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# STUDY OF PROPERTIES OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE COMPLEX PLANE

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**Speciality :** Functional Analysis

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THESIS

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# Abstract

In this thesis, we are interested in studying the growth of solutions of higher-order linear differential equations, specifically focusing on conditions on coefficients under which the solutions of these equations are of infinite order.

Firstly, we investigate the iterated order and iterated type of solutions of these equations where their coefficients are entire and meromorphic functions.

Secondly, we study the hyper-order of analytic solutions of linear differential equations whose coefficients are analytic near an isolated singular point. We also consider the non-homogeneous case.

Finally, we use a new idea to estimate the growth of solutions of linear differential equations. We consider the coefficients of these equations as solutions of certain second-order linear differential equations.
**Key words:** Nevanlinna theory, Linear diffrential equation, Meromorphic function, entire function, order of growth, an isolated singuler point.

**Résumé**

Dans cette thèse, nous nous intéressons à l'étude de la croissance des solutions des équations différentielles linéaires d'ordre supérieur, en nous concentrant spécifiquement sur les conditions portant sur les coefficients pour lesquelles les solutions de ces équations sont d'ordre infini.

Tout d'abord, nous étudions l'ordre itératif et le type itératif des solutions de ces équations lorsque leurs coefficients sont des fonctions entières et méromorphes.

Ensuite, nous étudions l'hyper-ordre des solutions analytiques des équations différentielles linéaires dont les coefficients sont analytiques au voisinage d'un point singulier isolé. Nous considérons également le cas non homogène.

Enfin, nous utilisons une nouvelle approche pour estimer la croissance des solutions des équations différentielles linéaires. Nous considérons les coefficients de ces équations comme des solutions de certaines équations différentielles linéaires du second ordre.

**Mots-clés :** Théorie de Nevanlinna, Équation différentielle linéaire, Fonction méromorphe, Fonction entière, Ordre de croissance, Point singulier isolé.

**الملخص:**
في هذه الرسالة، نهتم بدراسة تزايد الحلول للمعادلات التفاضلية الخطية من الرتب العالية، مركزين بشكل خاص على الشروط المتعلقة بالمعاملات التي تجعل حلول هذه المعادلات ذات تزايد لانهائي.
أولاً، ندرس الرتبة التكرارية والنوع التكراري لحلول هذه المعادلات عندما تكون معاملاتها دوال تحليلية وميرومورفية.
ثانياً، ندرس الرتبة الثانية التكرارية للحلول التحليلية للمعادلات التفاضلية الخطية التي تكون معاملاتها تحليلية في المستوى المركب المغلق باستثناء نقطة شاذة معزولة. ننظر أيضاً إلى الحالة غير المتجانسة.
وأخيراً، نستخدم نهجًا جديدًا لتقدير تزايد الحلول للمعادلات التفاضلية الخطية. نعتبر المعاملات في هذه المعادلات كحلول لبعض المعادلات التفاضلية الخطية من الرتبة الثانية.
**الكلمات المفتاحية:** نظرية نيفلينا، معادلات تفاضلية خطية، دالة ميرومورفية, دالة تحليلية، رتبة التزايد، نقطة شاذة معزولة.

# Introduction

Nevanlinna theory of value distribution is concerned with the density of points where a meromorphic function takes on a certain value in the complex plane. This theory plays a very important role in the study of the growth and oscillation of solutions of linear differential equations with complex coefficient functions.

The studies of the following linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z)$ are entire functions, have been continuously pursued over the years from various perspectives. Gundersen [21] shows that that if $f \not\equiv 0$ is a finite order solution of (1), where the growth of $A(z)$ dominates the growth of $B(z)$ in some angle, then $f$ will satisfy certain growth conditions in the angle. In [6], Hamani and Belaïdi generalized the result of Gundersen to the higher order linear differential equation and Belaïdi in [4] extended the result to the nonhomogeneous linear differential equation. In the same paper, Gundersen treated the equation (1) with conditions that contrasted those of the first result, where $B(z)$ dominates $A(z)$, and concluded that every nontrivial solution $f$ is of infinite order. Kwon in [30] addressed estimating the lower bound for the order of infinite-order solutions of (1), while Chen and Yang [12] provided a precise estimate for the hyper-order of solutions of (1). Under similar conditions, Belaïdi [7] treated the higher-order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{2}$$

where $k \geq 2$ is an integer and $A_0(z), \ldots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. In [53], Zemirni and Belaïdi extended the result by considering the $p$-iterated order and $p$-iterated type. They also explored the case when the coefficients $A_j(z)(j = 1, \ldots, k-1)$ are meromorphic functions. Later, different approaches are used to study equation (1). One involves extremal functions. It is assumed that either $A(z)$ is extremal for Yang's inequality (see [39, 35]) or $B(z)$ is extremal for Denjoy's conjecture [38]. The second approach, as discussed in [50], assumes that the coefficient $A(z)$ itself is a solution of another second-order linear differential equation of the form

$$w'' + P(z)w = 0, \tag{3}$$

where $P(z) = a_n z^n + ... + a_0, a_n \neq 0$. This assumption yields stability in the behavior of $A(z)$ via Hill's classical method of asymptotic integration. In this case, $A(z)$ is a special function, of which the Airy integral is one example. A combination of these two approaches was also discussed in [51]. Very recent papers have employed new ideas to solve the same problem, such as considering two coefficients $A(z)$ and $B(z)$ as solutions of (3) as seen in [41].

The linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{4}$$

where $A(z)$ and $B(z)$ are entire functions, $a$ and $b$ are complex numbers, has been extensively studied by various authors [1, 10, 11, 31, 32]. Kwon [31] proved that if $a$ and $b$ are complex numbers satisfying $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$

with $0 < c < 1$, then every nontrivial solution $f$ of equation (4) is of infinite order. In [23], Hamouda proved results similar to those in [31] in the unit disc concerning the differential equation

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^{\mu}}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^{\mu}}\right\} f = 0,$$

where $\mu > 0$ and $z_0, a, b$ are complex numbers such that $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Additionally, Fettouch and Hamouda [17] investigated the counterpart of these results near an isolated singular point $z_0$ for equations of the form:

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^{n}}\right\} f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^{n}}\right\} f = 0, \tag{5}$$

where $A(z), B(z) \not\equiv 0$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}, n \in \mathbb{N}^*$. Under certain conditions, they proved that every solution $f \not\equiv 0$ of (5) that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, is of infinite order and of hyper-order equal to $n$. In [14], Cherief and Hamouda extended the above results to the higher-order linear differential equation

$$f^{(k)} + A_{k-1}(z)\exp\left\{\frac{a_{k-1}}{(z_0 - z)^{n}}\right\} f^{(k-1)} + \ldots + A_0(z)\exp\left\{\frac{a_0}{(z_0 - z)^{n}}\right\} f = 0, \tag{6}$$

where $k \geq 2$ is an integer and $A_j(z)$ $(j = 0, \ldots, k-1)$ are analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ and $a_j (j = 0, ..., k-1)$ are complex numbers, $n \in \mathbb{N}^*$ . Under similar conditions, the conclusion in this case is that every solution $f \not\equiv 0$ of (6), that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ is of infinite order.

This thesis aims to study the growth of solutions of homogeneous and nonhomogeneous higher-order linear differential equations with entire and meromorphic functions in two different domains: the entire complex plane and the closed complex plane except for an isolated singular point.

The first chapter covers the fundamental concepts and key results related to the Nevanlinna theory of meromorphic functions, both in the complex plane and near an isolated singular point.

The second chapter generalizes the results given by Belaïdi and Zemirni in [53] by replacing the coefficient $A_0(z)$ by an arbitrary coefficient $A_s(z)$, where $s = 1, \ldots, k-1$, for higher-order linear differential equation of the form (2). This generalization is examined for both cases of entire and meromorphic coefficients.

The third chapter improves upon the previous results presented by Cherief and Hamouda [14] by estimating the hyper-order of solutions of equations of the form (6). Additionally, exploration of nonhomogeneous linear differential equations is conducted.

The fourth chapter investigates the growth of analytic solutions of the linear differential equation (6). Under certain conditions, it is proven that these solutions are of infinite order and hyper-order equal to $n$. Additionally, consideration of nonhomogeneous linear differential equations is made.

In the last chapter, we study the growth of solutions of higher-order linear differential equations in which certain coefficients are non-trivial solutions of second-order linear differential equations of the form (3).

Chapter 1

# Preliminaries

In this chapter, we present the basic definitions and properties of Nevanlinna theory used in this thesis. For more detailed information, readers can refer to [5, 25, 32].

## 1 Nevanlinna's notions in the complex plane

### 1.1 Functions and Concepts

There are basic functions, which define the whole Nevanlinna theory. we will define them successively :

**Definition 1.1** *[32] For any strictly positive real number $x$, we define $\log^+ x$ by*

$$\log^+ x = \max\{0, \log x\}.$$

The positive logarithmic function satisfies the following properties

- $\log x \leq \log^+ x$ .
- $\log^+ x \leq \log^+ y$, for $x \leq y$ .

- $\log x = \log^+ x - \log^+ \frac{1}{x}$ .

- $|\log x| = \log^+ x + \log^+ \frac{1}{x}$ .

- $\log^+ \left( \prod_{i=1}^{n} x_i \right) \leq \sum_{i=1}^{n} \log^+ x_i$ .

- $\log^+ \left( \sum_{i=1}^{n} x_i \right) \leq \sum_{i=1}^{n} \log^+ x_i + \log n, n \in \mathbb{N}^*$,

where $x > 0$, $y > 0$ and $x_i > 0 (i = 1, ..., n)$.

**Definition 1.2** *[25, 32] Let $f$ be a meromorphic function. For any complex number $a$, we define the counting function by*

$$N(r, a, f) = N(r, \frac{1}{f - a}) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r$$

*and*

$$N(r, \infty, f) = N(r, f) = \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r,$$

*where $n(r, a, f), n(r, \infty, f)$ respectively denotes the number of zeros of $f - a$ and the number of poles of $f$ according to its multiplicities in the disc $|z| \leq r$.*

We have the following properties :

- $N\Big(r, \sum_{i=1}^{n} f_i\Big) \leq \sum_{i=1}^{n} N(r, f_i)$.

- $N\Big(r, \prod_{i=1}^{n} f_i\Big) \leq \sum_{i=1}^{n} N(r, f_i)$, where $f_i$ are meromorphic functions and $n \in \mathbb{N}^*$.

**Definition 1.3** *[25, 32] Let $f$ be a meromorphic function. For any complex number $a$, we define the proximity function by*

$$m(r, a, f) = m\Big(r, \frac{1}{f - a}\Big) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \Big| \frac{1}{f(re^{i\phi}) - a} \Big| d\phi$$

*and*

$$m(r, \infty, f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi.$$

We have the following properties :

- $m\big(r, \sum_{i=1}^{n} f_i\big) \leq \sum_{i=1}^{n} m(r, f_i) + \log n.$
- $m\big(r, \prod_{i=1}^{n} f_i\big) \leq \sum_{i=1}^{n} m(r, f_i)$, where $f_i$ are meromorphic functions and $n \in \mathbb{N}^*$.

**Definition 1.4** *[32] The characteristic function of a meromorphic function $f$ is given by :*

$$T(r, f) = m(r, f) + N(r, f).$$

We have the following properties :

- $T\big(r, \sum_{i=1}^{n} f_i\big) \leq \sum_{i=1}^{n} T(r, f_i) + \log n.$
- $T\big(r, \prod_{i=1}^{n} f_i\big) \leq \sum_{i=1}^{n} T(r, f_i)$, where $f_i$ are meromorphic functions and $n \in \mathbb{N}^*$.

**Example 1.1** *For the function $f(z) = e^{az}, a \in \mathbb{C}^*$, we have $N(r, f) = 0$ and $m(r, f) = \frac{|a|}{\pi} r$. Then $T(r, f) = \frac{|a|}{\pi} r$.*

**Definition 1.5** *[8, 29] The order of growth of a meromorphic function $f$ is defined by*

$$\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r}.$$

*If $f$ is an entire function, then the order of $f$ is defined by*

$$\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log M(r, f)}{\log r},$$

*where $M(r, f) = \max_{|z|=r} |f(z)|$.*

**Example 1.2** *For the function $f(z) = e^z$, we have $T(r, f) = \frac{r}{\pi}$ and $M(r, f) = e^r$. Then $\sigma(f) = 1$.*

**Definition 1.6** *[41] Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote*

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$$

$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \cap \{z : |z| < r\}$$

*Let $\bar{F}$ denote the closure of $F$. Let $A(z)$ be an entire function of order $\sigma(A) \in (0, \infty)$. For simplicity, set $\sigma = \sigma(A)$ and $S = S(\alpha, \beta)$. We say that $A(z)$ blows up exponentially in $\bar{S}$ if for any $\theta \in (\alpha, \beta)$, the relation*

$$\lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \sigma$$

*holds. We also say that $A(z)$ decays to zero exponentially in $\bar{S}$ if for any $\theta \in (\alpha, \beta)$ the relation*

$$\lim_{r \to \infty} \frac{\log \log |A(re^{i\theta})|^{-1}}{\log r} = \sigma$$

*holds.*

**Definition 1.7** *[9] The type of a meromorphic function $f$, where $0 < \sigma(f) < \infty$ is defined by*

$$\tau(f) = \limsup_{r \to +\infty} \frac{T(r, f)}{r^{\sigma(f)}}.$$

*If $f$ is an entire function, then the type of $f$ is given by*

$$\tau_M(f) = \limsup_{r \to +\infty} \frac{\log M(r, f)}{r^{\sigma(f)}}.$$

**Example 1.3** *For the function $f(z) = e^z$, we have $\tau(f) = \frac{1}{\pi}$ and $\tau_M(f) = 1$.*

**Definition 1.8** *[29] The exponent of convergence of a sequence of the zeros of a meromorphic function $f$ is defined by*

$$\lambda(f) = \limsup_{r \to +\infty} \frac{\log N(r, \frac{1}{f})}{\log r}.$$

*Similarly, the exposant of convergence of a sequence of the poles of $f$ is defined by*

$$\lambda\left(\frac{1}{f}\right) = \limsup_{r\to+\infty} \frac{\log N(r,f)}{\log r},$$

**Example 1.4** *For the function $f(z) = e^z + b,\ b \in \mathbb{C}^*$, we have $\lambda(f) = 1$.*

**Definition 1.9** *[42, 45, 47, 48] Let $g(z)$ be a meromorphic function and let $\arg z = \theta \in \mathbb{R}$ be a ray from the origin. We denote, for each $\varepsilon > 0$, the exponent of convergence of the zero sequence of $g(z)$ at the ray $\arg z = \theta$ by $\lambda_{\theta,\varepsilon}(g)$ and by $\lambda_\theta(g) = \lim_{\varepsilon\to 0^+} \lambda_{\theta,\varepsilon}(g)$, where*

$$\lambda_\theta(g) = \lim_{\varepsilon\to 0^+} \limsup_{r\to+\infty} \frac{\log^+ n_{\theta-\varepsilon,\theta+\varepsilon}(r,0,g)}{\log r},$$

*here $n_{\alpha,\beta}(r,0,f)$ is the number of zeros of $f$ counting multiplicity in $\{z : \alpha < \arg z < \beta\} \cap \{|z| < r\}$.*

**Definition 1.10** *[41] We call the ray $\arg z = \theta$ which has the property $\lambda_\theta(g) = \sigma(g)$ an accumulation ray of the zero sequence of a meromorphic function $g$.*

**Definition 1.11** *[41]Let $w(z)$ be a non-trivial solution of equation $w'' + P(z)w = 0$, where $P(z) = a_n z^n + ... + a_0, a_n \neq 0$. We denote $p(w)$ the number of rays $\arg \theta_j$, which are not accumulation rays of the zero sequence of $w(z)$, where $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}, j = 0, 1, ..., n+1$.*

**Definition 1.12** *[52] For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of $a$ with respect to a meromorphic function $f$ is defined as follows :*

$$\delta(a,f) = \liminf_{r\to+\infty} \frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}, \quad \text{for} \quad a \in \mathbb{C}$$

*and*

$$\delta(\infty,f) = \liminf_{r\to+\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to+\infty} \frac{N(r,f)}{T(r,f)}.$$

Let $\mathbb{N} = \{0, 1, 2, ...\}$ denotes the set of natural numbers. Let us define inductively for $r \in \mathbb{R}$, $\exp_0 r := r$, $\exp_1 r := e^r$, and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all $r \in (0, +\infty)$ sufficiently large, we define $\log_0 r := r$, $\log_1 r := \log r$, and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r := \log r$ and $\log_{-1} r := \exp r$.

**Definition 1.13** *[8, 29] For $p \in \mathbb{N} - \{0\}$, the iterated $p$-order $\sigma_p(f)$ of a meromorphic function $f$ is defined by*

$$\sigma_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r, f)}{\log r},$$

*If $f$ is an entire function, then the iterated $p$-order of $f$ is defined by*

$$\sigma_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log_{p+1} M(r, f)}{\log r}.$$

**Example 1.5** *For the function $f(z) = e^{e^z}$, we have $\sigma_2(f) = 1$.*

**Definition 1.14** *[29] The finiteness degree of the order of a meromorphic function $f$ is defined by*

$$i(f) = \begin{cases} 0, & \textit{for } f \textit{ polynomial,} \\ \min\{j \in \mathbb{N} : \sigma_j(f) < +\infty\}, & \textit{for } f \textit{ transcendental for which} \\ & \textit{some } j \in \mathbb{N} \textit{ with } \rho_j(f) < +\infty \textit{ exists,} \\ +\infty, & \textit{for } f \textit{ with } \sigma_j(f) = +\infty \textit{ for all } j \in \mathbb{N}. \end{cases}$$

**Definition 1.15** *[9] For $p \in \mathbb{N} - \{0\}$, the iterated $p$-type of a meromorphic function $f$, where $0 < \sigma_p(f) < \infty$ is defined by*

$$\tau_p(f) = \limsup_{r \to +\infty} \frac{\log_{p-1} T(r, f)}{r^{\sigma_p(f)}}.$$

*If $f$ is an entire function, then the iterated $p$-type of $f$ is given by*

$$\tau_{M,p}(f) = \limsup_{r \to +\infty} \frac{\log_p M(r, f)}{r^{\sigma_p(f)}}.$$

**Definition 1.16** *[29] For $p \in \mathbb{N} - \{0\}$, the iterated $p$-exponent of convergence of a sequence of the zeros of a meromorphic function $f$ is defined by*

$$\lambda_p(f) = \limsup_{r \to +\infty} \frac{\log_p N(r, \frac{1}{f})}{\log r}.$$

*Similarly, the iterated $p$-exponent of convergence of a sequence of the poles of $f$ is defined by*

$$\lambda_p\left(\frac{1}{f}\right) = \limsup_{r \to +\infty} \frac{\log_p N(r, f)}{\log r}.$$

## 1.2 Measures

**Definition 1.17** *[25, 32] Let $E \subset (0, \infty)$ be a set and $\chi_E$ the characteristic function of $E$. The linear measure of $E$ is defined by*

$$m(E) = \int_0^{+\infty} \chi_E(t)dt.$$

**Definition 1.18** *[25, 32] Let $E \subset (1, \infty)$ be a set. The logarithmic measure of $E$ is defined by*

$$lm(E) = \int_1^{+\infty} \frac{\chi_E(t)}{t}dt.$$

## 1.3 Wiman Valiron Theorem

**Definition 1.19** *[26, 32]Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. The maximum term of $f$ is defined by*

$$\mu(r) = \mu(r, f) = \max_{m \geq 0}\{|a_m|r^m\}.$$

**Definition 1.20** *[26, 32] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. The central index of $f$ is defined by*

$$V(r) = V(r, f) = \max_{m \geq 0}\{m : |a_m|r^m = \mu(r, f)\}.$$

**Theorem 1.1** *[26, 32] Let $f$ be a transcendental entire function. Then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all $j \in \mathbb{N}$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\Big(\frac{V(r)}{z_r}\Big)^j$$

*as $r \to +\infty$, $r \notin E$, where $z_r$ is a point on the circle $|z| = r$ that satisfies $|f(z_r)| = M(r, f) = \max_{|z|=r} |f(z)|$.*

# 2 Nevanlinna's notions near an isolated Singular Point

In this section, we give some definitions which are also important in studying the growth and value distribution of meromorphic functions near an isolated singular point $z_0 \in \mathbb{C}$.

## 2.1 Functions and Concepts

**Definition 1.21** *[17]Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and let $f$ be a meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. We define the counting function near $z_0$ by*

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n_{z_0}(t, f) - n_{z_0}(\infty, f)}{t} dt - n_{z_0}(\infty, f) \log r,$$

*where $n_{z_0}(t, f)$ denotes the number of poles of $f$ in the region $\{z \in \mathbb{C} : t \leq |z_0 - z|\} \cup \{\infty\}$, each pole according to its multiplicity.*

**Definition 1.22** *[17]Let $f$ be a meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. We define the proximity function near $z_0$ by*

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f(z_0 - re^{i\phi})\right| d\phi.$$

**Definition 1.23** *[17] Let $f$ be a meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. We define the characteristic function near $z_0$ by*

$$T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f).$$

**Definition 1.24** *[17] Let $f$ be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. The order of $f$ near $z_0$ is defined by*

$$\sigma_T(f, z_0) = \limsup_{r \to 0} \frac{\log^+ T_{z_0}(r, f)}{-\log r}.$$

*For an analytic function $f$ in $\overline{\mathbb{C}}\backslash\{z_0\}$, we have also the definition*

$$\sigma_M(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{-\log r},$$

*where $M_{z_0}(r, f) = \max_{|z_0 - z| = r} |f(z)|$.*

**Definition 1.25** *[17]Let $f$ be a meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. The hyper-order of $f$ near $z_0$ is defined by*

$$\sigma_{2,T}(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ T_{z_0}(r, f)}{-\log r}.$$

*For an analytic function $f$ in $\overline{\mathbb{C}}\backslash\{z_0\}$, we have also the definition*

$$\sigma_{2,M}(f, z_0) = \limsup_{r \to 0} \frac{\log^+ \log^+ \log^+ M_{z_0}(r, f)}{-\log r}.$$

**Remark 1.1** *It is shown in [17] that if $f(z)$ is a non-constant meromorphic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ and $g(\omega) = f(z_0 - \frac{1}{\omega})$, then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have*

$$T(R, g) = T_{z_0}(\frac{1}{R}, f),$$

*where $R > 0$ and so $\sigma_T(f, z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $g(\omega)$ is entire and thus, $\sigma_T(f, z_0) = \sigma_M(f, z_0)$ and $\sigma_{2,T}(f, z_0) = \sigma_{2,M}(f, z_0)$. Then we can use the notations $\sigma(f, z_0)$ and $\sigma_2(f, z_0)$ without any ambiguity.*

**Example 1.6** *For $f(z) = \exp\left\{\frac{a}{(z_0 - z)^k}\right\}$, where $a \in \mathbb{C}^*$, $z_0 \in \mathbb{C}$ and $k \in \mathbb{N}^*$, we have $\sigma(f, z_0) = k$.*

## 2.2 Wiman Valiron Theorem

**Definition 1.26** *[24] Let $f$ be an analytic function $\overline{\mathbb{C}}\backslash\{z_0\}$ such that $f(z) = \sum_{n=0}^{+\infty} \frac{a_n}{(z-z_0)^n}$. Then for all given $|z_0 - z| = r > 0$, the maximum term of $f$ is defined by*

$$\mu_{z_0}(r) = \mu_{z_0}(r, f) = \max_{m\geq 0}\Big\{\frac{|a_m|}{r^m}\Big\}.$$

**Definition 1.27** *[24] Let $f$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. The central index of $f$ is defined by*

$$V_{z_0}(r) = V_{z_0}(r, f) = \max_{m\geq 0}\Big\{m : \frac{|a_m|}{r^m} = \mu_{z_0}(r, f)\Big\}.$$

**Theorem 1.2** *[24] Let $f$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then there exists a set $E \subset (0, 1)$ that has finite logarithmic measure, that is $\int_0^1 \frac{\chi_E(t)}{t}dt < +\infty$, such that for all $j \in \mathbb{N}$, we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\Big(\frac{V_{z_0}(r)}{z_0 - z_r}\Big)^j$$

*as $r \to 0$, $r \notin E$, where $z_r$ is a point on the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = M_{z_0}(r, f) = \max_{|z_0-z|=r} |f(z)|$.*

# 3 Key Theorems

In this section, we present the central theorems utilized throughout this thesis.

## 3.1 The First Fundamental Theorem

**Theorem 1.3** *[25] Let $f$ be a non constant meromorphic function and $a \in \mathbb{C}$. Then*

$$T(R, \frac{1}{f-a}) = T(R, f) + O(1)$$

*as $R \longrightarrow +\infty$.*

## 3.2 Phragmén-Lindelöf Theorem

Let $\alpha > \frac{1}{2}$. We set

$$S_\alpha = \{z \in \mathbb{C} : -\frac{\pi}{2\alpha} < \arg z < \frac{\pi}{2\alpha}\}$$

$$\gamma_r = \{r : z = re^{i\theta}, z \in S_\alpha\} \quad \textsf{and} \quad M(r, \gamma_r, f) = \max_{r \in \gamma_r} |f(z)|.$$

**Theorem 1.4** *[43] Let $f$ be an analytic function in $S_\alpha$ and continious in $\partial S_\alpha$ such that*

$$|f(z)| \leq M, \; \forall z \in \partial S_\alpha,$$

*where $M(>0)$ is a constant.*
*If*

$$\limsup_{r \to +\infty} \frac{\log \log M(r, \gamma_r, f)}{\log r} < \alpha,$$

*then*

$$|f(z)| \leq M, \; \forall z \in S_\alpha.$$

## 3.3 Liouville's Theorem

In complex analysis, Liouville's theorem is one of the immediate consequences of Cauchy's integral formula. This theorem is given as follows :

**Theorem 1.5** *If $f$ is a bounded entire function, then $f$ is constant.*

Chapter 2

# Solutions of Complex Linear Differential Equations With Fast-Growing Coefficients

## 1 Introduction and Main Results

In this chapter, we investigate the fast growth of solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \qquad (2.1)$$

where ($k \geq 2$) is an integer, the coefficients $A_j$ are entire or meromorphic functions in the complex plane. In [53], Zemirni and Belaïdi have estimated the iterated $p$-order and iterated $p$-type of solutions of (2.1) and obtained the following results :

**Theorem 2.1** *[53] Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be entire functions satisfying $\max\{\sigma_p(A_j) : j = 1, ..., k-1\} \leq \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$ and $\max\{\tau_p(A_j) : j = 1, ..., k-1\} \leq \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$ for $p \in \mathbb{N} - \{0, 1\}$. Suppose that there exist two* *positive real numbers* $\alpha$ *and* $\beta$ *with* $0 \leq \beta < \alpha$, *such that*

$$|A_0(z)| \geq \exp_{p-1}(\alpha e^{\tau r^\sigma})$$

*and*

$$|A_j(z)| \leq \exp_{p-1}(\beta e^{\tau r^\sigma}),\ j = 1, ..., k-1$$

*as* $|z| = r \to +\infty$ *for* $r \in E$ *(E is of infinite logarithmic measure). Then every solution* $f \not\equiv 0$ *of equation* (2.1) *satisfies* $\sigma_{p+1}(f) = \sigma$ *and* $\tau_{p+1}(f) = \tau$.

**Theorem 2.2** *[53] Let* $\{A_j(z)\}_{0\leq j\leq k-1}$ *be entire functions satisfying* $\max\{\sigma_p(A_j) : j = 1, ..., k-1\} \leq \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$ *and* $\max\{\tau_p(A_j) : j = 1, ..., k-1\} \leq \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$ *for* $p \in \mathbb{N} - \{0, 1\}$. *Suppose that there exist two positive real numbers* $\alpha$ *and* $\beta$ *with* $0 \leq \beta < \alpha$, *such that*

$$m(r, A_0) \geq \exp_{p-2}(\alpha e^{\tau r^\sigma})$$

*and*

$$m(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^\sigma}),\ j = 1, ..., k-1$$

*as* $|z| = r \to +\infty$ *for* $r \in E$ *(E is of infinite logarithmic measure). Then every solution* $f \not\equiv 0$ *of equation* (2.1) *satisfies* $\sigma_{p+1}(f) = \sigma$ *and* $\tau_{p+1}(f) = \tau$.

**Theorem 2.3** *[53] Let* $\{A_j(z)\}_{0\leq j\leq k-1}$ *be meromorphic functions satisfying* $\delta(\infty, A_0) = \delta > 0$, $\max\{\sigma_p(A_j) : j = 1, ..., k-1\} \leq \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$ *and* $\max\{\tau_p(A_j) : j = 1, ..., k-1\} \leq \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$ *for* $p \in \mathbb{N} - \{0, 1\}$. *Suppose that there exist two positive real numbers* $\alpha$ *and* $\beta$ *with* $0 \leq \beta < \alpha$, *such that*

$$T(r, A_0) \geq \exp_{p-2}(\alpha e^{\tau r^\sigma})$$

*and*

$$T(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^\sigma}),\ j = 1, ..., k-1$$

*as* $|z| = r \to +\infty$ *for* $r \in E$ *(E is of infinite logarithmic measure ). Then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities of equation* (2.1) *satisfies* $\sigma_{p+1}(f) = \sigma$ *and* $\tau_{p+1}(f) = \tau$.

**Theorem 2.4** *[53] Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be meromorphic functions satisfying $\lambda_p\left(\frac{1}{A_0}\right) < \sigma_p(A_0) = \sigma$, $\max\{\sigma_p(A_j) : j = 1, ..., k-1\} \leq \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$ and $\max\{\tau_p(A_j) : j = 1, ..., k-1\} \leq \tau_p(A_0) = \tau$ $(0 < \tau < +\infty)$ for $p \in \mathbb{N} - \{0, 1\}$. Suppose that there exist two positive real numbers $\alpha$ and $\beta$ with $0 \leq \beta < \alpha$, such that*

$$T(r, A_0) \geq \exp_{p-2}(\alpha e^{\tau r^{\sigma}})$$

*and*

$$T(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^{\sigma}}),\ j = 1, ..., k-1$$

*as $|z| = r \to +\infty$ for $r \in E$ ($E$ is of infinite logarithmic measure ). Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of equation (2.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.*

We continue to consider the above results by considering an arbitrary coefficient $A_s(z)$ $(1 \leq s \leq k-1)$ instead of $A_0(z)$. We will prove the following results:

**Theorem 2.5** *Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that there exists $s \in \{1, ..., k-1\}$ satisfying $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ and $\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau$ $(0 < \tau < +\infty)$ for $p \in \mathbb{N} - \{0, 1\}$. Suppose that there exist two positive real numbers $\alpha$ and $\beta$ with $0 \leq \beta < \alpha$, such that*

$$|A_s(z)| \geq \exp_{p-1}(\alpha e^{\tau r^{\sigma}}) \tag{2.2}$$

*and*

$$|A_j(z)| \leq \exp_{p-1}(\beta e^{\tau r^{\sigma}}),\ j \neq s \tag{2.3}$$

*as $|z| = r \to +\infty$ for $r \in E$ ($E$ is of infinite logarithmic measure). Then every transcendental solution $f$ of equation (2.1) satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.*

**Theorem 2.6** *Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that there exists $s \in \{1, ..., k-1\}$ satisfying $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ and*

$\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau \ (0 < \tau < +\infty)$ *for* $p \in \mathbb{N} - \{0, 1\}$. *Suppose that there exist two positive real numbers* $\alpha$ *and* $\beta$ *with* $0 \leq \beta < \alpha$, *such that*

$$m(r, A_s) \geq \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \tag{2.4}$$

*and*

$$m(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^{\sigma}}), \ j \neq s \tag{2.5}$$

*as* $|z| = r \to +\infty$ *for* $r \in E$ *(*$E$ *is of infinite logarithmic measure). Then every transcendental solution* $f$ *of equation* (2.1) , *in which* $f^{(n)}(z)$ *just has finite many zeros for all* $n < s \, (n = 0, ..., s-1)$, *satisfies* $\sigma_{p+1}(f) = \sigma$ *and* $\tau_{p+1}(f) = \tau$.

When the coefficients $\{A_j(z)\}_{j=0,1...,k-1}$ are meromorphic functions, we obtain the following two results :

**Theorem 2.7** *Let* $\{A_j(z)\}_{0 \leq j \leq k-1}$ *be meromorphic functions such that there exists* $s \in \{1, ..., k-1\}$ *satisfying* $\delta(\infty, A_s) = \delta > 0$, $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ *and* $\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau \ (0 < \tau < +\infty)$ *for* $p \in \mathbb{N} - \{0, 1\}$. *Suppose that there exist two positive real numbers* $\alpha$ *and* $\beta$ *with* $0 \leq \beta < \alpha$, *such that*

$$T(r, A_s) \geq \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \tag{2.6}$$

*and*

$$T(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^{\sigma}}), \ j \neq s \tag{2.7}$$

*as* $|z| = r \to +\infty$ *for* $r \in E$ *(*$E$ *is of infinite logarithmic measure). Then every transcendental meromorphic solution* $f$ *whose poles are of uniformly bounded multiplicities of equation* (2.1), *in which* $f^{(n)}(z)$ *just has finite many zeros for all* $n < s \, (n = 0, ..., s-1)$, *satisfies* $\sigma_{p+1}(f) = \sigma$ *and* $\tau_{p+1}(f) = \tau$.

**Theorem 2.8** *Let* $\{A_j(z)\}_{0 \leq j \leq k-1}$ *be meromorphic functions such that there exists* $s \in \{1, ..., k-1\}$ *satisfying* $\lambda_p(\frac{1}{A_s}) < \sigma_p(A_s) = \sigma$, $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ *and* $\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau \ (0 < \tau < +\infty)$

*for $p \in \mathbb{N} - \{0, 1\}$. Suppose that there exist two positive real numbers $\alpha$ and $\beta$ with $0 \leq \beta < \alpha$, such that*

$$T(r, A_s) \geq \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \tag{2.8}$$

*and*

$$T(r, A_j) \leq \exp_{p-2}(\beta e^{\tau r^{\sigma}}), \ j \neq s \tag{2.9}$$

*as $|z| = r \to +\infty$ for $r \in E$ ($E$ is of infinite logarithmic measure). Then every transcendental meromorphic solution $f$ whose poles are of uniformly bounded multiplicities of equation (2.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s$ $(n = 0, ..., s-1)$, satisfies $\sigma_{p+1}(f) = \sigma$ and $\tau_{p+1}(f) = \tau$.*

**Remark 2.1** *The proofs of Theorems 2.6, 2.7 and 2.8 are quite different from the proofs of Theorems 2.2, 2.3 and 2.4 in which we have added an essential condition for every transcendental (entire) meromorphic solution $f$ of equation (2.1).*

# 2 Auxiliary Results

To avoid some problems of the exceptional sets, we need the following lemma.

**Lemma 2.1** *[2, 21] Let $\varphi : [0, +\infty) \longmapsto \mathbb{R}$ and $\psi : [0, +\infty) \longmapsto \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin F_1 \cup [0, 1]$, where $F_1 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists $R = R(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r \geq R$.*

**Lemma 2.2** *[53] Let $f$ be a transcendental meromorphic function with $\sigma_p(f) = \sigma < +\infty$ for some $p \in \mathbb{N} - \{0\}$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $F_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all $z$*

*satisfying* $|z| = r \notin F_2 \cup [0,1]$ *and for all integer* $j \geq 1$*, we have*

*1. If* $p = 1$*, then*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r^{i(\sigma-1+\varepsilon)}.$$

*2. If* $p \geq 2$*, then*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}(r^{\sigma+\varepsilon}).$$

**Lemma 2.3** *[20] Let* $f$ *be a transcendental meromorphic function, and let* $\mu > 1$ *be a given constant. Then there exists a set* $F_3 \subset (1, +\infty)$ *of finite logarithmic measure and a constant* $B > 0$ *that depends only on* $\mu$ *and* $i$*,* $j (j > i \geq 0)$*, such that for all* $z$ *satisfiying* $|z| = r \notin F_3 \cup [0, 1]$*, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B \left[\frac{T(\mu r, f)}{r} (\log^\mu r) \log T(\mu r, f)\right]^{j-i}.$$

**Lemma 2.4** *[25] Let* $f$ *be a meromorphic function and let* $k \in \mathbb{N}$*. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log T(r, f) + \log r\right),$$

*possibly outside of an exceptional set* $H_1 \subset (0, +\infty)$ *of finite linear measure.*

**Lemma 2.5** *[29] Let* $f$ *be a meromorphic function for which* $i(f) = p \geq 1$ *and* $\sigma_p(f) = \sigma$*, and let* $k \geq 1$ *be an integer. Then for any* $\varepsilon > 0$*, there holds*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}(r^{\sigma+\varepsilon})\right)$$

*outside of a possible set* $H_2$ *of finite linear measure.*

**Lemma 2.6** *[11] Let* $f$ *be a transcendental entire function. Then, there exists a set* $F_4 \subset (1, +\infty)$ *of finite logarithmic measure such that for all* $z$ *satisfying* $|z| = r \notin F_4 \cup [0, 1]$ *and* $|f(z)| = M(r, f)$*, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s,$$

*where $s \geq 1$ is an integer.*

**Lemma 2.7** *[8] Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that $0 < p < +\infty$ and $\max\{\sigma_p(A_j) : j = 0, 1, ..., k-1\} \leq \sigma < +\infty$. Then every solution $f \not\equiv 0$ of equation (2.1) satisfies $\sigma_{p+1}(f) \leq \sigma$.*

**Lemma 2.8** *[9] Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be meromorphic functions such that $0 < p < +\infty$ and $\max\{\sigma_p(A_j) : j = 0, 1, ..., k-1\} \leq \sigma < +\infty$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of equation (2.1) satisfies $\sigma_{p+1}(f) \leq \sigma$.*

By using similar arguments as in the proof of Lemma 2.5 and Lemma 2.7 in [53], we can obtain the following two lemmas :

**Lemma 2.9** *Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be entire functions such that $1 < p < +\infty$. Suppose that there exists $s \in \{1, ..., k-1\}$ such that $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ and $\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau$ $(0 < \tau < +\infty)$. Then every transcendental solution $f$ of equation (2.1) with $\sigma_{p+1}(f) = \sigma$ satisfies $\tau_{p+1}(f) \leq \tau$.*

**Lemma 2.10** *Let $\{A_j(z)\}_{0\leq j\leq k-1}$ be meromorphic functions such that $1 < p < +\infty$. Suppose that there exists $s \in \{1, ..., k-1\}$ such that $0 < \max\{\sigma_p(A_j) : j \neq s\} \leq \sigma_p(A_s) = \sigma < +\infty$ and $\max\{\tau_p(A_j) : j \neq s\} \leq \tau_p(A_s) = \tau$ $(0 < \tau < +\infty)$. Then every transcendental meromorphic solution $f$ whose poles are of uniformly bounded multiplicities of equation (2.1) with $\sigma_{p+1}(f) = \sigma$ satisfies $\tau_{p+1}(f) \leq \tau$.*

# 3 Proof of Main Results

**Proof of Theorem 2.5**. Let $f$ be a transcendental entire solution of equation (2.1). By (2.1), it follows that

$$|A_s(z)| \leq \left|\frac{f(z)}{f^{(s)}(z)}\right| \left(\left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_{s+1}(z)| \left|\frac{f^{(s+1)}(z)}{f(z)}\right|\right.$$

$$+|A_{s-1}(z)|\left|\frac{f^{(s-1)}(z)}{f(z)}\right|+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|\Bigg). \tag{2.10}$$

By Lemma 2.7, we know that $\sigma_{p+1}(f) \leq \sigma$. Suppose that $\sigma_{p+1}(f) = \sigma_1 < \sigma$. Then by Lemma 2.2, for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \sigma_1$, we have for $p \geq 1$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_p(r^{\sigma_1+\varepsilon}),\ \ j = 1, ..., k, \tag{2.11}$$

where $|z| = r \notin F_2 \cup [0, 1]$ and by Lemma 2.6, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s \tag{2.12}$$

for all $|z| = r \notin F_4 \cup [0, 1]$. By substituting (2.2), (2.3), (2.11) and (2.12) into (2.10), we obtain

$$\exp_{p-1}(\alpha e^{\tau r^\sigma}) \leq 2kr^s \exp_{p-1}(\beta e^{\tau r^\sigma}) \exp_p(r^{\sigma_1+\varepsilon})$$

for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \sigma_1$ and all $r \in E - (F_2 \cup F_4 \cup [0, 1])$. Hence, we get

$$(\alpha - \beta)e^{\tau r^\sigma} \leq e^{r^{\sigma_1+\varepsilon}} + \log_{p-1} r^s + C$$

which is a contradiction, since $\alpha > \beta$ and $\sigma > \sigma_1 + \varepsilon$, where $C$ is some positive constant. Thus $\sigma_{p+1}(f) = \sigma$.

Now, by Lemma 2.3 we have for $j = 1, ..., k$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B[T(2r, f)]^{j+1} \leq B[T(2r, f)]^{k+1} \tag{2.13}$$

for all $|z| = r \notin F_3 \cup [0, 1]$. By substituting (2.2), (2.3), (2.12) and (2.13) into (2.10), we obtain

$$\exp_{p-1}(\alpha e^{\tau r^\sigma}) \leq 2kBr^s \exp_{p-1}(\beta e^{\tau r^\sigma})[T(2r, f)]^{k+1}$$

for all $r \in E - (F_3 \cup F_4 \cup [0, 1])$. Hence

$$\log(\alpha - \beta) + \tau r^\sigma \leq \log_p T(2r, f) + \log_p r^s + C_1$$

for some constant $C_1 > 0$ and for all $r \in E - (F_3 \cup F_4 \cup [0,1])$. Then by Lemma 3.14, and because $0 < \sigma_{p+1}(f) = \sigma < +\infty$ we deduce that $\tau_{p+1}(f) \geq \tau$. By Lemma 2.9 we know that $\tau_{p+1}(f) \leq \tau$, and thus $\tau_{p+1}(f) = \tau$.

**Proof of Theorem 2.6**. Let $f$ be a transcendental entire solution of the equation (2.1) . If $s+1 \leq j \leq k$, we use the properties of the proximity function of Nevanlinna, we have

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq m\left(r, \frac{f^{(j)}}{f}\right) + m\left(r, \frac{f}{f^{(s)}}\right).$$

According to the definition of the counting function such that $f$ has just finite many zeros, we obtain

$$N\left(r, \frac{f^{(s)}}{f}\right) = O\left(\log r\right),$$

so from the first fundamental theorem of Nevanlinna, we have

$$m\left(r, \frac{f}{f^{(s)}}\right) \leq T\left(r, \frac{f}{f^{(s)}}\right) = T\left(r, \frac{f^{(s)}}{f}\right) + O\left(1\right)$$

$$= m\left(r, \frac{f^{(s)}}{f}\right) + N\left(r, \frac{f^{(s)}}{f}\right) + O(1) = m\left(r, \frac{f^{(s)}}{f}\right) + O\left(\log r\right),$$

then

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq m\left(r, \frac{f^{(j)}}{f}\right) + m\left(r, \frac{f^{(s)}}{f}\right) + O\left(\log r\right). \tag{2.14}$$

If $0 \leq j \leq s-1$, we use the first fundamental theorem of Nevanlinna, we obtain

$$T\left(r, \frac{f^{(j)}}{f^{(s)}}\right) = T\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + O\left(1\right) = m\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + N\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + O\left(1\right).$$

According to the definition of the counting function such that $f^{(j)}$ has just finite many zeros, we have

$$N\left(r, \frac{f^{(s)}}{f^{(j)}}\right) = O(\log r)$$

so

$$m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)\leq T\left(r,\frac{f^{(j)}}{f^{(s)}}\right)=m\left(r,\frac{f^{(s)}}{f^{(j)}}\right)+O(\log r). \tag{2.15}$$

It follows from (2.1) that

$$A_s(z)=-\left(\frac{f^{(k)}(z)}{f^{(s)}(z)}+A_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(s)}(z)}+\cdots+A_{s+1}(z)\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right.$$
$$\left.+A_{s-1}(z)\frac{f^{(s-1)}(z)}{f^{(s)}(z)}+\cdots+A_1(z)\frac{f'(z)}{f^{(s)}(z)}+A_0(z)\frac{f(z)}{f^{(s)}(z)}\right)$$

which implies

$$m(r,A_s)\leq\sum_{i\neq s}m(r,A_i)+\sum_{s+1\leq j\leq k}m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)+\sum_{0\leq j\leq s-1}m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)+O(1). \tag{2.16}$$

By Lemma 2.7, we know that $\sigma_{p+1}(f)\leq\sigma$. Suppose that $\sigma_{p+1}(f)=\sigma_1<\sigma$. Then by Lemma 2.5, for any given $\varepsilon$ with $0<\varepsilon<\sigma-\sigma_1$, and for all sufficiently large $|z|=r\notin H_2$, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\left(\exp_{p-1}\{r^{\sigma_1+\varepsilon}\}\right)\leq C_2\exp_{p-1}\{r^{\sigma_1+\varepsilon}\},\quad j=1,...,k-1, \tag{2.17}$$

where $C_2$ is some positive constant. Now let $p\geq 2$, by substituting (2.4), (2.5), (2.14), (2.15) and (2.17) into (2.16), it follows that

$$\exp_{p-2}(\alpha e^{\tau r^{\sigma}})\leq(k-1)\exp_{p-2}(\beta e^{\tau r^{\sigma}})+C_2\exp_{p-1}(r^{\sigma_1+\varepsilon})+O(\log r) \tag{2.18}$$

holds for any given $\varepsilon$ with $0<\varepsilon<\sigma-\sigma_1$ and all $z$ satisfying $|z|=r\in E-H_2$ as $r\to+\infty$. Hence from (2.18) we get

$$(\alpha-\beta)e^{\tau r^{\sigma}}\leq e^{r^{\sigma_1+\varepsilon}}+\log_{p-1}r+C_3$$

which is a contradiction as $|z|=r\to+\infty$, $r\in E-H_2$, since $\alpha>\beta$ and $\sigma>\sigma_1+\varepsilon$, where $C_3$ is some positive constant. Thus $\sigma_{p+1}(f)=\sigma$.

Now, by using Lemma 2.4 and substituting (2.4), (2.5), (2.14), (2.15) into (2.16), it follows that

$$\exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \leq (k-1)\exp_{p-2}(\beta e^{\tau r^{\sigma}}) + O(\log T(r,f) + \log r) + O(\log r) \quad (2.19)$$

for all $z$ satisfying large $|z| = r \in E - H_1$. It follows from (2.19) that

$$\exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \leq (k-1)\exp_{p-2}(\beta e^{\tau r^{\sigma}}) + O(\log T(r,f) + \log r)$$

for all sufficiently large $|z| = r \in E - H_1$. Then we obtain

$$\log(\alpha - \beta) + \tau r^{\sigma} \leq \log_p T(r,f) + \log_p r + C_4 \quad (2.20)$$

for all sufficiently large $|z| = r \in E - H_1$, where $C_4$ is some positive constant. Hence by (2.20) and Lemma 3.14, we get $\tau \leq \tau_{p+1}(f)$. On the other hand, by Lemma 2.9, we have $\tau_{p+1}(f) \leq \tau$. Hence $\tau_{p+1}(f) = \tau$.

**Proof of Theorem 2.7.** Let $f$ be a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of equation (2.1) , in which $f^{(n)}(z)$ just has finite many zeros for all $n < s\,(n = 0, ..., s-1)$. By Lemma 2.8, we know that $\sigma_{p+1}(f) \leq \sigma$. Suppose that $\sigma_{p+1}(f) = \sigma_1 < \sigma$. Set

$$\delta(\infty, A_s) = \liminf_{r\to+\infty} \frac{m(r,A_s)}{T(r,A_s)} = \delta > 0. \quad (2.21)$$

Thus from (2.21), we have for sufficiently large $r$

$$m(r,A_s) > \frac{1}{2}\delta T(r,A_s). \quad (2.22)$$

By substituting (2.14), (2.15), (2.17) and (2.22) into (2.16), we obtain

$$\frac{1}{2}\delta T(r,A_s) < m(r,A_s) \leq \sum_{i\neq s} m(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + O(1)$$

$$\leq \sum_{i\neq s} T(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + O(1)$$

$$\leq \sum_{i\neq s} T(r,A_i) + C_2 \exp_{p-1}(r^{\sigma_1+\varepsilon}) + O(\log r) \tag{2.23}$$

for any given $\varepsilon$ with $0<\varepsilon<\sigma-\sigma_1$ and all $z$ satisfying $|z|=r\in E-H_2$ as $r\to+\infty$. Now let $p\geq 2$. It follows by (2.6), (2.7) and (2.23) that

$$\frac{1}{2}\delta \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \leq (k-1)\exp_{p-2}(\beta e^{\tau r^{\sigma}}) + C_2 \exp_{p-1}(r^{\sigma_1+\varepsilon}) + O(\log r) \tag{2.24}$$

for any given $\varepsilon$ with $0<\varepsilon<\sigma-\sigma_1$ and all $z$ satisfying $|z|=r\in E-H_2$ as $r\to+\infty$. Hence from (2.24), we get

$$(\alpha-\beta)e^{\tau r^{\sigma}} \leq e^{r^{\sigma_1+\varepsilon}} + \log_{p-1} r + C_5$$

which is a contradiction as $|z|=r\to+\infty$, $r\in E-H_2$, since $\alpha>\beta$ and $\sigma>\sigma_1+\varepsilon$, where $C_5$ is some positive constant. Thus $\sigma_{p+1}(f)=\sigma$.
Now by using Lemma 2.4 and substituting (2.6), (2.7), (2.14), (2.15), (2.22) into (2.16), it follows that

$$\frac{1}{2}\delta \exp_{p-2}(\alpha e^{\tau r^{\sigma}}) \leq \frac{1}{2}\delta T(r,A_s) < m(r,A_s)$$

$$\leq \sum_{i\neq s} m(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + O(1)$$

$$\leq \sum_{i\neq s} T(r,A_i) + \sum_{s+1\leq j\leq k} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r,\frac{f^{(j)}}{f^{(s)}}\right) + O(1)$$

$$\leq (k-1)\exp_{p-2}(\beta e^{\tau r^{\sigma}}) + O(\log T(r,f) + \log r) \tag{2.25}$$

for all sufficiently large $|z|=r\in E-H_1$. Then

$$\log(\alpha-\beta) + \tau r^{\sigma} \leq \log_p T(r,f) + \log_p r + C_6 \tag{2.26}$$

for all sufficiently large $|z|=r\in E-H_1$, where $C_6$ is some positive constant. Hence by (2.26) and Lemma 3.14, we get $\tau\leq\tau_{p+1}(f)$. On the other hand, by

Lemma 2.10, we have $\tau_{p+1}(f) \leq \tau$. Hence $\tau_{p+1}(f) = \tau$.

**Proof of Theorem 2.8**. Let $f$ be a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of equation (2.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s\,(n = 0, ..., s-1)$. By Lemma 2.8, we know that $\sigma_{p+1}(f) \leq \sigma$. Suppose that $\sigma_{p+1}(f) = \sigma_1 < \sigma$. Then by Lemma 2.5, for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \sigma_1$ and for sufficiently large $|z| = r \notin H_2$, we have (2.17).

Now let $p \geq 2$, by substituting (2.14), (2.15), (2.17) into (2.16), we have

$$\begin{aligned} m(r, A_s) &\leq \sum_{i\neq s} m(r, A_i) + \sum_{s+1\leq j\leq k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + O(1) \\ &\leq \sum_{i\neq s} T(r, A_i) + \sum_{s+1\leq j\leq k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + O(1) \\ &\leq \sum_{i\neq s} T(r, A_i) + C_2 \exp_{p-1}(r^{\sigma_1+\varepsilon}) + O(\log r) \end{aligned} \tag{2.27}$$

holds for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \sigma_1$ and all $z$ satisfying $|z| = r \in E - H_2$ as $r \to +\infty$. Since $\lambda_p\left(\frac{1}{A_s}\right) < \sigma_p(A_s) = \sigma$, we have for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \lambda_p(\frac{1}{A_s})$ and sufficiently large $r$

$$N(r, A_s) \leq \exp_{p-1}(r^{\lambda_p(\frac{1}{A_s})+\varepsilon}). \tag{2.28}$$

By (2.8), (2.9), (2.27) and (2.28), for any given $\varepsilon$ with $0 < \varepsilon < \min\left\{\sigma - \lambda_p(\frac{1}{A_s}), \sigma - \sigma_1\right\}$ and all $z$ satisfying $|z| = r \in E - H_2$ as $r \to +\infty$, we obtain

$$\exp_{p-2}(\alpha e^{\tau r^\sigma}) \leq T(r, A_s) = m(r, A_s) + N(r, A_s)$$

$$\leq (k-1)\exp_{p-2}(\beta e^{\tau r^\sigma}) + C_2 \exp_{p-1}(r^{\sigma_1+\varepsilon}) + \exp_{p-1}(r^{\lambda_p(\frac{1}{A_s})+\varepsilon}) + O(\log r). \tag{2.29}$$

Hence from (2.29), we get

$$(\alpha - \beta)e^{\tau r^\sigma} \leq e^{r^{\sigma_1+\varepsilon}} + e^{r^{\lambda_p(\frac{1}{A_s})+\varepsilon}} + \log_{p-1} r + C_7,$$

which is a contradiction as $|z| = r \to +\infty$, $r \in E - H_2$, since $\alpha > \beta$ and $0 < \varepsilon < \min\left\{\sigma - \lambda_p(\frac{1}{A_s}), \sigma - \sigma_1\right\}$, where $C_7$ is some positive constant. Thus $\sigma_{p+1}(f) = \sigma$.
Now it follows by (2.8), (2.9), (2.16), (2.28) and Lemma 2.4 that

$$\exp_{p-2}(\alpha e^{\tau r^\sigma}) \leq T(r, A_s) = m(r, A_s) + N(r, A_s)$$

$$\begin{aligned}
&\leq \sum_{i\neq s} m(r, A_i) + \sum_{s+1\leq j\leq k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + N(r, A_s) + O(1) \\
&\leq \sum_{i\neq s} T(r, A_i) + \sum_{s+1\leq j\leq k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\leq j\leq s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + N(r, A_s) + O(1) \\
&\leq (k-1)\exp_{p-2}(\beta e^{\tau r^\sigma}) + \exp_{p-1}(r^{\lambda_p(\frac{1}{A_s})+\varepsilon}) + O(\log T(r, f) + \log r) \qquad (2.30)
\end{aligned}$$

for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \lambda_p(\frac{1}{A_s})$ and all sufficiently large $|z| = r \in E - H_1$. Then by (2.30), we obtain

$$\log(\alpha - \beta) + \tau r^\sigma \leq r^{\lambda_p(\frac{1}{A_s})+\varepsilon} + \log_p T(r, f) + \log_p r + C_8 \qquad (2.31)$$

for any given $\varepsilon$ with $0 < \varepsilon < \sigma - \lambda_p(\frac{1}{A_s})$ and all sufficiently large $|z| = r \in E - H_1$, where $C_8$ is some positive constant. Hence by (2.31) and Lemma 3.14, we get $\tau \leq \tau_{p+1}(f)$. On the other hand, by Lemma 2.10, we have $\tau_{p+1}(f) \leq \tau$. Hence $\tau_{p+1}(f) = \tau$.

Chapter 3

# On the Hyper-order of Analytic Solutions of Linear Differential Equations near an isolated Singular Point

## 1 Introduction and Main Results

The aim of this chapter is to investigate the hyper-order of analytic solutions of the following linear differential equations :

$$f^{(k)} + A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\} f^{(k-1)} + \cdots + A_0(z) \exp\left\{\frac{a_0}{(z_0 - z)^n}\right\} f = 0 \quad (3.1)$$

$$f^{(k)} + A_{k-1}(z) \exp\left\{\frac{a_{k-1}}{(z_0 - z)^n}\right\} f^{(k-1)} + \cdots + A_0(z) \exp\left\{\frac{a_0}{(z_0 - z)^n}\right\} f = F, \quad (3.2)$$

where $k \geq 2$ is an integer, $n \in \mathbb{N} \backslash \{0\}$ and $z_0, a_j (j = 0, ..., k - 1)$ are complex numbers, $A_0(z) \not\equiv 0, \ldots, A_{k-1}(z)$ and $F \not\equiv 0$ are analytic functions near an isolated singular point $z_0$ . In [17], Fettouch and Hamouda proved the following result :

**Theorem 3.1** *[17] Let $z_0$ $a$, $b$ be complex constants, such that $\arg a \neq \arg b$ or $a = cb$ with $(0 < c < 1)$ and $n \in \mathbb{N}\backslash\{0\}$. Let $A(z), B(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\max\{\sigma(A, z_0), \sigma(B, z_0)\} < n$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + A(z)\exp\left\{\frac{a}{(z_0 - z)^n}\right\}f' + B(z)\exp\left\{\frac{b}{(z_0 - z)^n}\right\}f = 0$$

*satisfies $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) = n$.*

Cherief and Hamouda have extended Theorem 3.1 to higher order linear differential equations and proved the following two results :

**Theorem 3.2** *[14] Let $n \in \mathbb{N}\backslash\{0\}, k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$ and $A_0(z) \not\equiv 0$. If $a_j(j = 0, ..., k-1)$ are distinct complex numbers, then every solution $f \not\equiv 0$ of the differential equation (3.1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, satisfies $\sigma(f, z_0) = +\infty$.*

**Theorem 3.3** *[14] Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$ and $A_0(z) \not\equiv 0$. Let $a_j(j = 0, ..., k-1)$ be complex constants. Suppose that there exist nonzero numbers $a_0$ and $a_s$, such that $0 < s \leq k-1$, $a_0 = |a_0|e^{i\theta_s}$,$a_s = |a_s|e^{i\theta_s}$,$\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0A_s \not\equiv 0$ and for $j \neq 0, s$, $a_j$ satisfies either $a_j = d_ja_0$ $(0 < d_j < 1)$ or $a_j = d_ja_s$ $(0 < d_j < 1)$. Then every solution $f \not\equiv 0$ of equation (3.1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, satisfies $\sigma(f, z_0) = +\infty$.*

We continue to consider these above theorems and investigate the hyper-order of analytic solutions of equation (3.1). We will prove the following results :

**Theorem 3.4** *Let $n \in \mathbb{N} \backslash \{0\}, k \geq 2$ be an integer and $A_j(z), a_j (j = 0, ..., k - 1)$ satisfy the additional hypotheses of Theorem 3.2. Then every non-constant solution $f$ of equation (3.1) that is analytic in $\overline{\mathbb{C}} \backslash \{z_0\}$ satisfies $\sigma_2(f, z_0) = n$, where $z_0$ is an essential singular point for $f$.*

**Example 3.1** *Consider the differential equation*

$$f''' + \frac{3}{z}(2 + \frac{1}{z})f'' - \frac{1}{z^4} \exp\left\{\frac{2}{z}\right\} f' - \frac{2}{z^4}(3 + \frac{3}{z} + \frac{1}{z^2}) \exp\left\{\frac{1}{z}\right\} f = 0. \tag{3.3}$$

*Obiouvisly, the conditions of Theorem 3.4 are satisfied. Hence every non-constant solution $f$ of equation (3.3) that is analytic $\overline{\mathbb{C}} \backslash \{0\}$ satisfies $\sigma_2(f, 0) = 1$, where $0$ is an essential singular point for $f$.*
*Remark that the function $f(z) = \exp\{\exp(\frac{1}{z})\}$ is a solution of equation (3.3) that is analytic in $\overline{\mathbb{C}} \backslash \{0\}$ with $\sigma_2(f, 0) = 1$.*

**Theorem 3.5** *Let $n \in \mathbb{N} \backslash \{0\}, k \geq 2$ be an integer and $A_j(z), a_j (j = 0, ..., k - 1)$ satisfy the additional hypotheses of Theorem 3.3. Then every non-constant solution $f$ of equation (3.1) that is analytic in $\overline{\mathbb{C}} \backslash \{z_0\}$ satisfies $\sigma_2(f, z_0) = n$, where $z_0$ is an essential singular point for $f$.*

**Theorem 3.6** *Let $n \in \mathbb{N} \backslash \{0\}, k \geq 2$ be an integer and $A_j(z), a_j (j = 0, ..., k - 1)$ satisfy hypotheses of Theorem 3.4 or those of Theorem 3.5 . Let $F \not\equiv 0$ be an analytic function in $\overline{\mathbb{C}} \backslash \{z_0\}$ of order $\sigma = \sigma(F, z_0) < n$. Then every solution $f$ of equation (3.2) that is analytic in $\overline{\mathbb{C}} \backslash \{z_0\}$ satisfies $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) = n$, with at most one exceptional analytic solution $f_0$ of finite order in $\overline{\mathbb{C}} \backslash \{z_0\}$, where $z_0$ is an essential singular point for $f_0$.*

# 2 Auxiliary Results

**Lemma 3.1** *[17] Let $f$ be a non-constant meromorphic function in $\overline{\mathbb{C}} \backslash \{z_0\}$. Let $\alpha > 0$ be a given real constant and $j \in \mathbb{N}$. Then there exists a set $E_1 \subset (0, 1)$ of* *finite logarithmic measure, that is* $\int_0^1 \chi_{E_1}(t)\frac{dt}{t} < +\infty$ *and a constant* $A > 0$ *that depends on* $\alpha$ *and* $j$*, such that for all* $r = |z - z_0|$ *satisfying* $r \notin E_1$*, we have*

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq A\left[\frac{1}{r^2}T_{z_0}(\alpha r, f)\log T_{z_0}(\alpha r, f)\right]^j,$$

*where* $\chi_{E_1}$ *is the characteristic function of the set* $E_1$*.*

**Lemma 3.2** *[32]Let* $g$ *be a transcendental entire function, let* $0 < \eta_1 < \frac{1}{4}$ *and* $\omega_R$ *be a point such that* $|\omega_R| = R$ *and* $|g(\omega_R)| > M(R, g)V(R)^{-\frac{1}{4}+\eta_1}$ *holds. Then there exists a set* $F_1 \subset (1, +\infty)$ *of finite logarithmic measure, such that*

$$\frac{g^{(j)}(\omega_R)}{g(\omega_R)} = \left(\frac{V(R)}{\omega_R}\right)^j (1 + o(1)) \ (j \in \mathbb{N})$$

*holds as* $R \to +\infty$ *and* $R \notin F_1$*, where* $V(R)$ *is the central index of* $g$ *and* $M(R, g) = \max_{|\omega|=R} |g(\omega)|$*.*

**Remark 3.1** *[24] If* $f$ *is a non-constant analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$*. Then the function* $g(\omega) = f(z_0 - \frac{1}{\omega})$ *is entire in* $\mathbb{C}$ *and* $V_{z_0}(r) = V(R)$*, where* $R = \frac{1}{r}$*,* $R > 0$*,* $V(R)$ *is the central index of* $g$ *in* $\mathbb{C}$ *and* $V_{z_0}(r)$ *is the central index of* $f$ *near the singular point* $z_0$*.*

By using Lemma 3.2, remark 3.1 and similar arguments as in the proof of theorem 8 in [24], we can obtain the following Lemma :

**Lemma 3.3** *Let* $f$ *be a non-constant analytic function in* $\overline{\mathbb{C}}\backslash\{z_0\}$*. Let* $0 < \eta_1 < \frac{1}{4}$ *and* $z_r$ *be a point such that* $|z_0 - z_r| = r$ *and* $|f(z_r)| > M_{z_0}(r, f)V_{z_0}(r)^{-\frac{1}{4}+\eta_1}$ *holds. Then there exists a set* $E_2 \subset (0, 1)$ *of finite logarithmic measure, such that*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j (1 + o(1)) \ (j \in \mathbb{N})$$

*holds as* $r \to 0$*,* $r \notin E_2$*, where* $V_{z_0}(r)$ *is the central index of* $f$ *near* $z_0$*,* $z_0$ *is an essential singular point for* $f$ *and* $M_{z_0}(r, f) = \max_{|z_0-z|=r} |f(z)|$*.*

**Lemma 3.4** *Let $f$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. For $|z_0 - z| = r$ sufficiently small, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = max_{|z_0-z|=r}|f(z)|$. Then there exist a constant $\delta_r > 0$ and a set $E_3 \subset (0,1)$ of finite logarithmic measure, such that for all $z$ satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{V_{z_0}(r)}{z_0 - z}\right)^j (1 + o(1)) \ (j \in \mathbb{N}),$$

*where $V_{z_0}(z)$ is the central index of $f$ near $z_0$, $z_0$ is an essential singular point for $f$.*

**Proof** If $z_r = z_0 - re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M_{z_0}(r, f)$, since $|f(z)|$ is continuous in $|z_0 - z| = r$, then there exists a constant $\delta_r(> 0)$, such that for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\big||f(z)| - |f(z_r)|\big| < \varepsilon$$

that is

$$|f(z)| > \frac{1}{2}|f(z_r)| = \frac{1}{2}M_{z_0}(r, f) > M_{z_0}(r, f)V_{z_0}(r)^{-\frac{1}{4}+\eta_1}.$$

By Lemma 3.3,

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{V_{z_0}(r)}{z_0 - z}\right)^j (1 + o(1)) \ (j \in \mathbb{N})$$

holds for all $z$ satisfying $|z_0 - z| = r \notin E_2$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

**Lemma 3.5** *Let $f$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. For $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = max_{|z_0-z|=r}|f(z)|$. Then there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure, such that for all $z$ satisfying $| z_0 - z |= r \notin E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j \ (j \in \mathbb{N}),$$

*where $z_0$ is an essential singular point for $f$.*

**Proof** Let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = max_{|z_0-z|=r}|f(z)|$. Then by Lemma 3.4 there exist a constant $\delta_r$ $(> 0)$ and a set $E_3 \subset (0,1)$ of finite logarithmic measure, such that for all $z$ satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{V_{z_0}(r)}{z_0 - z}\right)^j (1 + o(1))\ (j \in \mathbb{N}). \tag{3.4}$$

Since $g(\omega) = f(z_0 - \frac{1}{\omega})$ is a transcendental entire function, it follows that $V(R) \longrightarrow +\infty$ as $R \longrightarrow +\infty$. On the other hand, $V(R) = V_{z_0}(r)(R = \frac{1}{r})$. Hence $V_{z_0}(r) \longrightarrow +\infty$ as $r \longrightarrow 0$. Then by (3.4), for all $z$ satisfying $|z_0 - z| = r \notin E_3$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \geq \frac{1}{2} r^{-j}$$

that is

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j \quad (j \in \mathbb{N}).$$

**Lemma 3.6** *[17]Let $A(z)$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ with $\sigma(A, z_0) < n$ $(n \in \mathbb{N}\backslash\{0\})$. Set $g(z) = A(z)\exp\left\{\frac{a}{(z_0-z)^n}\right\}$, where $a = \alpha + i\beta \neq 0$ is a complex number, $z_0 - z = re^{i\phi}$, $\delta_a(\phi) = \alpha\cos(n\phi) + \beta\sin(n\phi)$, and $H = \{\phi \in [0, 2\pi) : \delta_a(\phi) = 0\}$. (obviously , $H$ is a finite set). Then for any given $\varepsilon > 0$ and for any $\phi \in [0, 2\pi)\backslash H$, there exists $r_0 > 0$, such that for $0 < r < r_0$, we have*
***(i)*** *if $\delta_a(\phi) > 0$, then*

$$\exp\left\{(1-\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\} \leqslant |g(z)| \leqslant \exp\left\{(1+\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\}, \tag{3.5}$$

***(ii)*** *if $\delta_a(\phi) < 0$, then*

$$\exp\left\{(1+\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\} \leqslant |g(z)| \leqslant \exp\left\{(1-\varepsilon)\delta_a(\phi)\frac{1}{r^n}\right\}. \tag{3.6}$$

**Lemma 3.7** *[14]Let $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) \leqslant \alpha < \infty$. If $f$ is a solution of equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + ... + A_1(z)f + A_0(z)f = 0 \tag{3.7}$$

*that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $\sigma_2(f, z_0) \leqslant \alpha$.*

**Lemma 3.8** *[24] Let $f$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$. Then there exists a set $E_5 \subset (0,1)$ of finite logarithmic measure, such that*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j (j \in \mathbb{N})$$

*holds as $r \to 0$, $r \notin E_5$, where $z_r$ is a a point on the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = M_{z_0}(r, f) = \max_{|z_0 - z| = r} |f(z)|$.*

**Lemma 3.9** *[15] Let $f$ be a non-constant analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ of infinite order with the hyper-order $\sigma_2(f, z_0) = \sigma$ and $V_{z_0}(r)$ be the central index of $f$. Then*

$$\limsup_{r \to 0} \frac{\log^+ \log^+ V_{z_0}(r)}{-\log r} = \sigma.$$

**Lemma 3.10** *Let $k \geq 2$ be an integer, $A_j(z)(j = 0, ..., k-1)$ and $F(\not\equiv 0)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\max\{\sigma(A_j, z_0), \sigma(F, z_0)\} \leq \alpha < \infty$. If $f$ is an infinite order solution of equation*

$$f^{(k)} + A_{k-1}(z)f^{k-1} + ... + A_1(z)f' + A_0(z)f = F \tag{3.8}$$

*that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, then $\sigma_2(f, z_0) \leq \alpha$.*

**Proof** Assume that $f$ is an infinite analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (3.8). By (3.8), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \left|A_{k-1}(z)\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + ... + \left|A_1(z)\right|\left|\frac{f'(z)}{f(z)}\right| + \left|\frac{F(z)}{f(z)}\right| + \left|A_0(z)\right|. \tag{3.9}$$

By Lemma 3.8, there exists a set $E_5 \subset (0,1)$ of finite logarithmic measure, such that for all $j = 0, 1, ..., k$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left( \frac{V_{z_0}(r)}{z_0 - z_r} \right)^j \tag{3.10}$$

as $r \to 0$, $r \notin E_5$, where $z_r$ is a point on the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = M_{z_0}(r, f) = \max_{|z_0 - z| = r} |f(z)|$.
For any given $\varepsilon > 0$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$ we have

$$\Big|A_j(z)\Big| \le \exp\{\frac{1}{r^{\alpha+\varepsilon}}\} (j = 0, 1, ..., k-1) \tag{3.11}$$

and

$$\Big|F(z)\Big| \le \exp\{\frac{1}{r^{\alpha+\varepsilon}}\}. \tag{3.12}$$

Since $M_{z_0}(r, f) \ge 1$ as $r \to 0$, it follows from (3.12) that

$$\frac{|F(z)|}{M_{z_0}(r, f)} \le \exp\{\frac{1}{r^{\alpha+\varepsilon}}\} \ \text{ as } \ r \to 0. \tag{3.13}$$

By substituting (3.10), (3.11) and (3.13) into (3.9), we obtain

$$\Big(\frac{V_{z_0}(r)}{r}\Big)^k |1 + o(1)| \le (k+1) \Big(\frac{V_{z_0}(r)}{r}\Big)^{k-1} |1 + o(1)| \exp\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\} \tag{3.14}$$

for all $|z_0 - z_r| = r \notin E_5$, $r \to 0$ and $|f(z_r)| = M_{z_0}(r, f)$.
By (3.14) and Lemma 3.10, we get

$$\sigma_2(f, z_0) \le \alpha.$$

# 3 Proof of Main Results

**Proof of Theorem 3.4** Assume that $f$ is a non constant an analytic solution of (3.1) in $\overline{\mathbb{C}} \backslash \{z_0\}$, where $z_0$ is an essential singular point for $f$.

By Lemma 3.1, there exist a set $E_1 \subset (0,1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j} \ (j = 1, ..., k). \tag{3.15}$$

For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|$.

By Lemma 3.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure such that for all $z$ satisfying $|z_0 - z| = r \notin E_4$, $r \to 0$, and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\Big|\frac{f(z)}{f^{(j)}(z)}\Big| \leq 2r^j \ (j = 1, ..., k) \tag{3.16}$$

Set $a_j = \alpha_j + i\beta_j, \quad \delta_{a_j}(\theta) = \alpha_j \cos(n\theta) + \beta_j \sin(n\theta), \quad z_0 - z = re^{i\theta}$,

$$H_1 = \cup_{j=0}^{k-1} \{\theta \in [0, 2\pi) : \delta_{a_j}(\theta) = 0\},$$

$$H_2 = \cup_{0 \leq i < j \leq k-1} \{\theta \in [0, 2\pi) : \delta_{a_j - a_i}(\theta) = 0\}.$$

Since $a_j$ are distinct complex numbers, then there exists only one $s \in \{0, ..., k-1\}$, such that for any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\delta_1 = \delta_{a_s}(\theta) = \max\{\delta_{a_j}(\theta) : j = 0, ..., k-1\}.$$

We have : $\delta_1 > 0$ or $\delta_1 < 0$.

**Case 1**. $\delta_1 > 0$. Set $\delta_2 = \max\{\delta_{a_j}(\theta) : j \neq s\}$. Then $\delta_2 < \delta_1$.

**Subcase 1.1.** $\delta_2 > 0$ then $0 < \delta_2 < \delta_1$. Thus by Lemma 3.6, for any given $\varepsilon (0 < 2\varepsilon < \frac{\delta_1 - \delta_2}{\delta_1 + \delta_2})$, for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\Big|A_s(z) \exp\Big\{\frac{a_s}{(z_0 - z)^n}\Big\}\Big| \geq \exp\Big\{(1 - \varepsilon)\frac{\delta_1}{r^n}\Big\} \tag{3.17}$$

and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\Big| \le \exp\Big\{(1+\varepsilon)\frac{\delta_2}{r^n}\Big\} \ \ (j\neq s). \tag{3.18}$$

By (3.1), it follows that

$$\begin{aligned}-A_s(z)\exp\Big\{\frac{a_s}{(z_0-z)^n}\Big\} &= \frac{f^{(k)}(z)}{f^{(s)}(z)} + \sum_{j=s+1}^{k-1} A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\frac{f^{(j)}(z)}{f^{(s)}(z)} \\ &\quad + \sum_{j=0}^{s-1} A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\frac{f^{(j)}(z)}{f(z)}\frac{f(z)}{f^{(s)}(z)}.\end{aligned} \tag{3.19}$$

Substituting (3.15), (3.16), (3.17), (3.18) into (3.19), for all $z$ satisfying $|z_0-z| = r \notin E_1\cup E_4$, $r\to 0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_1\cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \le M_1 r^s \exp\Big\{(1+\varepsilon)\frac{\delta_2}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r,f)}{r}\Big]^{2k}, \tag{3.20}$$

where $M_1(>0)$ is a constant. Hence by (3.20), we obtain $\sigma_2(f,z_0)\ge n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f,z_0)=n$.

**Subcase 1.2.** $\delta_2<0$. By Lemma 3.6, for any given $\varepsilon(0<2\varepsilon<1)$, for all $z$ satisfying $|z_0-z|=r\notin E_1\cup E_4$, $r\to 0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_1\cup H_2)$, we have (3.17) and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\Big| \le \exp\Big\{(1-\varepsilon)\frac{\delta_2}{r^n}\Big\} < 1 \ \ (j\neq s). \tag{3.21}$$

Substituting (3.15), (3.16), (3.17), (3.21) into (3.19), for all $z$ satisfying $|z_0-z| = r \notin E_1\cup E_4$, $r\to 0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus(H_1\cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} \le M_2 r^s \Big[\frac{T_{z_0}(\alpha r,f)}{r}\Big]^{2k}, \tag{3.22}$$

where $M_2(>0)$ is a constant. Hence by (3.22), we obtain $\sigma_2(f,z_0)\ge n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f,z_0)=n$.

**Case 2.** $\delta_1 < 0$. By Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1-\varepsilon)\frac{\delta_1}{r^n}\Big\} < 1 \quad (j = 0, ..., k-1). \tag{3.23}$$

By (3.1), we get

$$-1 = \sum_{j=1}^{k-1} A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\frac{f^{(j)}(z)}{f(z)}\frac{f(z)}{f^{(k)}(z)} + A_0(z)\exp\Big\{\frac{a_0}{(z_0 - z)^n}\Big\}\frac{f(z)}{f^{(k)}(z)}. \tag{3.24}$$

Substituting (3.15), (3.16), (3.17), (3.23) into (3.24), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leq M_3 r^k \exp\Big\{(1+\varepsilon)\frac{\delta_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.25}$$

where $M_3(> 0)$ is a constant. Hence by (3.25), we obtain $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) = n$.

**Proof of Theorem 3.5** Assume that $f$ is a non constant analytic solution of (3.1) in $\overline{\mathbb{C}}\setminus\{z_0\}$, where $z_0$ is an essential singular point for $f$.
By Lemma 3.1, there exist a set $E_1 \subset (0, 1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have (3.15).
For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|$.
By Lemma 3.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0, 1)$ of finite logarithmic measure such that for all $z$ satisfying $|z_0 - z| = r \notin E_4$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.16).
Set

$$H_3 = \Big\{\theta \in [0, 2\pi) : \delta_{a_s}(\theta) = 0 \text{ or } \delta_{a_l}(\theta) = 0\Big\}$$

and

$$H_4 = \Big\{\theta \in [0, 2\pi) : \delta_{a_s}(\theta) = \delta_{a_l}(\theta)\Big\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_3 \cup H_4)$, we have
$\delta_{a_s}(\theta) \neq 0$, $\delta_{a_l}(\theta) \neq 0$ and $\delta_{a_s}(\theta) > \delta_{a_l}(\theta)$ or $\delta_{a_s}(\theta) < \delta_{a_l}(\theta)$.
Set $c_1 = \delta_{a_s}(\theta)$ and $c_2 = \delta_{a_l}(\theta)$.

**Case 1.** $c_1 > c_2$. Here we also divide our proof in three subcases.

**Subcase 1.1** $c_1 > c_2 > 0$. Set $c_3 = \max\{\delta_{a_j}(\theta) : j \neq s\}$ Then $0 < c_3 < c_1$. Thus by Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < \frac{c_1 - c_3}{c_1 + c_3})$, for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_3 \cup H_4)$, we have

$$\Big|A_s(z) \exp\Big\{\frac{a_s}{(z_0 - z)^n}\Big\}\Big| \geq \exp\Big\{(1 - \varepsilon)\frac{c_1}{r^n}\Big\} \tag{3.26}$$

and

$$\Big|A_j(z) \exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1 + \varepsilon)\frac{c_3}{r^n}\Big\} \quad (j \neq s). \tag{3.27}$$

Substituting (3.15), (3.16), (3.26), (3.27) into (3.19), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\big\{(1 - \varepsilon)\frac{c_1}{r^n}\big\} \leq M_4 r^s \exp\big\{(1 + \varepsilon)\frac{c_3}{r^n}\big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.28}$$

where $M_4(> 0)$ is a constant. Hence by (3.28), we obtain $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) = n$.

**Subcase 1.2.** $c_1 > 0 > c_2$. Set $\gamma_1 = \max\{d_j : j \neq s, l\}$. Thus, by Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < \frac{1 - \gamma_1}{1 + \gamma_1})$, for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_3 \cup H_4)$, we have (3.26)

$$\Big|A_j(z) \exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1 + \varepsilon)\frac{\gamma_1 c_1}{r^n}\Big\} \quad (j \neq s). \tag{3.29}$$

Substituting (3.15), (3.16), (3.26), (3.29) into (3.19), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \leq M_5 r^s \exp\Big\{(1+\varepsilon)\frac{\gamma_1 c_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.30}$$

where $M_5(>0)$ is a constant. Hence by (3.30), we obtain $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) = n$.

**Subcase 1.3.** $0 > c_1 > c_2$. Set $\gamma_2 = \min\{d_j : j \neq s, l\}$. By Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_3 \cup H_4)$, we have

$$\Big|A_s(z)\exp\Big\{\frac{a_s}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1-\varepsilon)\frac{c_1}{r^n}\Big\} \tag{3.31}$$

and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\Big\} \quad (j \neq s). \tag{3.32}$$

Substituting (3.15), (3.16), (3.31), (3.32) into (3.24), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leq M_6 r^k \exp\Big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.33}$$

where $M_6(>0)$ is a constant. Hence by (3.33), we obtain $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) = n$.

**Case 2.** $c_1 < c_2$. Using the same reasoning as in case 1, we can also obtain $\sigma_0(f, z_0) = n$.

**Poof of Theorem 3.6** First we show that (3.2) can possess at most one exceptional analytic solution$f_0$ in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order.
In fact, if $f^*$ is another analytic solution of finite order of equation (3.2), then

$f_0 - f^*(\not\equiv 0)$ is an analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order of the corresponding homogeneous equation of (3.2). This contradicts Theorem 3.4 and Theorem 3.5. We assume that $f$ is an infinite order analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (3.2). By Lemma 3.10, it follows that $\sigma_2(f, z_0) \leq n$.
Now we prove that $\sigma_2(f, z_0) \geq n$. By Lemma 3.1, there exist a set $E_1 \subset (0, 1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $z$ satisfying $|z_0 - z| = r \notin E_1$, we have (3.15).
For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|$.
By Lemma 3.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0, 1)$ of finite logarithmic measure such that for all $z$ satisfying $|z_0 - z| = r \notin E_4$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.16).
Since $|f(z)|$ is continous in $|z_0 - z| = r$, then there exists a constant $\lambda_r > 0$ such that for all $z$ satisfying $|z_0 - z| = r$ sufficiently small and $\arg(z_0 - z) = \theta \in [\theta_r - \lambda_r, \theta_r + \lambda_r]$, we have

$$\frac{1}{2}|f(z_r)| < |f(z)| < \frac{3}{2}|f(z_r)|. \tag{3.34}$$

On the other hand, for any given $\varepsilon (0 < 2\varepsilon < n - \sigma)$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$, we have

$$|F(z)| \leq \exp\{\frac{1}{r^{\sigma+\varepsilon}}\}. \tag{3.35}$$

Since $M_{z_0}(r, f) \geq 1$ as $r \to 0$, it follows from (3.34) and (3.35) that

$$\Big|\frac{F(z)}{f(z)}\Big| \leq 2\exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \quad \text{as} \quad r \to 0. \tag{3.36}$$

Set $\gamma = \min\{\delta_r, \lambda_r\}$.

**(i)** Suppose that $a_j (j = 0, .., k-1)$ satisfy hypotheses of Theorem 3.4.
Since $a_j$ are distinct complex numbers, then there exists only $s \in \{0, ..., k-1\}$

such that for any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, where $H_1$ and $H_2$ are defined above, we have

$$\delta_1 = \delta_{a_s}(\theta) = \max\{\delta_{a_j}(\theta) : j = 0, ..., k-1\}.$$

We have : $\delta_1 > 0$ or $\delta_1 < 0$

**Case 1**. $\delta_1 > 0$. Set $\delta_2 = \max\{\delta_{a_j}(\theta) : j \neq s\}$. Then $\delta_2 < \delta_1$.

**Subcase 1.1.** $\delta_2 > 0$. From (3.15), (3.16), (3.17), (3.18), (3.36) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\left\{(1-\varepsilon)\frac{\delta_1}{r^n}\right\} \leq B_1 r^s \exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \exp\left\{(1+\varepsilon)\frac{\delta_2}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \tag{3.37}$$

where $B_1(> 0)$ is a constant. From (3.37), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Subcase 1.2**. $\delta_2 < 0$. From (3.15), (3.16), (3.17), (3.21), (3.36) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$\exp\left\{(1-\varepsilon)\frac{\delta_1}{r^n}\right\} \leq B_2 r^s \exp\{\frac{1}{r^{\sigma+\varepsilon}}\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \tag{3.38}$$

where $B_2(> 0)$ is a constant. From (3.38), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Case 2.** $\delta_1 < 0$. From (3.15), (3.16), (3.23), (3.36) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we have

$$1 \leq B_3 r^k \exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \exp\left\{(1+\varepsilon)\frac{\delta_1}{r^n}\right\}\left[\frac{T_{z_0}(\alpha r, f)}{r}\right]^{2k}, \tag{3.39}$$

where $B_3(>0)$ is a constant. From (3.39), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**(ii)** Suppose that $a_j (j = 0, .., k-1)$ satisfy hypotheses of Theorem 3.5.
For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, where $H_3$ and $H_4$ are defined above, we have
$\delta_{a_s}(\theta) \neq 0$, $\delta_{a_l}(\theta) \neq 0$ and $\delta_{a_s}(\theta) > \delta_{a_l}(\theta)$ or $\delta_{a_s}(\theta) < \delta_{a_l}(\theta)$.
Set $c_1 = \delta_{a_s}(\theta)$ and $c_2 = \delta_{a_l}(\theta)$.

**Case 1.** $c_1 > c_2$. Here we also divide our proof in three subcases.

**Subcase 1.1** $c_1 > c_2 > 0$. From (3.15), (3.16), (3.26), (3.27), (3.36) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$\exp\big\{(1-\varepsilon)\frac{c_1}{r^n}\big\} \leq B_4 r^s \exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \exp\big\{(1+\varepsilon)\frac{c_3}{r^n}\big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.40}$$

where $B_4(>0)$ is a constant. Hence by (3.40), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Subcase 1.2.** $c_1 > 0 > c_2$. From (3.15), (3.16), (3.26), (3.29), (3.36) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$\exp\big\{(1-\varepsilon)\frac{c_1}{r^n}\big\} \leq B_5 r^s \exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \exp\big\{(1+\varepsilon)\frac{\gamma_1 c_3}{r^n}\big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.41}$$

where $B_5(>0)$ is a constant. From (3.41), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Subcase 1.3.** $0 > c_1 > c_2$. From (3.15), (3.16), (3.31), (3.32), (3.36) and

(3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_3 \cup H_4)$, we obtain

$$1 \leq B_6 r^k \exp\{\frac{1}{r^{\sigma+\varepsilon}}\} \exp\big\{(1+\varepsilon)\frac{\gamma_2 c_1}{r^n}\big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{3.42}$$

where $B_6(> 0)$ is a constant. From (3.42), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

Chapter 4

# Growth of Analytic Solutions of Linear Differential Equations with Analytic Coefficients near an isolated Singular Point

## 1 Introduction and Main Results

In [14], Cherief and Hamouda have considered equation (3.1) and proved the following result :

**Theorem 4.1** *[14] Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$, and let $a_j(j = 0, ..., k-1)$ be complex constants. Suppose that there exist nonzero numbers $a_0$ and $a_s$, such that $0 < s \leq k-1$, $a_0 = |a_0|e^{i\theta_s}$, $a_s = |a_s|e^{i\theta_s}$, $\theta_0, \theta_s \in [0, 2\pi)$, $\theta_0 \neq \theta_s$, $A_0 A_s \not\equiv 0$ and for $j \neq 0, s$, $a_j$ satisfies either $a_j = d_j a_0$ $(d_j < 1)$ or $\arg a_j = \arg a_s$. Then every solution $f \not\equiv 0$ of equation (3.1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$, satisfies $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) = n$ .*

In this chapter, we continue to consider the two equations (3.1) and (3.2) from the preceding chapter by giving new conditions on the analytic functions $A_j(j = 0, ..., k-1)$ to estimate the order and the hyper-order of analytic solutions of these equations. We will prove the folllowing results :

**Theorem 4.2** *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exist $s, l \in \{1, ..., k-1\}$ such that $A_sA_l \not\equiv 0$, $a_s = d_se^{i\phi}$, $a_l = -d_le^{i\phi}, \phi \in [0, 2\pi)$, $d_s > 0$, $d_l > 0$ and for $j \neq s, l$, $a_j = d_je^{i\phi}$ or $a_j = -d_je^{i\phi}(d_j \geq 0)$ and $\max\{d_j : j \neq s, l\} = d < min\{d_s, d_l\}$. Then every non-constant solution $f$ of equation (3.1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where $z_0$ is an essential singular point for $f$.*

**Theorem 4.3** *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer and $A_j(z)(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\backslash\{z_0\}$, such that $\sigma(A_j, z_0) < n$. Suppose that there exists $\{a_{i_0}, a_{i_1}, ..., a_{i_m}\} \subset \{a_1, ..., a_{k-1}\}$ such that $\arg a_{i_j}(j = 1, .., m)$ are distinct and for every nonzero $a_l \in \{a_1, ..., a_{k-1}\} \setminus \{a_{i_1}, ..., a_{i_m}\}$, there exists some $a_{i_s} \in \{a_{i_1}, ..., a_{i_m}\}$ such that $a_l = c_l^{(i_s)}a_{i_s}(0 < c_l^{(i_s)} < 1)$. Then every non-constant solution $f$ of equation (3.1) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, where $z_0$ is an essential singular point for $f$.*

**Theorem 4.4** *Let $n \in \mathbb{N}\backslash\{0\}$, $k \geq 2$ be an integer, $A_j(z)$ and $a_j(j = 0, ..., k-1)$ satisfy hypotheses of Theorem 4.2 or those of Theorem 4.3 Let $F \not\equiv 0$ be an analytic function in $\overline{\mathbb{C}}\backslash\{z_0\}$ of order $\sigma = \sigma(F, z_0) < n$. Then every solution $f$ of equation (3.2) that is analytic in $\overline{\mathbb{C}}\backslash\{z_0\}$ is of infinite order and satisfies $\sigma_2(f, z_0) = n$, with at most one exceptional analytic solution $f_0$ of finite order in $\overline{\mathbb{C}}\backslash\{z_0\}$ , where $z_0$ is an essential singular point for $f_0$.*

## 2 Proof of Main Results

**Proof of Theorem 4.2** Assume that $f$ is a non constant analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (3.1), where $z_0$ is an essential singular point for $f$.
By Lemma 3.1, there exist a set $E_1 \subset (0,1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $r = |z_0 - z|$ satisfying $r \notin E_1$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq \lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2j} (j = 1, ..., k). \tag{4.1}$$

For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|$.
By Lemma 3.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure, such that for all $z$ satisfying $|z_0 - z| = r \notin E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\Big|\frac{f(z)}{f^{(j)}(z)}\Big| \leq 2r^j \ (j = 1, ..., k). \tag{4.2}$$

Set

$$H = \{\theta \ \in [0, 2\pi) \ : \cos(\phi + n\theta) = 0 \ \}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, we have

$$\cos(\phi + n\theta) > 0 \qquad \text{or} \qquad \cos(\phi + n\theta) < 0.$$

**Case 1.** $\cos(\phi + n\theta) > 0$. Thus by Lemma 3.6, for any given $\varepsilon (0 < 2\varepsilon < \frac{d_s - d}{d_s + d})$, for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, we have

$$\Big|A_s(z) \exp\Big\{\frac{a_s}{(z_0 - z)^n}\Big\}\Big| \geq \exp\Big\{(1 - \varepsilon)\frac{d_s \cos(\phi + n\theta)}{r^n}\Big\} \tag{4.3}$$

and

$$\Big|A_j(z) \exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1 + \varepsilon)\frac{d \cos(\phi + n\theta)}{r^n}\Big\} \ \ (j \neq s). \tag{4.4}$$

By (3.1), it follows that

$$-A_s(z)\exp\Big\{\frac{a_s}{(z_0-z)^n}\Big\}=\frac{f^{(k)}(z)}{f^{(s)}(z)}\ +\ \sum_{j=s+1}^{k-1}A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\ \frac{f^{(j)}(z)}{f^{(s)}(z)}$$
$$+\sum_{j=0}^{s-1}A_j(z)\exp\Big\{\frac{a_j}{(z_0-z)^n}\Big\}\ \frac{f^{(j)}(z)}{f(z)}\frac{f(z)}{f^{(s)}(z)}. \tag{4.5}$$

Substituting (4.1), (4.2), (4.3), (4.4) into (4.5), for all $z$ satisfying $|z_0-z|=r\notin E_1\cup E_4$, $r\to 0$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{d_s\cos(\phi+n\theta)}{r^n}\Big\}\leq M_1r^s\exp\Big\{(1+\varepsilon)\frac{d\cos(\phi+n\theta)}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r,f)}{r}\Big]^{2k}, \tag{4.6}$$

where $M_1(>0)$ is a constant. Hence by (4.6), we obtain $\sigma(f,z_0)=+\infty$ and $\sigma_2(f,z_0)\geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f,z_0)\leq n$. Hence $\sigma_2(f,z_0)=n$.

**Case 2.** $\cos(\phi+n\theta)<0$. We use the same reasoning as in the case 1 by replacing $A_s(z)\exp\big\{\frac{a_s}{(z_0-z)^n}\big\}$ by $A_l(z)\exp\big\{\frac{a_l}{(z_0-z)^n}\big\}$ to prove that $\sigma(f,z_0)=+\infty$ and $\sigma_2(f,z_0)\geq n$. From this and Lemma 3.7, we have $\sigma_2(f,z_0)=n$.

**Proof of Theorem 4.3** Assume that $f$ is a non constant analytic solution in $\overline{\mathbb{C}}\setminus\{z_0\}$ of equation (3.1), where $z_0$ is an essential singular point for $f$.
By Lemma 3.1, there exist a set $E_1\subset(0,1)$ of finite logarithmic measure and a constant $\lambda>0$, such that for all $r=|z_0-z|$ satisfying $r\notin E_1$, we have (4.1).
For each sufficiently small $|z_0-z|=r$, let $z_r=z_0-re^{i\theta_r}$ be a point satisfying $|f(z_r)|=\max_{|z_0-z|=r}|f(z)|$.
By Lemma 3.5, there exist a constant $\delta_r>0$ and a set $E_4\subset(0,1)$ of finite logarithmic measure, such that for all $z$ satisfying $|z_0-z|=r\notin E_4$, $r\to 0$ and $\arg(z_0-z)=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]$, we have (4.2).
Set

$$H_1=\cup_{j=0}^{k-1}\{\theta\in[0,2\pi):\delta_{a_j}(\theta)=0\}$$

and

$$H_2 = \cup_{1\le s<d\le m}\{\theta \in [0, 2\pi) : \delta_{a_s}(\theta) = \delta_{a_d}(\theta)\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have
$\delta_{a_j}(\theta) \neq 0$ $(j = 0, ..., k-1)$, $\quad \delta_{a_s}(\theta) \neq \delta_{a_d}(\theta)$ $\ (1 \le s < d \le m)$.
Since $a_{i_j}$ $(j = 1, ..., m)$ are distinct complex numbers, then there exists only one $t \in \{1, ..., m\}$, such that

$$\delta_t = \delta_{a_{i_t}}(\theta) = \max\{\delta_{a_{i_j}}(\theta) : j = 1, ..., m\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have $\ \delta_{a_{i_t}}(\theta) > 0\ $ or $\ \delta_{a_{i_t}}(\theta) < 0$.

**Case 1.** $\delta_t > 0$. For $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $a_l = c_l^{(i_t)} a_{i_t}$ or $a_l = c_l^{(i_s)} a_{i_s}$ $s \neq t$.
Hence for $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $\delta_l(\theta) < \delta_t$.
Set $\delta = \max\{\delta_{a_j}(\theta) : j \neq i_t\}$, thus $\delta < \delta_t$.

**Subcase 1.1**. $\delta > 0$. Thus, by Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < \frac{\delta_t - \delta}{\delta_t + \delta})$, for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\left|A_{i_t}(z) \exp\left\{\frac{a_{i_t}}{(z_0 - z)^n}\right\}\right| \ge \exp\left\{(1-\varepsilon)\frac{\delta_t}{r^n}\right\} \tag{4.7}$$

and

$$\left|A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\}\right| \le \exp\left\{(1+\varepsilon)\frac{\delta}{r^n}\right\} \quad (j \neq i_t). \tag{4.8}$$

We can rewrite (3.1) as

$$-A_{i_t}(z) \exp\left\{\frac{a_{i_t}}{(z_0 - z)^n}\right\} = \frac{f^{(k)}(z)}{f^{(i_t)}(z)} + \sum_{j=i_t+1}^{k-1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\} \frac{f^{(j)}(z)}{f^{(i_t)}(z)}$$

$$+ \sum_{j=0}^{i_t - 1} A_j(z) \exp\left\{\frac{a_j}{(z_0 - z)^n}\right\} \frac{f^{(j)}(z)}{f(z)} \frac{f(z)}{f^{(i_t)}(z)}. \tag{4.9}$$

Substituting (4.1), (4.2), (4.7), (4.8) into (4.9), for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1 \cup H_2$, we obtain

$$\exp\big\{(1-\varepsilon)\frac{\delta_t}{r^n}\big\} \leq M_2 r^{i_t} \exp\big\{(1+\varepsilon)\frac{\delta}{r^n}\big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.10}$$

where $M_2(>0)$ is a constant. Hence by (4.10), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) \leq n$. Hence $\sigma_2(f, z_0) = n$.

**Subcase 1.2**. $\delta < 0$. By Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all $z$ satisfying $|z_0 - z| = r, r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have (4.7) and

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1-\varepsilon)\frac{\delta_{a_j}(\theta)}{r^n}\Big\} < 1 \quad (j \neq i_t). \tag{4.11}$$

Substituting (4.1), (4.2), (4.7), (4.11) into (4.9), for all $z$ satisfying $|z_0 - z| = r, r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus \notin H_1 \cup H_2$, we obtain

$$\exp\big\{(1-\varepsilon)\frac{\delta_t}{r^n}\big\} \leq M_3 r^{i_t}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.12}$$

where $M_3(>0)$ is a constant. Hence by (4.12), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) \leq n$. Hence $\sigma_2(f, z_0) = n$.

**Case 2.** $\delta_t < 0$. Set $c = \min\{c_l^{(i_j)} : l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$ and $j = 1, ...m\}$. By Lemma 3.6, for any given $\varepsilon(0 < 2\varepsilon < 1)$, for all $z$ satisfying $|z_0 - z| = r, r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\Big|A_j(z)\exp\Big\{\frac{a_j}{(z_0 - z)^n}\Big\}\Big| \leq \exp\Big\{(1-\varepsilon)\frac{c\delta_t}{r^n}\Big\} \quad (j = 0, .., k-1). \tag{4.13}$$

By (3.1), we get

$$-1 = A_{k-1}(z)\exp\Big\{\frac{a_{k-1}}{(z_0-z)^n}\Big\}\frac{f^{(k-1)}(z)}{f(z)}\frac{f(z)}{f^{(k)}(z)} + \ldots + A_0(z)\exp\Big\{\frac{a_0}{(z_0-z)^n}\Big\}\frac{f(z)}{f^{(k)}(z)}. \tag{4.14}$$

Substituting (4.1), (4.2), (4.13) into (4.14), for all $z$ satisfying $|z_0-z| = r, r \to 0$ and $\arg(z_0-z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leq M_4 r^k \exp\Big\{(1+\varepsilon)\frac{c\delta_t}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.15}$$

where $M_4(>0)$ is a constant. Hence by (4.15), we obtain $\sigma(f, z_0) = +\infty$ and $\sigma_2(f, z_0) \geq n$. On the other hand, by Lemma 3.7, we have $\sigma_2(f, z_0) = n$.

**Proof of Theorem 4.4** First we show that (3.2) can possess at most one exceptional analytic solution $f_0$ of finite order in $\overline{\mathbb{C}}\backslash\{z_0\}$.
In fact, if $f^*$ is another analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order of equation (3.2), then $f_0 - f^*(\not\equiv 0)$ is an analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of finite order of the corresponding homogeneous equation of (3.2). This contradicts Theorem 4.2 and Theorem 4.3.
We assume that $f$ is an infinite order analytic solution in $\overline{\mathbb{C}}\backslash\{z_0\}$ of equation (3.2). By Lemma 3.10 , $\sigma_2(f, z_0) \leq n$.
By Lemma 3.1, there exist a set $E_1 \subset (0,1)$ of finite logarithmic measure and a constant $\lambda > 0$, such that for all $z$ satisfying $|z_0 - z| = r \notin E_1$, we have (4.1).
For each sufficiently small $|z_0 - z| = r$, let $z_r = z_0 - re^{i\theta_r}$ be a point satisfying $|f(z_r)| = \max_{|z_0-z|=r}|f(z)|$.
By Lemma 3.5, there exist a constant $\delta_r > 0$ and a set $E_4 \subset (0,1)$ of finite logarithmic measure such that for all $z$ satisfying $|z_0 - z| = r \notin E_4$ and $\arg(z_0 - z) = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (4.2).
Since $|f(z)|$ is continous in $|z_0 - z| = r$, then there exists a constant $\lambda_r > 0$ such that for all $z$ satisfying $|z_0 - z| = r$ sufficiently small and $\arg(z_0 - z) = \theta \in$

$[\theta_r - \lambda_r, \theta_r + \lambda_r]$, we have

$$\frac{1}{2}|f(z_r)| < |f(z)| < \frac{3}{2}|f(z_r)|. \tag{4.16}$$

On the other hand, for any given $\varepsilon (0 < 2\varepsilon < n - \sigma)$, there exists $r_0 > 0$, such that for all $0 < r = |z_0 - z| < r_0$, we have

$$|F(z)| \leq \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\}. \tag{4.17}$$

Since $M_{z_0}(r, f) \geq 1$ as $r \to 0$, it follows from (4.16) and (4.17) that

$$\Big|\frac{F(z)}{f(z)}\Big| \leq 2\exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \quad \text{as} \quad r \to 0. \tag{4.18}$$

Set $\gamma = \min\{\delta_r, \lambda_r\}$.

**(i)** Suppose that $a_j (j = 0, .., k-1)$ satisfy hypotheses of Theorem 4.2.

**Case 1.** $\cos(\phi + n\theta) > 0$. From (4.1), (4.2), (4.7), (4.8), (4.18) and (3.2), for all $z$ satisfying $|z_0 - z| = r \notin E_1 \cup E_4$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H (H$ is defined above), we obtain

$$\exp\Big\{(1-\varepsilon)\frac{d_s \cos(\phi + n\theta)}{r^n}\Big\} \leq$$

$$B_1 r^s \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{d\cos(\phi + n\theta)}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.19}$$

where $B_1 (> 0)$ is a constant. From (4.19), we get $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Case 2.** $\cos(\phi + n\theta) < 0$. We use the same reasoning as in the case 1 by replacing $A_s(z)\exp\{\frac{a_s}{(z_0 - z)^n}\}$ by $A_l(z)\exp\{\frac{a_l}{(z_0 - z)^n}\}$ to prove that $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**(ii)** Suppose that $a_j (j = 0, .., k-1)$ satisfy hypotheses of Theorem 4.3.
Since $a_{i_j}$ $(j = 1, ..., m)$ are distinct complex numbers, then there exists only one $t \in \{1, ..., m\}$ such that

$$\delta_t = \delta_{a_{i_t}}(\theta) = \max\{\delta_{a_{i_j}}(\theta) : j = 1, ..., m\}.$$

For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, where $H_1$ and $H_2$ are defined above, we have
$\delta_{a_{i_t}}(\theta) > 0$ or $\delta_{a_{i_t}}(\theta) < 0$.

**Case 1.** $\delta_t > 0$. For $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $a_l = c_l^{(i_t)} a_{i_t}$ or $a_l = c_l^{(i_s)} a_{i_s}$ $s \neq t$.
Hence for $l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$, we have $\delta_l < \delta_t$.
Set $\delta = \max\{\delta_{a_j}(\theta) : j \neq i_t\}$ thus $\delta < \delta_t$.

**Subcase 1.1.** $\delta > 0$. From (4.1), (4.2), (4.7), (4.8), (4.18) and (3.2) for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1 \cup H_2$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_t}{r^n}\Big\} \leq B_2 r^{i_t} \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{\delta}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.20}$$

where $B_2 (> 0)$ is a constant. Hence by (4.20), we obtain that $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Subcase 1.2.** $\delta < 0$. From (4.1), (4.2), (4.7), (4.13), (4.18) and (3.2) for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1 \cup H_2$, we obtain

$$\exp\Big\{(1-\varepsilon)\frac{\delta_t}{r^n}\Big\} \leq B_3 r^{i_t} \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.21}$$

where $B_3 (> 0)$ is a constant. Hence by (4.21), we obtain that $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

**Case 2.** $\delta_t < 0$. Set $c = \min\Big\{c_l^{(i_j)} : l \in \{0, ..., k-1\} \setminus \{i_1, ..., i_m\}$ and $j = 1, .., m\Big\}$.
From (4.1), (4.2), (4.13), (4.18) and (3.2) for all $z$ satisfying $|z_0 - z| = r$, $r \to 0$ and $\arg(z_0 - z) = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_1 \cup H_2)$, we obtain

$$1 \leq B_4 r^k \exp\Big\{\frac{1}{r^{\sigma+\varepsilon}}\Big\} \exp\Big\{(1+\varepsilon)\frac{c\delta_t}{r^n}\Big\}\Big[\frac{T_{z_0}(\alpha r, f)}{r}\Big]^{2k}, \tag{4.22}$$

where $B_4 (> 0)$ is a constant. Hence by (4.22), we obtain that $\sigma_2(f, z_0) \geq n$. This and the fact that $\sigma_2(f, z_0) \leq n$ yield $\sigma_2(f, z_0) = n$.

Chapter 5

# Growth of Solutions of Higher Order Linear Differential Equations with Solutions of Another Equation as Coefficients

## 1 Introduction and Main Results

For the linear differential equation:

$$f'' + A(z)f' + B(z)f = 0, \tag{5.1}$$

where $A(z)$ and $B(z)$ $(\not\equiv 0)$ are entire functions, it is well-known that each solution of the equation (5.1) is an entire function. If $B(z)$ is transcendental and $f_1, f_2$ are two linearly independent solutions of the equation (5.1), then at least one of $f_1, f_2$ must have an infinite order. Hence, most solutions of the equation (5.1) have infinite order. On the other hand, there are equations of the form (5.1) that possess a solution $f(\not\equiv 0)$ of finite order, for example, $f(z) = e^z$ satisfies

$f'' + e^{-z}f' - (e^{-z}+1)f = 0$. Thus, a natural question is : What conditions on $A(z)$ and $B(z)$ will guarantee that every solution $f(\not\equiv 0)$ of equation (5.1) has an infinite order ? There are many results in the literature about the order of growth of solutions of (5.1), see [10, 19, 21, 27, 31].
The following result is a summary of results derived from Gundersen [21], Hellerstein, Miles and Rossi [27], and Ozawa [44].

**Theorem 5.1** *Suppose that $A(z)$ and $B(z)$ are entire functions satisfying one of the following conditions :*

*(i)* $\sigma(A) < \sigma(B)$.

*(ii)* $A(z)$ *is a polynomial and* $B(z)$ *is a transcendental entire function.*

*(iii)* $\sigma(A) < \sigma(B) \leq \frac{1}{2}$.

*Then every non-trivial solution of (5.1) is of infinite order.*

By Theorem 5.1, the main problem left to consider is whether every non-trivial solution of (5.1) has infinite order if $\sigma(A) = \sigma(B)$ or if $\sigma(B) < \sigma(A)$ and $\sigma(A) > \frac{1}{2}$. In general, the conclusion is false for these situations, for example, $f(z) = \exp(P(z))$ satisfies the equation

$$f'' + A(z)f' + (-P'' - (P')^2 - A(z)P')f = 0,$$

where $A(z)$ is an entire function and $P(z)$ is a non-constant polynomial. For the case of $\sigma(B) < \sigma(A)$ and $\sigma(A) > \frac{1}{2}$, there are also some examples listed in [21] showing that a non-trivial solution of equation (5.1) has a finite order. Therefore, it is interesting to find conditions on $A(z)$ and $B(z)$ guaranteeing that every non-trivial solution of (5.1) is of infinite order. Many parallel results obtained after Theorem 5.1 focus on the case $\sigma(A) \geq \sigma(B)$ and $\sigma(A) > \frac{1}{2}$ and can be founed in [10, 34, 35, 37, 38, 39, 49, 51].

Recently, this problem was studied using a new idea that a coefficient of (5.1) is a non-trivial solution of the following equation

$$\omega'' + P(z)\omega = 0 \tag{5.2}$$

where $P(z) = a_n z^n + ... + a_0$, $a_n \neq 0, a_n \neq 0$, see for example [36, 40, 50, 51]. The following result shows that the idea is viable.

**Theorem 5.2** *[50] Let $A(z)$ be a non-trivial solution of (5.2) and let $B(z)$ be a transcendental entire function with $\sigma(B) < \frac{1}{2}$. Then every non-trivial solution of (5.1) is of infinite order.*

Now a new idea is used to study the growth of solutions of (5.1), in which two coefficients of (5.1) are non-trivial solutions (5.2). In [41], the authors proved the following result:

**Theorem 5.3** *Suppose that $A(z)$ and $B(z)$ are two linearly independent solutions of (5.2). If the number of accumulation rays of the zero sequence of $A(z)$ is less than n+2, then every non-trivial solution of (5.1) is of infinite order.*

The next result in [41] shows that two coefficients of (5.2) are non-trivial solutions of (5.3) and (5.4) respectively

$$\omega'' + Q_1(z)\omega = 0 \tag{5.3}$$

$$\omega'' + Q_2(z)\omega = 0, \tag{5.4}$$

where $Q_1(z) = a_n z^n + ... + a_0, a_n \neq 0$, $Q_2(z) = b_m z^m + ... + b_0, b_m \neq 0$.

**Theorem 5.4** *Suppose that $A(z)$ and $B(z)$ are non-trivial solutions of (5.3) and (5.4) respectively. Suppose that $A(z)$ and $B(z)$ satisfy one of the following conditions:*

*(i) $m > n$.*

*(ii)* $m < n$.

*(iii)* $m = n$, $\arg a_n \neq \arg b_m$, *the number of accumulation rays of the zero sequence of* $A(z)$ *is less than* $n+2$.

*(iv)* $m = n$, *and* $a_n = cb_m$, *where* $0 < c < 1$.

*Then every non-trivial solution of (5.1) is of infinite order.*

The main purpose of this chapter is to generalize the results in Theorems 5.3, 5.4 to the higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + ... + A_0(z)f = 0, \tag{5.5}$$

where $k \geq 2$ is an integer and $A_{k-1}(z), ..., A_0(z) \not\equiv 0$ are entire functions. We will prove the following results :

**Theorem 5.5** *Let* $k \geq 2$ *be an integer and let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be entire functions. Suppose that there exists* $s \in \{1, ..., k-1\}$ *such that* $A_0(z)$ *and* $A_s(z)$ *are two linearly independent solutions of (5.3), and for* $j \neq 0, s$, $\sigma(A_j) < \sigma(A_0)$. *If the number of accumulation rays of the zero sequence of* $A_s(z)$ *is less than* $n+2$, *then every non-trivial solution of (5.5) is of infinite order.*

**Theorem 5.6** *Let* $k \geq 2$ *be an integer and let* $A_0(z), A_1(z), ..., A_{k-1}(z)$ *be entire functions. Suppose that there exist* $s, d \in \{1, ..., k-1\}$ *such that* $A_s(z)$ *and* $A_d(z)$ *are two linearly independent solutions of (5.3) and* $A_0(z)$ *is a non-trivial solution of (5.4) such that* $\max\{\sigma(A_j) : j \neq 0, s, d\} < \sigma(A_0)$ . *Suppose that* $A_0(z)$ *and* $A_s(z)$ *satisfy one of the following conditions:*

*(i)* $m > n$.

*(ii)* $m < n$.

*(iii)* $m = n$, $\arg a_n \neq \arg b_m$, *the number of accumulation rays of the zero sequence of* $A_s(z)$ *is less than* $n+2$.

*(iv)* $m = n$, $a_n = cb_m$, *where* $0 < c < 1$.

*Then every transcendental solution of (5.5) is of infinite order.*

# 2 Auxiliary Results

**Lemma 5.1** *[20] Let* $f$ *be a transcendental meromorphic function of finite order* $\sigma(f)$, *Let* $\varepsilon > 0$ *be a given real constant and let* $k$ *and* $j$ *be two integers such that* $k > j \geqslant 0$. *Then the following statement hold.*
*(i) There exists a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E_1$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* $z$ *satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$,

$$\frac{f^{(k)}(z)}{f^{(j)}(z)} \leq |z|^{(k-j)(\sigma(f)-1+\varepsilon)}. \tag{5.6}$$

*(ii) There exists a set* $E_2 \subset (1, +\infty)$ *that has finite logarithmic measure, such that for all* $z$ *satisfying* $|z| \notin E_2 \cup [0, 1]$, *the inequality (5.6) holds.*

The following Lemma, originally due to Hille is curcial for the proof of our results:

**Lemma 5.2** *[22, 28, 46] Let* $A(z)$ *be a non-trivial solution of* $\omega'' + P(z)\omega = 0$, *where* $P(z) = a_n z^n + ... + a_0, a_n \neq 0$. *Set* $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ *and* $S_j = S(\theta_j, \theta_{j+1})$, *where* $j = 0, 1, ..., n+1$ *and* $\theta_{n+2} = \theta_0 + 2\pi$. *Then* $A(z)$ *has the following properties :*

*(i) In each sector* $S_j$, $A(z)$ *either blows up or decays to zero exponentially.*

*(ii) If for some $j$, $A(z)$ decays to zero in $S_j$, then it must blow up in $S_{j-1}$ and $S_{j+1}$. However it is possible for $A(z)$ to blow up in many adjacent sectors.*

*(iii) If $A(z)$ decays to zero in $S_j$, then $A(z)$ has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \bar{S}_j \cup S_{j+1}$.*

*(iv) If $A(z)$ blows up in $S_{j-1}$ and $S_j$, then for each $\varepsilon > 0$, $A(z)$ has infinitly many zeros in each sector $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$ and furthermore, as $r \to +\infty$,*

$$n\Big(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A\Big) = (1 + o(1))\frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

*where $n\Big(\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A\Big)$ is the number of zeros of $A(z)$ in the region $\bar{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$ counting multiplicity.*

**Remark 5.1** *[41] It follows from definition 1.9 and Lemma 5.2 that the number of accumulation rays of the zero sequence of every non-trivial solution of (5.2) is less than or equal to $n+2$, and the set of the accumulation rays of the zero sequence of every non-trivial solution of (5.2) is a subset of $\{\theta_j : 0 \leq j \leq n+1\}$, where $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}, j = 0, 1, ..., n+1$.*

**Lemma 5.3** *[18] Let $A(z)$ be defined as in lemma 5.2. Then the following equality holds:*

$$\log M(r, A) = (1 + o(1))\frac{2\sqrt{|a_n|}}{n+2} r^{\frac{n+2}{2}}, \quad \text{as } r \longrightarrow +\infty$$

**Lemma 5.4** *[33] Let $\theta_1 < \theta_2$ be given to fix a sector $S(0) : \theta_1 \leq \arg z \leq \theta_2$, let $k \geq 2$ be a natural number, and let $\delta > 0$ be any real number such that $k\delta < 1$. Suppose that $A_0(z), ..., A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ are entire functions such that for real constants $\alpha > 0$, $\beta > 0$, we have, for any some $s = 1, ..., k-1$,*

$$|A_s(z)| \geq \exp((1 + \delta)\alpha |z|^{\beta})$$

*and*

$$|A_j(z)| \leq \exp(\delta\alpha|z|^\beta)$$

*for all* $j = 0, ..., s-1, s+1, ..., k-1$ *whenever* $|z| = r \geq r_\delta$ *in the sector* $S(0)$. *Given* $\varepsilon > 0$ *smal enough, if* $f$ *is a transcendental solution of finite order* $\sigma < \infty$ *of the linear differential equation (5.5), then the following conditions hold:*
*(i) There exists* $t \in \{0, ..., s-1\}$ *and a complex constant* $b_t \neq 0$ *such that* $f^{(t)} \to b_t$ *as* $z \to +\infty$ *in the sector* $S(\varepsilon) : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. *More precisely,*

$$|f^{(t)}(z) - b_t| \leq \exp(-(1-k\delta)\alpha|z|^\beta)$$

*in* $S(\varepsilon)$, *provided* $|z|$ *is large enough.*
*(ii) For each integer* $q \geq t+1$,

$$|f^{(q)}(z)| \leq \exp(-(1-k\delta)\alpha|z|^\beta)$$

*in* $S(3\varepsilon)$, *for all* $|z|$ *large enough.*

**Lemma 5.5** *[33] Suppose that* $f(z)$ *is an entire function, and that* $|f^{(k)}(z)|$ *is unbounded on a ray* $\arg z = \theta$. *Then there exists a sequence* $z_n = r_n e^{i\theta}$ *tending to infinity such that* $f^{(k)}(z_n) \to +\infty$ *and that*

$$\Big|\frac{f^{(i)}(z_n)}{f^{(k)}(z_n)}\Big| \leq \frac{1}{(k-i)!}(1+o(1))|z_n|^{k-i}$$

*provided* $i < k$.

# 3 Proof of Main Results

**Proof of Theorem 5.5** Suppose on the contrary to the assertion that there exists a non-trivial solution $f$ of (5.5) with $\sigma(f) < \infty$, we aim for a contradiction. Using Lemma 5.2 and the condition of Theorem 5.5, set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and

$S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 0, ..., n+1$, $\theta_{n+2} = \theta_0 + 2\pi$.
By the condition of Theorem 5.5 and the definition of accumulation rays of the zero sequence of meromorphic functions, we know that $p(A_s) \geq 2$. It follows from Lemma 5.2 that there exists at least one sector of the $n+2$ sectors, such that $A_s(z)$ decays to zero exponentially, without loss of generality, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n+1$. That is for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$,

$$\lim_{r\to\infty} \frac{\log\log \frac{1}{|A_s(re^{i\theta})|}}{\log r} = \frac{n+2}{2}. \tag{5.7}$$

Next we claim that it is impossible that both $A_s(z)$ and $A_0(z)$ decay to zero exponentially in a common sector. To prove our claim, without loss of generality, we suppose that $A_s(z)$ and $A_0(z)$ decay to zero exponentially in $S_0$. Set $h = \frac{A_s}{A_0}$. It follows from ([18], Lem.3) that as $r \to +\infty$,

$$N(r, \frac{1}{h-b}) = (1+o(1))T(r,h) = (1+o(1))\frac{2\sqrt{|a_n|}}{\pi\alpha}r^{\alpha},$$

holds for any $b \in \mathbb{C}$, with at most finitely many exceptions, where $\alpha = \frac{n+2}{2}$ Set $\omega = A_s - bA_0$. It is easy to see that $\omega$ is a solution of (5.2). It follows from ([18], Thm.3) that

$$N(r, \frac{1}{h-b}) = N(r, \frac{1}{\omega}) = (1+o(1))\frac{2\alpha - p(\omega)}{\pi\alpha^2}\sqrt{|a_n|}r^{\alpha},$$

as $r \to +\infty$. Combining the two equalities mentioned above, we get $p(\omega) = 0$. Thus implies that $\omega$ blows up exponentialy in every sector $S_j$, $j = 0, 1, ..., n+1$. This contradicts the assumption that $\omega$ decay to zero exponentially in $S_0$. Hence, $A_0(z)$ blows up exponentially in $S_{j_0}$, that is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$,

$$\lim_{r\to\infty} \frac{\log\log |A_0(re^{i\theta})|}{\log r} = \frac{n+2}{2}. \tag{5.8}$$

By Lemma 5.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all $z$ satisfying $\arg z = \psi_0$ and $|z| \geq R_0$,

$$\Big|\frac{f^{(i)}(z)}{f(z)}\Big| \leq |z|^{k\sigma(f)},\ i = 1, ..., k. \tag{5.9}$$

Let $\varepsilon \in (0, \sigma(A_0)/2)$ be a given constant. Since $\sigma(A_i) < \sigma(A_0)$ for all $i \neq 0, s$ and $0 \leq i \leq k-1$, then there exists an $R_1 > 1$ such that

$$|A_i(z)| < \exp(r^{\frac{n+2}{2}-2\varepsilon}) \tag{5.10}$$

for all $|z| = r > R_1$.
Thus, there exists a sequence of points $z_l = r_l e^{i\theta}$, where $r_l \to +\infty$ as $l \to +\infty$ and $\theta \in (\theta_{j_0}, \theta_{j_0+1}) - E_1$, such that (5.7),(5.8) and (5.9),(5.10) hold. Combining (5.7)-(5.9), (5.10) and (5.5), for any $l > l_0$,

$$\begin{aligned}\exp(r_l^{\frac{n+2}{2}-\varepsilon}) &\leq |A_0(r_l e^{i\theta})| \\ &\leq \Big|\frac{f^{(k)}(r_l e^{i\theta})}{f(r_l e^{i\theta})}\Big| + \sum_{j=1}^{k-1} |A_j(r_l e^{i\theta})| \Big|\frac{f^{(j)}(r_l e^{i\theta})}{f(r_l e^{i\theta})}\Big| \\ &\leq r_l^{2\sigma(f)}\Big(1 + \frac{1}{\exp(r_l^{\frac{n+2}{2}-\varepsilon})} + (k-2)\exp(r_l^{\frac{n+2}{2}-2\varepsilon})\Big).\end{aligned}$$

Obviously, that is a contradiction for sufficiently large $l$ and for any given $\varepsilon > 0$. Hence the conclusion of Theorem 5.5 holds.

**Proof of Theorem 5.6** Suppose the contrary to the assertion, that there exists a non-trivial solution $f$ of (5.5) with $\sigma(f) < \infty$, we aim for contradiction. It follows from ([3], Thm.1) that $\sigma(A_s) = \frac{n+2}{2}$ and $\sigma(A_0) = \frac{m+2}{2}$.
(1) Suppose that the condition (i) holds.
Then $\max\{\sigma(A_i) : i = 1, ..., k-1\} < \sigma(A_0)$. Therefore, the conclusion of Theorem 5.6 is deduced from [13].
(2) Suppose that the condition (ii) holds. Set

$$F_{A_0} = \Big\{\theta_j \in [0, 2\pi) : \theta_j = \frac{2j\pi - \arg b_m}{m+2}\Big\},\quad j = 0, 1, ..., m+1,$$

$$F_{A_s} = \Big\{\theta_j \in [0, 2\pi) : \theta_j = \frac{2j\pi - \arg a_n}{n+2}\Big\}, \quad j = 0, 1, ..., n+1.$$

By Lemma 5.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all $z$ satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, (5.9) holds. Set $E = E_1 \cup F_{A_0} \cup F_{A_s}$. Then for any $\theta \in [0, 2\pi) - E$, $A_0(z), A_s(z)$ have four possible growth types on the ray $\arg z = \theta$ :

(a) $A_s(re^{i\theta})$ satisfies

$$\lim_{r\to+\infty} \frac{\log\log|A_s(re^{i\theta})|^{-1}}{\log r} = \frac{n+2}{2} \tag{5.11}$$

and $A_0(re^{i\theta})$ satisfies

$$\lim_{r\to+\infty} \frac{\log\log|A_0(re^{i\theta})|}{\log r} = \frac{m+2}{2} \tag{5.12}$$

(b) $A_s(re^{i\theta})$ satisfies (5.11) and $A_0(re^{i\theta})$ satisfies

$$\lim_{r\to+\infty} \frac{\log\log|A_0(re^{i\theta})|^{-1}}{\log r} = \frac{m+2}{2} \tag{5.13}$$

(c) $A_s(re^{i\theta})$ satisfies

$$\lim_{r\to+\infty} \frac{\log\log|A_s(re^{i\theta})|}{\log r} = \frac{n+2}{2} \tag{5.14}$$

and $A_0(re^{i\theta})$ satisfies (5.12).

(d) $A_s(re^{i\theta})$ satisfies (5.14) and $A_0(re^{i\theta})$ satisfies (5.13).

(a') If $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (a), then using similar reasoning as in the proof of Theorem 5.5, we get a contradiction.

(b') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (b). Suppose that $|f^{(d)}(z)|$ is unbounded on the ray $\arg z = \theta$. Using Lemma 5.5, there exists an infinite sequence of points $z_l = r_l e^{i\theta}$ tending to infinity such that $f^{(d)}(z) \longrightarrow \infty$ and

$$\Big|\frac{f^{(i)}(z_l)}{f^{(d)}(z_l)}\Big| \leq \frac{1}{(d-i)!}(1+o(1))|z_l|^{d-i}, \quad i = 0, 1, ..., d-1 \tag{5.15}$$

as $l \to +\infty$.

By Lemma 5.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there exist a constant $R_0 = R_0(\psi_0) > 1$ such that for all $z$ satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have (5.9).

It follows from the proof of Theorem 5.5 that $A_d(z)$ blows up exponentially, that is on the ray $\arg z = \theta$, we have

$$\lim_{r \to +\infty} \frac{\log\log|A_d(re^{i\theta})|}{\log r} = \frac{n+2}{2}$$

It follows from (5.5), (5.9), (5.10) and (5.15) that

$$\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\} \leq |A_d(r_l e^{i\theta})| \leq$$

$$\Big|\frac{f^{(k)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| + |A_{k-1}(r_l e^{i\theta})|\Big|\frac{f^{(k-1)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| + ...$$

$$... + |A_s(r_l e^{i\theta})|\Big|\frac{f^{(s)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| + ... + |A_0(r_l e^{i\theta})|\Big|\frac{f(z_l)}{f^{(d)}(z_l)}\Big| \leq$$

$$M_1 r_l^{d+2\sigma(f)}\Big(1 + \frac{1}{\exp\{r_l^{\sigma(A_0)+\varepsilon}\}} + \frac{1}{\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\}} + (k-3)\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\}\Big)$$

Where $M_1 (> 0)$ is a constant. That is a contradiction for sufficiently large $l$ and for $\varepsilon \in \big(0, \frac{\sigma(A_d)}{2}\big)$. Hence $|f^{(d)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle.

(c') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (c). From Bank and Laine's results ([3], Thm. 1) we get $\sigma(A_s)=\frac{n+2}{2}>\frac{m+2}{2}=\sigma(A_0)$, there exists a real number $l>0$, such that $\sigma(A_s)=\frac{n+2}{2}>\frac{m+2+l}{2}>\frac{m+2}{2}=\sigma(A_0)$. Then for any given $\varepsilon\in(0,\frac{\pi}{8\sigma(A_s)})$ and $\eta\in(0,\frac{\sigma(A_s)-\sigma(A_0)}{4})$, we have

$$|A_s(z)|\geq\exp\Big\{(1+\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

and

$$|A_0(z)|\leq\exp\Big\{|z|^{\sigma(A_0)+\eta}\Big\}\leq\exp\Big\{|z|^{\sigma(A_s)-2\eta}\Big\}\leq\exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z\to+\infty$ in $\bar{S}(\frac{\varepsilon}{2})=\{z:\theta_j+\frac{\varepsilon}{2}\leq\arg z\leq\theta_{j+1}-\frac{\varepsilon}{2}\}, j=0,1,...,n+1$, where $\alpha$ and $\delta$ are positive constants satisfying $\delta k<1$ and $\theta_j=\frac{2j\pi-\arg(a_n)}{n+2}$.
On the other hand, it follows from the proof of Theorem 5.5 that $A_d(z)$ decays to zero exponentialy in $S_j=\{z:\theta_j<\arg z<\theta_{j+1}\},\theta_j=\frac{2j\pi-\arg(a_n)}{n+2}$ and $\theta_{n+2}=\theta_0+2\pi$, that is for the $\arg z=\theta$, we have

$$\lim_{r\to+\infty}\frac{\log\log|A_d(re^{i\theta})|^{-1}}{\log r}=\frac{n+2}{2}$$

Hence

$$|A_d(re^{i\theta})|\leq\frac{1}{\exp\Big\{|z|^{\frac{n+2}{2}-\varepsilon}\Big\}}\leq\exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\varepsilon}\Big\}$$

we have also

$$|A_i(z)|\leq\exp\Big\{|z|^{\sigma(A_0)+\eta}\Big\}\leq\exp\Big\{|z|^{\sigma(A_s)-2\eta}\Big\}\leq\exp\Big\{\delta\alpha|z|^{\frac{n+2}{2}-\varepsilon}\Big\},\quad i\neq0,s,d$$

By Lemma 5.4 , there exists $t\in\{0,1,..,s-1\}$ and $b_t\neq0$ such that

$$|f^{(t)}(z)-b_t|\leq\exp\Big\{-(1-k\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z\to+\infty$ in $\bar{S}_j(\varepsilon)$. For each integer $i\geq t+1$,

$$|f^{(i)}(z)|\leq\exp\Big\{-(1-k\delta)\alpha|z|^{\frac{n+2}{2}-\eta}\Big\}$$

as $z \to +\infty$ in $\bar{S}_j(\frac{3\varepsilon}{2})$.
Hence $|f^{(s)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle.

(d') Suppose that $A_s(re^{i\theta})$ and $A_0(re^{i\theta})$ satisfy the growth type (d). Suppose that $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$. Using Lemma 5.5, there exists an infinite sequence of points $z_l = r_l e^{i\theta}$ tending to infinity such that $f^{(s)}(z) \to \infty$ and

$$\Big|\frac{f^{(i)}(z_l)}{f^{(s)}(z_l)}\Big| \leq \frac{1}{(s-i)!}(1+o(1))|z_l|^{s-i}, \quad i = 0, 1, ..., s-1 \tag{5.16}$$

as $l \to +\infty$.
It follows from the proof of Theorem 5.5 that $A_d(z)$ decays to zero exponentially, that is on the ray $\arg z = \theta$, we have

$$\lim_{r \to +\infty} \frac{\log\log |A_d(re^{i\theta})|^{-1}}{\log r} = \frac{n+2}{2}$$

It follows from (5.5), (5.9), (5.10) and (5.16) that

$$\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\} \leq |A_s(r_l e^{i\theta})| \leq$$

$$\Big|\frac{f^{(k)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| + |A_{k-1}(r_l e^{i\theta})|\Big|\frac{f^{(k-1)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| +$$

$$... + |A_s(r_l e^{i\theta})|\Big|\frac{f^{(d)}(z_l)}{f(z_l)}\Big|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| + .... + |A_0(r_l e^{i\theta})|\Big|\frac{f(z_l)}{f^{(s)}(z_l)}\Big| \leq$$

$$M_2 r_l^{s+2\sigma(f)}\Big(1 + \frac{1}{\exp\{r_l^{\sigma(A_0)+\varepsilon}\}} + \frac{1}{\exp\{r_l^{\frac{n+2}{2}-\varepsilon}\}} + (k-3)\exp\{r_l^{\frac{n+2}{2}-2\varepsilon}\}\Big)$$

as $l \to +\infty$, where $M_2$ is a positive constant.
Obviously, this is a contradiction for sufficiently large $l$ and for $\varepsilon \in (0, \frac{\sigma(A_s)}{2})$.
Hence $|f^{(s)}(z)|$ must be bounded in the whole complex plane by Phragmén-Lindelöf principle.

Combining the case of (b')-(d'), by the Liouville Theorem, $f$ has to be a polynomial. This contradicts with the fact that $f$ is transcendental.

(3) Suppose that the condition (iii) holds.

This implies that the set of accumulation rays of the zero sequence of $A_s(z)$ and $A_0(z)$ are not the same. Then there exists a sector $S(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$, such that for any $\theta\in(\alpha,\beta)$, (5.7) and (5.8) hold. Then using similar reasoning as in the proof of Theorem (5.5), we get a contradiction, and then the conclusion is obtained.

(4) Suppose that the condition (iv) holds.

By Lemma 5.1, there exists a set $E_2\subset(1,+\infty)$ that has finite logarithmic measure, such that for all $z$ satisfying $|z|\notin[0,1]\cup E_2$, (5.9) holds.

Since $A_s(z)$ and $A_0(z)$ are non trivial solutions of (5.3) and (5.4) respectively, by Lemma 5.3, as $r\to+\infty$, the following equalities hold,

$$\log M(r,A_s)=(1+o(1))\frac{\sqrt{|a_n|}}{\alpha}r^{\alpha} \tag{5.17}$$

and

$$\log M(r,A_0)=(1+o(1))\frac{\sqrt{|b_m|}}{\alpha}r^{\alpha} \tag{5.18}$$

Where $\alpha=\frac{n+2}{2}$. We choose a sequence of points $\{z_l\}$ tending to infinity, $|z_l|=r_l\in(1,+\infty)-E_2$, such that

$$|A_0(z_l)|=M(r_l,A_0). \tag{5.19}$$

Combining (5.5) , (5.9), (5.17)-(5.19), as $l\to+\infty$, we get

$$\begin{aligned}\exp\Big\{(1+o(1))\frac{\sqrt{|b_m|}}{\alpha}r^{\alpha}\Big\}&=M(r_l,A_0)\\&=|A_0(z_l)|\\&\le|z_l|^{k\sigma(f)}\Big(1+\sum_{j\neq0,s}^{k-1}|A_j(z_l)|+|A_s(z_l)|\Big)\end{aligned}$$

$$\leq |z_l|^{k\sigma(f)}\Big(1+(k-2)\exp\Big\{r_l^{\alpha-\varepsilon}\Big\}+\exp\Big\{(1+o(1))\frac{\sqrt{|a_n|}}{\alpha}r_l^{\alpha}\Big\}\Big).$$

This implies that $|b_m| \leq |a_n|$. This contradicts the condition $|b_m| > |a_n|$. The conclusion of Theorem 5.6 holds.

# Conclusion

In this thesis, we have studied the properties of growth of solutions of higher-order linear differential equations.
We extended some previous results on $p$-iterated order and $p$-iterated type of solutions of linear differential equations with entire and meromorphic coefficients.
We have also investigated the hyper-order of analytic solutions of linear differential equations whose coefficients are analytic near a singular points. The question here is what about the case when coefficients are meromorphic functions near a singular point?
Finally, we have considered the growth of solutions of linear differential equations whose certain coefficients are solutions to another second-order linear differential equation. The questions that arise are :
What can be the order of growth in the non-homogeneous case?
What can we obtain if we consider the second member in this case as a solution to another second-order linear differential equation?

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