boundary value problem for fractional q-difference equations with non-linear integral conditions in Banach spase, by applying the Kuratowski's measure of non-compactness and Mönch's fixed point theorem. To support our findings , we include illustrative examples.

Chapter 4

Existence and Ulam Stability of Initial Value Problem for Impulsive Fractional q-Difference Equations

1 Introduction and Motivation

Impulsive fractional differential equations have played an important role in certain mathematical models of real phenomena, especially in the domains of biology and medicine(such as observe blood flow phenomena). In recent years, many researchers have been interested in the impulsive fractional q-difference equations, so that they achieved the existence and stability of their solutions; see references [26, 13, 17, 61, 75, 93] for example.

Hammou and Hamani in [54] established the existence results for solutions of the initial value problems for impulsive fractional differential equations involving the Caputo-Hadamard's fractional derivative of the following form:

$$^{\mathrm{CH}}\mathrm{D}^{r}y(t)=f(t,y(t));\quad t\in[a,\mathrm{T}],\ t\neq t_k,\ k=1,\cdots,m,\quad 1<r\leq 2,$$

$$\Delta y\,|_{t=t_k}=\mathrm{I}_k(y(t_k^-)),\quad k=1,\cdots,m,$$

$$\Delta y'\,|_{t=t_k}=\bar{\mathrm{I}}_k(y(t_k^-)),\quad k=1,\cdots,m,$$

$$y(a)=y_1,\quad y'(a)=y_2,$$

where $^{\mathrm{CH}}\mathrm{D}_q^r$ is the Caputo-Hadamard's fractional derivative, $f:[a,\mathrm{T}]\times\mathbb{E}\to\mathbb{E}$ is a function, $\mathrm{I}_k,\bar{\mathrm{I}}_k:\mathbb{E}\to\mathbb{E}$, $k=1,\cdots,m$ are functions, $a=t_0<t_1<\cdots<t_m<t_{m+1}=\mathrm{T}$, $\Delta y\,|_{t=t_k}=y(t_k^+)-y(t_k^-)$, $\Delta y'\,|_{t=t_k}=y'(t_k^+)-y'(t_k^-)$, $y(t_k^+)=\lim_{\varepsilon\to 0^+}y(t_k+\varepsilon)$ and $y(t_k^-)=\lim_{\varepsilon\to 0^-}y(t_k+\varepsilon)$ represent the right and left limits of y at $t=t_k$, $k=1,\cdots,m$ and $\mathbb{E}$ is a Banach space. They used Mönch's fixed point theorem and Kuratowski's measure of non-compactness to study the existence of solutions.

Motivated by the aforementioned work, in this chapter, we investigate the existence, uniqueness and Ulam stability of solutions to the initial value problem for impulsive fractional q-difference equations involving Caputo's fractional q-derivative, which is given as follows:

$$\begin{cases} \left({}^{C}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t)); \quad 1 < \beta \leq 2, \; t \in J = [0, T], \; t \neq t_i, \; i = 1, \cdots, n, \\ \Delta z \mid_{t=t_i} = \mathscr{I}_i(z(t_i^-)), \; i = 1, \cdots, n, \\ \Delta z' \mid_{t=t_i} = \overline{\mathscr{I}}_i(z(t_i^-)), \; i = 1, \cdots, n, \\ z(0) = z_0, \quad z'(0) = z_0^*, \end{cases} \tag{4.1}$$

where $q \in (0,1)$, $T > 0$, ${}^{C}\mathfrak{D}_q^{\beta}$ is the Caputo's fractional q-derivative of order $\beta \in (1,2]$, and $\phi : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathscr{I}_i, \overline{\mathscr{I}}_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$ are given functions, and $z_0, z_0^* \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T < +\infty$, $\Delta z \mid_{t=t_i} = z(t_i^+) - z(t_i^-)$ and $\Delta z' \mid_{t=t_i} = z'(t_i^+) - z'(t_i^-)$, $z(t_i^+) = \lim_{\varepsilon \to 0^+} z(t_i + \varepsilon)$ and $z(t_i^-) = \lim_{\varepsilon \to 0^-} z(t_i + \varepsilon)$ represent the right and left limits of z at $t = t_i$, $i = 1, \cdots, n$.

The remainder of the chapter is organized in the following format: In Section 2, we give the integrable solution of the initial value problem (4.1). After that in Section 3, we present the main results regarding the existence and Ulam stability of solutions to the initial value problem for impulsive fractional q-difference equations, which are two results of existence: one depends on Banach contraction principal theorem and the other on Krasnoselskii's fixed point theorem. In addition to the stabilities results which are depend the techniques of Ulam-Hyers and Ulam-Hyers-Rassias stabilities. In Section 4, we finish by providing an example that illustrates our main results.

2 Representation of the Integrable Solution

[1] In this section, we introduce the definition and lemma of the integral solution to the initial value problem for impulsive fractional q-difference equations (4.1), which is necessary for the continuation of the chapter.

First of all, we introduce the Banach space $\mathbb{PC}$ defined by:

$$\mathbb{PC}(J, \mathbb{R}) = \left\{ z : J \to \mathbb{R} \mid z \in C^2(J_i, \mathbb{R}), \textit{ and } z(t_i^+), z(t_i^-) \textit{ exist, with } z(t_i^-) = z(t_i^+), \; i = 1, \cdots, n \right\},$$

where $J_0 = (t_0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_i = (t_i, t_{i+1}]$, $i = 1, \cdots, n$, with the norm

$$\|z\|_{\mathbb{PC}} = \sup_{t \in J} |z(t)|.$$

[1] **N. Allouch** and S. Hamani, *Existence and Ulam Stability of Initial Value Problem for Impulsive Fractional q-Difference Equations.* (Submitted)

Now, let's define what is meant by the integral solution to the initial value problem (4.1).

Definition 2.1 *A function $z \in \mathbb{PC}(\mathrm{J},\mathbb{R})$ is said to be a solution of the problem (4.1) if z satisfies the fractional q-difference equation $\left({}^{\mathrm{C}}\mathfrak{D}_q^{\beta} z\right)(t) = \phi(t, z(t))$ on J where $\beta \in (1,2]$, and satisfies the following conditions:*

$$\Delta z\,|_{t=t_i} = \mathscr{I}_i(z(t_i^-)), \qquad i = 1, \cdots, n,$$
$$\Delta z'\,|_{t=t_i} = \overline{\mathscr{I}}_i(z(t_i^-)), \qquad i = 1, \cdots, n,$$

$$z(0) = z_0, \quad z'(0) = z_0^*.$$

Next, we need the following lemma to determine the main results of the initial value problem (4.1).

Lemma 2.2 *Let $\theta : \mathrm{J} \to \mathbb{R}$ be a continuous function. The integral solution of the following initial value problem:*

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_q^{\beta} z\right)(t) = \theta(t); \quad 1 < \beta \le 2,\ t \in \mathrm{J} = [0, \mathrm{T}],\ t \neq t_i,\ i = 1, \cdots, n, \\ \Delta z\,|_{t=t_i} = \mathscr{I}_i(z(t_i^-)),\ i = 1, \cdots, n, \\ \Delta z'\,|_{t=t_i} = \overline{\mathscr{I}}_i(z(t_i^-)),\ i = 1, \cdots, n, \\ z(0) = z_0, \quad z'(0) = z_0^*, \end{cases} \tag{4.2}$$

Given as follows:

$$z(t) = \begin{cases} z_0 + z_0^* t + \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s; & t \in \mathrm{J}_0 = [0, t_1], \\ z_0 + z_0^* t + \sum_{i=1}^n \mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^n (t - t_i) \overline{\mathscr{I}}_i(z(t_i^-)) & \\ + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s & \\ + \sum_{i=1}^n (t - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \theta(s) d_q s & \\ + \int_{t_i}^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s; & t \in \mathrm{J}_i = (t_i, t_{i+1}],\ i = 1, \cdots, n. \end{cases} \tag{4.3}$$

Proof. By applying the Riemann-Liouville's fractional q-integral of order $\beta \in (1,2]$ on both sides of the equation for the problem (4.2), and according to Lemma 3.22.
If $t \in \mathrm{J}_0 = [0, t_1]$, then we have:

$$z(t) = \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + c_0 + c_1 t.$$

Thank to the initial conditions of the problem (4.2), we find:

$$z(0) = c_0 = z_0 \quad and \quad z'(0) = c_1 = z_0^*.$$

Thus,

$$z(t) = z_0 + z_0^* t + \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s. \tag{4.4}$$

If $t \in \mathrm{J}_1 = (t_1, t_2]$, then we have:

$$z(t) = \int_{t_1}^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + c_0 + c_1(t - t_1). \tag{4.5}$$

By impulsive conditions of the problem (4.2), we give:

$$\Delta z\,|_{t=t_1} = z(t_1^+) - z(t_1^-) = \mathscr{I}_1(z(t_1^-)).$$

This means that,

$$\mathscr{I}_1(z(t_1^-)) = c_0 - \left(z_0 + z_0^* t_1 + \int_0^{t_1} \frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s \right).$$

So,

$$c_0 = z_0 + z_0^* t_1 + \mathscr{I}_1(z(t_1^-)) + \int_0^{t_1} \frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s.$$

Also, we have:

$$\Delta z'\,|_{t=t_1} = z'(t_1^+) - z'(t_1^-) = \overline{\mathscr{I}}_1(z(t_1^-)).$$

This implies that,

$$\overline{\mathscr{I}}_1(z(t_1^-)) = c_1 - \left(z_0^* + \int_0^{t_1} \frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \theta(s) d_q s \right).$$

So,

$$c_1 = z_0^* + \overline{\mathscr{I}}_1(z(t_1^-)) + \int_0^{t_1} \frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \theta(s) d_q s.$$

Changing c_0, c_1 in equation (4.5), we get:

$$\begin{aligned} z(t) &= z_0 + z_0^* t + \mathscr{I}_1(z(t_1^-)) + (t-t_1)\overline{\mathscr{I}}_1(z(t_1^-)) + \int_0^{t_1} \frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s \\ &\quad + (t-t_1)\int_0^{t_1} \frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \theta(s) d_q s + \int_{t_1}^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s. \end{aligned}$$

If $t \in \mathrm{J}_2 = (t_2, t_3]$, then we have:

$$z(t) = \int_{t_2}^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s + c_0 + c_1(t - t_2). \tag{4.6}$$

From the impulsive conditions of the problem (4.2), we give:

$$\Delta z\mid_{t=t_2}=z(t_2^+)-z(t_2^-)=\mathscr{I}_2(z(t_2^-)).$$

This means that,

$$\begin{aligned}\mathscr{I}_2(z(t_2^-)) &= c_0-\Big(z_0+z_0^*t_2+\mathscr{I}_1(z(t_1^-))+(t_2-t_1)\overline{\mathscr{I}}_1(z(t_1^-))+\int_0^{t_1}\frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs\\ &\quad+(t_2-t_1)\int_0^{t_1}\frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs\Big).\end{aligned}$$

Thus,

$$\begin{aligned}c_0 &= z_0+z_0^*t_2+\mathscr{I}_1(z(t_1^-))+\mathscr{I}_2(z(t_2^-))+(t_2-t_1)\overline{\mathscr{I}}_1(z(t_1^-))+\int_0^{t_1}\frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs\\ &\quad+(t_2-t_1)\int_0^{t_1}\frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs.\end{aligned}$$

Also, we have:

$$\Delta z'\mid_{t=t_2}=z'(t_2^+)-z'(t_2^-)=\overline{\mathscr{I}}_2(z(t_2^-)).$$

This implies that,

$$\overline{\mathscr{I}}_2(z(t_2^-))=c_1-\Big(z_0^*+\overline{\mathscr{I}}_1(z(t_1^-))+\int_0^{t_1}\frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs\Big).$$

Thus,

$$c_1=z_0^*+\overline{\mathscr{I}}_1(z(t_1^-))+\overline{\mathscr{I}}_2(z(t_2^-))+\int_0^{t_1}\frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs.$$

Substituting c_0, c_1 into equation (4.6), we find:

$$\begin{aligned}z(t) &= z_0+z_0^*t+\mathscr{I}_1(z(t_1^-))+\mathscr{I}_2(z(t_2^-))+(t-t_1)\overline{\mathscr{I}}_1(z(t_1^-))+(t-t_2)\overline{\mathscr{I}}_2(z(t_2^-))\\ &\quad+\int_0^{t_1}\frac{(t_1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\theta(s)d_qs\\ &\quad+(t-t_1)\int_0^{t_1}\frac{(t_1-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs+(t-t_2)\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\theta(s)d_qs\\ &\quad+\int_{t_2}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}h(s)d_qs.\end{aligned}$$

Generalizing in this manner, if $t \in \mathrm{J}_n = (t_n, t_{n+1}]$, then we give:

$$\begin{aligned} z(t) &= z_0 + z_0^* t + \sum_{i=1}^{n} \mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^{n} (t - t_i)\overline{\mathscr{I}}_i(z(t_i^-)) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s \\ &+ \sum_{i=1}^{n} (t - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta - 1)} \theta(s) d_q s + \int_{t_n}^{t} \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \theta(s) d_q s. \end{aligned} \tag{4.7}$$

Consequently, according to the equations (4.4) and (4.7) we find the solution (4.3). The proof is finished. ■

3 Main Results

The purpose of this section is to provide results for the existence, uniqueness and stability of solutions to the initial value problem for impulsive fractional q-difference equations (4.1), by using the fixed point theorems and Ulam stability techniques.

The following assumptions are necessary in order to determine the main results:

(P_1) The function $\phi : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(P_2) There exists a positive constant $\mathfrak{L}_\phi$, such that for every $t \in \mathrm{J}$ and every $y, z \in \mathbb{R}$, we have:

$$|\phi(t, y) - \phi(t, z)| \leq \mathfrak{L}_\phi |y - z|.$$

(P_3) There exist positive constants $\mathfrak{L}_{\mathscr{I}_i}, \mathfrak{L}_{\overline{\mathscr{I}}_i}$, $i = 1, \cdots, n$, such that for every $y, z \in \mathbb{R}$, we have:

$$|\mathscr{I}_i(y) - \mathscr{I}_i(z)| \leq \mathfrak{L}_{\mathscr{I}_i} |y - z| \quad \text{and} \quad |\overline{\mathscr{I}}_i(y) - \overline{\mathscr{I}}_i(z)| \leq \mathfrak{L}_{\overline{\mathscr{I}}_i} |y - z|.$$

(P_4) There exists a positive constant $\mathscr{M}_\phi$, such that for every $t \in \mathrm{J}$ and every $z \in \mathbb{R}$, we have:

$$|\phi(t, z)| \leq \mathscr{M}_\phi.$$

(P_5) The functions $\mathscr{I}_i, \overline{\mathscr{I}}_i : \mathbb{R} \to \mathbb{R}$ are continuous and there exist positive constants $\mathscr{M}_{\mathscr{I}_i}, \mathscr{M}_{\overline{\mathscr{I}}_i}$, $i = 1, \cdots, n$, such that for every $z \in \mathbb{R}$, we have:

$$|\mathscr{I}_i(z)| \leq \mathscr{M}_{\mathscr{I}_i} \quad \text{and} \quad |\overline{\mathscr{I}}_i(z)| \leq \mathscr{M}_{\overline{\mathscr{I}}_i}.$$

3.1 Existence and Uniqueness Result

In the following part, we apply Banach contraction principle theorem (Theorem 5.2) to examine the uniqueness of solutions to the initial value problem (4.1).

Theorem 3.1 *Suppose that the hypotheses* (P_1)-(P_2) *and* (P_3) *hold. If*

$$0 < n\mathfrak{L}_{\mathscr{I}_i} + n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T} + \frac{n\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)} + (n+1)\frac{\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)} < 1. \tag{4.8}$$

Then, the initial value problem (4.1) has a unique solution on J.

Proof. In order to demonstrate this result, we first transform the problem (4.1) into a fixed point problem and define the operator

$$\mathfrak{T} : \mathbb{PC}(\mathrm{J},\mathbb{R}) \longrightarrow \mathbb{PC}(\mathrm{J},\mathbb{R})$$

Given by:

$$\begin{aligned}(\mathfrak{T}z)(t) &= z_0 + z_0^* t + \sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(z(t_i^-)) + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs \\ &\quad + \sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_qs + \int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs, \\ &\quad t\in \mathrm{J}=(t_i,t_{i+1}],\ i=0,\cdots,n.\end{aligned}$$

Thanks to Lemma 2.2, it's clear that the fixed points of the operator $\mathfrak{T}$ are solutions of the initial value problem (4.1).

Next, we will show that the operator $\mathfrak{T}$ is a contraction mapping on $\mathbb{PC}(\mathrm{J},\mathbb{R})$.
Let $y,z\in\mathbb{PC}(\mathrm{J},\mathbb{R})$ and for every $t\in\mathrm{J}$, we have:

$$\begin{aligned}|(\mathfrak{T}y)(t)-(\mathfrak{T}z)(t)| &= \Bigg|\sum_{i=1}^{n}\left(\mathscr{I}_i(y(t_i^-))-\mathscr{I}_i(z(t_i^-))\right) + \sum_{i=1}^{n}(t-t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right) \\ &\quad + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs \\ &\quad + \sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs \\ &\quad + \int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs\Bigg|.\end{aligned}$$

This implies that,

$$
\begin{aligned}
|(\mathfrak{T}y)(t)-(\mathfrak{T}z)(t)| \leq & \sum_{i=1}^{n}\left|\mathscr{I}_i(y(t_i^-))-\mathscr{I}_i(z(t_i^-))\right|+\sum_{i=1}^{n}(t-t_i)\left|\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right| \\
&+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs \\
&+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs \\
&+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs.
\end{aligned}
$$

Using the hypotheses (P_2)-(P_3), we obtain:

$$
\begin{aligned}
|(\mathfrak{T}y)(t)-(\mathfrak{T}z)(t)| \leq & \mathfrak{L}_{\mathscr{I}_i}\sum_{i=1}^{n}\left|y(t_i^-)-z(t_i^-)\right|+\mathfrak{L}_{\overline{\mathscr{I}}_i}\sum_{i=1}^{n}(t-t_i)\left|y(t_i^-)-z(t_i^-)\right| \\
&+\mathfrak{L}_{\phi}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|y(s)-z(s)\right|d_qs \\
&+\mathfrak{L}_{\phi}\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left|y(s)-z(s)\right|d_qs \\
&+\mathfrak{L}_{\phi}\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|y(s)-z(s)\right|d_qs.
\end{aligned}
$$

According to the formula (1.43) and for every $t \in \mathrm{J}$, we find:

$$
\begin{aligned}
|(\mathfrak{T}y)(t)-(\mathfrak{T}z)(t)| \leq & n\mathfrak{L}_{\mathscr{I}_i}\|y-z\|_{\mathbb{PC}}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}\|y-z\|_{\mathbb{PC}}+\frac{n\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|y-z\|_{\mathbb{PC}} \\
&+\frac{n\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}\|y-z\|_{\mathbb{PC}}+\frac{\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\|y-z\|_{\mathbb{PC}}.
\end{aligned}
$$

So,

$$
\|\mathfrak{T}(y)-\mathfrak{T}(z)\|_{\mathbb{PC}} \leq \left(n\mathfrak{L}_{\mathscr{I}_i}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}+\frac{n\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+(n+1)\frac{\mathfrak{L}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\right)\|y-z\|_{\mathbb{PC}}.
$$

Thus, from the condition (4.8), the operator $\mathfrak{T}$ is a contraction, and according to Banach contraction principle theorem, we conclude that the operator $\mathfrak{T}$ has a unique fixed point, which is the unique solution to the initial value problem (4.1). ■

3.2 Existence Result

This part discusses the existence of solutions to the initial value problem (4.1), through the use of Krasnoselskii's fixed point theorem (Theorem 5.5).

Theorem 3.2 *Assume that the hypotheses* (P_1), (P_3) *and* (P_4), (P_5) *are satisfied. If*

$$n\left(\mathfrak{L}_{\mathscr{I}_i} + \mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}\right) < 1. \tag{4.9}$$

Then, the initial value problem (4.1) has at least one solution on J.

Proof. To illustrate this result, we will employ Krasnoselskii's fixed point theorem. Firstly, we consider the set:

$$\mathfrak{B}_{\xi} = \{z \in \mathbb{PC}(\mathrm{J}, \mathbb{R}) : \|z\|_{\mathbb{PC}} \leq \xi\},$$

where

$$\xi \geq (n+1)\frac{\mathcal{M}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)} + \frac{n\mathcal{M}_{\phi}\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)} + |z_0| + |z_0^*|\mathrm{T} + n\left(\mathcal{M}_{\mathscr{I}_i} + \mathcal{M}_{\overline{\mathscr{I}}_i}\mathrm{T}\right).$$

Also, we define the following operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on $\mathfrak{B}_{\xi}$:

$$\begin{aligned}
(\mathfrak{T}_1 z)(t) &= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s, z(s))d_q s + \sum_{i=1}^{n}(t - t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s, z(s))d_q s \\
&\quad + \int_{t_i}^{t}\frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s, z(s))d_q s. \\
(\mathfrak{T}_2 z)(t) &= z_0 + z_0^* t + \sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^{n}(t - t_i)\overline{\mathscr{I}}_i(z(t_i^-)).
\end{aligned}$$

Next, we will give the proof in steps.

Step 1: $\mathfrak{T}_1 y + \mathfrak{T}_2 z \in \mathfrak{B}_{\xi}$ for any $y, z \in \mathfrak{B}_{\xi}$.

Let $y, z \in \mathfrak{B}_{\xi}$ and for each $t \in \mathrm{J}$, then we have:

$$\begin{aligned}
|(\mathfrak{T}_1 y)(t) + (\mathfrak{T}_2 z)(t)| &= \Bigg|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s, y(s))d_q s + \sum_{i=1}^{n}(t - t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
&\quad \times\phi(s, y(s))d_q s + \int_{t_i}^{t}\frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s, y(s))d_q s + z_0 + z_0^* t \\
&\quad + \sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^{n}(t - t_i)\overline{\mathscr{I}}_i(z(t_i^-))\Bigg|.
\end{aligned}$$

This means that,

$$\begin{aligned}|(\mathfrak{T}_1 y)(t)+(\mathfrak{T}_2 z)(t)| &\leq \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))\right| d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\\ &\times\left|\phi(s,y(s))\right| d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))\right| d_qs+|z_0|+|z_0^*|t\\ &+\sum_{i=1}^{n}\left|\mathscr{I}_i(z(t_i^-))\right|+\sum_{i=1}^{n}(t-t_i)\left|\overline{\mathscr{I}}_i(z(t_i^-))\right|.\end{aligned}$$

By hypotheses (P_4)- (P_5), we get:

$$\begin{aligned}|(\mathfrak{T}_1 y)(t)+(\mathfrak{T}_2 z)(t)| &\leq \mathscr{M}_\phi\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs+\mathscr{M}_\phi\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}d_qs\\ &+\mathscr{M}_\phi\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs+|z_0|+|z_0^*|t+\sum_{i=1}^{n}\mathscr{M}_{\mathscr{I}_i}+\sum_{i=1}^{n}\mathscr{M}_{\overline{\mathscr{I}}_i}(t-t_i).\end{aligned}$$

Using the formula (1.43) and for each $t\in \mathrm{J}$, we find:

$$|(\mathfrak{T}_1 y)(t)+(\mathfrak{T}_2 z)(t)| \leq \frac{n\mathscr{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+\frac{n\mathscr{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+\frac{\mathscr{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+|z_0|+|z_0^*|\mathrm{T}+n\mathscr{M}_{\mathscr{I}_i}+n\mathscr{M}_{\overline{\mathscr{I}}_i}\mathrm{T}.$$

Thus,

$$\begin{aligned}\|\mathfrak{T}_1 y+\mathfrak{T}_2 z\|_{\mathbb{PC}} &\leq (n+1)\frac{\mathscr{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+\frac{n\mathscr{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+|z_0|+|z_0^*|\mathrm{T}+n\left(\mathscr{M}_{\mathscr{I}_i}+\mathscr{M}_{\overline{\mathscr{I}}_i}\mathrm{T}\right),\\ &\leq \xi.\end{aligned}$$

Hence, $\mathfrak{T}_1 y+\mathfrak{T}_2 z\in\mathfrak{B}_\xi$ for any $y,z\in\mathfrak{B}_\xi$.

Step 2: $\mathfrak{T}_1$ is a continuous and compact operator.

Now, we shall show that the operator $\mathfrak{T}_1$ is continuous.

Let $\{z_m\}_{m\in\mathbb{N}}$ be a sequence such that $z_m\to z$ in $\mathfrak{B}_\xi$. Then, for every $t\in \mathrm{J}$, we have:

$$\begin{aligned}|(\mathfrak{T}_1 z_m)(t)-(\mathfrak{T}_1 z)(t)| &\leq \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z_m(s))-\phi(s,z(s))\right| d_qs\\ &+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left|\phi(s,z_m(s))-\phi(s,z(s))\right| d_qs\\ &+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z_m(s))-\phi(s,z(s))\right| d_qs.\end{aligned}$$

By formula (1.43) and for each $t\in \mathrm{J}$, we obtain:

$$\|\mathfrak{T}_1(z_m)-\mathfrak{T}_1(z)\|_{\mathbb{PC}} \leq \left((n+1)\frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+\frac{n\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}\right)\left\|\phi(.,z_m(.))-\phi(.,z(.))\right\|_{\mathbb{PC}}.$$

Since ϕ is a continuous function, i.e.:

$$\|\phi(.,z_m(.))-\phi(.,z(.))\|_{\mathbb{PC}}\to 0 \quad as \quad m\to\infty.$$

Hence,

$$\|\mathfrak{T}_1(z_m)-\mathfrak{T}_1(z)\|_{\mathbb{PC}}\to 0 \ \ as \ \ m\to\infty.$$

Thus, $\mathfrak{T}_1$ is a continuous operator on $\mathfrak{B}_\xi$.

Next, we will prove that the operator $\mathfrak{T}_1$ is uniformly bounded on $\mathfrak{B}_\xi$.
Let $z\in\mathfrak{B}_\xi$ and for every $t\in \mathrm{J}$, then we have:

$$\begin{aligned}
|(\mathfrak{T}_1 z)(t)| &= \Bigg|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_qs\\
&\quad+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\Bigg|,\\
&\leq \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z(s))\right|d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left|\phi(s,z(s))\right|d_qs\\
&\quad+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z(s))\right|d_qs.
\end{aligned}$$

Applying the hypothesis (P_4), we get:

$$\begin{aligned}
|(\mathfrak{T}_1 z)(t)| &\leq \mathcal{M}_\phi\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs+\mathcal{M}_\phi\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}d_qs\\
&\quad+\mathcal{M}_\phi\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}d_qs.
\end{aligned}$$

From the formula (1.43) and for each $t\in\mathrm{J}$, we find:

$$|(\mathfrak{T}_1 z)(t)| \leq \frac{n\mathcal{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+\frac{n\mathcal{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+\frac{\mathcal{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}.$$

So,

$$\|\mathfrak{T}_1(z)\|_{\mathbb{PC}} \leq (n+1)\frac{\mathcal{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}+\frac{n\mathcal{M}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}.$$

Thus, $\mathfrak{T}_1$ is an uniformly bounded operator on $\mathfrak{B}_\xi$.

Finally, we will show that the operator $\mathfrak{T}_1$ is equi-continuous.
Let $t_1, t_2 \in$ J such that $t_1 < t_2$ and for $z \in \mathfrak{B}_\xi$, then we have:

$$\begin{aligned}
|(\mathfrak{T}_1 z)(t_2) - (\mathfrak{T}_1 z)(t_1)| &= \Big|\sum_{i=1}^{n}(t_2 - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_q s + \int_{t_i}^{t_2} \frac{(t_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
&\quad \times \phi(s,z(s))d_q s - \sum_{i=1}^{n}(t_1 - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_q s \\
&\quad - \int_{t_i}^{t_1} \frac{(t_1 - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_q s\Big|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(\mathfrak{T}_1 z)(t_2) - (\mathfrak{T}_1 z)(t_1)| &\leq \sum_{i=1}^{n}(t_2 - t_1)\int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\left|\phi(s,z(s))\right| d_q s \\
&\quad + \int_{t_i}^{t_1} \frac{\left((t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)}\right)}{\Gamma_q(\beta)}\left|\phi(s,z(s))\right| d_q s \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,z(s))\right| d_q s.
\end{aligned}$$

By hypothesis (P_4), we obtain:

$$\begin{aligned}
|(\mathfrak{T}_1 z)(t_2) - (\mathfrak{T}_1 z)(t_1)| &\leq \mathscr{M}_\phi \sum_{i=1}^{n}(t_2 - t_1)\int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} d_q s \\
&\quad + \mathscr{M}_\phi \int_{t_i}^{t_1} \frac{\left((t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)}\right)}{\Gamma_q(\beta)} d_q s \\
&\quad + \mathscr{M}_\phi \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)} d_q s.
\end{aligned}$$

After calculating the integrals, we find:

$$|(\mathfrak{T}_1 z)(t_2) - (\mathfrak{T}_1 z)(t_1)| \leq \frac{\mathscr{M}_\phi}{\Gamma_q(\beta)}\sum_{i=1}^{n}(t_2 - t_1)(t_i - t_{i-1})^{(\beta-1)} + \frac{\mathscr{M}_\phi}{\Gamma_q(\beta+1)}\left[(t_2 - t_i)^{(\beta)} - (t_1 - t_i)^{(\beta)}\right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero, i.e.:

$$|(\mathfrak{T}_1 z)(t_2) - (\mathfrak{T}_1 z)(t_1)| \to 0 \quad as \quad t_1 \to t_2.$$

Thus, $\mathfrak{T}_1$ is an equi-continuous operator. So, $\mathfrak{T}_1$ is a relatively compact operator on $\mathfrak{B}_\xi$. Consequently, thanks to Arzela-Ascoli's theorem (Theorem 1.9), we conclude that the operator $\mathfrak{T}_1$ is completely continuous on $\mathfrak{B}_\xi$.

Step 3: $\mathfrak{T}_2$ is a contraction operator on $\mathfrak{B}_\xi$.
Let $y, z \in \mathfrak{B}_\xi$ and for every $t \in \mathrm{J}$, then we have:

$$|(\mathfrak{T}_2 y)(t) - (\mathfrak{T}_2 z)(t)| = \left| \sum_{i=1}^{n} \left(\mathscr{I}_i(y(t_i^-)) - (\mathscr{I}_i(z(t_i^-))\right) + \sum_{i=1}^{n} (t - t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-)) - \overline{\mathscr{I}}_i(z(t_i^-))\right)\right|.$$

This impulse that,

$$|(\mathfrak{T}_2 y)(t) - (\mathfrak{T}_2 z)(t)| \leq \sum_{i=1}^{n} \left|\mathscr{I}_i(y(t_i^-)) - (\mathscr{I}_i(z(t_i^-))\right| + \sum_{i=1}^{n} (t - t_i)\left|\overline{\mathscr{I}}_i(y(t_i^-)) - \overline{\mathscr{I}}_i(z(t_i^-))\right|.$$

Thanks to the hypothesis (P_3), we obtain:

$$|(\mathfrak{T}_2 y)(t) - (\mathfrak{T}_2 z)(t)| \leq \mathfrak{L}_{\mathscr{I}_i} \sum_{i=1}^{n} \left|y(t_i^-) - z(t_i^-)\right| + \mathfrak{L}_{\overline{\mathscr{I}}_i} \sum_{i=1}^{n} (t - t_i)\left|y(t_i^-) - z(t_i^-)\right|.$$

Then, for every $t \in \mathrm{J}$, we get:

$$|(\mathfrak{T}_2 y)(t) - (\mathfrak{T}_2 z)(t)| \leq n\mathfrak{L}_{\mathscr{I}_i} \|y - z\|_{\mathbb{PC}} + n\mathfrak{L}_{\overline{\mathscr{I}}_i} \mathrm{T} \|y - z\|_{\mathbb{PC}}.$$

Thus,

$$\|\mathfrak{T}_2(y) - \mathfrak{T}_2(z)\|_{\mathbb{PC}} \leq n\left(\mathfrak{L}_{\mathscr{I}_i} + \mathfrak{L}_{\overline{\mathscr{I}}_i} \mathrm{T}\right) \|y - z\|_{\mathbb{PC}}.$$

Hence, from the condition (4.9), the operator $\mathfrak{T}_2$ is contraction on $\mathfrak{B}_\xi$.
Consequently, according to Krasnoselskii's fixed point theorem, we deduce that the operator $\mathfrak{T}$ has at least one fixed point which is the solution to the initial value problem (4.1).
■

3.3 Ulam Stability Results

In this part, we will define and investigate different types of Ulam stability for the initial value problem (4.1), through the use of Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Based on the references [1, 6, 23, 46, 56, 82, 83, 91], we provide the following definitions:

Definition 3.3 *The problem (4.1) is Ulam-Hyers stable if there exists a real number* $\eta > 0$, *such that for every* $\epsilon > 0$ *and for every solution* $y \in \mathbb{PC}(\mathrm{J}, \mathbb{R})$ *of the following inequality:*

$$|({}^{\mathrm{C}}\mathfrak{D}_q^{\beta} y)(t) - \phi(t, y(t))| \leq \epsilon, \quad 1 < \beta \leq 2,\ t \in \mathrm{J},\ t \neq t_i,\ i = 1, \cdots, n, \tag{4.10}$$

there exists a solution $z \in \mathbb{PC}(\mathrm{J}, \mathbb{R})$ *of the problem (4.1) with the norm*

$$\|y - z\|_{\mathbb{PC}} \leq \eta\epsilon.$$

Definition 3.4 *The problem (4.1) is generalized Ulam-Hyers stable if there exists* $\kappa \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\kappa(0) = 0$*, such that for every* $\epsilon > 0$ *and for every solution* $y \in \mathbb{PC}(J, \mathbb{R})$ *of the inequality (4.10), there exists a solution* $z \in \mathbb{PC}(J, \mathbb{R})$ *of the problem (4.1) with the norm*

$$\| y - z \|_{\mathbb{PC}} \leq \kappa(\epsilon).$$

Definition 3.5 *The problem (4.1) is Ulam-Hyers-Rassias stable with respect to* σ *if there exists* $\eta_\sigma > 0$*, such that for every* $\epsilon > 0$ *and for every solution* $y \in \mathbb{PC}(J, \mathbb{R})$ *of the following inequality:*

$$|({}^{C}\mathfrak{D}_q^{\beta} y)(t) - \phi(t, y(t))| \leq \epsilon\sigma(t), \quad 1 < \beta \leq 2,\ t \in J,\ t \neq t_i,\ i = 1, \cdots, n, \tag{4.11}$$

there exists a solution $z \in \mathbb{PC}(J, \mathbb{R})$ *of the problem (4.1) with the norm*

$$\| y - z \|_{\mathbb{PC}} \leq \eta_\sigma \epsilon\sigma(t),\ t \in J.$$

Remark 3.6 *Clearly : Definition 3.3* $\Rightarrow$ *Definition 3.4.*

Following that, we introduce the main results of Ulam stabilities for the initial value problem (4.1).

Theorem 3.7 *Suppose that the hypotheses* (P_1)-(P_2)-(P_3) *and condition (4.8) hold. Then, the initial value problem (4.1) is Ulam-Hyers stable.*

Proof. Let $y \in \mathbb{PC}(J, \mathbb{R})$ be a solution of the inequality (4.10) and let $z \in \mathbb{PC}(J, \mathbb{R})$ be the unique solution of the initial value problem (4.1). Then, according to Lemma 2.2, we give:

$$\begin{aligned} z(t) &= z_0 + z_0^* t + \sum_{i=1}^{n} \mathscr{I}_i(z(t_i^-)) + \sum_{i=1}^{n} (t - t_i)\overline{\mathscr{I}}_i(z(t_i^-)) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \phi(s, z(s)) d_q s \\ &\quad + \sum_{i=1}^{n} (t - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta - 1)} \phi(s, z(s)) d_q s + \int_{t_i}^{t} \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \phi(s, z(s)) d_q s, \\ &\quad t \in J_i = (t_i, t_{i+1}],\ i = 0, \cdots, n. \end{aligned}$$

Through integration of the inequality (4.10) and for every $t \in J$, we find:

$$\begin{aligned} \left| \begin{array}{c} y(t) - z_0 - z_0^* t - \sum_{i=1}^{n} \mathscr{I}_i(y(t_i^-)) - \sum_{i=1}^{n} (t - t_i)\overline{\mathscr{I}}_i(y(t_i^-)) \\ - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \phi(s, y(s)) d_q s - \int_{t_i}^{t} \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} \phi(s, y(s)) d_q s \\ - \sum_{i=1}^{n} (t - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \phi(s, y(s)) d_q s \end{array} \right| &\leq \mathfrak{I}_q^{\beta} \epsilon, \\ &\leq \frac{t^{(\beta)}}{\Gamma_q(\beta + 1)} \epsilon. \end{aligned}$$

Then, we can write:

$$
\begin{aligned}
|y(t)-z(t)| \leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(z(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,z(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_qs \\
& -\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\Big|, \\
\leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(y(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(y(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,y(s))d_qs-\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs+\sum_{i=1}^{n}\left(\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right)+\sum_{i=1}^{n}(t-t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right) \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs\Big|, \\
\leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(y(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(y(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,y(s))d_qs-\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs\Big|+\Big|\sum_{i=1}^{n}\left(\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right)+\sum_{i=1}^{n}(t-t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right) \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs\Big|.
\end{aligned}
$$

Therefore, for each $t \in \mathrm{J}$, we get:

$$
\begin{aligned}
|y(t)-z(t)| \leq\ & \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}\epsilon+\sum_{i=1}^{n}\left|\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right|+\sum_{i=1}^{n}(t-t_i)\left|\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right| \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs.
\end{aligned}
$$

Thank to hypotheses (P_2)-(P_3) and for every $t \in \mathrm{J}$, we obtain:

$$
\|y-z\|_{\mathbb{PC}} \leq \frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\epsilon+\left(n\mathfrak{L}_{\mathscr{I}_i}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}+\frac{n\mathfrak{L}_\phi\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+(n+1)\frac{\mathfrak{L}_\phi\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\right)\|y-z\|_{\mathbb{PC}}.
$$

From the condition (4.8), we find:

$$\begin{aligned}\|y-z\|_{\mathbb{PC}} &\leq \frac{\frac{\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}}{1-\left(n\mathfrak{L}_{\mathscr{I}_i}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}+\frac{n\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+(n+1)\frac{\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\right)}\epsilon,\\ &:= \eta\epsilon.\end{aligned}$$

Thus, the initial value problem (4.1) is Ulam-Hyers stable. ■

Corollary 3.8 *If we take* $\kappa(\epsilon)=\eta\epsilon$; $\kappa(0)=0$, *we conclude that the initial value problem (4.1) is generalized Ulam-Hyers stable.*

Theorem 3.9 *Suppose that the hypotheses* (P_1)-(P_2)-(P_3) *and condition (4.8) are satisfied and the following hypothesis holds:*

(P_6) *Let* $\sigma \in C(\mathrm{J},\mathbb{R}_+)$ *be an increasing function. There exists* $\lambda_\sigma > 0$, *such that for every* $t \in \mathrm{J}$, *we have:*

$$\mathfrak{I}_q^{\beta}\sigma(t) \leq \lambda_\sigma \sigma(t).$$

Then, the initial value problem (4.1) is Ulam-Hyers-Rassias stable.

Proof. Let $y \in \mathbb{PC}(\mathrm{J},\mathbb{R})$ be a solution of the inequality (4.11) and let $z \in \mathbb{PC}(\mathrm{J},\mathbb{R})$ be the unique solution of the initial value problem (4.1). Then, from Lemma 2.2, we have:

$$\begin{aligned} z(t) &= z_0+z_0^*t+\sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-))+\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(z(t_i^-))+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\\ &\quad+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs,\\ &\quad t\in \mathrm{J}_i=(t_i,t_{i+1}],\ i=0,\cdots,n.\end{aligned}$$

By integration of the inequality (4.11) and for every $t \in \mathrm{J}$, we get:

$$\begin{aligned}\left|\begin{array}{c} y(t)-z_0-z_0^*t-\sum_{i=1}^{n}\mathscr{I}_i(y(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(y(t_i^-))\\ -\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,y(s))d_qs-\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,y(s))d_qs\\ -\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,y(s))d_qs\end{array}\right| &\leq \mathfrak{I}_q^{\beta}\epsilon\sigma(t),\\ &\leq \epsilon\lambda_\sigma\sigma(t).\end{aligned}$$

Therefore, we can write:

$$
\begin{aligned}
|y(t)-z(t)| \leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(z(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(z(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,z(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,z(s))d_qs \\
& -\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\phi(s,z(s))d_qs\Big|, \\
\leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(y(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(y(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,y(s))d_qs-\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs+\sum_{i=1}^{n}\left(\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right)+\sum_{i=1}^{n}(t-t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right) \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs\Big|, \\
\leq\ & \Big| y(t)-z_0-z_0^* t-\sum_{i=1}^{n}\mathscr{I}_i(y(t_i^-))-\sum_{i=1}^{n}(t-t_i)\overline{\mathscr{I}}_i(y(t_i^-))-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs-\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)}\phi(s,y(s))d_qs-\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} \\
& \times\phi(s,y(s))d_qs\Big|+\Big|\sum_{i=1}^{n}\left(\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right)+\sum_{i=1}^{n}(t-t_i)\left(\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right) \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left(\phi(s,y(s))-\phi(s,z(s))\right)d_qs\Big|.
\end{aligned}
$$

Then, for every $t\in \mathrm{J}$, we obtain:

$$
\begin{aligned}
|y(t)-z(t)| \leq\ & \epsilon\lambda_\sigma\sigma(t)+\sum_{i=1}^{n}\left|\mathscr{I}_i(y(t_i^-))-(\mathscr{I}_i(z(t_i^-))\right|+\sum_{i=1}^{n}(t-t_i)\left|\overline{\mathscr{I}}_i(y(t_i^-))-\overline{\mathscr{I}}_i(z(t_i^-))\right| \\
& +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs+\sum_{i=1}^{n}(t-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\beta-2)}}{\Gamma_q(\beta-1)} \\
& \times\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs+\int_{t_i}^{t}\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}\left|\phi(s,y(s))-\phi(s,z(s))\right|d_qs.
\end{aligned}
$$

Applying the hypotheses (P_2)-(P_3) and for each $t\in \mathrm{J}$, we find:

$$
\|y-z\|_{\mathbb{PC}} \leq \epsilon\lambda_\sigma\sigma(t)+\left(n\mathfrak{L}_{\mathscr{I}_i}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}+\frac{n\mathfrak{L}_\phi\mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+(n+1)\frac{\mathfrak{L}_\phi\mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\right)\|y-z\|_{\mathbb{PC}}.
$$

By condition (4.8), we get:

$$\begin{aligned} \|y-z\|_{\mathbb{PC}} &\leq \frac{\epsilon\lambda_\sigma\sigma(t)}{1-\left(n\mathfrak{L}_{\mathscr{I}_i}+n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}+\frac{n\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)}+(n+1)\frac{\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)}\right)}, \\ &:= \eta_\sigma\epsilon\sigma(t). \end{aligned}$$

Thus, the initial value problem (4.1) is Ulam-Hyers-Rassias stable. ■

4 An Example

Consider the following initial value problem for impulsive fractional q-difference equation:

$$\begin{cases} \left({}^{\mathrm{C}}\mathfrak{D}_{1/6}^{4/3}z\right)(t)=\frac{\cos(z(t))}{6\ln(3t+6)}; \quad t\in \mathrm{J}=[0,1],\ t\neq\frac{1}{3}, \\ \Delta z\,|_{t=\frac{1}{3}}=\frac{\cos(z(\frac{1}{3}^-))}{9}, \\ \Delta z'\,|_{t=\frac{1}{3}}=\frac{\sin(z(\frac{1}{3}^-))}{6}, \\ z(0)=0, \quad z'(0)=0, \end{cases} \tag{4.12}$$

where $q=\frac{1}{6}$, $\beta=\frac{4}{3}$, $z_0=z_0^*=0$, $n=1$, $\mathrm{T}=1$ and

$$\phi(t,z)=\frac{\cos(z)}{6\ln(3t+6)}; \quad (t,z)\in \mathrm{J}\times\mathbb{R},$$

and

$$\mathscr{I}_i(z)=\frac{\cos(z)}{9}, \quad \overline{\mathscr{I}}_i(z)=\frac{\sin(z)}{6}; \quad z\in\mathbb{R}.$$

Obviously, the functions ϕ and $\mathscr{I}_i,\overline{\mathscr{I}}_i$ are continuous.
Let $y,z\in\mathbb{R}$ and $t\in \mathrm{J}=[0,1]$. Then, we have:

$$\begin{aligned} |\phi(t,y)-\phi(t,z)| &\leq \frac{1}{6\ln(3t+6)}|\cos(y)-\cos(z)|, \\ &\leq \frac{1}{6\ln(6)}|y-z|, \end{aligned}$$

and

$$|\mathscr{I}_i(y)-\mathscr{I}_i(z)|\leq\frac{1}{9}|y-z|,$$

$$|\overline{\mathscr{I}}_i(y)-\overline{\mathscr{I}}_i(z)|\leq\frac{1}{6}|y-z|.$$

Therefore, the hypotheses (P_2)-(P_3) are satisfied with $\mathfrak{L}_\phi=\frac{1}{6\ln(6)}$ and $\mathfrak{L}_{\mathscr{I}_i}=\frac{1}{9}$, $\mathfrak{L}_{\overline{\mathscr{I}}_i}=\frac{1}{6}$.

Now, we will confirm that the condition (4.8) holds with $n=1, \mathrm{T}=1$. In fact,

$$\begin{aligned} n\mathfrak{L}_{\mathscr{I}_i} + n\mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T} + \frac{n\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta)} + (n+1)\frac{\mathfrak{L}_\phi \mathrm{T}^{(\beta)}}{\Gamma_q(\beta+1)} &= \frac{1}{9} + \frac{1}{6} + \frac{1}{6\ln(6)\Gamma_{\frac{1}{6}}\left(\frac{4}{3}\right)} + \frac{2}{6\ln(6)\Gamma_{\frac{1}{6}}\left(\frac{7}{3}\right)}, \\ &\simeq 0.5524 < 1. \end{aligned}$$

Hence, thanks to Theorem 3.1, the initial value problem (4.12) has a unique solution on $[0,1]$, and all the conditions of Theorem 3.7 hold, thus, the initial value problem (4.12) is Ulam-Hyers stable.

Next, let $\sigma(t) = t^2$ for every $t \in \mathrm{J} = [0,1]$, we have:

$$\mathfrak{I}_{1/6}^{4/3}\sigma(t) = \frac{\Gamma_{1/6}(3)}{\Gamma_{1/6}(\frac{13}{3})}t^{2+\frac{4}{3}} \leq \frac{\Gamma_{1/6}(3)}{\Gamma_{1/6}(\frac{13}{3})}t^2 = \lambda_\sigma\sigma(t). \tag{4.13}$$

So, the hypothesis (P_6) holds with $\sigma(t) = t^2$ and $\lambda_\sigma = \frac{\Gamma_{1/6}(3)}{\Gamma_{1/6}(\frac{13}{3})}$. Then, all the conditions of Theorem 3.9 are satisfied, thus, the initial value problem (4.12) is Ulam-Hyers-Rassias stable.

On the other hand, let $z \in \mathbb{R}$ and for each $t \in \mathrm{J} = [0,1]$, then we have:

$$|\phi(t,z)| \leq \frac{1}{6\ln(6)},$$

and

$$|\mathscr{I}_i(z)| \leq \frac{1}{9}, \quad |\overline{\mathscr{I}}_i(z)| \leq \frac{1}{6}.$$

Therefore, the hypotheses (P_4)-(P_5) are satisfied with $\mathscr{M}_\phi = \frac{1}{6\ln(6)}$ and $\mathscr{M}_{\mathscr{I}_i} = \frac{1}{9}$, $\mathscr{M}_{\overline{\mathscr{I}}_i} = \frac{1}{6}$.

Next, we will verify that the condition (4.9) holds with $n=1, \mathrm{T}=1$. In effect,

$$n\left(\mathfrak{L}_{\mathscr{I}_i} + \mathfrak{L}_{\overline{\mathscr{I}}_i}\mathrm{T}\right) = \frac{1}{9} + \frac{1}{6} = \frac{15}{54} < 1.$$

Thus, all the conditions of Theorem 3.2 hold. Consequently, the initial value problem (4.12) has at least one solution on $[0,1]$.

5 Conclusion

In this work, we have gave sufficient conditions for the existence of solutions to the initial value problem for impulsive fractional q-difference equations involving Caputo's fractional q-derivative of order $\beta \in (1,2]$. Thus, we were able to obtain the results of the existence and uniqueness of solutions to the initial value problem (4.1) by applying some fixed point theorems (Banach contraction principle, Krasnoselskii). Additionally, we have defined and examines the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the initial value problem (4.1). To support our results, we have provide an illustrative example.

Conclusion and Perspectives

In this project, our primary scientific contributions have focused on providing sufficient conditions for the existence, uniqueness and stability of solutions to boundary value problems for fractional q-difference equations (order $\beta \in (0,1]$ and $\beta \in (1,2]$) and initial value problem for impulsive fractional q-difference equations involving Caputo's fractional q-derivative. Consequently, we obtained the existence results using various fixed point theorems (Banach, Schaefer, Krasnoselskii, Non-linear alternative of Leray-Schauder) and Mönch's fixed point theorem combined with the notion of Kuratowski's measures of noncompactness. In addition, we have discussed the stability results by applying Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

For the perspective and future research, it would be interesting to expand on the findings of the thesis by considering fractional q-difference inclusions and systems of fractional q-difference equations. Also, we will apply numerical methods to solve problems of fractional q-difference equations and take into account their applications in various fields of science and engineering.

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