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Abstract

In this thesis, a new approach for analysis of linear singular fractional order systems is introduced. Necessary and sufficient conditions for the admissibility for both cases $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$ are established. The problem of admissibility of closed-loop systems is then treated. The given results are derived in terms of linear matrix inequalities $\mathcal{LMI}$s without using either the decomposition on the matrices of the original system or the normalization of the system. Then, an observer-based controller is designed to guarantee the admissibility for the closed-loop system with fractional derivative α belonging to $]\,0,\,1]$ and for the case $1 \leq \alpha < 2$ admissibility condition has been also proposed to design a static output controller for closed-loop systems. Finally, numerical examples are proposed to demonstrate the validity of the proposed approach.

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Publications and communications

International Publications

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Notation

$\mathbb{R}$	:	Field of real numbers
$\mathbb{C}$	:	Field of complex numbers
$\mathbb{C}^-$	:	$\{s/s \in \mathbb{C}, \mathrm{Re}(s) < 0\}$
$\mathbb{R}^{n\times m}$	:	Space of $n \times m$real matrices
$\mathbb{R}^n$	:	Space of n-dimensional real vectors
$\mathbb{C}^n$	:	Space of n-dimensional complex vectors
$\mathbb{C}^{n\times m}$	:	Space of $n \times m$complex matrices
$\in$	:	belongs to
$\times$	:	Inner product
$*$	:	Convolution product
$\otimes$	:	Kronecker product
$p \Leftrightarrow q$	:	Statements p and q are equivalent
$p \Rightarrow q$	:	Statement p implies statement q
$\mathrm{rank}(.)$	:	Rank of a matrix
$\det(.)$	:	Determinant of a matrix
$\deg$	:	Degree of a polynomial
Re	:	Real part of a complex number
I_n	:	Identity matrix of the size$n \times n$
$0_{n\times m}$	:	Zero matrix of the size$n \times m$
X^{T}	:	Transpose of matrix$X$
$X_{\perp}$	:	Orthogonal of matrix$X$
X^{-1}	:	Inverse of matrix$X$
X^*	:	Conjugate transpose of matrix$X$
$\mathrm{Sym}\{X\}$	:	$X + X^{\mathrm{T}}$, X real matrix
$\mathrm{Her}\{X\}$	:	$X + X^*$, X complex matrix
$\mathrm{diag}(X1, \ldots, Xm)$	:	Block diagonal matrix with blocks$X_1, \ldots, X_m$
$X \succeq 0$	:	X is real symmetric (or hermitian) positive semi-definite

$X \succ 0$	:	X is real symmetric (or hermitian) positive definite
$\mathrm{spec}(X) \equiv \sigma(A)$	:	Spectrum of a matrix X : set of eigenvalues of X
$\sigma(E, A)$	:	$\{s/s \in \mathbb{C}, s$ finite, $\det(sE - A) = 0\}$
X^{-1}	:	Inverse matrix of X, X must be square with $\det(X) \neq 0$
$\|.\|$	:	Modulus of a complex number
$\arg(.)$	:	Argument of a complex number
$n!$	:	Factorial(n), $n \in \mathbb{N}$: The product of all the integers from1to n
$[.]$	:	The integer part of a real number
D^{α}	:	Fractional order derivative
$\mathcal{LMI}$	:	Linear matrix inequality
$\mathcal{GLMI}$	:	Generalized linear matrix inequality
$SISO$	:	Single input single output
LTI	:	Linear time invariant
FOS	:	Fractional order system
$SFOS$	:	Singular fractional order system
$\mathcal{L}(.)$	:	Laplace transform of an argument
SVD	:	Singular value decomposition

Introduction

In the understanding and development of large class of systems it is now a well realized and accepted fact that the researchers have taken their initiation from nature. Natural things can be well understood in two possible ways, quantitative and qualitative. Mathematics plays a central role in this direction. It is the science of patterns and relationships. When we go back to understand the quantitative and qualitative behavior of nature, it seems that evolution is from integer to fraction. Quantitative behavior can be well explained using number theory, which started from integer and reached to fractional due to division operation and finally converged to real numbers. Calculus is a branch of mathematics describing how things change. It provides a framework for modeling systems undergoing change, and a way to deduce the predictions of such models. All these resulted in pointing a fact that integer order calculus is a subset of fractional calculus.

Fractional calculus can be defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer, goes back to the initiative of the philosopher and creator of modern calculus G. W. Leibniz, who made some remarks on the meaning and possibility of fractional derivative of order $\frac{1}{2}$ in the late 17:th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Surveys of the history of the fractional theory derivative can be found in [37, 87, 100, 108, 114].

For three centuries, the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. Starting from the sixties, the researches in this domain pointed out that the non-integer order derivative revealed to be more adequate tool for the description of properties of various real materials as polymers. Various types of physical phenomena, in favor of the use of models with the help of fractional derivative, that is, fractality, recursivity, diffusion and / or relaxation phenomena are given in [20]. Recent books [44, 57, 107, 115] provide a rich source of information on fractional-order calculus and its applications. The book by M. Caputo [19], published in 1969, in which he systematically used his original definition of fractional differentiation for formulating and solving problems of viscoelasticity and his lectures on seismology [18] must also added to this gallery as well as a series of A. Oustaloup's books on applications of fractional derivatives in control theory [101, 102, 103, 104].

Nowadays, interest in fractional differentiation keeps growing. Fractional tools also appear in automatic, particularly in control of dynamic systems where the system to be controlled and / or the controller are governed by fractional differential equations. The introduction of these instruments is driven by the robust character that provides Crone control (robust-control of non-integer order), introduced by A. Oustaloup in 1983. The main advantage of fractional derivative is that it provides an excellent tool for the description of memory and hereditary properties of various materials and processes in comparison with classical integer-order models.

As we all know, the problem of stability is very essential and crucial issue in control theory, especially on control of fractional-order systems. Very recently, stability and robustness of such class of systems have been investigated extensively both from an algebraic and an analytic point of view [1, 13, 73, 83] and references therein. In spite of intensive researches, many challenging and unsolved problems related to control theory of fractional order systems remain an open problems. The main contribution of this dissertation include the analysis of the admissibility and stabilization condition for singular fractional-order linear time-invariant systems. The developments summarized above are the three chapters of this thesis. The main aspects will be described. In order to facilitate the reading of this thesis, some notions and developments are recalled in the appendix.

Chapter 1: Singular Systems

This chapter discusses the state response structure for singular systems and the state space equivalent forms needed for later discussion. Some fundamental concepts in the system analysis are provided such as regularity, absence of impulsiveness and stability which together constitute the crucial criterion for singular systems that is the admissibility. Necessary and sufficient conditions in terms of linear matrix inequalities are presented. By employing these fundamental results, the closed-loop behavior is given under state feedback and static output feedback.

Chapter 2: Fractional-order systems

This chapter is devoted to the presentation of the fractional-order systems. Historical background and a comprehensive description of the theory of fractional derivation are offered: the different definitions of the fractional derivation proposed in literature (Grünwald-Letnikov, Riemann-Liouville and Caputo), Laplace transform, the functions of Mittag-Leffler, ... The representation of fractional state in the state space is given (in fact, it should rather speak of pseudo-state). The choice of the approach of Caputo for further developments in this manuscript is justified. The most important results existing in the literature concerning the stability of linear fractional-order systems are recalled, namely the famous result on the localization poles of the system and those obtained by the resolution of the linear matrix inequalities in a convex region in the complex plane for the case where the fractional-order derivative α satisfies $1 \leq \alpha < 2$ and in a non-convex region for the case $0 < \alpha < 1$. These results are used in the discussion concerning the stabilization, by both state feedback and static output feedback, for such class of systems, not forgetting to mention the results that extend the Kalman criterion for controllability and observability, then the minimum energy control problem for the standard case is formulated and solved. This part is enriched by numerical examples and simulations.

Chapter 3: Singular Fractional-Order Linear Continuous-Time Systems

This chapter covers a new class of dynamic systems, it is about singular fractional-order linear systems. In first time, The model is presented and the solution to the state equation is derived with some examples. As we all know that for singular systems, we need to consider not only stability but also the regularity and the non-impulsiveness. Specifically, regularity guarantees the existence and the uniqueness of a solution to a given singular system, while non-impulsiveness ensures no infinite dynamical modes in such system. Analysis and synthesis for singular fractional order systems were investigated in some papers. For example in [97], singular fractional order systems are considered with differentiation order between 1 and 2 and the obtained results in terms of $\mathcal{LMI}$s , under the assumption that the system is regular and impulse free, are only sufficient conditions to get asymptotic stabilization. These results are derived using the decomposition of the original system with Weierstrass canonical form. For the same class of systems with alpha between 0 and 2, results derived for the stability and stabilization problem are also just sufficient conditions in [118]. In [94], the robust stabilization of uncertain descriptor systems with the fractional order derivative belonging $(0, 2)$ was treated using the concept of the normalization to check sufficient conditions. Improvements in our work compared to that shown previously are such that our result ensures the three criteria to get admissibility and stabilization of singular fractional order systems. Necessary and sufficient conditions are derived in terms of $\mathcal{LMI}$s where the matrices of the original system are involved. Using the obtained result, to ensure the admissibility of the closed-loop system is determined with the help of a static output feedback controller for the case $1 \leq \alpha < 2$ in [79]and an observer based control for the case $0 < \beta < 1$ in [80].

Chapter 1

Singular Linear Systems

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1.1 Introduction

In this chapter, we define the class of singular linear invariant-time systems that will be used in our study. Some fundamental results will be reminded. Singular systems are a powerful tool for modeling insofar as they can describe processes governed simultaneously by dynamic and static equations. Such formalism is thus particularly suited to the study of interconnected systems, subjected to physical constraints with static and impulsive behavior. In order to enlighten the reader, we recall some basic properties of singular systems such the necessity and sufficient condition for the existence of a single trajectory system to an input and initial condition data. We give here a quick reminder of the fundamental useful results as the equivalence between representation of state, regularity, impulsiveness, the analysis and the controller design for such systems. All these results will be used to derive the main results of the third chapter.

1.2 Overview of Singular Linear Invariant-Time Systems

Modeling a complex physical process usually starts with the choice of variables used for its description and by the choice of magnitudes allowing to act on the evolution of the system. These variable, called state variables and control, are selected as far as possible to have a physical signification (position, speed, acceleration, temperature, pressure, etc...). After the choice of these variables, the mathematical relationships connecting the selected variables are dictated by the laws of the behavior of the considered system. These relationships can be of two types: dynamic (i.e. involved the variations of the variables over the time) or purely static. We arrive at a setting in equation of the form

$$\begin{aligned} 0 &= f(\dot{x}(t), u(t), y(t)) \\ 0 &= g(x(t), u(t), y(t)) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector grouping the state variables, $\dot{x}(t)$ is the derivative with respect to time, $u(t) \in \mathbb{R}^m$ means the control vector and $y(t) \in \mathbb{R}^p$ is the vector of the measured outputs. After linearization around an operating point (for example by the tangent linearized, using as variables, deviations point functioning) we obtain the following formalism

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1.2}$$

where E, A, B, C, D are real constant matrices of compatible dimensions with those of $x(t), u(t)$ and $y(t)$. The derivatives of several state variables can be involved in the same relation, therefore E has not necessarily a diagonal structure. All the relationships of the behavior are not necessarily dynamic, E is not necessarily of full line rank. It can, moreover, be considered without loss of generality that E, and A are square matrices, this can be done by completing with zero lines until obtaining matrices of $n \times n$ dimensions. It is not restrictive to assume zero the direct term transfer of the command to the output, in fact it suffices to increase the state vector to include $u(t)$ and annul the matrix D of the direct transfer of $u(t)$ to $y(t)$ in equation (1.2). Indeed, equation (1.2) can be expressed as

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \dot{\begin{bmatrix} x \\ \xi \end{bmatrix}} &= \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} u \\ y &= \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \end{aligned} \tag{1.3}$$

So, in the following we will adapt for linear singular continuous time systems this formalism

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.4}$$

and for linear singular discrete time system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.5}$$

Singular systems are also known as descriptor systems, implicit systems, semi-state systems, differential-algebraic systems, generalized state-space systems, constrained systems and so on. Compared with normal systems (i.e. state-space systems), singular systems contain both differential and algebraic equations, therefore they can describe dynamic and algebraic constraints simultaneously. Due to the more general descriptions than normal systems, singular systems have been widely studied by many authors in the past decades [34, 64]. This is due not only to the theoretical interest but also to the extensive application of such systems in different research areas such as economic systems [74], electrical networks [98], chemical process [57] and highly interconnected large-scale systems [34], etc.

A singular system has important specific characteristics compared with a state-space system [11, 125, 132].

When the matrix E is invertible, however in this case it is possible to reduce the common state representation by pre-multiplying the state equation by E^{-1} then we get

$$\begin{aligned} \dot{x}(t) &= E^{-1}Ax(t) + E^{-1}Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.6}$$

But it is noted that even when E is invertible it is preferable to use the singular representation due to the eventual bad conditioning of $E^{-1}A$. Furthermore, the matrix E is not necessarily full rank, in which case $\operatorname{rank}(E) = r \leq n$.

1.2.1 Regularity of Singular System

In the case of normal[1] systems, for any initial condition x_0 and for any input $u(t)$ known on the interval $[0, t]$, the state response is unique and is given by the formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1.7}$$

[1]The classic linear system in linear system theory is termed the normal system here for the sake of distinction with singular systems

For singular systems, the response is uniquely defined for a sufficiently differentiable input and a given initial condition if and only if the pencil (E, A) is regular.

Definition 1.2.1. *[43] A pencil of matrices (E, A) or $sE - A$ is called regular if*
(i) E and A are square matrices of the same order n.
(ii) The determinant $|sE - A|$ does not vanish identically,
or equivalently if there exist a scalar $s \in \mathbb{C}$ such that $|sE - A| \neq 0$.

In other words, the regularity means also the solvability as used by Yip and Sincovec [132]. To illustrate the necessity of the regularity, let us apply the Laplace transformation to the dynamic equation in (1.6), we get

$$E\mathcal{L}(\dot{x}) = A\mathcal{L}(x) + B\mathcal{L}(u) \tag{1.8}$$

i.e.,

$$sE\mathcal{L}(x) - Ex_0 = A\mathcal{L}(x) + B\mathcal{L}(u) \tag{1.9}$$

which can be rewritten as

$$(sE - A)\mathcal{L}(x) = B\mathcal{L}(u) + Ex_0 \tag{1.10}$$

The equation (1.10) has a unique solution $\mathcal{L}(x)$ for any initial condition and any continuous input if and only if the matrix $sE - A$ is invertible which means that the pencil(E, A) must be necessarily regular. Indeed, if the pencil(E, A) is not regular then it will exist non-zero vector v such that

$$(sE - A)v = 0$$

It is clear that if $\mathcal{L}(x)$ is a solution of (1.10), then all vectors $\mathcal{L}(x) + \alpha v$ are also solutions for any α. Consequently, the system has not a unique solution, and it is also obvious that there may be no solution for this system. The following lemma in [132] allows to check the regularity of the pencil (E, A) (or the pair (E, A)).

Lemma 1.2.1. *The following proposals are equivalent.*
1) The pair (E, A) is regular.
2) There exist two non-singular matrices P and Q such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{1.11}$$

where $n_1 + n_2 = n$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent with h index of nilpotence (i.e., $N^h = 0, N^{h-1} \neq 0$).

3) The matrix $F(k) \in \mathbb{R}^{nk \times n(k+1)}$ *is of full rows rank for* $k \geq 1$ *with*

$$F(k) = \begin{bmatrix} E & A & & & & \\ & E & A & & & \\ & & E & A & & \\ & & & \ddots & \ddots & \\ & & & & E & A \end{bmatrix}$$

Remark 1.2.1. *The easiest to be implemented to checking is the latest, but requires the computation of the rank of a large matrix. In order to overcome this difficulty, Luenberger proposes, in [75], an algorithm requiring only manipulation of rows and columns of the matrix* $[E\ A]$ *called "Shuffle algorithm".*

1.2.2 Equivalent Singular Systems

The choice of variables used to describe a process is generally not unique, subsequently the model describing the process is not unique. Moreover, it is often useful to change the space to derive interesting structural properties or to simplify the implementation of the corrector or the associated observer. It is therefore pertinent to determine an equivalence relation between state representations modeling a same system.

For a given singular system (E, A, B, C), two equivalent forms have a particular interest and will often used for analysis and control. It is about the Weierstrass-Kronecker form and the decomposition by singular values of the matrix E. The first form uses a result established in [43], stated in Lemma 1.2.1. We can therefore define the Weierstrass-Kronecker decomposition which is also called standard decomposition.

Definition 1.2.2. *(Weierstrass-Kronecker decomposition) For any regular system* (1.4)*, there exist non-singular matrices* Q *and* P *such that* (1.4) *is equivalent to*

$$\begin{cases} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ y_1(t) &= C_1 x_1(t) \end{cases} \tag{1.12}$$

$$\begin{cases} N\dot{x}_2(t) &= x_2(t) + B_2 u(t) \\ y_2(t) &= C_2 x_2(t) \end{cases} \tag{1.13}$$

$$y(t) = C_1 x_1(t) + C_2 x_2(t) \tag{1.14}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, N is nilpotent matrix with h the index of nilpotence and

$$\begin{aligned} QEP &= \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \\ QB &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, P^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{1.15}$$

The matrices Q and P are not unique. Indeed, if Q and P define a standard form for the system (1.4), then for any non-singular matrices T_1, T_2, $\bar{Q} = \mathrm{diag}(T_1, T_2)Q$ and $\bar{P} = P\,\mathrm{diag}(T_1^{-1}, T_2^{-1})$ also define a standard form for (1.4). If the regularity of the system is not known, then this form cannot be applied. Moreover, the Kronecker-Weierstrass decomposition is sometimes numerically unreliable, especially in the case where the order of the system is relatively large. Another decomposition which does not depend on the regularity of systems is called the singular value decomposition form of the matrix E, which consists of separating the dynamic relationships from the static relationships.

Definition 1.2.3. *For any matrix $E \in \mathbb{R}^{n\times n}$, there exist two non-singular matrices Q and P such that*

$$QEP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

By taking the coordinate transformation $P^{-1}x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 \in \mathbb{R}^r, x_2 \in \mathbb{R}^{n-r}$, the system (1.4) *is equivalent to*

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ 0 &= A_{21}x_1 + A_{22}x_2 + B_2u \\ y &= C_1x_1 + C_2x_2 \end{aligned} \tag{1.16}$$

where

$$QAP = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, CP = \begin{bmatrix} C_1 & c_2 \end{bmatrix} \tag{1.17}$$

In (1.16), the first equation is a differential one composed of the dynamics of the system, whereas the second equation is algebraic which encompasses the interconnections and static constraints.

1.2.3 Temporal Response

Under the regularity assumption, the system (1.4) is defined by both equations (1.12)-(1.14), where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and N is nilpotent with h its index of nilpotence.

Note that the subsystem (1.12) is an usual linear differential equation, it has then a unique solution with any initial condition $x_1(0)$ and for any continuous input $u(t)$

$$x_1(t) = e^{A_1 t}x_1(0)\int_0^t e^{A_1(t-\tau)}B_1u(\tau)d\tau \tag{1.18}$$

Thus, $y_1(t)$ is completely determined by $x_1(0)$ and $u(\tau), 0 \leq \tau \leq t$.

To obtain the substate of the subsystem (1.13), the Laplace transformation may be applied. We obtain

$$(sN - I)X_2(s) = Nx_2(0) + B_2U(s) \tag{1.19}$$

where $X_2(s)$ and $U(s)$ stand for the Laplace transform of $x_2(t)$ and $u(t)$ respectively. From the equation (1.19) and taking account the nilpotence of the matrix N we obtain

$$X_2(s) = (sN - I)^{-1}(Nx_2(0) + B_2U(s)) = -\sum_{k=0}^{h-1} N^k s^k (Nx_2(0) + B_2U(s)) \tag{1.20}$$

Note that the Laplace transform of the Dirac function $\delta(t)$ is as follows

$$\mathcal{L}(\delta^k(t)) = s^k \tag{1.21}$$

Hence the inverse Laplace transform of $X_2(s)$ yields

$$x_2(t) = -\sum_{k=0}^{h-1}\delta^k(t)N^{k+1}x_2(0) - \sum_{k=0}^{h-1}N^k B_2 u^{(k)}(t) \tag{1.22}$$

In this case, the state response takes the form

$$\begin{aligned} x(t) = P\begin{bmatrix} I \\ 0 \end{bmatrix}\left(e^{A_1 t}x_1(0) + \textstyle\int_0^t e^{A_1(t-\tau)}B_1u(\tau)d\tau\right) + \\ P\begin{bmatrix} 0 \\ I \end{bmatrix}\left(-\sum_{k=0}^{h-1}\delta^k(t)N^{k+1}x_2(0) - \sum_{k=0}^{h-1}N^k B_2 u^{(k)}(t)\right) \end{aligned} \tag{1.23}$$

Particularly by setting $t > 0$, $t \to 0^+$, we must have

$$x(0^+) = P\begin{bmatrix} I \\ 0 \end{bmatrix}x_1(0) - P\begin{bmatrix} 0 \\ I \end{bmatrix}\sum_{k=0}^{h-1}N^k B_2 u^{(k)}(0^+) \tag{1.24}$$

Initial conditions satisfying the constraint (1.24) are called admissible conditions.

$x_1(t)$ in (1.18) represents a cumulative effect of $u(\tau), 0 \leq \tau \leq t$ with no relation to $u(t)$, while the response $x_2(t)$ is so fast it insistently reflects the properties of u at time t. This is why the subsystems (1.12) and (1.13) are called the slow and fast subsystems, respectively.

The formalism (1.23) can be used to represent the systems whose initial conditions are not admissible or those containing "jump" behaviors. Usually the jumps are undesirable or dangerous for the system security. So, we will distinguish systems called impulsive and impulse-free for continuous systems. For discrete systems, we talk about causal system.

1.2.4 Impulse Free System

Definition 1.2.4. *(Impulse free system) The singular system* (1.4) *is impulse free (without pulses) if its response is continuous for any initial condition and any control* $u(t)$ *which is* $(h-1)$ *piecewise continuously differentiable.*

From (1.22), we can deduce the following lemma.

Lemma 1.2.2. *The following assumptions are equivalent.*
1) The system (1.4) *or the pencil of the pair* (E, A) *is impulse free.*
2) The system (1.4) *is regular and the matrix* N *in the Weierstrass-Kronecker decomposition is zero.*
3) The matrix A_{22} *in the singular values decomposition is invertible.*
4) The equality $\deg(\det(sE-A)) = \operatorname{rank}(E)$ *is verified.*
5) $\operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \end{bmatrix} - \operatorname{rank}(E) = n$

Furthermore, provided that the descriptor system (1.4) is regular and invertible matrices Q and P exist to make it Weierstrass-Kronecker form. The transfer function $G(s)$ of this system can be written as

$$G(s) = C_1(sI - A_1)^{-1}B_1 - C_2(sN - I)^{-1}B_2 \tag{1.25}$$

For an impulse free system, that $N = 0$, we have

$$G(s) = C_1(sI - A_1)^{-1}B_1 + C_2B_2 \tag{1.26}$$

It is noted that the term $C_2(sN-I)^{-1}B_2$ creates polynomial terms of s if both B_2 and C_2 are non zero. Hence the impulse free assumption guarantees to the transfer function $G(s)$ to be proper. The converse statement is, however, not true. Clearly if either B_2 or C_2 vanishes, the transfer function is still proper[2], even if the system is impulsive.

[2] The transfer function of any linear system is a rational function $G(s) = \frac{n(s)}{d(s)}$ where $n(s)$, $d(s)$ are polynomials. $G(s)$ is proper if $m = \deg n(s) \leq n = \deg d(s)$. $G(s)$ is strictly proper if $m < n$.

1.3 Singular Systems Analysis

1.3.1 Stability of Singular Systems

Stability is a fundamental property of a dynamic system because it guarantees that the response of the system does not diverge in response to an input and a finite initial condition. The definition of the exponential stability of a singular system is identical to that of normal systems.

Definition 1.3.1. *Singular system* (1.4) *is called exponentially stable if there exist scalars* α, $\beta > 0$ *such that when for* $u(t) = 0$, $t > 0$ *its state satisfies*

$$\|x(t)\|_2 \leq \alpha e^{-\beta t} \|x(0)\|_2 \, , t > 0 \tag{1.27}$$

It is evident that if the system (1.4) is exponentially stable, then $\lim_{t\to\infty} x(t) = 0$ which is called the asymptotic stability and is the most frequently used. Consider the system (1.4) in the form of Weierstrass-Kronecker. For $u(t) = 0$, $t > 0$, the state vector of the system is $x(t) = e^{A_1 t} x_{10}$ since $x_2(t) = 0$. So, the stability of (1.4) essentially depends on that of the slow subsystem (1.12) and consequently on the location of the eigenvalues of the matrix A_1.
We will use

$$\sigma(E, A) = \{s/s \in \mathbb{C}, s \text{ finite}, \ \det(sE - A) = 0\} \tag{1.28}$$

to denote the finite pole set for the system (1.4) and $\sigma(A_1) = \sigma(I, A_1)$ to the set of the eigenvalues of the matrix A_1. Since any nilpotent matrix has all its eigenvalues equal to zero, it is easy to show, using the Weierstrass-Kronecker equivalent form, that

$$\sigma(E, A) = \sigma(A_1)$$

We can then give an algebraic characterization for the stability of singular systems.

Theorem 1.3.1. *The singular system* (1.4) *or* (1.5) *is stable if and only if*

$$\sigma(E, A) \subset \mathbb{C}^-$$

i.e.,

$$\text{Re}(\sigma(E, A)) < 0$$

$\mathbb{C}^-$ *represents the open left half complex plane.*

The stability condition given in the previous theorem depends only on the finite poles of the system, or precisely on the stability of the slow subsystem, but does not imposes the impulsiveness of the system. Although, the system is stable but the output energy can be infinite for a finite energy input. To avoid this paradox, an additional concept called admissibility is defined for singular systems which plays the same role as stability for state-space systems.

1.3.2 Admissibility of Singular Systems

Definition 1.3.2. *Consider the descriptor system* (1.4).
1) The system (1.4) *is said to be regular if* $\det(sE - A)$ *is not identically zero.*
2) The system (1.4) *is said to be impulse free if* $\deg(\det(sE - A)) = \text{rank}(E)$.
3) The system (1.4) *is said to be stable if all roots of* $\det(sE - A)$ *have negative parts.*
4)The system (1.4) *is said to be admissible if it is regular, impulse free and stable.*

Furthermore, it can be deduced that if a descriptor system is impulse free, then it is regular. There exist some conditions equivalent to the admissibility.

Lemma 1.3.1. *Suppose that the descriptor system* (1.4) *is regular then there exist non-singular matrices* Q *and* P *such that the Weierstrass-Kronecker equivalent form holds. Then,*
1) (1.4) *is impulse free if and only if* $N = 0$.
2) (1.4) *is stable if and only if* $\text{Re}(\sigma(A_1) < 0$.
3) (1.4) *is admissible if and only if* $N = 0$ *and* $\text{Re}(\sigma(A_1) < 0$.

When the regularity of the system (1.4) is not known, it is always possible to choose two non-singular matrices Q and P such that the decomposition via singular values can be obtained (see (1.16)). Then, we have the following result.

Lemma 1.3.2. *1) The system* (1.4) *is impulse free if and only if the matrix* A_{22} *is non-singular.*
2) The system (1.4) *is admissible if and only if* A_{22} *is non-singular and* $\text{Re}(\sigma(A_{11} - A_{12}A_{22}^{-1}A_{21})) < 0$.

Both Lemmas 1.3.1 and 1.3.2 provide equivalent conditions for the admissibility of the system (1.4). It is noted that Lemma 1.3.1 is based on the assumption of the regularity of (1.4) and the conditions involve the decomposition of the matrices of the original system.
In the following, necessary and sufficient conditions for the admissibility of linear singular system involved the matrices of the original system via $\mathcal{LMI}$s (Linear Matrix Inequalities) and without

assuming the regularity of the system. An overview of the $\mathcal{LMI}$s is given in Appendix B
For linear singular continuous invariant-time system, we have the following results.

Theorem 1.3.2. *[81, 130] Consider the system* (1.4)*. The following statements are equivalent.*
1) The unforced system of (1.4) *is admissible.*
2) There exists a matrix $X \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{array}{rcl} E^T X & = & X^T E \succeq 0 \\ X^T A & + & A^T X \prec 0 \end{array} \tag{1.29}$$

3) There exists a matrix $X \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{array}{rcl} EX & = & X^T E^T \succeq 0 \\ X^T A^T & + & AX \prec 0 \end{array} \tag{1.30}$$

Proof. For the proof, see both papers [130] and [81] whose offer different derivation. □

It is noted that conditions (1.29) and (1.30) developed in 1.3.2 are non-strict $\mathcal{LMI}$s, it may result in numerical problems when checking since equality constraints are fragile and usually not satisfied perfectly. In most cases the non-strict $\mathcal{LMI}$s have non feasible solution and the equality constraints can not be directly solved with $\mathcal{LMI}$s. Therefore strict $\mathcal{LMI}$ conditions are more tractable and numerically reliable. To present a Lyapunov-type stability condition (see B), two adjective parameters V, U are introduced.

Theorem 1.3.3. *[123]The following statements are equivalent.*
1) The unforced system of (1.4) *is admissible.*
2) There exist matrices $P \in \mathbb{R}^{n\times n} \succ 0$*,* $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ *such that*

$$A(PE^T + VQU^T) + (PE^T + VQU^T)^T A^T \prec 0 \tag{1.31}$$

where $V, U \in \mathbb{R}^{n\times(n-r)}$ *are any matrices of full column rank and verify* $EV = 0$ *and* $E^T U = 0$

A similar result was established in [130] where only one adjective parameter is introduced.

Theorem 1.3.4. *[130]The following statements are equivalent.*
1) The unforced system of (1.4) *is admissible.*
2) There exist matrices $P \in \mathbb{R}^{n\times n} \succ 0$*,* $Q \in \mathbb{R}^{(n-r)\times n}$ *such that*

$$(PE + SQ)^T A + A^T (PE + SQ) \prec 0 \tag{1.32}$$

where $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^T S = 0$
3) There exist matrices $P \in \mathbb{R}^{n\times n} \succ 0$, $Q \in \mathbb{R}^{(n-r)\times n}$ such that

$$(PE^T + SQ)^T A^T + A(PE^T + SQ) \prec 0 \tag{1.33}$$

where $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $ES = 0$

Proof. For the proof, see [130]. □

There exists an other result where a matrix P is chosen as parameter depended only on the two matrices related on restricted equivalent transform of the system (1.4).

Theorem 1.3.5. *[131] The system* (1.4) *is admissible if for two chosen non-singular matrices M and N satisfying $MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $M, N \in {}^{n\times n}$ such that*

$$P^T A + A^T P \prec 0 \tag{1.34}$$

where $r = \operatorname{rank}(E)$ and $P = M^T N^{-1}$.

Condition of admissibility in Theorem 1.3.5 is in strict $\mathcal{LMI}$, does not need equality constraint condition , does not need to introduce the basis of null space of the matrix E but is only a sufficient condition but not necessary condition.

The following Theorem gives necessary and sufficient conditions for descriptor discrete-time system (1.5), with $u(t) = 0$, to be admissible.

Theorem 1.3.6. *The discrete-time descriptor system* (1.5) *or the pair (E, A) is admissible if and only if the following equivalent statements hold.*
(i) There exists a matrix $X = X^T$ satisfying the following $\mathcal{LMI}$:

$$E^T X E \succeq 0, \quad A^T X A - E^T X E \prec 0 \tag{1.35}$$

(ii) There exist matrices $P \succ 0$ and $Q = Q^T$ satisfying the following $\mathcal{LMI}$:

$$A^T(P - E^{\perp T} Q E^{\perp})A - E^T P E \prec 0 \tag{1.36}$$

(iii) There exist matrices $P \succ 0$ and $Q = Q^T$ satisfying the following $\mathcal{LMI}$:

$$A(P - E^{\perp T \perp} Q E^{\perp T})A^T - EPE^T \prec 0 \tag{1.37}$$

(iv) There exist matrices $P \succ 0$, $Q = Q^T$, F and G satisfying the following $\mathcal{LMI}$:

$$\begin{bmatrix} -E^T PE + A^T F^T + FA & -F + A^T G^T \\ -F^T + GA & P - E^{\perp T} QE^{\perp} - G - G^T \end{bmatrix} \prec 0 \tag{1.38}$$

(v) There exist matrices $P \succ 0$, $Q = Q^T$, F and G satisfying the following $\mathcal{LMI}$:

$$\begin{bmatrix} -E^T PE + AF^T + FA^T & -F + AG^T \\ -F^T + GA^T & P - E^{\perp T \perp} QE^{\perp T} - G - G^T \end{bmatrix} \prec 0 \tag{1.39}$$

where $E^{\perp}$ is any matrix such that $E^{\perp}E = 0$ and $E^{\perp}E^{\perp T} \succ 0$.

Proof. For the proof, see [23] □

Remark 1.3.1. *Note that for the standard case, i.e. when $E = I$, we get $E^{\perp} = 0$ and $\mathcal{LMI}$ conditions* (1.38) *or* (1.39) *are reduced to the existence of matrices $P \succ 0$, F and G as given in [106] or with $F = 0$ as it is stated in the earlier work of Oliveira and al. in [105].*

1.4 Singular System Synthesis

We will use the term feedback control to refer to state feedback and static output feedback, one of the commonly used methods to change the system's dynamic or static properties. The basic objectives of the singular control systems are that the controlled system is stable without impulsiveness and its dynamics fixed arbitrary.

1.4.1 State feedback control

Here, we assume that all the state variables are available for a state feedback. Consider the singular linear system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1.40}$$

and consider the following state feedback

$$u(t) = Kx(t) \tag{1.41}$$

where $K \in \mathbb{R}^{m \times n}$ is a gain matrix to be determined.
Applying the controller (1.41) to (1.40), we obtain the the closed-loop system as follows

$$E\dot{x}(t) = (A + BK)x(t) \tag{1.42}$$

Definition 1.4.1. *The singular system* (1.40) *is called stabilizable if there exists a state feedback* (1.41) *such that the closed-loop system* (1.42) *is stable.*

Conditions of the existence of a state feedback control making the system (1.42) stable and impulse free are cited in the following theorem.

Theorem 1.4.1. *[34]*
1) There exists a state feedback control (1.41) *such that the system* (1.42) *is stable if and only if for any* $s \in \mathbb{C}, \mathrm{Re}(s) \geq 0$

$$\mathrm{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n, \tag{1.43}$$

2) There exists a state feedback control (1.41) *such that the system* (1.42) *is impulse free if and only if*

$$\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \mathrm{rank}(E) \tag{1.44}$$

3) There exists a state feedback control (1.41) *such that the system* (1.42) *is admissible if and only if*

$$\mathrm{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = n + \mathrm{rank}(E) \tag{1.45}$$

and for any $s \in \mathbb{C}, \mathrm{Re}(s) \geq 0$

$$\mathrm{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n, \tag{1.46}$$

Based on the stability conditions presented in both Theorems 1.3.3 and 1.3.4, the stabilizing controller design can be formulated as a convex optimization problem characterized by linear matrix inequalities $\mathcal{LMI}$.

Theorem 1.4.2. *[123]There exists a state feedback controller* (1.41) *such that the closed-loop system* (1.42) *is admissible if and only if there exist matrices* $P \in \mathbb{R}^{n\times n} \succ 0$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $L \in \mathbb{R}^{m\times n}$ *and* $H \in \mathbb{R}^{m\times(n-r)}$ *which satisfy the following* $\mathcal{LMI}$

$$\mathrm{Sym}\left\{A(PE^T + VSU^T) + B(LE^T + HU^T)\right\} \prec 0 \tag{1.47}$$

where V *and* U *are any matrices of full column rank and satisfying* $EV = 0$ *and* $E^TU = 0$ *Then, a stabilizing feedback gain is given by*

$$K = (LE^T + HU^T)(PE^T + VSU^T)^{-1} \tag{1.48}$$

Similar result is derived in [130]

Theorem 1.4.3. *Consider the continuous singular system* (1.40)*. There exists a state feedback controller* (1.41) *such that the closed-loop system* (1.42) *is admissible if and only if there exist matrices* $P \in \mathbb{R}^{n\times n} \succ 0$, $Q \in \mathbb{R}^{(n-r)\times n}$ *and* $Y \in \mathbb{R}^{m\times n}$ *such that*

$$\text{Sym}\left\{A(PE^{T} + SQ) + BY\right\} \prec 0 \tag{1.49}$$

where S is any matrix with full column rank and satisfies $ES = 0$. In this case, we can assume that the matrix $PE^{T} + SQ$ is non-singular (if this is not the case, then we can chose some $\theta \in (0, 1)$ such that $PE^{T} + SQ + \theta\tilde{P}$ is non-singular and satisfies (1.49)*, in which $\tilde{P}$ is any non-singular matrix satisfying $E\tilde{P} = \tilde{P}^{T}E^{T} \succeq 0$), then a stabilizing state feedback controller can be chosen as*

$$u(t) = Y(PE^{T} + SQ)^{-1}x(t) \tag{1.50}$$

Remark 1.4.1. *In the proof of Theorem* (1.4.3)*, it is assumed that the matrix $PE^{T} + SQ$ is non-singular. If this is not the case, a small perturbation in $PE^{T} + SQ$ to make it nonsingular without violating* (1.49).

1.4.2 Output Feedback Control

In practical applications, usually not all the state variables are available for feedback. In this sub-section we address the problem of admissibility by static output-feedback for the descriptor system given by (1.4). So, we need to define its dual system.
Consider the singular system defined by (1.4),

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

The system

$$\begin{aligned} E^{T}\dot{z}(t) &= A^{T}z(t) + C^{T}u(t) \\ y(t) &= B^{T}z(t) \end{aligned}$$

is called its dual system.
The control law given by a static output-feedback is given by

$$u(t) = Ky(t) \tag{1.51}$$

We obtain the closed-loop system of (1.4)

$$\begin{aligned} E\dot{x}(t) &= (A + BKC)x(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.52}$$

where the constant gain K, of appropriate dimensions, is computed in such a way that the singular closed-loop system (1.52) is admissible. The stabilizability characterizes the controllability of the system's stability. The dual concept of stabilizability is detectability which is defined as follows.

Definition 1.4.2. *The singular linear system* (1.4) *is called detectable if its dual system* (E^T, A^T, C^T, B^T) *is stabilizable.*

Using the properties of the transpose of the matrices, we get an equivalent definition

Definition 1.4.3. *The singular linear system* (1.4) *is called detectable if there exists a matrix* $G \in \mathbb{R}^{n\times p}$ *such that the pair* $(E, A + GC)$ *is stable, i.e.* $\sigma(E, A + GC) \subset \mathbb{C}^-$

Therefore, the impulse terms should be also eliminated in the state response, in other terms, the closed-loop system must be not only stable but also impulse free so we have the following theorem.

Theorem 1.4.4. *The following statements are true.*
1) The system (1.4) *is detectable if and only if for any* $s \in \mathbb{C}$ *finite,* $\mathrm{Re}(s) \geq 0$,

$$\mathrm{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n \tag{1.53}$$

2) The closed-loop system $(E, A + GC)$ *is impulse free if and only if*

$$\mathrm{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \mathrm{rank}(E) \tag{1.54}$$

3) The closed-loop system $(E, A + GC)$ *is admissible if and only if*

$$\mathrm{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \mathrm{rank}(E) \tag{1.55}$$

and for any finite $s \in \mathbb{C}$, $\mathrm{Re}(s) \geq 0$

$$\mathrm{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n \tag{1.56}$$

To solve the static output-feedback admissibility problem, the following theorem is introduced in [21]

Theorem 1.4.5. *The continuous singular system* (1.4) *is admissible if and only if there exist three matrices X, Y, and Z such that*

$$EXE^{T} + \mathrm{Sym}\left\{E^{\dagger}Z\right\} \succ 0 \tag{1.57}$$

$$\mathrm{Sym}\left\{A(XE^{T} + E^{\perp}Y)\right\} \prec 0 \tag{1.58}$$

where $E^{\dagger} = U^{-1}(I - UEV)U$ and $E^{\perp} = V(I - UEV)U$, that fulfill $EE^{\perp} = 0$ and $E^{\dagger}E = 0$ with U and V are non-singular matrices satisfying $UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

It must be pointed out that if the singular system defined by the pair (E, A) is admissible, then its dual, defined by the pair $(E^{\mathrm{T}}, A^{\mathrm{T}})$, is also admissible; so Theorem 1.4.5 can also be written as follows.

Corollary 1.4.1. *The continuous singular system (E^{T}, A^{T}) is admissible if and only if there exist matrices X, Y and Z such that*

$$E^{T}XE + \mathrm{Sym}\left\{E^{\dagger}Z\right\} \succ 0 \tag{1.59}$$

$$\mathrm{Sym}\left\{A^{T}(XE + E^{\ddagger}Y)\right\} \prec 0 \tag{1.60}$$

with $E^{\ddagger} = U^{T}(I - UEV)U^{-T}$ and $E^{\dagger} = U^{T}(I - UEV)V^{T}$ that fulfill $E^{T}E^{\ddagger} = 0$ and $E^{\dagger}E^{T} = 0$ with U and V are non-singular matrices satisfying $UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Then, the closed-loop system (1.52) is admissible if the following inequality is verified:

$$\mathrm{Sym}\left\{(A + BKC)^{\mathrm{T}}(XE + E^{\ddagger}Y)\right\} \prec 0 \tag{1.61}$$

where the matrix X satisfies condition (1.59).
Solution to this problem was proposed by M. Chaabane and all in [21] in two steps given in lemmas 3.1 and 3.2 respectively. The first step is devoted to establishing a relation between a classical state feedback controller design and static output feedback. In the second step some relaxed variables are included and so the design of the controller gain is formulated as an LMI problem.

1.5 Conclusion

In this chapter, We presented the state of the art on linear singular continuous invariant time systems. Some basic concepts are recalled as equivalent realizations and decomposition of the system. Important results for this class of systems such the regularity, the impulsiveness, the stability and the admissibility are reviewed. The concepts of controllability and observability of linear singular systems Were also presented with their extension to the stabilization by state feedback and static output feedback. These results will be used at the last chapter to obtain conditions for admissibility and stabilization for singular fractional order systems.

Chapter 2

Fundamentals of Fractional Order Systems

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2.1 Introduction

Fractional calculus can be defined as the generalization of classical calculus to order of integration and differentiation not necessary integer. The theory of derivatives of non-integer order goes back to the Leibniz's note in his letter to l'Hospital [61], dated 30 September 1695, in which the meaning of the derivative of order one half is discussed. The question raised by Leibniz for a non-integer-order derivative was an ongoing topic for more than 300 years. Thereafter, the theory of derivatives and integrals of arbitrary order has appeared, which by the end of XIX century took more or less form due primarily to Liouville (1832,1837), Riemann (1847), Grünwald (1867), Letnikov (1868) [62]. For three centuries the theory of fractional derivatives was developed mainly as a pure theoretical field of mathematics useful only for mathematicians.

However, during the past few years, the fractional calculus has aroused a growing interest taking benefit of the fractional operator compacity for description of memory and hereditary properties of various materials and physical systems which are often neglected in the classical integer-order models and plays a significant role in modeling of real-word phenomena as electromagnetic systems [38], dielectric polarization [121], viscoelastic systems [9]. In this Chapter we will give in first time a brief overview of the fractional order calculus, then the fractional order dynamical systems and their behavior will be also discussed.

2.2 Fractional order operators and properties

2.2.1 Special Functions of the Fractional Calculus

Here, we give some information on the Euler's gamma and the Mittag-Leffler functions which play the most important role in the theory of the differentiation of arbitrary order.

- **The Gamma Function**
  One of the basic function of the fractional calculus is Euler' s gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

  Definition 2.2.1. *(Gamma Function) The gamma function* $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{2.1}$$

  which converges in the right half of the complex plane $\mathrm{Re}(z) > 0$.

  The Figure (2.1) represents the Gamma function.

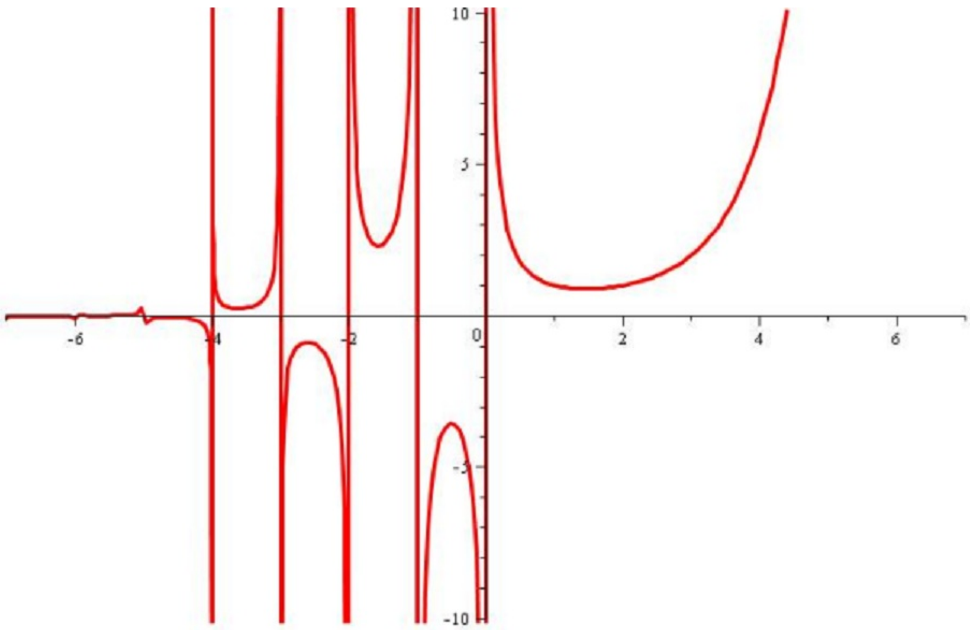

Figure 2.1: Representation of the Gamma Function on $\mathbb{R}$.

One of the basic properties of the gamma function is that it satisfies the following function equation

$$\Gamma(z+1) = z\Gamma(z)$$

which can be easily proved by integrating by parts

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t} t^z dt \\ &= [-e^{-t}t^z]_0^\infty + \int_0^\infty z e^{-t} t^{z-1} dt \\ &= z\Gamma(z)\end{aligned}$$

- **The Mittag-Leffler Function**

  The exponential function, e^z, plays a very important role in the theory of integer-order differential equations. Its one parameter generalization, called the Mittag-Leffler function, is the function which is now denoted by [41]

  $$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

  It was introduced by G. M. Mittag-Leffler [88, 89, 90] and studied also by A. Wiman [128]. For $\alpha = 1$, we obtain

  $$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

  - The two-parameter function of the Mittag-Leffler type, which plays a very important role in the fractional calculus, was introduced by Agarwal in [3]. This function could have been called the Agarwal function. However, Humbert and Agarwal obtain a number of relationships for this function and generously left the same notation as for the one parameter Mittag-Leffler function, and that is the reason that now the two-parameter function is also called the Mittag-Leffler function.

    Definition 2.2.2. *A two-parameter function of the Mittag-Leffler type is defined by the series expansion [41]*

    $$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0$$

It follows from the definition

for $\beta = 1, \alpha > 0,\ E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} = E_\alpha(z)$

for $\beta = \alpha = 1,\ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$

for $\beta = 2, \alpha = 1,\ E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z-1}{z}$

for $\beta = 3, \alpha = 1,\ E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z-z-1}{z^2}$

The Figure (2.2) represents the Mittag-Leffler function.

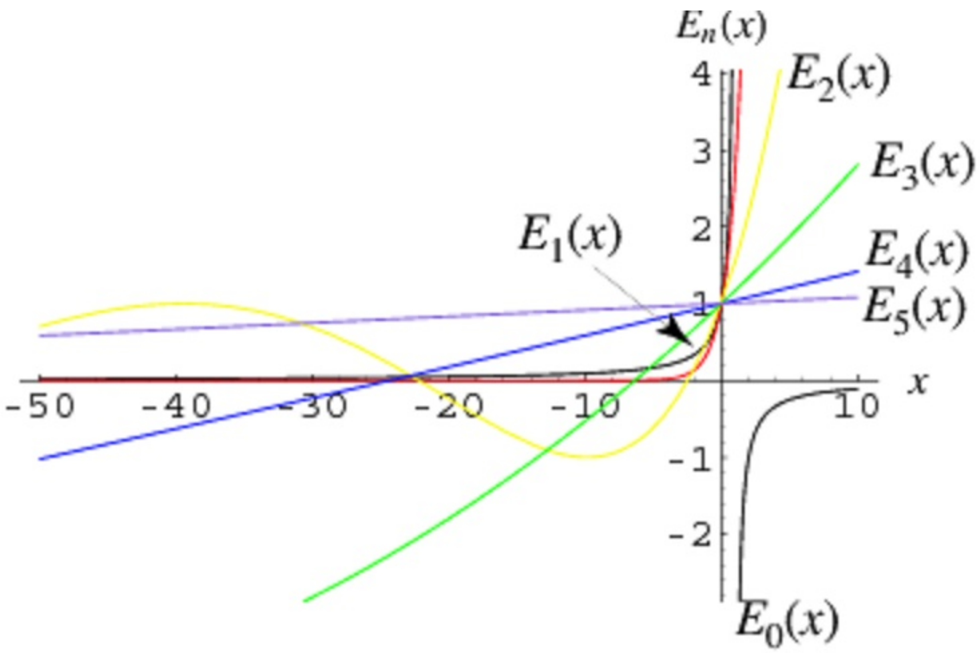

Figure 2.2: Representation of the Mittag-Leffler for different values of α.

2.2.2 Fractional-Order Integration

According to the Riemann-Liouville approach to fractional calculus, the notion of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy), that reduces the calculation of the n-fold primitive of a continuous function f(t) to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau, t > 0, n \in \mathbb{N}^* \tag{2.2}$$

In a natural way we are led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function and introducing the arbitrary positive real number α, we define the Fractional Integral of order $\alpha \geq 0$:

$$J^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \mathrm{t} > 0, & \alpha > 0 \\ f(t), & \alpha = 0 \end{cases} \tag{2.3}$$

We note the semi-group property

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha, \beta \geq 0 \tag{2.4}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t), \alpha, \beta \geq 0 \tag{2.5}$$

and the effect of the operator J^α on the power functions is (see [115]) :

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \alpha > 0, \gamma > -1, t > 0 \tag{2.6}$$

For $\alpha = 1$ and $\gamma = n \in \mathbb{N}$, the usual result is recovered, namely

$$J t^n = \frac{t^{n+1}}{n+1} \tag{2.7}$$

2.2.3 Riemann-Liouville Fractional Derivative

Denoting by D^n with $n \in \mathbb{N}$, the operator of the derivative of order n , we first note that

$$D^n J^n = I, J^n D^n \neq I, n \in \mathbb{N} \tag{2.8}$$

i.e. D^n is left-inverse (and not right-inverse) to the corresponding integral operator J^n.
As a consequence we expect that D^α is defined as left-inverse to J^α. For this purpose, introducing

the smallest positive integer m such that $0 \leq m-1 < \alpha < m$, we define the Riemann-Liouville fractional derivative of order $\alpha > 0$ as

$$^{rl}D^{\alpha}f(t) = D^{m}J^{m-\alpha} \tag{2.9}$$

namely,

$$^{rl}D^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)d\tau, & m-1 \leq \alpha < m \\ \frac{d^m}{dt^m}f(t), & \alpha = m \end{cases} \tag{2.10}$$

Defining for complement $D^0 f(t) = f(t)$, then we easily recognize that

$$^{rl}D^{\alpha}J^{\alpha}f(t) = f(t) \tag{2.11}$$

and

$$^{rl}D^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}, \alpha > 0, \gamma > -1, t > 0 \tag{2.12}$$

Note the remarkable fact that the fractional derivative $D^{\alpha}f(t)$ is not zero for the constant function $f(t) \equiv 1$ if α is not integer. indeed, if we replace γ by zero in (2.12), we obtain

$$D^{\alpha}1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{2.13}$$

2.2.4 Caputo Fractional Derivative

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(0)$, $f'(0)$, $f"(0)$, etc. Unfortunately, the Riemann-Liouville approach leads to initial conditions containing the limit values of the Riemann-Liouville derivative at the lower terminal $\lim_{t\to 0} D^{\alpha-1}f(t)$, $\lim_{t\to 0} D^{\alpha-2}f(t)$, $\cdots$ $\lim_{t\to 0} D^{\alpha-n}f(t)$ and there is no physical interpretation for such types of initial conditions.

A certain solution to this conflict was proposed by the so-called Caputo fractional derivative definition for $\alpha > 0$

$$^{c}D^{\alpha}f(t) = J^{(m-\alpha)}D^{m}f(t), \;\; m-1 < \alpha < m \tag{2.14}$$

namely,

$$^{c}D^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, & m-1 < \alpha < m \\ \frac{d^{(m)}}{dt^m}f(t), & \alpha = m \end{cases} \tag{2.15}$$

This definition is of course more restrictive than Riemann-Liouville definition, since it requires the absolute integrability of the derivative of order m. The main advantage of Caputo's approach is that

the initial conditions for fractional differential equation with Caputo derivative take the same form of integer-order.
In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero, i.e.

$$^{c}D^{\alpha}1 = 0,\ \alpha > 0 \tag{2.16}$$

2.2.5 Grünwald-Letnikov Fractional Derivative

The derivative of a continuous function proposed by Grünwald-Letnikov can be obtained intuitively from the definition of the usual derivative (the derivative of integer order):

$$f^{(n)}(t) = \frac{d^n}{dt^n}f(t) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} f(t-kh)$$

where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}$ is the usual binomial coefficients. A generalization of the backward difference by allowing the derivative order to be an arbitrary positive real was proposed by Grünwald-Letnikov

$$^{gl}_{0}D^{\alpha}_{t}f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{[\frac{t}{h}]}(-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t-kh), \alpha > 0 \tag{2.17}$$

where $[\frac{t}{h}]$ denotes the integer part and $\begin{pmatrix} \alpha \\ k \end{pmatrix}$ represents binomial coefficient generalized to real numbers. Namely,

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} \tag{2.18}$$

2.2.6 Laplace Transforms of Fractional Order Derivatives

The Laplace transform of the function $f(t)$ is the function $F(s)$ of the complex variable s defined by

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st}f(t)dt \tag{2.19}$$

For the existence of the integral (2.19), we need to define the exponential order function.

Definition 2.2.3. *A function f has exponential order a if there exist constants $M > 0$ such that for some $T \geq 0$,*

$$|f(t)| \leq Me^{at}, t \geq T.$$

For the existence of the integral (2.19), we have this theorem.

Theorem 2.2.1. *If f is piecewise continuous on $[0, \infty)$ and of exponential order a, then the Laplace transform $\mathcal{L}(f(t))$ exists for $\mathrm{Re}(s) > a$ and converges absolutely.*

In order to apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform. If $\mathcal{L}(f(t)) = F(s)$, then the inverse Laplace transform is denoted by

$$\mathcal{L}^{-1}(F(s)) = f(t), t \geq 0$$

which maps the Laplace transform of a function back to the original function and is defined by

$$f(t) = \mathcal{L}^{-1}(F(s)) = \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, c = \mathrm{Re}(s) > c_0 \tag{2.20}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral.
we mention above some properties of the Laplace transform which we will need later.

- The Laplace transform is a linear mapping, i.e. for all functions f and g admitting Laplace transforms and for all real α and β:

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f(t)) + \beta \mathcal{L}(g(t)) \tag{2.21}$$

- Under the same conditions we have

$$\mathcal{L}((f * g)(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t)) \tag{2.22}$$

  where $*$ denotes the convolution product defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$$

  with f and g are continuous functions on $[0, \infty)$.

- An other useful property which we need is the Laplace transform of the derivative of an integer order n of a function $f(t)$:

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \tag{2.23}$$

  where $F(s) = \mathcal{L}(f(t))$.

At present we will give the Laplace transform of fractional derivative of order $\alpha > 0$ view previously

- The writing of the fractional integral of order $\alpha > 0$ (2.3) as a convolution product

$$J^{\alpha}f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) \tag{2.24}$$

allows to calculate its Laplace transform [100]:

$$\mathcal{L}(J^{\alpha}f(t)) = \frac{F(s)}{s^{\alpha}} \tag{2.25}$$

Indeed,

$$\mathcal{L}(J^{\alpha}f(t)) = \mathcal{L}(\frac{t^{\alpha-1}}{\Gamma(\alpha)})\mathcal{L}(f(t))$$

let $F(s) = \mathcal{L}(f(t))$ and

$$\begin{aligned} \mathcal{L}(\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}) &= \tfrac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-st}dt \\ &= \tfrac{1}{\Gamma(\alpha)}\int_0^{\infty} \tfrac{x^{\alpha-1}}{s^{\alpha}}e^{-x}dx \\ &= \tfrac{1}{\Gamma(\alpha)}\tfrac{\Gamma(\alpha)}{s^{\alpha}} \\ &= \tfrac{1}{s^{\alpha}} \end{aligned}$$

then we get the following property

$$\mathcal{L}(\frac{t^{\alpha-1}}{\Gamma(\alpha)}) = \frac{1}{s^{\alpha}} \tag{2.26}$$

Thus the formula (2.25) is derived.

- The expression of the Laplace transform of the Riemann-Liouville fractional derivative of order $\alpha > 0$ is given by

$$\mathcal{L}({}^{rl}D^{\alpha}f(t)) = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^k \left[D^{(\alpha-k-1)}f(t)\right]_{t=0} \tag{2.27}$$

where $m-1 \leq \alpha < m, m \in \mathbb{N}^*$.

Indeed, according to the Riemann-Liouville fractional derivative definition in the form (2.9) and by putting

$$g(t) = J^{m-\alpha}f(t)$$

we obtain

$${}^{rl}D^{\alpha}f(t) = D^m(g(t))$$

then the application of the Laplace transform of an integer order derivative (2.23) gives

$$\mathcal{L}({}^{rl}D^{\alpha}f(t)) = s^m G(s) - \sum_{k=0}^{m-1} s^k \left[\frac{d^{(m-k-1)}}{dt^{(m-k-1)}}g(t)\right]_{t=0}$$

with $G(s) = \mathcal{L}(g(t))$. Taking in to account the Laplace transform of the fractional integral (2.25) with $F(s) = \mathcal{L}(f(t))$ then

$$G(s) = \frac{F(s)}{s^{m-\alpha}}$$

In the other hand, we have

$$\begin{aligned} \tfrac{d^{(m-k-1)}}{dt^{(m-k-1)}}g(t) &= \tfrac{d^{(m-k-1)}}{dt^{(m-k-1)}}(J^{m-\alpha}f(t)) \\ &= \tfrac{d^{(m-k-1)}}{dt^{(m-k-1)}}J^{m-k-1-(\alpha-k-1)}f(t) \\ &= {}^{rl}D^{(\alpha-k-1)}(f(t)) \end{aligned}$$

Finally, we obtain

$$\mathcal{L}({}^{rl}D^{\alpha}f(t)) = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^k \left[{}^{rl}D^{(\alpha-k-1)}f(t)\right]_{t=0}$$

As can be seen there is the limit values of fractional derivatives at the lower terminate $t = 0$ which have not physical interpretation, that is why the practical applicability of the Laplace transform of the Riemann-Liouville is limited.

- By writing the Caputo fractional derivative as

$$^{c}D^{\alpha}f(t) = J^{m-\alpha}(f^{m}(t)), m-1 < \alpha < m \tag{2.28}$$

and by using the Laplace transform of the fractional order integral (2.25) and the integer-order derivative (2.21), we get the Laplace transform of the Caputo fractional derivative for $m-1 < \alpha \leq m$

$$\mathcal{L}({}^{c}D^{\alpha}f(t)) = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\left[D^{(k)}f(t)\right]_{t=0} \tag{2.29}$$

- Assuming that $m-1 < \alpha < m$ and using the Laplace transform of the power function (2.26), the formula of the Laplace transform of the convolution and the Laplace transform of the integer-order derivative (2.21), we obtain the Laplace transform of the Grünwald-Letnikov derivative as

$$\mathcal{L}({}^{gl}D^{\alpha}f(t)) = s^{\alpha}F(s) \tag{2.30}$$

Remark 2.2.1. *It can be noticed that Laplace transformation of the RL derivative requires information of initial conditions of the function i.e.* $D^{\alpha-k-1}f(0)$ *(see(2.27)). From the mathematical point of view initial value problem is rigorous and elegant. However, This is found to be difficult to measure in term of physical quantities. In the Caputo' s definition initial conditions* $x(0), \dot{x}(0), \ldots, x^{(n-1)}(0)$ *are the same as for the integer derivatives of f, which has well known physical interpretation.For example, if one can interprets* $x(t)$ *as a position , then* $\dot{x}(t)$ *stands for speed and* $\ddot{x}(t)$ *expresses as an acceleration. This gives a strong support for acceptance of Caputo' s derivative to researchers and practicing engineers.*

In the remaining work, the fractional derivative of Caputo will be used and simply denoted by D.

2.2.7 Applications of Fractional Calculus

the basics of fractional calculus (integral and differential operations of noninteger order) were treated and improved long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and attracted the attention in various fields of science and engineering by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, robotics, signal processing, and electromagnetics. Here we state some of applications.

- **Electric transmission lines**
  During the last decades of the nineteenth century, Heaviside successfully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator form solution of heat equation due to Gregory(1846), Heaviside introduced the letter p for the differential operator $\frac{d}{dt}$ and gave the solution of the diffusion equation

  $$\frac{\partial^2 u}{\partial x^2} = a^2 p$$

  for the temperature distribution $u(x,t)$ in the symbolic form

  $$u(x,t) = Ae^{(ax\sqrt{p})} + Be^{(-ax\sqrt{p})}$$

  in which $p = \frac{d}{dx}$ was treated as constant, where a, A and B are also constant.

- **Application of Fractional Calculus to Fluid Mechanics**
  Vladimir V. Kulish and José L. Lage [58]
  Application of fractional calculus to the solution of time-dependent, viscous-diffusion fluid mechanics problems are presented. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous-diffusion equation in a semi-infinite space is shown to yield explicit analytical (fractional) solutions for the shearstress and fluid speed anywhere in the domain. Comparing the fractional results for boundary shear-stress and fluid speed to the existing analytical results for the first and second Stokes problems, the fractional methodology is validated and shown to be much simpler and more powerful than existing techniques.

- **Wave propagation in viscoelastic horns using a fractional calculus rheology model**
  Margulies, Timothy [78]
  The complex mechanical behavior of materials are characterized by fluid and solid models with fractional calculus differentials to relate stress and strain fields. Fractional derivatives have been shown to describe the viscoelastic stress from polymer chain theory for molecular solutions. Here the propagation of infinitesimal waves in one dimensional horns with a small cross-sectional area change along the longitudinal axis are examined. In particular, the linear, conical, exponential, and catenoidal shapes are studied. The wave amplitudes versus frequency are solved analytically and predicted with mathematical computation. Fractional rheology data from Bagley are incorporated in the simulations. Classical elastic and fluid "Webster equations" are recovered in the appropriate limits. Horns with real materials that employ fractional calculus representations can be modeled to examine design trade-offs for engineering or for scientific application.

- **Fractional differentiation for edge detection**
  B. Mathieu, P. Melchior, A. Oustaloup, Ch. Ceyral [82]
  In image processing, edge detection often makes use of integer-order differentiation operators, especially order 1 used by the gradient and order 2 by the Laplacian. This paper demonstrates how introducing an edge detector based on non-integer (fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

- **Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles**

J.I. Suárez, B.M. Vinagre , A.J. Calderón , C.A. Monje and Y.Q. Chen [119]
Here it is presented the use of Fractional Order Controllers (FOC) applied to the path-tracking problem in an autonomous electric vehicle. A lateral dynamic model of an industrial vehicle has been taken into account to implement conventional and Fractional Order Controllers. Several control schemes with these controllers have been simulated and compared.

2.3 Models and Representations of Fractional-Order Systems

In the following, we will focus only on the continuous-time representation.
The equations for a continuous-time dynamic system of fractional order can be written as follows

$$H(D^{\alpha_0\alpha_1\alpha_2...\alpha_m})(y_1, y_2, ..., y_p) = G(D^{\beta_0\beta_1\beta_2...\beta_n})(u_1, u_2, ..., u_k) \tag{2.31}$$

where y_i, u_j functions of time representative the outputs and the input respectively of the system described by (2.31). $H(.), G(.)$are combinations (not necessarily linear) laws of the fractional-order derivative operator and $\alpha_i, \beta_j > 0$ are fractional order derivatives relating respectively to the output and input to the system.
Let us now consider a SISO LTI FOS. By means of its dynamic input-output relation , we can derive its continuous-time models. In all what follows, we use Caputo' s definition of a fractional derivative with initial time $t = 0$. The derived differential equation is then expressed by [107]

$$\sum_{i=0}^{n} a_i D^{\alpha_i} y(t) = \sum_{j=0}^{m} b_j D^{\beta_j} u(t) \tag{2.32}$$

where a_i, b_j are real constants. In equation (2.32), which describes the dynamics of a LTI system mono-variable of fractional order, two cases arise and lead to two types of systems: commensurate systems (systems with commensurate orders) and non- commensurate systems (systems with non commensurate orders). A system is with commensurate order if all the differentiation orders of the fractional differential equation are integer multiples of a basic order, α, that is, $\alpha_i = i\alpha, \beta_i = i\alpha$, the equation (2.32) becomes

$$\sum_{i=0}^{n} a_i D^{i\alpha} y(t) = \sum_{j=0}^{m} b_j D^{j\alpha} u(t) \tag{2.33}$$

If in (2.33), $\alpha = \frac{1}{q}, q \in \mathbb{N}^*$, the system will be of rational order. The equation (2.32), in which α_i, β_j are the fractional orders that can be either commensurate or noncommensurate, can be written as

$$D^{\alpha_n} y(t) = -\frac{a_{n-1}}{a_n} D^{\alpha_{n-1}} y(t) - \frac{a_{n-2}}{a_n} D^{\alpha_{n-2}} y(t) - \cdots \frac{a_0}{a_n} D^{\alpha_0} y(t) + \sum_{j=0}^{m} \frac{b_j}{a_n} D^{\beta_j} u(t) \tag{2.34}$$

Assume that $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$. The procedure for obtaining a state-space representation from (2.34) is as follows: firstly, let us introduce an intermediate variable

$$x_i(t) = D^{\alpha_{i-1}} y(t), i = 1, 2, \ldots, n \tag{2.35}$$

Then we obtain, with the help of the semi-group property of the fractional-order derivative, successively by induction

$$\begin{array}{rcl} x_1(t) & = & D^{\alpha_0} y(t) \\ x_2(t) & = & D^{\alpha_1} y(t) = D^{\alpha_1-\alpha_0+\alpha_0} y(t) = D^{\alpha_1-\alpha_0}(D^{\alpha_0} y(t)) = D^{\alpha_1-\alpha_0} x_1(t) \\ x_3(t) & = & D^{\alpha_2} y(t) = D^{\alpha_2-\alpha_1+\alpha_1} y(t) = D^{\alpha_2-\alpha_1}(D^{\alpha_1} y(t)) = D^{\alpha_2-\alpha_1} x_2(t) \\ & \vdots & \\ x_n(t) & = & D^{\alpha_{n-1}} y(t) = D^{\alpha_{n-1}-\alpha_{n-2}+\alpha_{n-2}} y(t) = D^{\alpha_{n-1}-\alpha_{n-2}}(D^{\alpha_{n-1}} y(t)) = D^{\alpha_{n-1}-\alpha_{n-2}} x_{n-1}(t) \\ D^{\alpha_n} y(t) & = & D^{\alpha_n-\alpha_{n-1}+\alpha_{n-1}} y(t) = D^{\alpha_n-\alpha_{n-1}}(D^{\alpha_{n-1}} y(t)) = D^{\alpha_n-\alpha_{n-1}} x_n \end{array} \tag{2.36}$$

With (2.34) and (2.36), it is easy to build the following group of equations

$$\begin{array}{rcl} D^{\alpha_1-\alpha_0} x_1(t) & = & x_2(t) \\ D^{\alpha_2-\alpha_1} x_2(t) & = & x_3(t) \\ & \vdots & \\ D^{\alpha_i-\alpha_{i-1}} x_i(t) & = & x_{i+1}(t) \\ & \vdots & \\ D^{\alpha_n-\alpha_{n-1}} x_n & = & -\frac{a_{n-1}}{a_n} x_n - \frac{a_{n-2}}{a_n} x_{n-1} - \ldots - \frac{a_0}{a_n} x_1 + \sum_{j=0}^{m} \frac{b_j}{a_n} D^{\beta_j} u(t) \end{array} \tag{2.37}$$

which can be expressed in matrix form

$$D^{\alpha} x(t) = Ax(t) + BD^{\beta} u(t) \tag{2.38}$$

with $\alpha = (\alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \ldots, \alpha_n - \alpha_{n-1})$, $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$,
$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \end{bmatrix}^{\mathrm{T}}$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-3}}{a_n} & -\frac{a_{n-4}}{a_n} & \ldots & -\frac{a_0}{a_n} \end{bmatrix}, B = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ \frac{b_0}{a_n} & \ldots & \frac{b_m}{a_n} \end{bmatrix} \tag{2.39}$$

With the given of the input $u(t)$, we can determine the second term of the right member in (2.32), leading to a new state representation where the matrix B would be a simple column matrix of the same size as the state vector $x(t)$. Indeed, by posing

$$\sum_{j=0}^{m} b_j D^{\beta_j} u(t) = e(t) \tag{2.40}$$

the representation (2.39) becomes

$$D^{\alpha}x(t) = Ax(t) + Be(t), B = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times 1} \tag{2.41}$$

For a fractional order system described by n-term fractional differential equation

$$\sum_{i=0}^{n} a_i D^{\alpha_i} y(t) = e(t) \tag{2.42}$$

The legitimate choice of the derivative order $\alpha_0 = 0$ of the output corresponding to the coefficient a_0 allows us to establish a state equation augmented by an output equation of the form $y(t) = Cx(t)$, hence the following realization is obtained

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + Be(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.43}$$

In the case of commensurate systems, the differentiation fractional-order $\alpha_i - \alpha_{i-1}$ are equal to a unique value, say α. In this case, (2.43) becomes

$$\begin{aligned} D^{\alpha}x(t) &= \begin{bmatrix} D^{\alpha}x_1(t) \\ D^{\alpha}x_2(t) \\ \vdots \\ D^{\alpha}x_n(t) \end{bmatrix} = Ax(t) + Be(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.44}$$

Applying the Laplace transform to (2.32) with zero initial conditions, the input-output representations of fractional-order systems can be obtained. In the case of continuous models, a fractional-order system will be given by a transfer function of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} \dots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} \dots + a_0 s^{\alpha_0}} \tag{2.45}$$

As can be seen in the previous equations, a fractional-order system has an irrational-order transfer function in Laplace' s domain. Because of this, it can be said that a fractional-order system has an

unlimited memory, and obviously the systems of integer-order are just particular cases.
In the case of a commensurate-order system, the continuous-time transfer function is given by

$$G(s) = \frac{\sum_{k=0}^{m} b_k (s^\alpha)^k}{\sum_{k=0}^{n} a_k (s^\alpha)^k} \tag{2.46}$$

which can be considered as a pseudo-rational function, $H(\lambda)$, of the variable $\lambda = s^\alpha$,

$$H(\lambda) = \frac{\sum_{k=0}^{m} b_k \lambda^k}{\sum_{k=0}^{n} a_k \lambda^k} \tag{2.47}$$

In this thesis, the behavior of the fractional systems studied will be approached by commensurate models with the properties of linearity and invariance in continuous-time where the Caputo's fractional derivative is suitable.

2.3.1 Temporal Response of Fractional Continuous-Time Linear System

Consider the continuous-time linear system described by the equation

$$D^\alpha x(t) = Ax(t) + Bu(t), 0 < \alpha \leq 1 \tag{2.48a}$$

$$y(t) = Cx(t) + Du(t) \tag{2.48b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Theorem 2.3.1. *The solution of the equation* (2.48a) *has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau,\ x(0) = x_0 \tag{2.49}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.50}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \tag{2.51}$$

$E_\alpha(At^\alpha)$ *is the Mittag-Leffler function to one parameter and* $\Gamma(x)$ *is the Euler gamma function.*

Proof. Applying the Laplace transform (2.19) to (2.48a) and taking in to account (2.29) and that $0 < \alpha \leq 1$, we obtain

$$s^{\alpha}X(s) - s^{\alpha-1}x_0 = AX(s) + BU(s) \tag{2.52}$$

where

$$X(s) = \mathcal{L}(x(t)),\ U(s) = \mathcal{L}(u(t))$$

Then we get

$$(s^{\alpha}I_n - A)X(s) = s^{\alpha-1}x_0 I_n + BU(s) \tag{2.53}$$

Since the pencil of matrices $s^{\alpha}I_n - A$ is regular then (2.53) implies that

$$X(s) = (s^{\alpha}I_n - A)^{-1}(s^{\alpha-1}x_0 I_n + BU(s)) \tag{2.54}$$

As

$$\begin{aligned}(s^{\alpha}I_n - A)(\sum_{k=0}^{\infty}\tfrac{A^k}{s^{(k+1)\alpha}}) &= \sum_{k=0}^{\infty}\tfrac{A^k s^{\alpha}}{s^{(k+1)\alpha}} - \sum_{k=0}^{\infty}\tfrac{A^{k+1}}{s^{(k+1)\alpha}} \\ &= \sum_{k=0}^{\infty}\tfrac{A^k}{s^{k\alpha}} - \sum_{k=0}^{\infty}\tfrac{A^{k+1}}{s^{(k+1)\alpha}} \\ &= \tfrac{A^0}{s^0} + \sum_{k=1}^{\infty}\tfrac{A^k}{s^{k\alpha}} - \sum_{h=1}^{\infty}\tfrac{A^h}{s^{h\alpha}} \\ &= I_n\end{aligned}$$

then

$$(s^{\alpha}I_n - A)^{-1} = \sum_{k=0}^{\infty}\frac{A^k}{s^{(k+1)\alpha}} \tag{2.55}$$

by substitution of (2.55) in (2.54), we get

$$X(s) = \sum_{k=0}^{\infty}\frac{A^k}{s^{k\alpha+1}}x_0 + \sum_{k=0}^{\infty}\frac{A^k}{s^{(k+1)\alpha}}BU(s) \tag{2.56}$$

Applying the inverse Laplace transform and taking into account its linearity, the equation (2.57) is written as

$$\mathcal{L}^{-1}(X(s)) = \sum_{k=0}^{\infty}A^k\mathcal{L}^{-1}\left(\frac{1}{s^{k\alpha+1}}\right)x_0 + \sum_{k=0}^{\infty}A^k B\mathcal{L}^{-1}\left(\frac{U(s)}{s^{(k+1)\alpha}}\right) \tag{2.57}$$

According to to the effect of the Laplace transform on the power function in (2.26) as well as on the fractional integral in (2.25), the equation (2.57) becomes

$$\begin{aligned}x(t) &= \sum_{k=0}^{\infty}A^k\tfrac{t^{k\alpha}}{\Gamma(k\alpha+1)}x_0 + \sum_{k=0}^{\infty}A^k BJ^{(k+1)\alpha}u(t) \\ &= \sum_{k=0}^{\infty}\tfrac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0 + \sum_{k=0}^{\infty}A^k B\tfrac{1}{\Gamma((k+1)\alpha)}\int_0^t(t-\tau)^{(k+1)\alpha-1}u(\tau)d\tau \\ &= \sum_{k=0}^{\infty}\tfrac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0 + \int_0^t\sum_{k=0}^{\infty}\tfrac{A^k(t-\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}Bu(\tau)d\tau\end{aligned} \tag{2.58}$$

Defining

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k (t-\tau)^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)}$$

the state response in (2.49) is derived. □

In the same way, the state response is obtained for $m - 1 < \alpha \leq m$.

Theorem 2.3.2. *The solution of the equation* (2.48a) *for* $m - 1 < \alpha \leq m$ *has the form*

$$x(t) = \sum_{h=1}^{m} \Phi_h(t) x^{(m-h)}(0^+) + \int_0^t \Phi(t-\tau) B u(\tau) d\tau \tag{2.59}$$

where

$$\Phi_h(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha + h) - 1}}{\Gamma(k\alpha + h)}, \; \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)}$$

Proof. Applying the Laplace transform to (2.48a) while considering (2.29) for $m - 1 < \alpha \leq m$ yields

$$s^\alpha X(s) - \sum_{h=1}^{m} s^{\alpha - h} x^{(m-h)}(0^+) = AX(s) + BU(s) \tag{2.60}$$

where

$$X(s) = \mathcal{L}(x(t)), \; U(s) = \mathcal{L}(u(t))$$

Using (2.55), the equation (2.60) can be written as

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} \frac{A^k}{s^{(k+1)\alpha}} \left(\sum_{h=1}^{m} s^{\alpha-h} x^{(m-h)}(0^+) + BU(s) \right) \\ &= \sum_{k=0}^{\infty} \sum_{h=1}^{m} \frac{A^k}{s^{(k\alpha+h)}} x^{(m-h)}(0^+) + \sum_{k=0}^{\infty} \frac{A^k}{s^{(k+1)\alpha}} BU(s) \\ &= \sum_{h=1}^{m} \sum_{k=0}^{\infty} \frac{A^k}{s^{(k\alpha+h)}} x^{(m-h)}(0^+) + \sum_{k=0}^{\infty} \frac{A^k}{s^{(k+1)\alpha}} BU(s) \end{aligned} \tag{2.61}$$

Applying the inverse Laplace transform enables us to obtain

$$x(t) = \sum_{h=1}^{m} \Phi_h(t) x^{(m-h)}(0^+) + \int_0^t \Phi(t-\tau) B u(\tau) d\tau \tag{2.62}$$

with

$$\begin{aligned}\Phi_h(t) &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}\left(\tfrac{1}{s^{(k\alpha+h)}}\right)\\ &= \sum_{k=0}^{\infty} \tfrac{A^k t^{(k\alpha+h)-1}}{\Gamma(k\alpha+h)}\end{aligned}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k (t-\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$$

□

Remark 2.3.1. *As the fractional-order derivative of x depends on the "history" of x from the lower limit of the integral t_0 that defines this operator until the present instant t, the knowledge of $x(t_0)$ is not sufficient to determine the future behavior of the system [72]. Consequently, vector $x(t)$ does not strictly represent the state of the system. Then we should denote $x(t)$ as a pseudo-state. However, we call this vector a state only for lexical simplifying purposes.*

2.4 Stability of Linear Fractional-Order Continuous-Time Systems

As in classical calculus, stability analysis is a central task in the study of fractional differential system and fractional control. As for linear time invariant integer order systems, it is now well known that stability of a linear fractional order system depends on the location of the system poles in the complex plane.

Theorem 2.4.1. *[85]A fractional order system defined by its transfer function*

$$G(s) = \frac{Q(s)}{P(s)} \tag{2.63}$$

for $\mathrm{Re}(s) \geq 0$, *where* $P(s) = \sum_{k=0}^{p} P_k s^{\alpha_k}$ *with* $\alpha_{k+1} > \alpha_k \geq 0$ *and* $Q(s) = \sum_{l=0}^{q} q_l s^{\beta_l}$ *with* $\beta_{l+1} > \beta_l \geq 0$ *are no longer polynomials. The system has the main property of:*

$$BIBO\ stability \Leftrightarrow \exists M > 0, \|G(s)\| \leq M, \forall s, \mathrm{Re}(s) \geq 0$$

Moreover, in the case where no simplification occurs between P and Q, that is all the roots of $P(s) = 0$ not being roots of $Q(s) = 0$, the stability property then reads:

$$BIBO\ stability \Leftrightarrow P(s) \neq 0, \forall s, \mathrm{Re}(s) \geq 0$$

This statement appears for the first time in [85] as Theorem 2.24 (conjecture), which was fully proved, and solved later in [13].However such a theorem does not permit to conclude to system stability without system poles computation, which constitutes a tedious work. However, poles location analysis remains a difficult task in the general case. For commensurate fractional order systems, powerful criteria have been proposed. Using commensurate order hypothesis, the system (2.63) also admits a state space like representation:

$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.64}$$

In [85], the given result permits to check the system stability through the location in the complex plane of the dynamic matrix eigenvalues. This work is in fact the starting point of several results in the field. For $1 \leq \alpha < 2$, stability condition is given in [93].

Theorem 2.4.2. *[83, 85, 110] Autonomous system:*

$$D^{\alpha}x(t) = Ax(t), x(0) = x_0, 0 < \alpha < 2 \tag{2.65}$$

is asymptotically stable if and only if

$$|\arg(\text{spec}(A))| > \alpha\frac{\pi}{2} \tag{2.66}$$

where $\text{spec}(A)$ *is the spectrum (set of all eigenvalues) of A.*

For a minimal realization[1] of (2.64), in [83], the following result has been also demonstrated.

Theorem 2.4.3. *[83]If the triplet* (A, B, C) *is minimal[2], system* (2.64) *is BIBO stable if and only if* $|\arg(\text{spec}(A))| > \alpha\frac{\pi}{2}$

Based on the previous results, stable regions $D_s^{\alpha} = \left\{\lambda/\lambda \in \mathbb{C}, |\arg(\lambda)| > \alpha\frac{\pi}{2}\right\}$ for a fractional system depending on its differentiation order α and on the value of $|\arg(\text{spec}(A))|$ are illustrated in Figure(2.3).

The location of the eigenvalues of a matrix in a particular region of the complex plane can be solved using the formalism of $\mathcal{LMI}$ regions introduced in [31] which are the basis of the stability criteria

[1]A state space model (A, B, C, D) is a realization of a transfer function $G(s)$ if its transfer function coincides with G(s).

[2]A realization (A, B, C, D) of a transfer function/matrix $G(s)$ is said to be minimal if no other realization of $G(s)$ has smaller dimension.

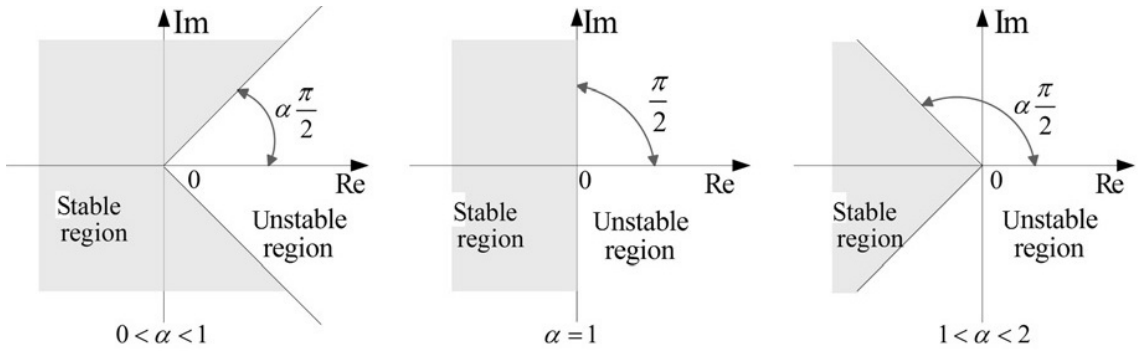

Figure 2.3: Stable regions D_s^α for $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$.

in terms of $\mathcal{LMI}$ for non-integer models presented in this section. A recall on the $\mathcal{LMI}$ areas is presented in Annex B.
In the first paragraph, the case of order belonging $1 \leq \alpha < 2$ is considered. The stability domain associated is then convex and can be described using $\mathcal{LMI}$ regions. The case $0 < \alpha < 1$ is then discussed. In this case, the stability domain is not convex and can not be described by $\mathcal{LMI}$ areas.

1. **Case** $1 \leq \alpha < 2$

   As can be shown in figure (2.3), the stability domain of a fractional system with order $1 \leq \alpha < 2$ is a convex set, $\mathcal{LMI}$ methods for defining such a region can be used. Hence a $\mathcal{LMI}$-based sufficient and necessary condition for the stability can be formulated in the following theorem.

   Theorem 2.4.4. *[110, 111]A fractional system described by $D^\alpha x(t) = Ax(t)$ with order $1 \leq \alpha < 2$ is asymptotically stable if and only if there exists a matrix $P \in \mathbb{R}^{n\times n} \succ 0$ such that*

$$\begin{bmatrix} (A^TP + PA)\sin(\alpha\frac{\pi}{2}) & (A^TP - PA)\cos(\alpha\frac{\pi}{2}) \\ (PA - A^TP)\cos(\alpha\frac{\pi}{2}) & (A^TP + PA)\sin(\alpha\frac{\pi}{2}) \end{bmatrix} \prec 0 \tag{2.67}$$

   Proof. From [29](see appendix B.4, conic sector), relation (B.40) is verified if and only if the following $\mathcal{LMI}$ feasibility problem is verified:

$$\exists P \in \mathbb{R}^{n\times n} \succ 0, (A^{\mathrm{T}}P + PA)\sin(\alpha\frac{\pi}{2}) + j(A^{\mathrm{T}}P - PA)\cos(\alpha\frac{\pi}{2}) \prec 0 \tag{2.68}$$

As an $\mathcal{LMI}$ involving real terms can be verified from a complex one, the problem becomes $\exists P \in \mathbb{R}^{n\times n} \succ 0$:

$$\begin{bmatrix} (A^{\mathrm{T}}P + PA)\sin(\alpha\frac{\pi}{2}) & (A^{\mathrm{T}}P - PA)\cos(\alpha\frac{\pi}{2}) \\ (PA - A^{\mathrm{T}}P)\cos(\alpha\frac{\pi}{2}) & (A^{\mathrm{T}}P + PA)\sin(\alpha\frac{\pi}{2}) \end{bmatrix} \prec 0 \tag{2.69}$$

This completes the proof. □

Remark 2.4.1. *In what follows, variants of the condition* (2.69) *will be presented.*

- *Using the Kronecker product,* (2.69) *can be written as*

$$\mathrm{Sym}\left\{\Theta \otimes (A^{T}P)\right\} \prec 0 \tag{2.70}$$

where $\Theta = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \\ -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}$

- *By pre- and post-multiplying the formula* (2.69) *by* $\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix}$*, an equivalent condition for the stability is then obtained*
$\exists P \in \mathbb{R}^{n\times n} \succ 0$

$$\begin{bmatrix} (PA^{T} + AP)\sin(\alpha\frac{\pi}{2}) & (PA^{T} - AP)\cos(\alpha\frac{\pi}{2}) \\ (AP - PA^{T})\cos(\alpha\frac{\pi}{2}) & (PA^{T} + AP)\sin(\alpha\frac{\pi}{2}) \end{bmatrix} \prec 0 \tag{2.71}$$

- *The formula* (2.71) *can also be written as*

$$\mathrm{Sym}\left\{\Theta^{T} \otimes (AP)\right\} \prec 0 \tag{2.72}$$

- *Since for any complex matrix* $M \prec 0$*, its conjugate verifies* $\bar{M} \prec 0$*,* (2.68) *is equivalent to*

$$(A^{T}P + PA)\sin(\alpha\frac{\pi}{2}) - j(A^{T}P - PA)\cos(\alpha\frac{\pi}{2}) \prec 0 \tag{2.73}$$

which can be expressed as

$$\mathrm{Sym}\left\{\Theta^{T} \otimes (A^{T}P)\right\} \prec 0 \tag{2.74}$$

2. **Case** $0 < \alpha < 1$
Figure (2.3) shows that the stability domain D_s^{α} of a linear fractional system is not convex when $0 < \alpha < 1$. Due to the absence of the convexity property, the $\mathcal{LMI}$ conditions can

not be derived directly as in the case of integer order or fractional order with $1 \leq \alpha < 2$. However, in literature different approaches are suggested to by pass this problem and $\mathcal{LMI}$ condition are derived indirectly. Some of the well recognized results which exist in literature are discussed here. An analysis via stability domain decomposition was proposed in [111] This analysis is based on the fact that the stability domainD_s^α given by (2.66) can be viewed as the union of two half planes, denoted D_{s_1} and D_{s_2}. They result from the rotation of the left-half plane with angles $\varphi_1 = \varphi$ and $\varphi_2 = -\varphi$ respectively, where $\varphi = (1-\alpha)\frac{\pi}{2}$, as shown in figure (2.4).

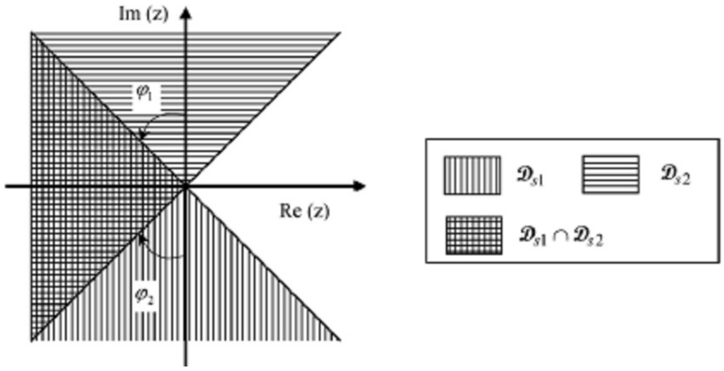

Figure 2.4: Stability region as a union of two half planes.

Consequently, stability region D_s^α can be defined by

$$D_s^\alpha = D_{s_1} \cup D_{s_2} \tag{2.75}$$

where

$$D_{s_i} = \left\{ z \in \mathbb{C} : \mathrm{Re}(ze^{j\varphi_i}) < 0 \right\}, \forall i \in \{1,2\} \tag{2.76}$$

Since D_{s_1} and D_{s_2} are not symmetrical with respect to the real axis, they do not constitute $\mathcal{LMI}$ regions. The formalism introduced in [31] and developed by in [8] permits to deal with this case using the concept of generalized $\mathcal{LMI}$ regions $\mathcal{GLMI}$, initially extended in the case of fractional order systems in [111]. In order not overburdening this section, a recall on the $\mathcal{GLMI}$ region, the application of the D-stability in such region have been developed in Appendix C. A necessary and sufficient condition in terms of $\mathcal{LMI}$ for stability of a non-integer system with order $0 < \alpha < 1$ can thus be obtained by using Theorem (C.2.1), see Appendix C.2, on $\mathcal{GLMI}$ region D of the complex plane,

Lemma 2.4.1. *[42] Let $A \in \mathbb{R}^{n\times n}$, $0 < \alpha < 1$ and $\theta = (1-\alpha)\frac{\pi}{2}$. The fractional-order system $D^{\alpha}x(t) = Ax(t)$ is asymptotically stable if and only if there exist two positive definite Hermitian matrices $X_1 = X_1^* \in \mathbb{C}^{n\times n}$, $X_2 = X_2^* \in \mathbb{C}^{n\times n}$ such that*

$$e^{-j\theta}X_1A^T + e^{j\theta}AX_1 + e^{j\theta}X_2A^T + e^{-j\theta}AX_2 \prec 0 \tag{2.77}$$

or

$$e^{-j\theta}X_1A + e^{j\theta}A^TX_1 + e^{j\theta}X_2A + e^{-j\theta}A^TX_2 \prec 0 \tag{2.78}$$

Proof. As each domain $\mathcal{D}_{s_i},\ i \in \{1,2\}$ has the form (C.10), then it is a $\mathcal{GLMI}$ region of first order with $m = 1,\ \alpha_i = 0,\ \beta_i = e^{j\varphi_i}$ and $\mathcal{D}_s$ has the form (C.13), based on the recall made in appendix C.3, we deduce that $\mathcal{D}_s$ is a $\mathcal{GLMI}$ region of order $l = m+1 = 3$ characterized by

$$\theta_1 = \frac{1}{2}\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \psi_1 = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_1 = -J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\theta_2 = \frac{1}{2}\begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ \psi_2 = \begin{bmatrix} \beta_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_2 = -J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\omega = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It is easy to see that conditions (C.1), (C.2) and (C.3) are verified. This allows us to apply the theorem (C.2.1), to get necessary and sufficient condition for the stability of the matrix A in the region $\mathcal{D}_s$. Therefore, there exist two matrices $X_1 \in \mathbb{C}^{n\times n}$ and $X_2 \in \mathbb{C}^{n\times n}$ such that (C.4) and (C.5) hold.

Condition (C.4) becomes

$$\begin{gathered} \theta_1 \otimes X_1 + \theta_1^* \otimes X_1^* + \psi_1 \otimes (AX_1) + \psi_1^* \otimes (AX_1)^* \\ +\theta_2 \otimes X_2 + \theta_2^* \otimes X_2^* + \psi_2 \otimes (AX_2) + \psi_2^* \otimes (AX_2)^* \prec 0 \end{gathered}$$

After few computations and taking into account that $\varphi_1 = \varphi$ and $\varphi_2 = -\varphi$ we obtain

$$\begin{bmatrix} e^{j\varphi}AX_1 + e^{-j\varphi}(AX_1)^* + e^{-j\varphi}AX_2 + e^{j\varphi}(AX_2)^* & 0 & 0 \\ 0 & -\frac{1}{2}(X_1 + X_1^*) & 0 \\ 0 & 0 & -\frac{1}{2}(X_2 + X_2^*) \end{bmatrix} \prec 0$$

Then we deduce

$$e^{j\varphi}AX_1 + e^{-j\varphi}(AX_1)^* + e^{-j\varphi}AX_2 + e^{j\varphi}(AX_2)^* \prec 0$$

We can also deduce that

$$X_1 + X_1^* \succ 0$$

and

$$X_2 + X_2^* \succ 0$$

In the other hand, Condition (C.5) becomes

$$H_1 \otimes X_1 + J_1 \otimes X_1^* + H_2 \otimes X_2 + J_2 \otimes X_2^* = 0_{3n}$$

After computation, we get

$$\begin{bmatrix} 0_n & 0_n & 0_n \\ 0_n & X_1 - X_1^* & 0_n \\ 0_n & 0_n & X_2 - X_2^* \end{bmatrix} = 0_{3n}$$

which means that $X_1 = X_1^*$ and $X_2 = X_2^*$. Finally, we obtain that the matrices X_1 and X_2 are hermitian definite positive and (2.77) is derived. The proof is then complete. □

The following Lemma proposed in [73] presents another version of Lemma (2.4.1), where the complex $\mathcal{LMI}$ (2.77) is replaced by a real linear matrix inequality through the use of the Kronecker product.

Lemma 2.4.2. *[73]Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable if and only if there exist two real symmetric positive definite matrices $X_{i1} \in \mathbb{R}^{n\times n}, i = 1, 2$, and two skew-symmetric matrices $X_{i2} \in \mathbb{R}^{n\times n}, i = 1, 2$, such that*

$$\begin{bmatrix} X_{11} & X_{12} \\ -X_{12} & X_{11} \end{bmatrix} \succ 0, \begin{bmatrix} X_{21} & X_{22} \\ -X_{22} & X_{21} \end{bmatrix} \succ 0 \tag{2.79}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \text{Sym}\left\{\Theta_{ij} \otimes (AX_{ij})\right\} \prec 0 \tag{2.80}$$

where

$$\Theta_{11} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & -\cos(\alpha\frac{\pi}{2}) \\ \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}$$

$$\Theta_{12} = \begin{bmatrix} \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \end{bmatrix}$$

$$\Theta_{21} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \\ -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}$$

$$\Theta_{22} = \begin{bmatrix} -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & -\cos(\alpha\frac{\pi}{2}) \end{bmatrix}$$

To prove the Lemma (2.4.2), we need the following result.

Lemma 2.4.3. *For complex matrix $M \in \mathbb{C}^{n\times n}$, define*

$$M = A + jB,\ A, B \in \mathbb{R}^{n\times n}$$

Then $M \succ 0$ if and only if

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \succ 0 \tag{2.81}$$

or

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \succ 0 \tag{2.82}$$

Proof. see Appendix A.2 □

Proof. From Lemma (2.4.1), it follows that the fractional-order system $D^{\alpha}x(t) = Ax(t)$ with $0 < \alpha < 1$ is asymptotically stable if and only if there exist two hermitian positive definite matrices $X_i, i = 1, 2$, such that the $\mathcal{LMI}$ (2.77) holds. Define $X_{i1} = \mathrm{Re}\,(X_i), X_{i2} = \mathrm{Im}\,(X_i), i = 1, 2$. From $X_i = X_{i1} + X_{i2}, i = 1, 2$ and$X_i = X_i^* \succ 0$, it follows that the matrices $X_{i1}, i = 1, 2$ are symmetric definite positive and the matrices $X_{i2}, i = 1, 2$ are skew matrices[3].According to Lemma (2.4.3), we have

$$\begin{bmatrix} X_{11} & X_{12} \\ -X_{12} & X_{11} \end{bmatrix} \succ 0, \begin{bmatrix} X_{21} & X_{22} \\ -X_{22} & X_{21} \end{bmatrix} \succ 0$$

[3]A skew-symmetric (or antisymmetric) matrix is a square matrix whose transpose is its negation; that is, it satisfies the condition $A^{\mathrm{T}} = -A$

so the inequality (2.79) holds. Substituting in the inequality (2.77) $X_i = X_{i1} + jX_{i2}, i = 1, 2$ and $e^{j\theta} = \sin(\alpha\frac{\pi}{2}) + j\cos(\alpha\frac{\pi}{2})$, we obtain

$$\begin{aligned}(\sin(\alpha\frac{\pi}{2}) + j\cos(\alpha\frac{\pi}{2}))(X_{11} + jX_{12})A^{\mathrm{T}} + (\sin(\alpha\frac{\pi}{2}) - j\cos(\alpha\frac{\pi}{2}))A(X_{11} + jX_{12}) \\ +(\sin(\alpha\frac{\pi}{2}) - j\cos(\alpha\frac{\pi}{2}))(X_{21} + jX_{22})A^{\mathrm{T}} + (\sin(\alpha\frac{\pi}{2}) + j\cos(\alpha\frac{\pi}{2}))A(X_{21} + jX_{22}) \prec 0\end{aligned} \quad (2.83)$$

which can be rewritten as follows

$$\begin{aligned}(\sin(\alpha\frac{\pi}{2})X_{11}A^{\mathrm{T}} - \cos(\alpha\frac{\pi}{2})X_{12}A^{\mathrm{T}}) + j(\sin(\alpha\frac{\pi}{2})X_{12}A^{\mathrm{T}} + \cos(\alpha\frac{\pi}{2})X_{11}A^{\mathrm{T}}) \\ +(\sin(\alpha\frac{\pi}{2})AX_{11} + \cos(\alpha\frac{\pi}{2})AX_{12}) + j((\sin(\alpha\frac{\pi}{2})AX_{12} - \cos(\alpha\frac{\pi}{2})AX_{11}) \\ +(\sin(\alpha\frac{\pi}{2})X_{21}A^{\mathrm{T}} + \cos(\alpha\frac{\pi}{2})X_{22}A^{\mathrm{T}}) + j(\sin(\alpha\frac{\pi}{2})X_{22}A^{\mathrm{T}} - \cos(\alpha\frac{\pi}{2})X_{21}A^{\mathrm{T}}) \\ +(\sin(\alpha\frac{\pi}{2})AX_{21} - \cos(\alpha\frac{\pi}{2})AX_{22}) + j(\sin(\alpha\frac{\pi}{2})AX_{22} + \cos(\alpha\frac{\pi}{2})AX_{21}) \prec 0\end{aligned} \quad (2.84)$$

Using the fact that $X_{12}^{\mathrm{T}} = -X_{12}$ and $X_{22}^{\mathrm{T}} = -X_{22}$, (2.84) is equivalent to

$$\begin{aligned}(\sin(\alpha\frac{\pi}{2})X_{11}A^{\mathrm{T}} + \cos(\alpha\frac{\pi}{2})X_{12}^{\mathrm{T}}A^{\mathrm{T}}) + j(-\sin(\alpha\frac{\pi}{2})X_{12}^{\mathrm{T}}A^{\mathrm{T}} + \cos(\alpha\frac{\pi}{2})X_{11}A^{\mathrm{T}}) \\ +(\sin(\alpha\frac{\pi}{2})AX_{11} + \cos(\alpha\frac{\pi}{2})AX_{12}) + j(\sin(\alpha\frac{\pi}{2})AX_{12} - \cos(\alpha\frac{\pi}{2})AX_{11}) \\ +(\sin(\alpha\frac{\pi}{2})X_{21}A^{\mathrm{T}} - \cos(\alpha\frac{\pi}{2})X_{22}^{\mathrm{T}}A^{\mathrm{T}}) + j(-\sin(\alpha\frac{\pi}{2})X_{22}^{\mathrm{T}}A^{\mathrm{T}} - \cos(\alpha\frac{\pi}{2})X_{21}A^{\mathrm{T}}) \\ +(\sin(\alpha\frac{\pi}{2})AX_{21} - \cos(\alpha\frac{\pi}{2})AX_{22}) + j(\sin(\alpha\frac{\pi}{2})AX_{22} + \cos(\alpha\frac{\pi}{2})AX_{21}) \prec 0\end{aligned} \quad (2.85)$$

which is the same as

$$\begin{aligned}\mathrm{Sym}\left\{\sin(\alpha\frac{\pi}{2})AX_{11} + \cos(\alpha\frac{\pi}{2})AX_{12} + \sin(\alpha\frac{\pi}{2})AX_{21} - \cos(\alpha\frac{\pi}{2})AX_{22}\right\} \\ +j(-\sin(\alpha\frac{\pi}{2})X_{12}^{\mathrm{T}}A^{\mathrm{T}} + \cos(\alpha\frac{\pi}{2})X_{11}A^{\mathrm{T}} + \sin(\alpha\frac{\pi}{2})AX_{12} - \cos(\alpha\frac{\pi}{2})AX_{11}) \\ +j(-\sin(\alpha\frac{\pi}{2})X_{22}^{\mathrm{T}}A^{\mathrm{T}} - \cos(\alpha\frac{\pi}{2})X_{21}A^{\mathrm{T}} + \sin(\alpha\frac{\pi}{2})AX_{22} + \cos(\alpha\frac{\pi}{2})AX_{21}) \prec 0\end{aligned} \quad (2.86)$$

or identically

$$\begin{aligned}&\mathrm{Sym}\left\{\sin(\alpha\frac{\pi}{2})AX_{11}\right\} + \mathrm{Sym}\left\{\cos(\alpha\frac{\pi}{2})AX_{12}\right\} \\ &+ \mathrm{Sym}\left\{\sin(\alpha\frac{\pi}{2})AX_{21}\right\} + \mathrm{Sym}\left\{-\cos(\alpha\frac{\pi}{2})AX_{22}\right\} \\ &+ j\cos(\alpha\frac{\pi}{2})(X_{11}A^{\mathrm{T}} - AX_{11}) + j\sin(\alpha\frac{\pi}{2})(AX_{12} - X_{12}^{T}A^{\mathrm{T}}) \\ &+ j\cos(\alpha\frac{\pi}{2})(AX_{21} - X_{21}A^{\mathrm{T}}) + j\sin(\alpha\frac{\pi}{2})(AX_{22} - X_{22}^{\mathrm{T}}A^{\mathrm{T}})\end{aligned} \quad (2.87)$$

According to Lemma (2.4.3), inequality (2.87) can be expressed in real terms as follows

$$\Xi = \begin{bmatrix} \Phi & \Psi \\ -\Psi & \Phi \end{bmatrix} \prec 0 \tag{2.88}$$

with

$$\begin{aligned} \Phi = {} & \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})AX_{11}\right\} + \mathrm{Sym}\left\{\cos(\alpha\tfrac{\pi}{2})AX_{12}\right\} \\ & + \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})X_{21}AX_{21}\right\} + \mathrm{Sym}\left\{-\cos(\alpha\tfrac{\pi}{2})AX_{22}\right\} \end{aligned}$$

$$\begin{aligned} \Psi = {} & \cos(\alpha\tfrac{\pi}{2})(X_{11}A^{\mathrm{T}} - AX_{11}) + \sin(\alpha\tfrac{\pi}{2})(AX_{12} - X_{12}^{T}A^{\mathrm{T}}) \\ & + \cos(\alpha\tfrac{\pi}{2})(AX_{21} - X_{21}A^{\mathrm{T}}) + \sin(\alpha\tfrac{\pi}{2})(AX_{22} - X_{22}^{\mathrm{T}}A^{\mathrm{T}}) \end{aligned}$$

So it is easy to see that the matrix Ξ is the same as

$$\Xi = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4$$

where

$$\Xi_1 = \begin{bmatrix} \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})AX_{11}\right\} & \cos(\alpha\tfrac{\pi}{2})(X_{11}A^{\mathrm{T}} - AX_{11}) \\ \cos(\alpha\tfrac{\pi}{2})(AX_{11} - X_{11}A^{\mathrm{T}}) & \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})AX_{11}\right\} \end{bmatrix}$$

$$\Xi_2 = \begin{bmatrix} \mathrm{Sym}\left\{\cos(\alpha\tfrac{\pi}{2})AX_{12}\right\} & \sin(\alpha\tfrac{\pi}{2})(AX_{12} - X_{12}^{T}A^{\mathrm{T}}) \\ \sin(\alpha\tfrac{\pi}{2})(X_{12}^{T}A^{\mathrm{T}} - AX_{12}) & \mathrm{Sym}\left\{\cos(\alpha\tfrac{\pi}{2})AX_{12}\right\} \end{bmatrix}$$

$$\Xi_3 = \begin{bmatrix} \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})AX_{21}\right\} & \cos(\alpha\tfrac{\pi}{2})(AX_{21} - X_{21}A^{\mathrm{T}}) \\ \cos(\alpha\tfrac{\pi}{2})(X_{21}A^{\mathrm{T}} - AX_{21}) & \mathrm{Sym}\left\{\sin(\alpha\tfrac{\pi}{2})AX_{21}\right\} \end{bmatrix}$$

$$\Xi_4 = \begin{bmatrix} \mathrm{Sym}\left\{-\cos(\alpha\tfrac{\pi}{2})AX_{22}\right\} & \sin(\alpha\tfrac{\pi}{2})(AX_{22} - X_{22}^{\mathrm{T}}A^{\mathrm{T}}) \\ \sin(\alpha\tfrac{\pi}{2})(X_{22}^{\mathrm{T}}A^{\mathrm{T}} - AX_{22}) & \mathrm{Sym}\left\{-\cos(\alpha\tfrac{\pi}{2})AX_{22}\right\} \end{bmatrix}$$

With the help of the Kronecker product, the matrices $\Xi_i, i = 1, 2, 3, 4$ can be written as

$$\Xi_1 = \mathrm{Sym}\left\{\begin{bmatrix} \sin(\alpha\tfrac{\pi}{2}) & -\cos(\alpha\tfrac{\pi}{2}) \\ \cos(\alpha\tfrac{\pi}{2}) & \sin(\alpha\tfrac{\pi}{2}) \end{bmatrix} \otimes (AX_{11})\right\}$$

$$\Xi_2 = \mathrm{Sym}\left\{\begin{bmatrix} \cos(\alpha\tfrac{\pi}{2}) & \sin(\alpha\tfrac{\pi}{2}) \\ -\sin(\alpha\tfrac{\pi}{2}) & \cos(\alpha\tfrac{\pi}{2}) \end{bmatrix} \otimes (AX_{12})\right\}$$

$$\Xi_3 = \mathrm{Sym}\left\{\begin{bmatrix} \sin(\alpha\tfrac{\pi}{2}) & \cos(\alpha\tfrac{\pi}{2}) \\ -\cos(\alpha\tfrac{\pi}{2}) & \sin(\alpha\tfrac{\pi}{2}) \end{bmatrix} \otimes (AX_{21})\right\}$$

$$\Xi_4 = \mathrm{Sym}\left\{ \begin{bmatrix} -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & cos(\alpha\frac{\pi}{2}) \end{bmatrix} \otimes (AX_{22}) \right\}$$

By defining

$$\Theta_{11} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & -\cos(\alpha\frac{\pi}{2}) \\ \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}, \Theta_{12} = \begin{bmatrix} \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \end{bmatrix}$$

$$\Theta_{21} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \\ -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}, \Theta_{22} = \begin{bmatrix} -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & cos(\alpha\frac{\pi}{2}) \end{bmatrix}$$

It is clear that to check the inequality (2.80), it suffices to substitute $\Theta_{ij}, i, j = 1, 2$ in the matrix Ξ. This ends the proof. □

An other interesting result published in [42] on the stability of fractional linear systems based on $\mathcal{LMI}$ formulation This $\mathcal{LMI}$ is a rewrite of the $\mathcal{LMI}$ (2.77) in Lemma (2.4.1) and is given by the following theorem.

Theorem 2.4.5. *[42]Let $A \in \mathbb{R}^{n\times n}$, $0 < \alpha < 1$ and $\theta = (1-\alpha)\frac{\pi}{2}$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable if and only if there exist a positive definite Hermitian matrix $X = X^* \in \mathbb{C}^{n\times n}$ such that*

$$(rX + \bar{r}\overline{X})^T A^T + A(rX + \bar{r}\overline{X}) \prec 0 \tag{2.89}$$

where $r = e^{j(1-\alpha)\frac{\pi}{2}}$.

Proof. We have to show that both $\mathcal{LMI}$ problem in Lemma (2.4.1) and Theorem (2.4.5) are equivalent. According to Lemma (2.4.1), The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable if and only if there exist two complex matrices $X_1 = X_1^* \succ 0, X_2 = X_2^* \succ 0$ such that the $\mathcal{LMI}$ (2.77) holds.
Define $L_1(X_1) = rX_1A^{\mathrm{T}} + \bar{r}AX_1$ and $L_2(X_2) = \bar{r}X_2A^{\mathrm{T}} + rAX_2$. It follows that $\mathcal{LMI}$ problem in Lemma (2.4.1) can be rewritten as

$$L_1(X_1) + L_2(X_2) \prec 0 \tag{2.90}$$

for some complex matrices $X_1 = X_1^* \succ 0, X_2 = X_2^* \succ 0$. Condition (2.89) is equivalent to

$$(rX^{\mathrm{T}} + \bar{r}\overline{X}^{\mathrm{T}})A^{\mathrm{T}} + A(rX + \bar{r}\overline{X}) \prec 0$$

Since $X = X^*$, then $X^{\mathrm{T}} = \overline{X}$. In this case the last inequality can be rewritten as

$$(r\overline{X}A^{\mathrm{T}} + \bar{r}A\overline{X}) + (\bar{r}XA^{\mathrm{T}} + rAX) \prec 0$$

which means that the $\mathcal{LMI}$ problem of Theorem (2.4.5) is equivalent to

$$L_1(\overline{X}) + L_2(X) \prec 0 \tag{2.91}$$

for some complex matrix $X = X^* \succ 0$. Consequently, we have to prove that (2.90) and (2.91) are equivalent.

- If (2.91) holds, it suffices to take $X_1 = \overline{X}$ and $X_2 = X$ to see that (2.90) also holds.
- Conversely, if there exist complex matrices $X_1 = X_1^* \succ 0, X_2 = X_2^* \succ 0$ such that (2.90) holds then $\overline{L_1(X_1) + L_2(X_2)} \prec 0$ and consequently

  $$L_1(X_1) + L_2(X_2) + \overline{L_1(X_1) + L_2(X_2)} \prec 0 \tag{2.92}$$

  Few computations show that (2.92) is equivalent to

  $$rX_1A^{\mathrm{T}} + \bar{r}AX_1 + \bar{r}X_2A^{\mathrm{T}} + rAX_2 + r\overline{X_1}A^{\mathrm{T}} + \bar{r}A\overline{X_1} + \bar{r}\overline{X_2}A^{\mathrm{T}} + rA\overline{X_2} \prec 0 \tag{2.93}$$

  i.e.,

  $$r(X_1 + \overline{X_2})A^{\mathrm{T}} + \bar{r}A(X_1 + \overline{X_2}) + \bar{r}(\overline{X_1} + X_2)A^{\mathrm{T}} + rA(\overline{X_1} + X_2) \prec 0$$

  It is easy to see that this last inequality is the same as

  $$L_1(\overline{\overline{X_1} + X_2}) + L_2(\overline{X_1} + X_2) \prec 0 \tag{2.94}$$

  From $X_1 \succ 0$ and $X_2 \succ 0$, we have $\overline{X_1} \succ 0$ and $\overline{X_1} + X_2 \succ 0$. So we get (2.91) for $X = \overline{X_1} + X_2$.

This ends the proof. □

To illustrate the results in Theorems (2.4.4) and (2.4.5), we now provide numerical examples

Example 2.4.1. *1) Consider the fractional order system described by*

$$D^{\alpha}x(t) = Ax(t)$$

with

$$A = \begin{bmatrix} -1 & 2 & 1 \\ -1 & -2 & 0.2 \\ 0.5 & 0 & -2 \end{bmatrix}, \ \alpha = 1.5$$

To verify the stability of this system, we will show two different ways.
The stability of the system can be studied by the localization of the eigenvalues of the matrix A.

$$\mathrm{spec}(A) = \{-1.5727 + 1.0923i, -1.5727 - 1.0923i, -1.8546\}$$

Each eigenvalue λ *of* A *satisfy the criteria* $|\arg(\lambda)| > \alpha\frac{\pi}{2}$*, which means that the system is asymptotically stable.*
According to Theorem (2.4.4)*, a feasible solution of the linear matrix inequality* (2.67) *is as follows*

$$P = \begin{bmatrix} 0.3112 & 0.1669 & 0.1384 \\ 0.1669 & 0.4876 & 0.1602 \\ 0.1384 & 0.1602 & 0.5463 \end{bmatrix}$$

2) Consider the fractional order system described by

$$D^{\alpha}x(t) = Ax(t)$$

with

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -6 & 0 \\ 0 & -1 & -4 \end{bmatrix}, \ \alpha = 0.5$$

The eigenvalues of the matrix$A$ *are as follows*

$$\mathrm{spec}(A) = \{-1, -4, -6\}$$

It is easy to verify that they lie in the stable region.
The stability of the system can also be verified through the resolution of the $\mathcal{LMI}$ (2.89) *in*

Theorem (2.4.5). *a feasible solution is*

$$X = \begin{bmatrix} 625.6851 & 170.3633 & 214.7760 \\ 170.3633 & 313.7579 & 83.7615 \\ 214.7760 & 83.7615 & 359.0291 \end{bmatrix}$$

The stability of fractional order system was studied in [122] where an other interesting result on this topic was derived. It's about an equivalent LTI system for an FO-LTI system in the sense of stability. To this end, we need the following theorem which states a necessary and sufficient condition to place the eigenvalues of a real matrix in a specified sector.

Theorem 2.4.6. *[6, 36]Eigenvalues of an $n \times n$ matrix A lie within the region Ω in Figure* (2.5) $\Omega = \left\{\lambda \in \mathbb{C} / \operatorname{Re}(\lambda)\cos(\delta) \pm \operatorname{Im}(\lambda)\sin(\delta) \leq 0; 0 \leq \delta < \frac{\pi}{2}\right\}$ *if and only if the eigenvalues of the $2n \times 2n$ matrix*

$$\tilde{A} = \begin{bmatrix} A\cos(\delta) & -A\sin(\delta) \\ A\sin(\delta) & A\cos(\delta) \end{bmatrix}$$

have negative real parts.

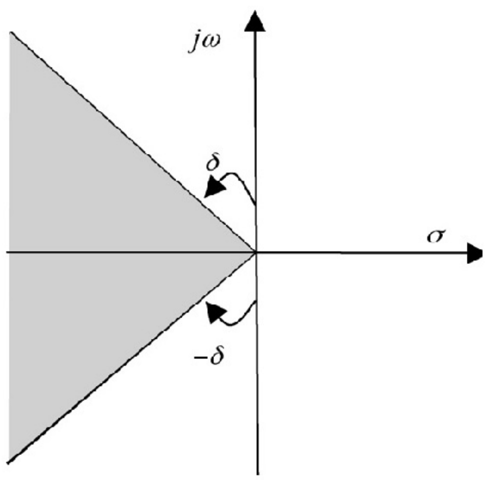

Figure 2.5: Region Ω in Theorem (2.4.6) is shown in gray.

In fact, the region Ω can be interpreted as stable region of FO-LTI system $D^{\alpha}x(t) = Ax(t)$ where $\delta = (\alpha - 1)\frac{\pi}{2}$. Now, the following two theorems are proposed.

Theorem 2.4.7. *[122] The FO-LTI system* $D^{\alpha}x(t) = Ax(t)$ *with* $1 \leq \alpha < 2$ *is asymptotically stable if and only if the LTI system,*

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A\sin(\alpha\frac{\pi}{2}) & A\cos(\alpha\frac{\pi}{2}) \\ -A\cos(\alpha\frac{\pi}{2}) & A\sin(\alpha\frac{\pi}{2}) \end{bmatrix} \tilde{x}(t)$$

is asymptotically stable.

Proof. According to Theorem (2.4.6) and the stability condition (2.66), the proof is obvious. □

Theorem 2.4.8. *[122] all eigenvalues FO-LTI system* $D^{\alpha}x(t) = Ax(t)$ *with* $0 < \alpha \leq 1$ *settle in the unstable region (Figure* (2.3)*) if and only if the LTI system,*

$$\dot{\tilde{x}}(t) = -\begin{bmatrix} A\sin(\alpha\frac{\pi}{2}) & -A\cos(\alpha\frac{\pi}{2}) \\ A\cos(\alpha\frac{\pi}{2}) & A\sin(\alpha\frac{\pi}{2}) \end{bmatrix} \tilde{x}(t) \tag{2.95}$$

is asymptotically stable.

Proof. (Necessity) Suppose that all the eigenvalues of the FO-LTI system $D^{\alpha}x(t) = Ax(t)$ with $0 < \alpha \leq 1$ are located in the unstable region ,shown in Figure (2.3),

$$\Omega_1 = \left\{\lambda/\lambda \in \mathbb{C}, |\arg(\lambda)| < \alpha\frac{\pi}{2}\right\}$$

We know that if λ is an eigenvalue of the matrix A, $-\lambda$ is an eigenvalue of the matrix $-A$ and then it lies in the symmetrical region with respect to the imaginary axis

$$\Omega_2 = \left\{\lambda/\lambda \in \mathbb{C}, |\arg(\lambda)| > (2-\alpha)\frac{\pi}{2}\right\}$$

It is clear that the region Ω_2 is the stability region of the fractional order system

$$D^{(2-\alpha)}x(t) = -Ax(t), 1 \leq 2-\alpha < 2$$

Thus, according to Theorem (2.4.7), the system

$$\dot{\tilde{x}}(t) = \begin{bmatrix} -A\sin((2-\alpha)\frac{\pi}{2}) & -A\cos((2-\alpha)\frac{\pi}{2}) \\ A\cos((2-\alpha)\frac{\pi}{2}) & -A\sin((2-\alpha)\frac{\pi}{2}) \end{bmatrix} \tilde{x}(t)$$

is asymptotically stable. After some computations, one can see that this last system is exactly the system (2.95), so we get our goal.
(Sufficiency) Just follow similar reasoning. □

Theorems (2.4.7) and (2.4.8) relate stability of a fractional order system to stability of its equivalent ordinary system. Based on these two theorems, most of the stability related analysis in the ordinary systems is applicable to the fractional systems with commensurate order as well.

2.5 Controllability and Observability of FO-LTI Systems

Controllability and observability represent two major concepts of modern control system theory. These concepts were introduced by R. Kalman in 1960. They can be roughly defined as follows.
Controllability: In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.
Observability: In order to see what is going on inside the system under observation, the system must be observable.
The controllability and observability conditions for commensurate-order systems can be seen in [83, 84, 126] in which tests for these concepts are connected to the rank tests of certain matrices: the controllability and observability matrices as in the integer case due to Kalman [55] and the proofs are given following a method similar to that used for integer-order systems [99, 137].
Consider the linear fractional-order invariant-time system described by

$$\begin{cases} D^{\alpha}x(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{cases} \qquad 0<\alpha<2 \tag{2.96}$$

where α is the fractional commensurate order of the system, $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the input vector $y(t)\in\mathbb{R}^p$ is the output vector and A, B, C are constant matrices with appropriate dimensions.
The definition of both the controllability and the observability given below are the same as for integer order case [27]. In what follows, the initial time can be taken zero.

Definition 2.5.1. *The system* (2.96) *is controllable if, for a given time t_0, there exists a finite time $t_f > t_0$ such that from any initial condition $x(t_0)=x_0$ and any $x(t_f)=x_f$ in the state-space, there exists an input $u(t), t\in[t_0,t_1]$ which can lead the system from x_0 to x_f at time t_f.*

Controllability is an important concept since it establishes the fact that we can control the system to modify its behavior (stabilizing a system unstable, change specific dynamics). This concept plays a

very important role in the synthesis theory of control systems in state-space.
A trivially uncontrollable system is the one whose input matrix is zero, $B = 0$.

Theorem 2.5.1. *(**Controllability Criterion**) The following statements are equivalent.*

- *The system given by* (2.96) *(or the pair* (A, B) *is controllable.*
- *The matrix* $\mathcal{C}$ *defined by*
$$\mathcal{C} = \left[B\ AB\ A^2B\ ...\ A^{n-1}B\right] \tag{2.97}$$
is full-rank.
- *For all* $\lambda \in \mathbb{C}$,
$$\text{rank}\left(\left[\begin{array}{cc} \lambda I_n - A & B \end{array}\right]\right) = n \tag{2.98}$$

Definition 2.5.2. *The system* (2.96) *is observable on* $[t_0, t_1]\,, t_1 > 0$, *if* $x(t_0)$ *can be deduced from the observation of the output* $y(t)$ *and the knowledge of the input* $u(t)$ *for* $t \in [t_0, t_1]$.

Clearly, the notion of observability is crucial for systems where the state vector is not completely accessible to the measure but must be rebuilt, estimated or filtered from data supplied by the output.

Theorem 2.5.2. *(**Observability Criterion**) The following statements are equivalent.*

- *The system given by* (2.96) *(or the pair* (C, A) *is observable if and only if the matrix* $\mathcal{O}$ *defined by*
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^{n-1} \end{bmatrix} \tag{2.99}$$
is full-rank.
- *For all* $\lambda \in \mathbb{C}$
$$\text{rank}\left(\begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix}\right) = n \tag{2.100}$$

Testing the rank of the matrix $\mathcal{O}$ may pose numerical difficulties, since it requires the computation of powers of A up to A^{n-1}, an operation which may be ill-conditioned. Testing the rank of the matrix $\begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix}$ for all $\lambda \in \text{spec}(A)$ is generally more robust numerically.

2.6 Minimum Energy Control For Linear Fractional-Order Systems

So far, in the control system design, the freedom should be used to improve performance of the system such as minimum energy consumption. this aim requires choosing a best control law from all feasible strategies. This is the optimal control problem, loosely speaking.
According to [10, 12], we have these results.

Theorem 2.6.1. *The linear system* (2.48) *is controllable on* $[0, t_f]$ *if and only if the controllability gramian matrix defined by*

$$W_c = \int_0^{t_f} (t_f - \tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f - \tau)^\alpha A) B][E_{\alpha,\alpha}((t_f - \tau)^\alpha A) B]^T d\tau \tag{2.101}$$

is a positive definite matrix for some $t_f > 0$.

Remark 2.6.1. *Furthermore, the input*

$$u(t) = [E_{\alpha,\alpha}((t_f - t)^\alpha A) B]^T \; W_c^{-1} [x_f - E_\alpha \left(t_f^\alpha A\right) x_0], t \in [0 \; t_f t] \tag{2.102}$$

steers $x(t)$ *from* $x_0 = x(0)$ *to* $x_f = x(t_f)$.
Here T *denotes the transpose matrix .*

If the system modeled by (2.48) is controllable, then there exist usually many input vectors $u(t) \in \mathbb{R}^m$ that steer the solution $x(t)$ of the system from $x_0 = x(0)$ to $x_f = x(t_f)$. Among these input vectors, we shall looking for such $u(t), t \in [0 \; t_f]$ which minimizes the performance index

$$I(u) = \int_0^{t_f} (t_f - \tau)^{\alpha-1} u^T(\tau) Q u(\tau) d\tau \tag{2.103}$$

where $Q \in \mathbb{R}_+^{m\times m}$ is a symmetric positive defined matrix. The minimum energy control problem for the system (2.48) can be formulated as follows. Given matrices A and B of (2.48), the symmetric matrix Q, vectors $x_0, x_f \in \mathbb{R}^n$ and $t_f > 0$, find input vector $u(t) \in \mathbb{R}^m, t \in [0, \; t_f]$ that steers $x(t)$ from $x(0) = x_0$ to $x(t_f) = x_f$ and minimizes the performance index (2.103). To solve this problem, we define the matrix

$$\begin{aligned} W_f &= W_f(t_f, Q) \\ &= \int_0^{t_f} (t_f - \tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f - \tau)^\alpha A) B] Q^{-1} [E_{\alpha,\alpha}((t_f - \tau)^\alpha A) B]^T d\tau \end{aligned} \tag{2.104}$$

Then, we derive the following result.

Theorem 2.6.2. *The fractional linear continuous time system* (2.48) *is controllable on* $[0,\ t_f]$ *if and only if the matrix defined by* (2.104) *is positive definite for some* $t_f > 0$.

Proof. (Sufficiency) Since W_f is positive definite, it is non-singular and therefore its inverse is well defined. Let x_f be the final pseudo-state to be reached.Take the control function

$$\hat{u}(t) = Q^{-1}[E_{\alpha,\alpha}((t_f - t)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0] \tag{2.105}$$

It will be shown that $\hat{u}(t)$ steers $x(t)$ from x_0 to x_f. Substituting (2.105) into (2.49) for $t = t_f$ and using (2.104) we get

$$\begin{aligned}
x(t_f) &= E_{\alpha}(t_f^{\alpha}A)x_0 \\
&+ \int_0^{t_f}(t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B] \\
&\times Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]d\tau \\
&= E_{\alpha}(t_f^{\alpha}A)x_0 + \int_0^{t_f}(t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B] \\
&\times Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T d\tau W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0] \\
&= E_{\alpha}(t_f^{\alpha}A)x_0 + W_f W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0] \\
&= x_f
\end{aligned}$$

We conclude that the system (2.48) is controllable.
(Necessity) Assume that the system (2.48) is controllable but the matrix W_f defined by (2.104) is not positive definite. It results that there exists a non-zero vector $x \in \mathbb{R}^n$ such that

$$x^T W_f x = 0_{\mathbb{R}^n}$$

i.e.,

$$x^T \int_0^{t_f}(t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]Q^{-1}[E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]^T d\tau x = 0 \tag{2.106}$$

as the matrix Q is symmetric positive definite, then there exists a matrix H non-singular such that

$$Q = H^T H$$

Substuting Q in (2.106), we then get

$$x^T \int_0^{t_f} (t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]H^{-1}[H^{-1}]^T[E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]^T d\tau x = 0 \text{ on } [0\ t_f]$$

$$\int_0^{t_f} (t_f-\tau)^{\alpha-1}[x^T E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)BH^{-1}][x^T E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)BH^{-1}]^T d\tau = 0 \text{ on } [0\ t_f]$$

$$\int_0^{t_f} (t_f-\tau)^{\alpha-1}\left\|[x^T E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)BH^{-1}]\right\|^2 = 0 \text{ on } [0\ t_f]$$

such $t_f - \tau \geq 0$, $\left\|x^T E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)BH^{-1}]\right\|^2 \geq 0$ on $[0\ t_f]$
then

$$x^T E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)BH^{-1} = 0$$

Consequently, we obtain

$$[x^T E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T = 0$$

then

$$(t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B][E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]^T x d\tau = 0$$

whence

$$\int_0^{t_f} (t_f-\tau)^{\alpha-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B][E_{\alpha,\alpha}\left((t_f-\tau)^{\alpha}A\right)B]^T x d\tau = 0$$

leading to the conclusion that

$$W_c x = 0$$

i.e., the system is not controllable. This is a contradiction, thus W_f is positive definite, which completes the proof. □

Theorem 2.6.3. *Let the system* (2.48) *be controllable on* $[0,\ t_f]$ *and* $u(t)$ *for* $t \in [0\ t_f]$ *be an input vector that steers the pseudo-state of the system from* $x(0) = x_0$ *to* $x_f = x(t_f)$*. The input vector* $\hat{u}(t)$ *defined by* (2.105) *steers also* $x(t)$ *from* x_0 *to* x_f *and minimizes the performance index* (2.103)

$$I(\hat{u}) \leq I(u) \tag{2.107}$$

The minimal value of (2.103) *for* (2.105) *is given by*

$$I(\hat{u}) = [x_f - E_{\alpha}(t_f^{\alpha}A)x_0]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0] \tag{2.108}$$

Proof. Let the system (2.48) be controllable on $[0,\ t_f]$.
Note that it was seen in theorem (2.6.2) that the input vector (2.105) steers $x(t)$ from x_0 to x_f. Since each of $u(t)$ and $\hat{u}(t)$ steers the solution of the system (2.48) from x_0 to x_f, then

$$\begin{aligned} x_f &= E_\alpha\left(t_f^\alpha A\right)x_0 + \int_0^{t_f}(t_f-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_f-\tau)^\alpha A)Bu(\tau)d\tau \\ &= E_\alpha\left(t_f^\alpha A\right)x_0 + \int_0^{t_f}(t_f-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_f-\tau)^\alpha A)B\hat{u}(\tau)d\tau \end{aligned}$$

i.e.,

$$\int_0^{t_f}(t_f-\tau)^{\alpha-1}E_{\alpha,\alpha}((t_f-\tau)^\alpha A)B[u(\tau)-\hat{u}(\tau)]d\tau = 0 \tag{2.109}$$

from (2.109) it follows that

$$\int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T[E_{\alpha,\alpha}((t_f-\tau)^\alpha A)B]^T d\tau = 0$$

then

$$\int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T[E_{\alpha,\alpha}((t_f-\tau)^\alpha A)B]^T d\tau W_f^{-1}[x_f - E_\alpha(t_f^\alpha A)x_0] = 0 \tag{2.110}$$

substitution of (2.105) into (2.110) yields

$$\int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T Q\hat{u}(\tau)d\tau = 0 \tag{2.111}$$

i.e.,

$$\int_0^{t_f}(t_f-\tau)^{\alpha-1}u^T(\tau)Q\hat{u}(\tau)d\tau = \int_0^{t_f}(t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau \tag{2.112}$$

It will be shown that

$$\begin{aligned} \int_0^{t_f}(t_f-\tau)^{\alpha-1}u^T(\tau)Qu(\tau)d\tau &= \int_0^{t_f}(t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau \\ &+ \int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T Q[u(\tau)-\hat{u}(\tau)]d\tau \end{aligned} \tag{2.113}$$

One denote by T_2 the second term in (2.113), then

$$\begin{aligned} T_2 = &\int_0^{t_f}(t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau \\ &+ \int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T Qu(\tau)d\tau \\ &- \int_0^{t_f}(t_f-\tau)^{\alpha-1}[u(\tau)-\hat{u}(\tau)]^T Q\hat{u}(\tau)d\tau \end{aligned}$$

Using (2.111), we obtain

$$
\begin{aligned}
T_2 = & \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau \\
& + \int_0^{t_f} (t_f-\tau)^{\alpha-1}u(\tau)^T Q u(\tau)d\tau \\
& - \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}(\tau)^T Q u(\tau)d\tau
\end{aligned} \tag{2.114}
$$

Applying the transpose at (2.112) gives

$$
\int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q^T u(\tau)d\tau = \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q^T \hat{u}(\tau)d\tau
$$

since Q is a symmetric matrix, then

$$
\int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q u(\tau)d\tau = \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q \hat{u}(\tau)d\tau \tag{2.115}
$$

Substituting (2.115) into (2.114), we obtain

$$
T_2 = \int_0^{t_f} (t_f-\tau)^{\alpha-1}u^T(\tau)Q u(\tau)d\tau \tag{2.116}
$$

we conclude that the formula (2.113) is derived. Let remark that the formula (2.113) is exactly

$$
I(u) = I(\hat{u}) + I(u-\hat{u})
$$

Since the performance index is positive, then we deduce

$$
I(\hat{u}) \leq I(u)
$$

To find the minimal value (2.108) of (2.103), we substitute (2.105) into (2.103)

$$
\begin{aligned}
I(\hat{u}) = & \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau \\
= & \int_0^{t_f} (t_f-\tau)^{\alpha-1}[Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]]^T Q \\
& [Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]]d\tau
\end{aligned}
$$

$$I(\hat{u}) = \int_0^{t_f} (t_f-\tau)^{\alpha-1}\hat{u}^T(\tau)Q\hat{u}(\tau)d\tau$$
$$= \int_0^{t_f} (t_f-\tau)^{\alpha-1}[Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]]^T Q$$
$$[Q^{-1}[E_{\alpha,\alpha}((t_f-\tau)^{\alpha}A)B]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]]d\tau$$

We use (2.104) and since Q and W_f are symmetric matrices, then

$$I(\hat{u}) = [x_f - E_{\alpha}(t_f^{\alpha}A)x_0]^T W_f^{-1}[x_f - E_{\alpha}(t_f^{\alpha}A)x_0]$$

Then the proof is complete. □

If the assumptions of the theorem (2.6.2) are satisfied, the optimal input vector (2.105) and the minimal value (2.108) of the performance index can be computed and compared to those of an input vector which satisfies the controllability of the system using the following procedure.

Procedure

setep 1 Knowing A find $E_{\alpha}(t^{\alpha}A)$, $E_{\alpha,\alpha}(t^{\alpha}A)$

setep 2 Using (2.101)-(2.104) find W_c and W_f

setep 3 Using (2.102)-(2.105) find $u(t)$ and $\hat{u}(t)$

setep 4 Using (2.103) find $I(u)$ and using (2.108) find $I(\hat{u})$

Example 2.6.1. *Consider the following continuous-time linear fractional-order linear system defined by the representation of dimension* $n = 2$*.*

$$\begin{cases} D^{\alpha}x_1 & = x_2 + u_1 \\ D^{\alpha}x_2 & = u_2 \end{cases}$$

With $\alpha = 0.1$ *and* $t_f = 1$

In this case, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find an input vector $\hat{u}(t)$ *for* $t \in [0, t_f]$ *that steers* $x(t)$ *from* $x_0 = 0$ *to* $x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and minimizes the performance index* (2.103).

The the controllability condition given in Theorem (2.5.1) *is verified*

$$rank[B\ AB] = 2$$

Using the procedure we calculate

Step 1. *The Mittag-Leffler matrix function for this problem is reduced to*

$$E_\alpha(t^\alpha A) = \begin{bmatrix} 1 & \frac{t^\alpha}{\Gamma(\alpha+1)} \\ 0 & 1 \end{bmatrix}, E_{\alpha,\alpha}(t^\alpha A) = \begin{bmatrix} \frac{1}{\Gamma(\alpha)} & \frac{t^\alpha}{\Gamma(2\alpha)} \\ 0 & \frac{1}{\Gamma(\alpha)} \end{bmatrix}$$

Remark that $A^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, k = 2, 3, \dots$

Step 2. *The gramian controllability is given by*

$$W_c = \int_0^{t_f} (t_f - \tau)^{\alpha-1} [E_{\alpha,\alpha}\left((t_f - \tau)^\alpha A\right) B][E_{\alpha,\alpha}\left((t_f - \tau)^\alpha A\right) B]^T d\tau$$

$$E_{\alpha,\alpha}\left((t_f - \tau)^\alpha A\right) B = \begin{bmatrix} \frac{1}{\Gamma(\alpha)} & \frac{(t_f-\tau)^\alpha}{\Gamma(2\alpha)} \\ 0 & \frac{1}{\Gamma(\alpha)} \end{bmatrix}$$

and

$$W_c = \begin{bmatrix} \frac{t_f^\alpha}{\alpha(\Gamma(\alpha))^2} + \frac{t_f^{3\alpha}}{3\alpha(\Gamma(2\alpha))^2} & \frac{t_f^{2\alpha}}{2\alpha\Gamma(\alpha)\Gamma(2\alpha)} \\ \frac{t_f^{2\alpha}}{2\alpha\Gamma(\alpha)\Gamma(2\alpha)} & \frac{t_f^\alpha}{\alpha(\Gamma(\alpha))^2} \end{bmatrix}$$

For $Q = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$*, the matrix* W_f *is given by*

$$\begin{aligned} W_f &= \int_0^{t_f} (t_f - \tau)^{\alpha-1} [E_{\alpha,\alpha}\left((t_f - \tau)^\alpha A\right) B] Q^{-1} [E_{\alpha,\alpha}\left((t_f - \tau)^\alpha A\right) B]^T d\tau \\ &= \begin{bmatrix} \frac{10t_f^\alpha}{\alpha(\Gamma(\alpha))^2} - \frac{6t_f^{2\alpha}}{2\alpha\Gamma(\alpha)\Gamma(2\alpha)} + \frac{t_f^{3\alpha}}{3\alpha(\Gamma(2\alpha))^2} & \frac{-3t_f^\alpha}{\alpha(\Gamma(\alpha))^2} + \frac{t_f^{2\alpha}}{2\alpha\Gamma(\alpha)\Gamma(2\alpha)} \\ \frac{-3t_f^\alpha}{\alpha(\Gamma(\alpha))^2} + \frac{t_f^{2\alpha}}{2\alpha\Gamma(\alpha)\Gamma(2\alpha)} & \frac{t_f^\alpha}{\alpha(\Gamma(\alpha))^2} \end{bmatrix} \end{aligned}$$

substitution of $t_f = 1$ *and* $\alpha = 0.1$ *gives*

$$W_c = \begin{bmatrix} 0.2686 & 0.1145 \\ 0.1145 & 0.1105 \end{bmatrix}, W_c^{-1} = \begin{bmatrix} 6.6654 & -6.9063 \\ -6.9063 & 16.2065 \end{bmatrix}$$

$$W_f = \begin{bmatrix} 0.5762 & -0.2170 \\ -0.2170 & 0.1105 \end{bmatrix}, W_f^{-1} = \begin{bmatrix} 6.6654 & 13.0899 \\ 13.0899 & 34.7574 \end{bmatrix}$$

Step 3. *Using the formula* (4)*, we obtain*

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -0.0253 \\ 0.9776 - 0.0525(1-t)^{0.1} \end{bmatrix}$$

Using the formula (2.105), *we obtain the desired input vector*

$$\hat{u}(t) = \begin{bmatrix} \hat{u_1}(t) \\ \hat{u_2}(t) \end{bmatrix} = \begin{bmatrix} 5.677 - 12.91(1-t)^{0.1} \\ 4.303(1-t)^{0.1} - 0.5 \end{bmatrix}$$

In figure (2.6) *are plotted the first and the second components for each control.*

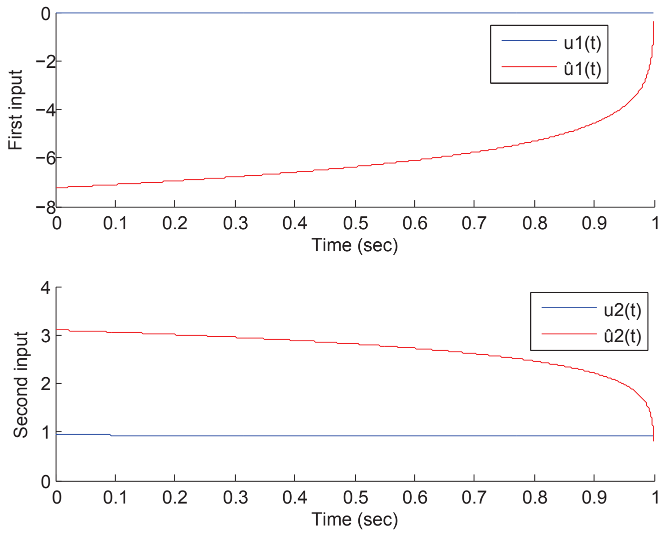

Figure 2.6: Input components versus optimal ones

Step 4. *Using* (2.103) *and* (2.108), *the performance indexes are*

$$I(u) = 89.09$$

$$I(\hat{u}) = 67.60$$

Effectively

$$I(\hat{u}) \leq I(u)$$

For more assessment of our work, first we evaluate the performance index for different values of fractional order α *by fixing the final time* $t_f = 1s$, *the results are showen in Figure* (2.7). *Secondly, we evaluate the performance index for different values of final time* t_f *with fixed fractional order* $\alpha = 0.1$, *see Figure* (2.8). *From figures* (2.7) *and* (2.8), *its clear that the performance index for the optimal control is lower then the performance index of an other control in all situations.*

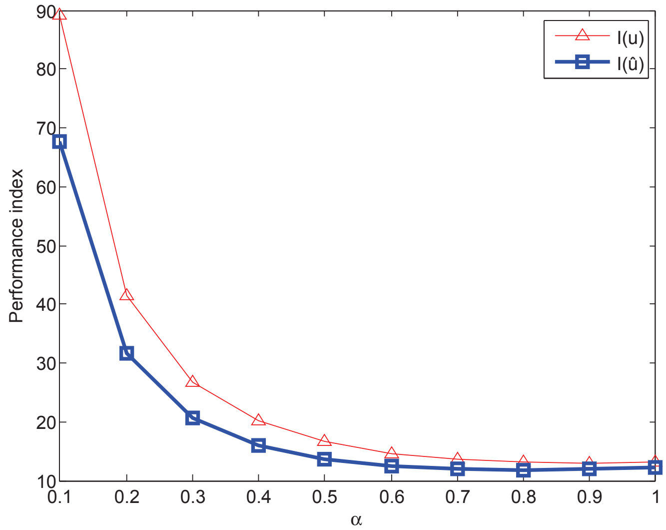

Figure 2.7: Performance index for different values of α with $t_f = 1s$

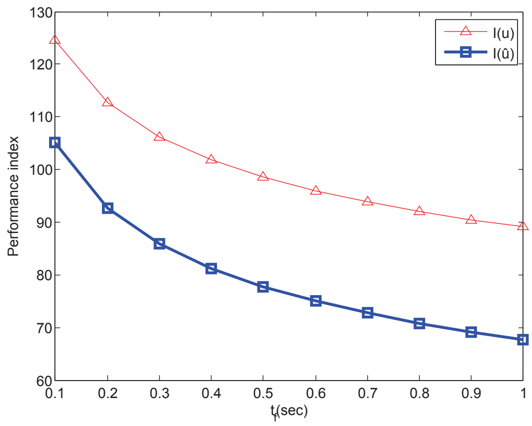

Figure 2.8: Performance index for different values of final time t_f and $\alpha = 0.1$

2.7 The Fractional-Order System Control

Control is an interdisciplinary branch of engineering and mathematics that deals with the modification of dynamic systems to obtain a desired behavior given in terms of a set of specifications or a reference model. Several control methods have been executed to reach system performances [39, 71]. Researches keep making immense efforts in developing various stabilizing control design schemes such as linear matrix inequality $\mathcal{LMI}$ based method which receives much attention in the last decades, see [69, 92, 120] and the references therein. Fractional-order systems represent a wider class of complex dynamical systems. This class of systems attracts much attention in both theory and applications in the latest four decades.

As for systems with integer order derivative, the stability of a fractional linear system depends on the location of the eigenvalues of the dynamic matrix in the complex plane [83, 84]. Recently, a considerable interest has been also devoted to the analysis of the stability and stabilization of linear systems derived from fractional order [93, 110, 111, 120], where some results have been fulfilled through $\mathcal{LMI}$ methods for mainly two types of fractional orders, $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$. It is notable that the problem of the synthesis for FOS is difficult, however it was treated in recent papers. The most obtained results were in terms of $\mathcal{LMI}$ approach. In fact, Based on the generalization of Gronwall-Bellman Lemma, the analytical stability conditions and state feedback stabilization problem of non linear fractional-order differentiation equation have been studied in [95, 96]. the problem of stabilization and robust stabilization by pseudo-state feedback of commensurate polytopic systems, particularly on the case $0 < \alpha < 1$ was investigated in [42]. Stabilization of fractional order systems with uncertainties was also highlighted in [60], where the method of observer-based control and static output feedback control in both cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$ was treated and the formalism via $\mathcal{LMI}$ was derived. The robust asymptotical stability and stabilization problem for a class of uncertain FO-LTI interval systems with the fractional order α belonging to $0 < \alpha < 1$ and linear coupling relationships between the fractional order α and the system parameters were investigated in [65]. Accordingly, study on fractional-order systems has been extended to T-S fuzzy models. In [66, 67], the stability, robust stability and stabilization were investigated for such class of systems. Recently, the static output feedback stabilization for fractional-order systems in T-S fuzzy models has been developed in both papers [52, 70].
In this section, we present some existing results concerning the problem of stabilization of linear fractional-order systems. In first time, the problem is processed via the pole placement approach with $0 < \alpha < 2$ and secondly, by an $\mathcal{LMI}$ approach where $1 \leq \alpha < 2$ and a $\mathcal{GLMI}$ method in the case where $0 < \alpha \leq 1$.

2.7.1 Eigenvalue Assignment

Consider the linear fractional-order invariant-time system described by (2.96).

1. **Stabilization by State Feedback**

   To begin, it is assumed that all components of the state vector are measured which means that $C = I_n$. The problem we wish to consider now is to determine a state feedback control law $u(t) = Kx(t)$ having the property that the resulting closed-loop system is asymptotically stable.

Definition 2.7.1. *The fractional-order system* (2.96) *is said to be stabilizable if there exists a state feedback control law defined by* $u(t) = Kx(t), K \in \mathbb{R}^{m \times n}$ *such that the closed-loop system* $D^{\alpha}x(t) = (A + BK)x(t)$ *is asymptotically stable, or equivalently*

$$|\arg(\text{spec}(A + BK)| > \alpha\frac{\pi}{2}$$

The matrix K is known as the state feedback gain matrix, and it affects the closed-loop system behavior. The main influence of K is through the matrix A, resulting in the matrix $A + BK$ of the closed-loop system. Clearly, K affects the eigenvalues of $A + BK$, and therefore, the modes of the closed-loop system. In other words, "the eigenvalues of the original system can arbitrarily be changed in this case." This last statement, commonly used in the literature, is rather confusing: The eigenvalues of a given system are not physically changed by the use of feedback. They are the same as they used to be before the introduction of feedback. Instead, the feedback law $u(t) = Kx(t)$, generates an input $u(t)$ that, when fed back to the system, makes it behave as if the eigenvalues of the system were at different locations [i.e., the input $u(t)$ makes it behave as a different system, the behavior of which is, we hope, more desirable than the behavior of the original system].

Theorem 2.7.1. *For the system* (2.96)*, there exists a control law* $u(t) = Kx(t)$*,* $K \in \mathbb{R}^{m \times n}$ *such that the* n *eigenvalues of* $A + BK$ *can be assigned to arbitrary, real or complex-conjugate, locations if and only if the pair* (A, B) *is controllable.*

Since, we can displace with a state feedback law only the controllable eigenvalues of the pair (A, B) from A to $A + BK$, we have the following corollary.

Corollary 2.7.1. *The uncontrollable eigenvalues of* (A, B) *cannot be shifted by state feedback.*

It is now quite clear that a given system (2.96) can be made asymptotically stable via the state feedback control law $u(t) = Kx(t)$ only when all the uncontrollable eigenvalues of A are already in the stable region. This is so because state feedback can alter only the controllable eigenvalues.

Definition 2.7.2. *The pair* (A, B) *is called stabilizable if and only if all its uncontrollable eigenvalues verify* $|\arg(\sec(A))| > \alpha\frac{\pi}{2}$.

Example 2.7.1. *Consider the stabilization of the following fractional order system described in* (2.96) *for the fractional order* $0 < \alpha < 1$ *and with parameters as follows*

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 5 & 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; \alpha = 0.5$$

The eigenvalues of the matrix A are $\lambda_1 = 3.6139, \lambda_2 = -2.8108, \lambda_3 = 0.1969$. According to Theorem (2.4.2)*, the system is unstable as can be seen in Figure* (2.9)*. The controllability*

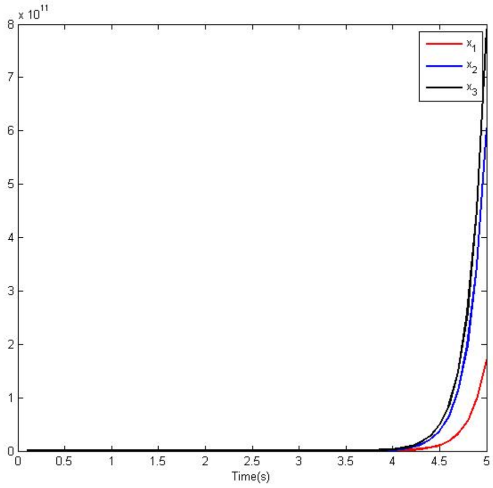

Figure 2.9: State responses of the open-loop system with $\alpha = 0.5$.

matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -1 & 0 & 6 & 2 \\ 0 & 1 & 0 & 1 & 10 & 5 \\ 0 & 0 & 5 & 2 & 0 & 4 \end{bmatrix}$$

and its rank is

$$\text{rank}(\mathcal{C}) = 3$$

The system is then stabilizable. We wish to determine K so that the eigenvalues of $A + BK$ are at $-1, -2.8108, -2$. We obtain

$$K = \begin{bmatrix} 3.4337 & 0.9728 & 2.3199 \\ -0.9433 & -3.3771 & -2.5122 \end{bmatrix}$$

as the appropriate state feedback matrix. The resulting closed-loop system is then asymptotically stable as shown in Figure (2.10)*.*

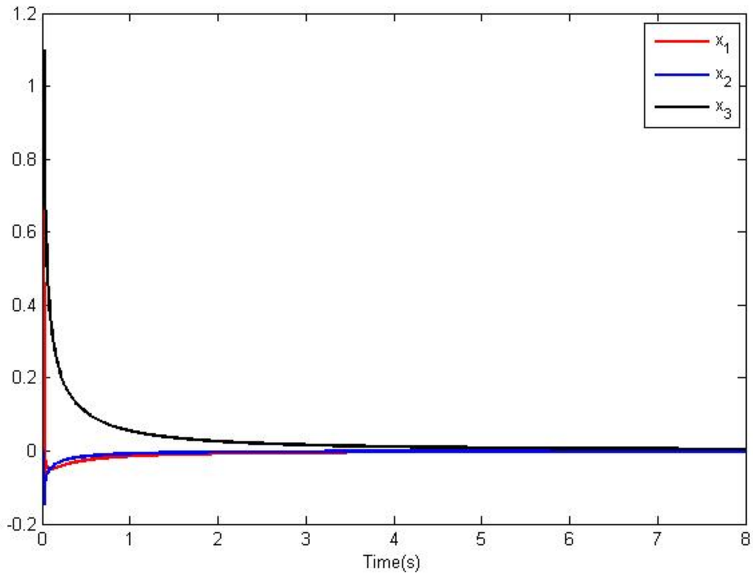

Figure 2.10: State responses of the closed-loop system with $\alpha = 0.5$.

Since it is highly unrealistic to assume that all states are measured, then the stabilization by state feedback is not always possible. In the following, we will focus to find an feedback which gives desired closed loop eigenvalues for system where only outputs are available for feedback.

2. **Stabilization by Static Output Feedback**
   The static output feedback problem concerns finding a static or constant feedback gain to achieve certain desired closed-loop characteristics. The output feedback control problem is much more difficult to solve when compared to state feedback control problem. Consider the system (2.96).

Definition 2.7.3. *The fractional-order system* (2.96) *is said to be detectable, or the pair* (A, C) *is detectable if there exists a gain* $L \in \mathbb{R}^{n \times p}$ *such that*

$$|\arg(\text{spec}(A + LC)| > \alpha\frac{\pi}{2}$$

Theorem 2.7.2. *For the system* (2.96)*, there exists a gain* L *such that the* n *eigenvalues of* $A + LC$ *can be assigned to arbitrary, real or complex-conjugate, locations if and only if the pair* (A, C) *is observable.*

Since, we can displace only the observable eigenvalues of the pair (A, C) from A to $A + LC$, we have the following corollary.

Corollary 2.7.2. *The unobservable eigenvalues of* (A, C) *cannot be shifted from* A *to* $A+LC$*.*

and then we can make the following definition.

Definition 2.7.4. *The pair* (A, C) *is called detectable if and only if all its unobservable eigenvalues verify* $|\arg(\sec(A))| > \alpha\frac{\pi}{2}$*.*

The stabilization of (2.96) by static output feedback is now introduced.

Definition 2.7.5. *The system* (2.96) *is stabilizable by static output feedback if there exists a control law* $u(t) = Ly(t), L \in \mathbb{R}^{m\times p}$ *such that the closed-loop system*

$$D^{\alpha}x(t) = (A + BLC)x(t)$$

is asymptotically stable, or equivalently
i.e.,

$$|\arg(\text{spec}(A + BLC)| > \alpha\frac{\pi}{2}$$

It is now quite clear that a given system (2.96) can be made asymptotically stable via the static output feedback control law $u(t) = Ly(t)$ only when all the unobservable and the uncontrollable eigenvalues of A are already in the stable region. This is so because static output feedback can alter only the observable and the controllable eigenvalues.

Example 2.7.2. *Consider the fractional-order system* (2.96) *with the following parameters*

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 1 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \alpha = 1.5$$

Obviously, when $u(t) = 0$*, the system* (2.96) *is unstable because the eigenvalues of the matrix* A *are 2 and 3, which are outside the stability area as can be shown in Figure* (2.11).

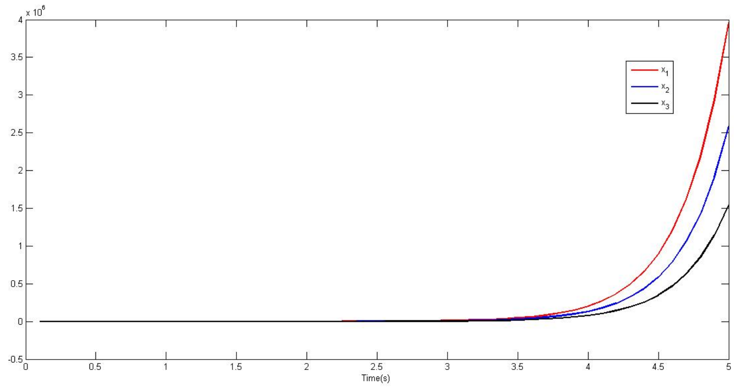

Figure 2.11: State responses of the open-loop system with $\alpha = 1.5$.

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 4 & 5 \\ 0 & 2 & 6 \\ 6 & 12 & 44 \\ 6 & 4 & 30 \end{bmatrix}$$

and the controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & 1 & 5 & 1 & 18 & 6 \\ 1 & 0 & 2 & 0 & 10 & 6 \\ 0 & 0 & 1 & 1 & 8 & 4 \end{bmatrix}.$$

Since $\mathrm{rank}(\mathcal{O}) = 3$ *and* $\mathrm{rank}(\mathcal{C}) = 3$*, the system is observable and controllable, so all its eigenvalues can be shifted by static output feedback* $u(t) = Ly(t)$*. To get the eigenvalues of the closed-loop system at* $\{-2.9847, -2.0251, -0.9902\}$*, the static output feedback gain can be computed in several different ways. For simple problems it is convenient to introduce the elements of* L *as unknown parameters, determine the characteristic polynomial of the closed-loop system and identify it with the desired characteristic polynomial. the resulting static output feedback gain is*

$$L = \begin{bmatrix} 147.1667 & -155 \\ -4.1667 & 1 \end{bmatrix}.$$

The closed-loop system $D^\alpha x(t) = (A + BLC)x(t)$ *is then asymptotically stable. This result is shown in Figure* (2.12)

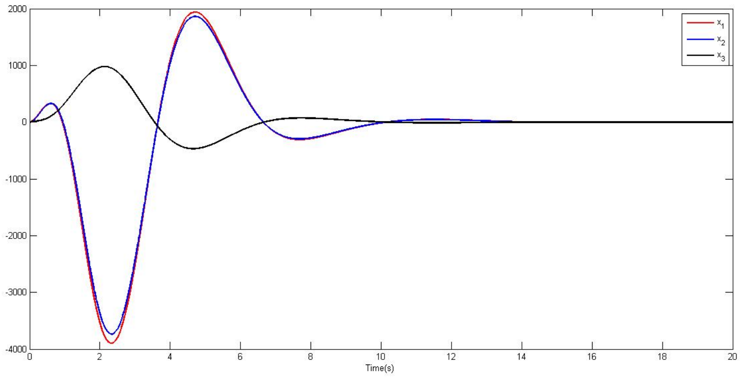

Figure 2.12: State responses of the closed-loop system with $\alpha = 1.5$.

2.7.2 LMI Conditions

According to section 2.5 of Chapter 2, we have seen that there are two $\mathcal{LMI}$ conditions for stability of fractional order linear system $0 < \alpha < 2$. Which leads us to study stabilization of the system (2.96) in both cases $1 \leq \alpha < 2$ and $0 < \alpha < 1$. The aim is to compute a static state feedback control law of the form $u(t) = Kx(t)$ where $K \in \mathbb{R}^{m \times n}$ is a constant matrix gain such that the stability of the closed-loop system

$$D^\alpha x(t) = (A + BK)x(t) \tag{2.117}$$

is ensured, which means that

$$|\arg(\text{spec}(A + BK)| > \alpha\frac{\pi}{2}$$

1. **Case** $1 \leq \alpha < 2$

   The problem, when $1 < \alpha < 2$, is a well known problem in $\mathcal{LMI}$ control theory. A solution of this problem is provided by the $\mathcal{LMI}$ region framework [29]. In order to obtain an $\mathcal{LMI}$ condition, a linearising change is applied and leads to Theorem (2.7.3).

   Theorem 2.7.3. *[42]The fractional-order system* (2.96) *with* $1 < \alpha < 2$ *is stabilizable by state feedback control* $u(t) = Kx(t)$ *if and only if there exist matrices* $X \in \mathbb{R}^{n \times n} \succ 0, Y \in \mathbb{R}^{m \times n}$

such that

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \prec 0 \tag{2.118}$$

with

$\Sigma_{11} = \Sigma_{22} = \left(AX + XA^T + BY + Y^TB^T\right)\sin(\alpha\frac{\pi}{2})$

$\Sigma_{12} = \left(AX - XA^T + BY - Y^TB^T\right)\cos(\alpha\frac{\pi}{2})$

In this case, a stabilizing controller is given by:

$$K = YX^{-1} \tag{2.119}$$

Proof. (Necessity) Suppose there exist a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system (2.117) is asymptotically stable. So there exists $X \succ 0$ such that the condition (2.69) holds since it is equivalent to that in Theorem (2.4.4) taking $A + BK$ instead of Awhich leads to a nonlinear matrix inequality. Using the linearising change $Y = KX$, the relation (2.118) is obtained. (Sufficiency) By substituting $Y = KX$ in (2.118), the asymptotic stability of the closed-loop system (2.117) is easy to check. □

Example 2.7.3. *Consider the fractional order system* (2.96) *with the following coefficients:* $\alpha = 1.5$*, and*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix A are $\{1, 2, -3, -4\}$, thus they don't all lie in the stable region described by (2.66) *which concludes the corresponding system is not asymptotically stable. In this example, the objective is to design pseudo-state feedback controller $u(t) = Kx(t)$ such that the closed-loop system* (2.117) *is asymptotically stable. solving $\mathcal{LMI}$* (2.118) *in Theorem* (2.7.3)*, it can be obtained that*

$$X = 10^4 \begin{bmatrix} 4.9707 & 0.0643 & 0.9094 & -1.0784 \\ 0.0643 & 4.5115 & 1.2248 & -1.5149 \\ 0.9094 & 1.2248 & 4.3854 & 0.6673 \\ -1.0784 & -1.5149 & 0.6673 & 3.6336 \end{bmatrix}$$

$$Y = 10^5 \begin{bmatrix} -0.8244 & 0.0295 & -0.1614 & 0.1995 \\ -0.0094 & -1.2251 & -0.3042 & 0.3779 \\ 0.3321 & 0.4702 & 0.8586 & 0.3587 \end{bmatrix}$$

The state feedback controller is given by

$$K = \begin{bmatrix} -1.6053 & 0.1730 & -0.1084 & 0.1647 \\ -0.0402 & -2.8028 & 0.1222 & -0.1627 \\ 0.7443 & 1.1880 & 1.2474 & 1.4743 \end{bmatrix}$$

The eigenvalues of the matrix $A+BK$ are as follows $\{-3.6076, -0.6328, -0.7220, -0.7220\}$ *which confirms that the resulting closed-loop system is asymptotically stable, its state responses are illustrated in Figure* (2.13)

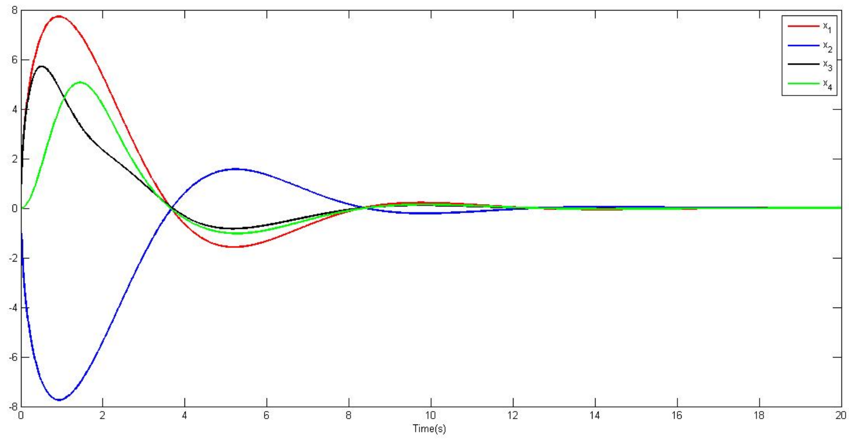

Figure 2.13: State responses of the closed-loop system with $\alpha = 1.5$.

2. **Case** $0 < \alpha < 1$

   In order to extend Theorem (2.4.5) to synthesis, a linearising change of variable must be found. Finding such a change of variable may appear more difficult than in the previous cases as variable X involved in $\mathcal{LMI}$ (2.89) is complex when the controller gain K to be found is real. However, $\mathcal{LMI}$ (2.89) of Theorem (2.4.5) has been formulated such that $rX + \bar{r}\bar{X}$ is real. Replacing A by $A + BK$ and applying the change of variables $Y = K(rX + \bar{r}\bar{X})$ thus allows to obtain the following necessary and sufficient condition for the design of a pseudo-state feedback controller with real gain.

Theorem 2.7.4. *[42] Fractional system* (2.96) *of order* $0 < \alpha < 1$ *controlled by the static state feedback* $u(t) = Kx(t)$ *is asymptotically stable if and only if there exists matrices* $X \in$

$\mathbb{C}^{n\times n} \succ 0$, $Y \in \mathbb{R}^{m\times n}$ *such that*

$$(rX + \bar{r}\bar{X})^T A^T + A(rX + \bar{r}\bar{X}) + Y^T B^T + BY \prec 0 \tag{2.120}$$

where $r = e^{j(1-\alpha)\frac{\pi}{2}}$. *A stabilizing controller gain is then:*

$$K = Y(rX + \bar{r}\bar{X})^{-1} \tag{2.121}$$

Proof. $\mathcal{LMI}$ (2.120) together with relation (2.121) follow directly from discussion above. To prove the regularity of the matrix $rX + \bar{r}\bar{X}$ involved in (2.121), we will show that it is positive definite, which means that all its eigenvalues are strictly positive. From the fact that the matrix $rX + \bar{r}\bar{X}$ is real, then it is positive definite if and only if

$$\forall x \in \mathbb{R}^n, x^T(rX + \bar{r}\bar{X})x > 0$$

Since $x^{\mathrm{T}}Xx$ is a positive real, then

$$x^{\mathrm{T}}Xx = \overline{x^{\mathrm{T}}Xx} = x^{\mathrm{T}}\overline{X}x$$

then

$$x^T(rX + \bar{r}\bar{X})x = (r + \overline{r})x^T Xx$$

When $0 < \alpha < 1$, $r + \overline{r} = 2\sin(\alpha\frac{\pi}{2}) > 0$ which implies that

$$x^T(rX + \bar{r}\bar{X})x > 0$$

Therefore $rX + \bar{r}\bar{X}$ is invertible. Consequently, relation (2.121) always holds.

□

Example 2.7.4. *Considering the following continuous fractional-order system* (2.96) *with* $\alpha = 0.5$ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3.46 & 1.4 \\ 2.8 & 0.2 \end{bmatrix}$
We can easily get that the open loop of the system (2.96) *is unstable because all of the eigenvalues of the matrix* A *are* $\{0.3177, 1.3412 + j1.1615, 1.3412 - j1.1615\}$ *and do not lie in the stable area described in Figure* (2.3). *This can be confirmed in Figure* (2.14).

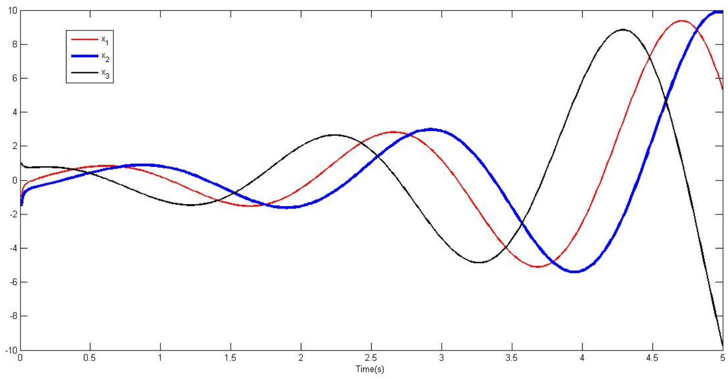

Figure 2.14: State responses of the open-loop system with $\alpha = 0.5$.

Using MATLAB LMI Control Toolbox to solve (2.120) *in Theorem* (2.7.4)*, we can obtain the feasible solutions as follows*

$$X = \begin{bmatrix} 504.4855 & 240.3975 & 241.2208 \\ 240.3975 & 543.1452 & 541.8596 \\ 241.2208 & 541.8596 & 687.5943 \end{bmatrix}$$

$$Y = \begin{bmatrix} -593.1776 & -314.2401 & 133.7964 \\ 803.5579 & -507.7170 & -94.1131 \end{bmatrix}$$

Furthermore, we can get that

$$K = \begin{bmatrix} -1.1514 & -3.1151 & 3.0534 \\ 2.5733 & -4.8479 & 2.7808 \end{bmatrix}$$

It is easy to verify that system (2.96) *is asymptotically stable because all of the eigenvalues of the matrix $A + BK$ are*

$$\{-1.2850, -3.2387 + j7.3369, -3.2387 - j7.3369\}$$

and verify the condition (2.66)*.*

With the control gain matrix K obtained, we can draw the state figure of the closed-loop system shown as in Figure (2.15)*. It is easily to see that although the open loop fractional order system is unstable, by Figure* (2.15)*, the closed-loop fractional order system can be stabilized by the control law $u(t) = Kx(t)$.*

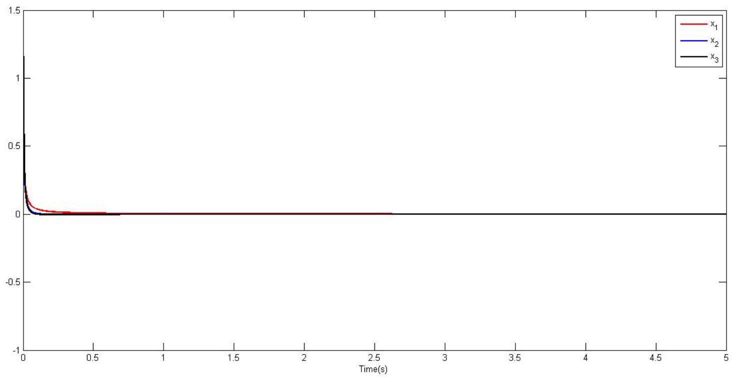

Figure 2.15: State responses of the closed-loop system with $\alpha = 0.5$.

2.8 Conclusion

The preceding paragraphs have been devoted to the description of the non-integer derivation and its use in modeling. After introducing specific functions for non-integer differentiation, the gamma function and the Mittag-Leffler function, the operator of non-integer integration and its properties was presented. Then, three approaches more popular and more practicable, which are those of Grünwald-Letnikov, Riemann-Liouville and Caputo are highlighted. Note that only the commensurable non-integer models admit a pseudo state representation whose form is comparable to that of the integer models. The solution of the state space equation hes been derived using the Mittag-Leffler by the Laplace transform. We presented the basic results on the stability of fractional linear systems in the case where the stability domain is a convex domain (that is, when the fractional order derivative α is between $1 < \alpha < 2$) and in the case where the stability domain is a non-convex domain (with $0 < \alpha < 1$). The results obtained in the literature are cited with their detailed proofs. The controllability and the observability have also been defined with some existing algebraic criteria in the literature. Sufficient and necessary controllability condition has been given and demonstrated for fractional continuous time linear system. We analyzed and solved the minimum energy control problem for this class of systems. The theoretical proposed solution is assessed by some simulation results. The focus is subsequently on the stabilization problem of fractional systems.The stabilization by state feedback and static output feedback of FOS was highlighted.

The following Chapter deals with the admissibility of fractional singular systems, and new necessary and sufficient conditions are proposed for open-loop and closed loop systems in $\mathcal{LMI}$ forms.

Chapter 3

Singular Fractional-Order Linear Continuous-Time Systems

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3.1 Introduction

Research on dynamic systems often requires mathematical modeling of system behavior. A large class of physical systems can be modeled by Differential-Algebraic equations known also as singular systems. This chapter is addressed to a new family of so-called singular linear systems with non-integer order derivatives. These systems are described by the following fractional differential equation

$$ED^{\alpha}x(t) = Ax(t) \quad 0 < \alpha < 2 \tag{3.1}$$

The models of systems of the form (3.1), which are called fractional singular systems in [54, 94, 97], are of great interest for modeling many practical processes like their counterparts the singular linear systems of the form $(E\dot{x}(t) = Ax(t))$. To begin a motivation example of singular fractional electrical circuit is presented, then analysis of the singular fractional-order systems in the continuous context is initiated. As for integer-order systems systems, it was proved that the stability of linear fractional-order systems depends on the localization of the eigenvalues in the complex plan. Consequently, many works, specially those derived via linear matrix inequalities ($\mathcal{LMI}$s), have been proposed (see for example [42, 50, 65, 94, 111]). Therefore, for a singular fractional-order system, it is important to develop conditions which guarantee that the given singular system is not only stable but also regular and impulse-free. In the literature, it is reported that such conditions can usually be obtained by decomposing singular systems into slow and fast sub-systems [130]. However, this may lead to some numerical problems. Furthermore, from the mathematical point of view, the decomposition approach is not elegant. Thus, attention in this chapter will be focused on the derivation of such conditions without decomposing the original singular system, and a linear matrix inequality ($\mathcal{LMI}$) approach will be developed. Necessary and sufficient conditions of admissibility for singular fractional-order continuous-time systems are proposed in strict $\mathcal{LMI}$ terms. By employing the derived results, the behavior of the closed-loop system is then determined by static output feedback controller for the case $1 \leq \alpha < 2$ and by observer based controller for the case $0 < \alpha < 1$.

3.2 Illustrative example

Consider electrical circuit [54] shown on Figure (3.1) with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2. Using the Kirchhoff's laws.

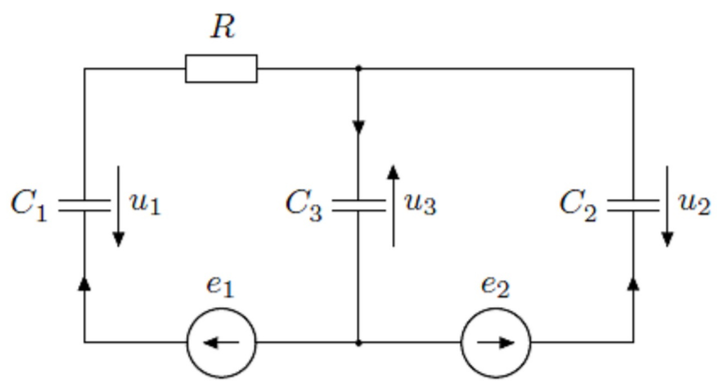

Figure 3.1: Electrical circuit

We can write for the electrical circuit the equations

$$e_1 = RC_1 + D^\alpha u_1 + u_1 + u_3 \tag{E1}$$

$$0 = C_1 D^\alpha u_1 + C_2 D^\alpha u_2 - C_3 D^\alpha u_3 \tag{E2}$$

$$e_2 = u_2 + u_3 \tag{E3}$$

The equations (E1)-(E2)-(E3) can be written as

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix} D^\alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.2}$$

which has the form

$$ED^\alpha u = Au + Be$$

with

$$E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.3}$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.4}$$

Note that the matrix E is singular since $\det(E) = 0$ but the pencil

$$\begin{aligned} \det(s^{\alpha}E - A) &= \begin{bmatrix} RC_1s^{\alpha} + 1 & 0 & 1 \\ C_1s^{\alpha} & C_2s^{\alpha} & -C_3s^{\alpha} \\ 0 & 1 & 1 \end{bmatrix} \\ &= (RC_1s^{\alpha} + 1)(C_2s^{\alpha} + C_3s^{\alpha}) + C_1s^{\alpha} \end{aligned} \tag{3.5}$$

is not identically zero, then it is regular. Therefore, the electrical circuit is a singular fractional linear system.

Remark 3.2.1. *If the electrical circuit contains at least one mesh consisting of branches with only ideal supercondensators and voltage sources then its matrix E is singular since the row corresponding to this mesh is zero row. This follows from the fact that the equation written by the use of the voltage Kirchhoff's law is algebraic one.*

3.3 Generalities About Singular Fractional-Order Systems

3.3.1 Preliminaries Results

Considering the following singular fractional-order (SFO) continuous time system described by

$$\begin{aligned} ED^{\alpha}x(t) &= Ax(t) + Bu(t), 0 < \alpha < 2, \\ y(t) &= Cx(t). \end{aligned} \tag{3.6}$$

where α is the time fractional derivative order, $x(t) \in \mathbb{R}^n$ is the pseudo-state, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ with $\operatorname{rank} C = p$ and matrix $E \in \mathbb{R}^{n\times n}$ is singular with $\operatorname{rank} E = r < n$.

The unforced SFO system of (3.6) can be written as

$$ED^{\alpha}x(t) = Ax(t). \tag{3.7}$$

First, we present some preliminaries results on singular fractional-order systems

Definition 3.3.1. *The SFO system* (3.7) *or the pair* (E, A) *is said to be regular if there exists an unique solution* $x(t)$ *for a given initial conditions.*

As in [34], the following lemmas which generalize well known results of singular "integer-order" systems to singular fractional-order singular systems were proposed in [94, 97].

Lemma 3.3.1. *The SFO system* (3.7) *or the pair* (E, A) *is said to be regular if and only if* $\det(s^{\alpha}E - A)$ *is not identically zero, where* $s \in \mathbb{C}$.

Lemma 3.3.2. *The SFO system* (3.7) *or the pair* (E, A) *is said to be impulse free if* (3.7) *is regular and* $\deg(\det(\lambda E - A)) = \operatorname{rank} E$*, where* $\lambda \in \mathbb{C}$.

Lemma 3.3.3. *The SFO system* (3.7) *or the pair* (E, A) *is said to be stable if*

$$|\arg(\operatorname{spec}(E, A))| > \alpha\frac{\pi}{2},$$

where $\operatorname{spec}(E, A) = \{\lambda/\lambda \in \mathbb{C}, \lambda \text{ finite}, \det(\lambda E - A) = 0\}$ denotes the set of finite modes for the pair (E, A).

Lemma 3.3.4. *The singular linear fractional-order system* (3.7) *is admissible if*
(i) the pair (E, A) *is regular.*
(ii) the pair (E, A) *is impulse free.*
(iii) the pair (E, A) *is stable.*

Let the pair (E, A) be given, then it is always possible to find non-singular matrices M and N such that $MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and $MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. This decomposition can be obtained via singular value decomposition of matrix E followed by scaling of the bases. Then we have the following lemmas.

Lemma 3.3.5. *Let the pair* (E, A) *is regular. Then, the pair* (E, A) *is impulse free if and only if* A_{22} *is invertible.*

Lemma 3.3.6. *Assume that the pair* (E, A) *is regular. Then*

(i) There exist two invertible matrices M *and* N *satisfying*

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & F \end{bmatrix}, MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{3.8}$$

where $A_1 \in \mathbb{R}^{r\times r}$ and $F \in \mathbb{R}^{(n-r)\times(n-r)}$ is nilpotent.

(ii) The pair (E, A) is impulse free if and only if $F = 0$.

Lemma 3.3.7. *The SFO system* (3.7) *is admissible if and only if*

(i) The pair (E, A) is regular, that is $\det(\lambda E - A)$ is not identically zero, where $\lambda \in \mathbb{C}$.

(ii) the pair (E, A) is impulse free, that is $\deg\det(\lambda E - A) = \operatorname{rank} E$, where $\lambda \in \mathbb{C}$.

(iii) the finite modes of the pair (E, A), which are the eigenvalues λ_i of matrix A_1 in system (3.8)*, satisfy $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$ for $i = 1, \ldots, r$.*

3.3.2 Solution of Singular Fractional Linear Systems

For the regular system (3.6) with $0 < \alpha \leq 1$, the solution was derived in first time by T. Kaczorek in [54] and later by D. Bouagada and Van Dooren in [14]. Our contribution is a new approach to compute solution of state space singular fractional-order linear continuous-time system. $x(t)$ and $u(t)$ are supposed continuously derivable. The boundary conditions are given by $x(0) = x_0$. We assume that the pencil (E, A) is regular, then there exists non-singular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & M \end{bmatrix}, \; PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{3.9}$$

where n_1 is equal to the degree of the polynomial $\det(E\lambda - A)$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $M \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index δ i.e.,

$$M^{\delta} = 0 \; and \; M^{\delta-1} \neq 0$$

and $n_1 + n_2 = n$ We have assumed that the pencil $(E\lambda - A)$ is regular, than the pencil $(E\lambda^{\alpha} - A)$ is also regular. As in [34], we can then apply the weierstrass decomposition to the dynamical equation of the system (3.6) with

$$E = P^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & M \end{bmatrix} Q^{-1}, A = P^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} Q^{-1} \tag{3.10}$$

which yields

$$P^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & M \end{bmatrix} Q^{-1c}d^{\alpha}x\left(t\right) = P^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} Q^{-1}x\left(t\right) + Bu\left(t\right) \tag{3.11}$$

premultiplying (3.11) by the matrix $P \in \mathbb{R}^{n\times n}$ and introducing a new state vector $\tilde{x}\left(t\right)$

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & M \end{bmatrix} d^{\alpha}\tilde{x}\left(t\right) = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tilde{x}\left(t\right) + PBu\left(t\right) \tag{3.12}$$

where

$$\tilde{x}\left(t\right) = Q^{-1}x\left(t\right) = \begin{bmatrix} x_1\left(t\right) \\ x_2\left(t\right) \end{bmatrix}, x_1\left(t\right) \in \mathbb{R}^{n_1}, x_2\left(t\right) \in \mathbb{R}^{n_2} \tag{3.13}$$

is partitioned according to the matrices.

and

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, B_1 \in \mathbb{R}^{n_1\times m}, B_2 \in \mathbb{R}^{n_2\times m} \tag{3.14}$$

then we obtain two new subsystems

$$d^{\alpha}x_1\left(t\right) = A_1x_1\left(t\right) + B_1u\left(t\right) \tag{3.15}$$

$$Md^{\alpha}x_2\left(t\right) = x_2\left(t\right) + B_2u\left(t\right) \tag{3.16}$$

Note that

$$Q^{-1}x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, x_{10} \in \mathbb{R}^{n_1}, x_{20} \in \mathbb{R}^{n_2}$$

By Theorem (2.3.1), the solution of the subsystem (3.15) is given by

$$x_1\left(t\right) = \Phi_{10}\left(t\right)x_{10} + \int_0^t \Phi_{11}\left(t-\tau\right)B_1u\left(\tau\right)d\tau \tag{3.17}$$

where

$$\Phi_{10}\left(t\right) = \sum_{k=0}^{\infty}\frac{A_1^k t^{k\alpha}}{\Gamma\left(k\alpha+1\right)},\ \Phi_{11}\left(t\right) = \sum_{k=0}^{\infty}\frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma\left(\left(k+1\right)\alpha\right)}$$

Based on the fact that the matrix M is nilpotent, we deduced the solution of the subsytem (3.16).
If $M = 0$, we have that

$$x_2\left(t\right) = -B_2u\left(t\right) \tag{3.18}$$

and we are done. If not, premultiplying (3.16) by M gives

$$M^2\, d^{\alpha}x_2\,(t) = Mx_2\,(t) + MB_2u\,(t) \tag{3.19}$$

consequently, the derivative of order α of (3.19) gives

$$M^2d^{2\alpha}x_2\,(t) = M\, d^{\alpha}x_2\,(t) + MB_2\, d^{\alpha}u\,(t) \tag{3.20}$$

Now we insert the equation (3.16) into (3.20).
This gives

$$x_2\,(t) = -B_2u\,(t) - MB_2\, d^{\alpha}u\,(t) + M^2d^{2\alpha}x_2\,(t) \tag{3.21}$$

If $M^2 = 0$ we derive the solution, otherwise we continue until $M^{\delta} = 0$ for some δ since M is nilpotent. By continuously taking derivatives with respect on both sides of equation (3.16), and consider as in [17] the following analogue of the derivative of first order with the derivative of non integer order

$$d^{i\alpha} = \underbrace{d^{\alpha}d^{\alpha}\ldots d^{\alpha}}_{i\ times} \tag{3.22}$$

we readily deduce that the solution x_2 is given by

$$x_2\,(t) = -B_2u\,(t) - \sum_{i=1}^{\mu-1} M^i\, B_2\, d^{i\alpha}u\,(t) \tag{3.23}$$

Note that x_2 is a linear combination of derivatives of $u(t)$ at time t. Therefore, the following theorem has been proved.

Theorem 3.3.1. *Consider a system described by* (3.6) *with* $0 < \alpha \leq 1$ *and the initial condition given by* $x\,(0) = x_0$. *If the system* (3.6) *is regular, its solution can be described by*

$$\begin{aligned} x\,(t) \;&=\; Q\begin{bmatrix} I_{n_1} \\ 0_{n_2\times n_1}\end{bmatrix}\left(\Phi_{10}\,(t)\,x_{10} + \int_0^t \Phi_{11}\,(t-\tau)\,B_1u\,(\tau)\,d\tau\right) \\ &\quad + Q\begin{bmatrix} 0_{n_1\times n_2} \\ I_{n_2}\end{bmatrix}\left(-B_2u\,(t) - \sum_{i=1}^{\mu-1} M^i\, B_2\, d^{i\alpha}u\,(t)\right) \end{aligned}$$

and the output is given by the formula

$$\begin{aligned} y\,(t) \;&=\; CQ\begin{bmatrix} I_{n_1} \\ 0_{n_2\times n_1}\end{bmatrix}\left(\Phi_{10}\,(t)\,x_{10} + \int_0^t \Phi_{11}\,(t-\tau)\,B_1u\,(\tau)\,d\tau\right) \\ &\quad + CQ\begin{bmatrix} 0_{n_1\times n_2} \\ I_{n_2}\end{bmatrix}\left(-B_2u\,(t) - \sum_{i=1}^{\mu-1} M^i\, B_2\, d^{i\alpha}u\,(t)\right) \end{aligned}$$

where

$$\Phi_{10}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi_{11}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$$

We conclude the section with an example.

Example 3.3.1. *Consider the system* (3.6) *with* $0 < \alpha \leq 1$ *and the following data*

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}, D = 0 \end{aligned}$$

and the initial condition

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to make sure that

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{E} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{A} = PAQ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{B} = PB = B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

By making the following change of variable, we obtain

$$\tilde{X}(t) = Q^{-1}x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \end{bmatrix}$$

Note that

$$\tilde{X}_1(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \tilde{X}_2(t) = \begin{bmatrix} \tilde{x}_3(t) \\ \tilde{x}_4(t) \end{bmatrix}$$

It follows

$$\begin{cases} d^{\alpha}\tilde{X}_1(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tilde{X}_1(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \\ 0 &= \tilde{X}_2(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \end{cases}$$

and the initial condition

$$\begin{aligned} \tilde{X}_0 &= Q^{-1}x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \\ x_{10} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_{20} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

remark that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, k = 2, 3, \cdots$$

then

$$\begin{aligned} \tilde{X}_1(t) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{t}{\Gamma(\alpha+1)} & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{t}{\Gamma(\alpha+1)} \end{bmatrix} \end{aligned}$$

and

$$\tilde{X}_2(t) = \begin{bmatrix} 0 \\ -u(t) \end{bmatrix}$$

Finally, we get

$$x(t) = Q \begin{bmatrix} 1 \\ \frac{t}{\Gamma(\alpha+1)} \\ 0 \\ -u(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{t}{\Gamma(\alpha+1)} \\ -u(t) \end{bmatrix}$$

and the output result is

$$y(t) = Cx(t) = 1 - u(t)$$

3.4 Admissibility of singular fractional-order systems

The stability and the admissibility of singular fractional-order systems have been proposed in few works notably in [118, 97] where conditions are given in terms of linear matrix inequalities, which are derived using the decomposition on the matrices of the original. In [133], results derived for the admissibility were obtained under the regularity assumption.
In this section, strict $\mathcal{LMI}$ admissibility conditions are given for open-loop system (3.7), using the matrices of the original system. Both cases $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$ are treated separately.

3.4.1 Admissibility of singular fractional-order systems, case $1 \leq \alpha < 2$

The following theorem gives necessary and sufficient conditions for system (3.7) with fractional-order derivative α satisfying $1 \leq \alpha < 2$ to be admissible.

Theorem 3.4.1. *The SFO system* (3.7) *is admissible if and only if the following equivalent statements hold:*
(i) There exist matrices $X \succ 0$ and Y satisfying

$$\mathrm{Sym}\{\Theta \otimes A^T(XE + E_0Y^T)\} \prec 0, \tag{3.24}$$

where $E_0 \in \mathbb{R}^{n\times(n-r)}$ is any matrix of full column rank and satisfies $E^TE_0 = 0$.
(ii) There exist matrices $X \succ 0$ and Y satisfying

$$\mathrm{Sym}\{\Theta \otimes A(XE^T + E_0Y^T)\} \prec 0, \tag{3.25}$$

where $E_0 \in \mathbb{R}^{n\times(n-r)}$ *is any matrix of full column rank and satisfies* $EE_0 = 0$.
(iii) There exists a matrix P *satisfying*

$$E^TP = P^TE, E^TP \succeq 0, \ \mathrm{Sym}\{\Theta \otimes A^TP\} \prec 0. \tag{3.26}$$

avec

$$\Theta = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}, \theta = \pi - \alpha\frac{\pi}{2}$$

Proof. First, we prove that the admissibility and (i) are equivalent.

Sufficiency - Assume that the inequality (3.24) holds for some matrices $X \succ 0$ and Y. Since $\mathrm{rank}\, E = r < n$, invertible matrices M and N can always be found by rank decomposition or by singular value decomposition such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.27}$$

Let

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, E_0 = M^T \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}. \tag{3.28}$$

From the invertibility of the matrix M we have that the matrix E_0 is of full column rank and satisfies $E^TE_0 = 0$. Let

$$X = M^T \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} M, Y = N^{-T} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. \tag{3.29}$$

From (3.27), (3.28) and (3.29), we obtain

$$A^T\left(XE + E_0Y^T\right) = N^{-T}\overline{A}N^{-1}$$

with

$$\overline{A} = \begin{bmatrix} A_{11}^TX_{11} + A_{21}^TY_1^T & A_{21}^TY_2 \\ A_{12}^TX_{11} + A_{22}^TX_{12}^T + A_{22}^TY_1^T & A_{22}^TY_2^T \end{bmatrix}$$

By using the property of the Kronecker product, we get

$$\begin{aligned} \Theta \otimes \left(A^T\left(XE + E_0Y^T\right)\right) &= (I_2.\Theta) \otimes \left(N^{-T}.\left(\overline{A}N^{-1}\right)\right) \\ &= \begin{bmatrix} N^{-T} & 0 \\ 0 & N^{-T} \end{bmatrix} \left(\Theta \otimes \overline{A}\right) \begin{bmatrix} N^{-1} & 0 \\ 0 & N^{-1} \end{bmatrix} \end{aligned}$$

with

$$\Theta \otimes \overline{A} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ -\overline{A}_{12} & \overline{A}_{11} \end{bmatrix},$$

and

$$\overline{A}_{11} = \begin{bmatrix} \left(A_{11}^T X_{11} + A_{21}^T Y_1^T\right)\sin\theta & A_{21}^T Y_2 \sin\theta \\ \left(A_{12}^T X_{11} + A_{22}^T X_{12}^T + A_{22}^T Y_1^T\right)\sin\theta & A_{22}^T Y_2^T \sin\theta \end{bmatrix}$$

$$\overline{A}_{12} = \begin{bmatrix} A_{11}^T X_{11} + A_{21}^T Y_1^T)\cos\theta & A_{21}^T Y_2 \cos\theta \\ \left(A_{12}^T X_{11} + A_{22}^T X_{12}^T + A_{22}^T Y_1^T\right)\cos\theta & A_{22}^T Y_2^T \cos\theta \end{bmatrix}$$

Then

$\mathrm{Sym}\{\Theta \otimes (A^T(XE + E_0 Y^T))\} =$

$$\begin{bmatrix} N^{-T} & 0 \\ 0 & N^{-T} \end{bmatrix} \begin{bmatrix} \overline{A}_{11} + \overline{A}_{11}^T & \overline{A}_{12} - \overline{A}_{12}^T \\ \overline{A}_{12}^T - \overline{A}_{12} & \overline{A}_{11} + \overline{A}_{11}^T \end{bmatrix} \begin{bmatrix} N^{-1} & 0 \\ 0 & N^{-1} \end{bmatrix}$$

Consequently

$$\mathrm{Sym}\{\Theta \otimes (A^T(XE + E_0 Y^T))\} \prec 0 \Longrightarrow \overline{A}_{11} + \overline{A}_{11}^T \prec 0$$

Knowing that

$$\overline{A}_{11} + \overline{A}_{11}^T = \begin{bmatrix} * & * \\ * & \left(A_{22}^T Y_2^T + Y_2 A_{22}\right)\sin\theta \end{bmatrix}$$

we deduce that

$$A_{22}^T Y_2^T + Y_2 A_{22} \prec 0$$

since $\sin\theta > 0$. Therefore A_{22} invertible, which means that the system (3.7) is regular and impulse free (see Lemma 3.3.5).

Since the system (3.7) is regular and impulse free, invertible matrices L and R can always be found such that

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \; LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{3.30}$$

Let

$$X = L^T \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} L, Y = R^{-T} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, E_0 = L^T \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}. \tag{3.31}$$

From (3.30) and (3.31), inequality (3.25) is equivalent to

$$\begin{bmatrix} R^{-T} & 0 \\ 0 & R^{-T} \end{bmatrix} \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{12}^T & \delta_{11} \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \prec 0, \tag{3.32}$$

with

$$\delta_{11} = \begin{bmatrix} \left(A_1^T X_{11} + X_{11}^T A_1\right) \sin\theta & \left(X_{12} + Y_1\right) \sin\theta \\ \left(X_{12}^T + Y_1^T\right) \sin\theta & \left(Y_2^T + Y_2\right) \sin\theta \end{bmatrix}$$

$$\delta_{12} = \begin{bmatrix} \left(A_1^T X_{11} - X_{11}^T A_1\right) \cos\theta & -\left(X_{12} + Y_1\right) \cos\theta \\ \left(X_{12}^T + Y_1^T\right) \cos\theta & \left(Y_2^T - Y_2\right) \cos\theta \end{bmatrix}$$

The inequality (3.32) is equivalent to

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{12}^T & \delta_{11} \end{bmatrix} \prec 0, \tag{3.33}$$

which is equivalent to

$$\begin{bmatrix} \left(A_1^T X_{11} + X_{11}^T A_1\right) \sin\theta & \left(A_1^T X_{11} - X_{11}^T A_1\right) \sin\theta & \left(X_{12} + Y_1\right) \sin\theta & -\left(X_{12} + Y_1\right) \cos\theta \\ \left(X_{11}^T A_1 - A_1^T X_{11}\right) \cos\theta & \left(A_1^T X_{11} + X_{11}^T A_1\right) \sin\theta & \left(X_{12} + Y_1\right) \cos\theta & \left(X_{12} + Y_1\right) \sin\theta \\ \left(X_{12}^T + Y_1^T\right) \sin\theta & \left(X_{12}^T + Y_1^T\right) \cos\theta & \left(Y_2^T + Y_2\right) \sin\theta & \left(Y_2^T - Y_2\right) \cos\theta \\ -\left(X_{12}^T + Y_1^T\right) \cos\theta & \left(X_{12}^T + Y_1^T\right) \sin\theta & \left(Y_2 - Y_2^T\right) \cos\theta & \left(Y_2^T + Y_2\right) \sin\theta \end{bmatrix} \prec 0.$$

Then, we deduce that

$$\begin{bmatrix} \left(A_1^T X_{11} + X_{11}^T A_1\right) \sin\theta & \left(A_1^T X_{11} - X_{11}^T A_1\right) \cos\theta \\ \left(X_{11}^T A_1 - A_1^T X_{11}\right) \cos\theta & \left(A_1^T X_{11} + X_{11}^T A_1\right) \sin\theta \end{bmatrix} \prec 0 \tag{3.34}$$

From Theorem (2.4.2) and Theorem (2.4.4), Since $X_{11} \succ 0$ the inequality (3.34) guarantees the asymptotic stability of the system $D^\alpha x_1(t) = A_1 x_1(t),\ x_1(t) \in \mathbb{R}^r$. Then system (3.7) is stable. Consequently system (3.7) is admissible since it is regular, impulse free and stable.

Necessity - Assume that the system (3.7) is admissible, then it is always possible to find non-singular matrices L and R such that (3.30) holds. From Lemma 3.3.3, we have $\mathrm{spec}(E,A) = \mathrm{spec}(A_1)$. Then we get

$$|\arg(\mathrm{spec}(A_1))| > \frac{\alpha\pi}{2}$$

According to Theorem 1.3.1, there exists a matrix $X_1 \in \mathbb{R}^{r\times r} \succ 0$ such that

$$\mathrm{Sym}\{\Theta \otimes (A_1^T X_1)\} \prec 0. \tag{3.35}$$

Let

$$E_0 = L^T \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}, Y = R^{-T} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, X = L^T \begin{bmatrix} X_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} L, \tag{3.36}$$

where $E^T E_0 = 0_n$.

Using (3.30) and (3.36) and some direct computation, we obtain

$$\mathrm{Sym}\{\Theta \otimes (A^T(XE + E_0Y^T))\} = \begin{bmatrix} R^{-T} & 0 \\ 0 & R^{-T} \end{bmatrix} \left[\overline{A} + \overline{A}^T\right] \begin{bmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix}$$

where

$$\overline{A} = \begin{bmatrix} A_1^T X_1 \sin\theta & 0 & A_1^T X_1 \cos\theta & 0 \\ 0 & -I_{n-r}\sin\theta & 0 & -I_{n-r}\cos\theta \\ -A_1^T X_1 \cos\theta & 0 & A_1^T X_1 \sin\theta & 0 \\ 0 & I_{n-r}\cos\theta & 0 & -I_{n-r}\sin\theta \end{bmatrix},$$

and

$$\overline{A} + \overline{A}^T = \begin{bmatrix} \chi_1 & \chi_2 \\ \chi_2^T & \chi_1 \end{bmatrix},$$

with

$$\chi_1 = \begin{bmatrix} (A_1^T X_1 + X_1 A_1)\sin\theta & 0 \\ 0 & -2I_{n-r}\sin\theta \end{bmatrix}, \chi_2 = \begin{bmatrix} (A_1^T X_1 - X_1 A_1)\cos\theta & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, to prove $\mathrm{Sym}\{\Theta\otimes(A^T(XE+E_0Y^T))\} \prec 0$, it suffices to prove $\chi_1 \prec 0$ and $\chi_1 - \chi_2\chi_1^{-1}\chi_2^T \prec 0$. Indeed, from inequality (3.35), we have $\chi_1 \prec 0$. We have also

$$\chi_1 - \chi_2\chi_1^{-1}\chi_2^T = \begin{bmatrix} (A_1^T X_1 + X_1 A_1)\sin\theta & 0 \\ 0 & -2I_{n-r}\sin\theta \end{bmatrix} - \begin{bmatrix} (A_1^T X_1 - X_1 A_1)\cos\theta & 0 \\ 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} (A_1^T X_1 + X_1 A_1)^{-1}\frac{1}{\sin\theta} & 0 \\ 0 & \frac{-1}{2\sin\theta} I_{n-r} \end{bmatrix} \begin{bmatrix} (X_1 A_1 - A_1^T X_1)\cos\theta & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma & 0 \\ 0 & -2I_{n-r}\sin\theta \end{bmatrix}$$

where

$$\Sigma = (A_1^TX_1 + X_1A_1)\sin\theta - (A_1^TX_1 - X_1A_1)(A_1^TX_1 + X_1A_1)^{-1}(X_1A_1 - A_1^TX_1)\frac{\cos^2\theta}{\sin\theta}$$

From (3.35), we get $\Sigma \prec 0$ and then $\chi_1 - \chi_2\chi_1^{-1}\chi_2^T \prec 0$ which guarantees

$$\mathrm{Sym}\{\Theta \otimes (A^T(XE + E_0Y^T))\} \prec 0.$$

This completes the proof of the equivalence between the admissibility of the system (3.7) and (i). The equivalence between items (i) and (ii) can be obtained from the fact that the pair (E, A) is admissible if and only if the pair $\left(E^T, A^T\right)$ is admissible.

Indeed, we have

$$\det(s^\alpha E - A) = \det(s^\alpha E - A)^T = \det(s^\alpha E^T - A^T)$$

and

$$\deg\left(\det(\lambda E - A)\right) = \deg\left(\det(\lambda E - A)^T\right) = \deg\left(\det(\lambda E^T - A^T)\right)$$

which shows that the pair (E, A) is regular and impulse free if and only if the pair $\left(E^T, A^T\right)$ is regular and impulse free. On the other hand, from Lemma 3.3.6, if the pair (E, A) is regular and impulse free, there exist two non-singular matrices M and N such that

$$E = M\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}N,\ A = M\begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}N,$$

then we have

$$E^T = N^T\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}M^T,\ A^T = N^T\begin{bmatrix} A_1^T & 0 \\ 0 & I_{n-r} \end{bmatrix}M^T.$$

In that case, the stability of the pair (E, A) depends entirely on A_1 or equivalently on A_1^T. This can be shown from Theorem (2.4.4). Consequently, the stability of the pair (E, A) is equivalent to that of the pair $\left(E^T, A^T\right)$. This proves the equivalence between items (i) and (ii).

The proof that (i) is equivalent to (iii) results directly from the equality between the following sets

$$\begin{aligned} \Xi &= \left\{X \in \mathbb{R}^{n\times n} :\ E^TX = X^TE,\ E^TX \geq 0, \mathrm{rank}\, E^TX = r\right\}, \\ \Lambda &= \left\{X = PE + E_0Q : P \in \mathbb{R}^{n\times n},\ P > 0,\ Q \in \mathbb{R}^{(n-r)\times n}\right\}. \end{aligned}$$

where $E_0 \in \mathbb{R}^{n\times(n-r)}$ is a matrix of full-column rank such that $E^TE_0 = 0$, and $r = \mathrm{rank}\, E$. This completes the proof of Theorem 3.4.1. □

Remark 3.4.1. *Notice that for the case* $E = I_n$, *we get* $E_0 = 0_n$ *and* $\mathcal{LMI}$ *conditions given in Theorem 3.4.1 are reduced to* $\mathcal{LMI}$ *conditions given in Theorem 2.4.4.*
On the other hand, for $\alpha = 1$, *the obtained* $\mathcal{LMI}$ *admissibility conditions of Theorem 3.4.1 are only the admissibility conditions for singular systems . From this viewpoint our result is a generalization of the results obtained in previous works for integer-order singular systems (see [130]).*

3.4.2 Numerical Examples

Consider the linear singular fractional-order system (3.7) with the following data:

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \; \alpha = 1.5 \tag{3.37}$$

The system (3.37) is admissible. Indeed, $det(s^{\alpha}E - A) = 38s^{\alpha} + 20$ which is not identically zero and impulse free with $s^{\alpha} = \lambda$. The finite mode of the pair(E, A) lives in the stability region since it is equal to $-\frac{10}{19}$. This can also be verified by solving the linear matrix inequality (3.25) in Theorem 1.3.2 which is feasible, and concludes that the singular fractional-order system (3.37) is admissible. A feasible solution solution of (3.25) is as follows

$$X = \begin{bmatrix} 4.0117 & -1.5249 & -1.3670 \\ -1.5249 & 1.5061 & -0.4056 \\ -1.3670 & -0.4056 & 1.4156 \end{bmatrix}, \; Y = \begin{bmatrix} 0.5736 & 0.3607 \\ 1.1342 & 1.3679 \\ 1.3225 & 1.9889 \end{bmatrix}$$

The state response of system (3.37) is shown in figure 3.2 which confirms that it is asymptotically stable and its states converge to zero.

3.4.3 Admissibility of Singular Fractional-Order Systems, case $0 < \alpha \leq 1$

Here, we focus with the admissibility problem of the system (3.7) with $0 < \alpha \leq 1$. The following theorem gives necessary and sufficient conditions for the admissibility of (3.7).

Theorem 3.4.2. *The singular fractional-order linear continuous system* (3.7) *with order* $0 < \alpha \leq 1$ *is admissible if and only if there exist matrices* $X = X^* \in \mathbb{C}^{n\times n} \succ 0$, $Y \in \mathbb{R}^{n\times(n-r)}$ *such that*

$$\mathrm{Sym}\left\{A\left(\left(zX + \overline{z}\overline{X}\right)E^T + E_0Y^T\right)\right\} \prec 0 \tag{3.38}$$

where E_0 *is any matrix of full rank column that satisfies* $EE_0 = 0$ *and* $z = e^{j(1-\alpha)\frac{\pi}{2}}$.

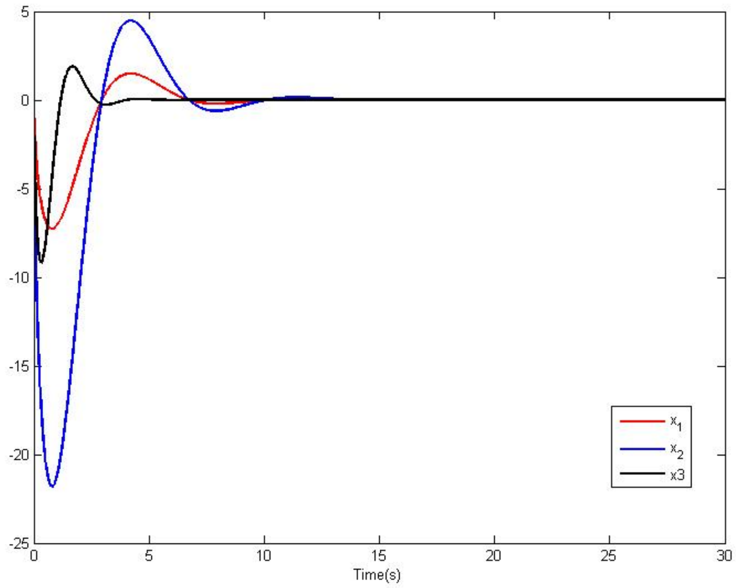

Figure 3.2: Time response of $x(t)$ with $x(0) = [-1, -3, -2]^T$ and fractional order $\alpha = 1.5$.

Proof. Sufficiency - Assume that the inequality (3.38) holds for some matrices $X = X^* \in \mathbb{C}^{n\times n} \succ 0$ and $Y \in \mathbb{R}^{n\times(n-r)}$. Since $\operatorname{rank}\ E = r < n$, invertible matrices M, N can always be found such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.39}$$

The matrices M and N are not unique and can be obtained either by rank decomposition or by singular value decomposition.

Let

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, E_0 = N \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}. \tag{3.40}$$

Then from (3.39), (3.40) and the invertibility of the matrix N, we have

$$EE_0 = M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} N \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0_{n\times(n-r)}.$$

Let us now define matrices X and Y as

$$X = N \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} N^T, Y = M^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. \tag{3.41}$$

Tacking account (3.39), (3.40) and (3.41) into (3.38), we get $\Sigma + \Sigma^T \prec 0$ with

$$\begin{aligned} \Sigma &= A\left(\left(zX + \overline{z}\overline{X}\right)E^T + E_0 Y^T\right) \\ &= M^{-1} \times \begin{bmatrix} A_{11}\left(zX_{11} + \overline{z}\overline{X_{11}}\right) + A_{12}\left(zX_{21} + \overline{z}\overline{X_{21}} + Y_1^T\right) & A_{12}Y_2^T \\ A_{21}\left(zX_{11} + \overline{z}\overline{X_{11}}\right) + A_{22}\left(zX_{21} + \overline{z}\overline{X_{21}} + Y_1^T\right) & A_{22}Y_2^T \end{bmatrix} M^{-T} \end{aligned}$$

which implies

$$A_{22}Y_2^T + Y_2 A_{22}^T \prec 0.$$

Therefore the matrix A_{22} is invertible which means that the system (3.7) is regular and impulse free. Since the system (3.7) is regular and impulse free, invertible matrices L and R can always be found such that

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{3.42}$$

Let now,

$$X = R \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} R^T, Y = L^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, E_0 = R \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}. \tag{3.43}$$

From (3.42) and (3.43) and the invertibility of the matrix L, the condition (3.38) is equivalent to

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{22} \end{bmatrix} \prec 0,$$

where

$$\begin{aligned} \Omega_{11} &= A_1\left(zX_{11} + \overline{z}\overline{X}_{11}\right) + \left(zX_{11} + \overline{z}\overline{X}_{11}\right)^T A_1^T, \\ \Omega_{12} &= \left(zX_{21} + \overline{z}\overline{X}_{21}\right)^T + Y_1, \\ \Omega_{22} &= Y_2 + Y_2^T. \end{aligned}$$

It follows that $\Omega_{11} \prec 0$. From Theorem (2.4.5), we deduce that the system $D^\alpha x_1(t) = A_1 x(t)$ is stable, and consequently the system (3.7) is stable. We conclude that the system (3.7) is admissible since it is regular, impulse free and stable.

Necessity - Assume that the system (3.7) is admissible. Then invertible matrices can always be found such that (3.42) holds and the system $D^{\alpha}x_1(t) = A_1x(t)$ is stable. Then according to Theorem (2.4.5), there exist a matrix $X_1 = X_1^* \in \mathbb{C}^{r\times r} \succ 0$ such that

$$A_1\left(zX_1 + \overline{z}\overline{X}_1\right) + \left(zX_1 + \overline{z}\overline{X}_1\right)^T A_1^T \prec 0, \tag{3.44}$$

with $z = e^{j(1-\alpha)\frac{\pi}{2}}$.
Let matrices,

$$X = R\begin{bmatrix} X_1 & 0 \\ 0 & I_{n-r}\end{bmatrix}R^T, Y = L^{-1}\begin{bmatrix} 0 \\ -I_{n-r}\end{bmatrix}, E_0 = R\begin{bmatrix} 0 \\ I_{n-r}\end{bmatrix}. \tag{3.45}$$

It is easy to show that $X = X^*$ and the matrix E_0 is of full rank column and satisfies $EE_0 = 0$. From (3.42) and (3.45) and some computation, we have

$$\mathrm{Sym}\left\{A\left(\left(zX + \overline{z}\overline{X}\right)E^T + E_0Y^T\right)\right\} = L^{-1}\begin{bmatrix} A_1\left(zX_1 + \overline{z}\overline{X}_1\right) + \left(zX_1 + \overline{z}\overline{X}_1\right)^T A_1^T & 0 \\ 0 & -2I_{n-r}\end{bmatrix}L^{-T}$$

Tacking into account (3.44), we get

$$\mathrm{Sym}\left\{A\left(\left(zX + \overline{z}\overline{X}\right)E^T + E_0Y^T\right)\right\} \prec 0,$$

which completes the proof of the Theorem (3.4.2). □

Note that, since the admissibility of the pair (E, A) is equivalent to the admissibility of the pair $\left(E^T, A^T\right)$, we have

Corollary 3.4.1. *The singular fractional-order linear continuous system* (3.7) *with order* $0 < \alpha \leq 1$ *is admissible if and only if there exist matrices* $X = X^* \in \mathbb{C}^{n\times n} \succ 0$, $Y \in \mathbb{R}^{n\times(n-r)}$ *such that*

$$\mathrm{Sym}\left\{A^T\left(\left(zX + \overline{z}\overline{X}\right)E + E_0Y^T\right)\right\} \prec 0 \tag{3.46}$$

where E_0 *is any matrix of full rank column that satisfies* $E^TE_0 = 0$ *and* $z = e^{j(1-\alpha)\frac{\pi}{2}}$.

Remark 3.4.2. *Let remark that for* $E = I_n$, $E_0 = 0$ *and the* $\mathcal{LMI}$ *condition* (3.38) *in Theorem* (3.4.2) *is the same as the* $\mathcal{LMI}$ *condition of stability for FOS given in Theorem* (2.4.5). *If we take* $\alpha = 1$, (3.38) *becomes* $\mathrm{Sym}\left\{A\left(\left(X+\overline{X}\right)E^T + E_0Y^T\right)\right\} \prec 0$ *with* $X + \bar{X} \succ 0$ *as* $X \succ 0$ *which means that we recover the admissibility condition* (1.33) *for singular system (see Theorem* (1.3.4). *Now, if we take* $E = I_n$ *and* $\alpha = 1$ *we obtain the Lyapunov stability for the system* $\dot{x}(t) = Ax(t)$. *According to this discussion, we deduce that the derived result in Theorem* (3.4.2) *is more general.*

3.4.4 Numerical Example

Example 3.4.1. *Consider the linear singular fractional-order system* (3.7) *with* $\alpha = 0.3$ *and*

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ -1 & 0.5 & 0.3 \end{bmatrix}. \tag{3.47}$$

A feasible solution of the linear matrix inequality (3.38) *of Theorem 3.4.1 is as follows:*

$$X = \begin{bmatrix} 0.4651 & -0.5025 & -0.1057 \\ -0.5025 & 0.6017 & 0.0932 \\ -0.1057 & 0.0932 & 0.1454 \end{bmatrix}, Y = \begin{bmatrix} 1.0968 \\ 1.3352 \\ 0.8230 \end{bmatrix}$$

We conclude that the singular fractional-order system (3.47) *is admissible. The state responses of the selected system is shown in Figure 3.3, which confirms that the system is asymptotically stable and its states converge to zero.*

Example 3.4.2. *Now, we will consider the same system taken in Example1 in [135]*

$$E = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \ A = \begin{bmatrix} -1 & 1 & 0 \\ -3 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix},$$

with $\alpha = 0.5$.
Result of Corollary (3.4.1) *is applied.* $\mathcal{LMI}$ (3.46) *leads to*

$$X = \begin{bmatrix} 12.4104 & -1.2979 \\ -1.2979 & 0.1395 \end{bmatrix}, Y = \begin{bmatrix} 1.4322 \\ 1.8691 \end{bmatrix}.$$

Improvements over result obtained in [135] is that no assumption is required and the number of variables involved in the LMI condition is really smaller than in [135] where there are five variables

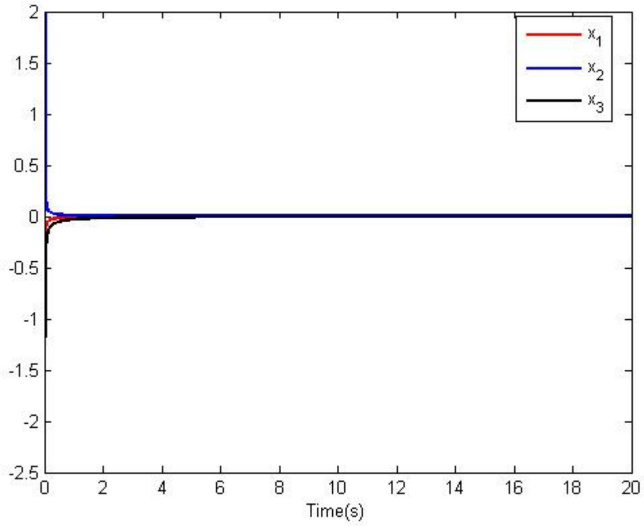

Figure 3.3: State responses of the selected system (3.47)

to be determined. This confirms that the admissibility result in Theorem(3.4.2) *and Corollary*(3.4.1) *is indeed less conservative than that derived in [135].*

3.5 Admissibility of Closed-loop Systems

In this section, we shall deal with the stabilization problem for singular fractional-order systems. The purpose is the design of controllers such that the closed-loop system is regular, stable and impulse-free. Both state feedback and output feedback controllers are considered in the case $1 \leq \alpha < 2$ and observer-based controller for the case $0 < \alpha \leq 1$. Based on the admissibility conditions presented in the previous section, necessary and sufficient conditions for the existence of stabilizing controllers are obtained and the stabilizing controllers design can be formulated.

3.5.1 Static output feedback controller design, case $1 \leq \alpha < 2$

As the state variables are rarely available in practical applications, an output feedback controller is often applied. In the following, a static output feedback controller is proposed to ensure the admissibility of the closed-loop system. Indeed, consider the static output feedback controller:

$$u(t) = Ky(t), \tag{3.48}$$

where K is the control gain matrix to be designed. Then the closed-loop system of the system (3.6) is

$$ED^{\alpha}x(t) = (A + BKC)\,x(t). \tag{3.49}$$

The following lemma will be used in the sequel.

Lemma 3.5.1. *Given a matrix $\Pi \in \mathbb{R}^{p\times n}$ with $\operatorname{rank}\Pi = p$, then there exists a singular value decomposition of the matrix Π as follows*

$$\Pi = U\left[S\;0_{p\times(n-p)}\right]V^T,$$

where $S = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p > 0$, $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{n\times n}$ are unitary[1] *matrices.*

The following theorem gives necessary and sufficient conditions for system (3.6) to be admissible.

Theorem 3.5.1. *The closed-loop system* (3.49) *with order $1 \leq \alpha < 2$ is admissible if and only if there exist matrices $X \succ 0$, Y and scalar $\gamma > 0$ satisfying*

$$\operatorname{Sym}\left\{\Theta \otimes \left(A^T\left(XE + E_0Y^T\right)\right)\right\} - \frac{1}{\gamma}Q^TQ \prec 0. \tag{3.50}$$

where $E_0 \in \mathbb{R}^{n\times(n-r)}$ is any matrix of full column rank and satisfies $E^TE_0 = 0$ and

$$Q = \begin{bmatrix} B^T\left(XE + E_0Y^T\right)\sin\theta & 0 \\ 0 & B^T\left(XE + E_0Y^T\right)\sin\theta \end{bmatrix}.$$

The controller gain K is given by

$$K = -\frac{\sin\theta}{\gamma}B^T\left(XE + E_0Y^T\right)V\begin{bmatrix} S^{-1}U^{-1} \\ 0_{(n-p)\times p} \end{bmatrix}, \tag{3.51}$$

with U and V are unitary matrices such that $C = U\,[S\;0]\,V^T$.

Proof. Necessity - By theorem (3.4.1), we know that the system (3.49) is admissible if and only if there exist matrices $X \succ 0$ and Y satisfying

$$\operatorname{Sym}\left\{\Theta \otimes \left((A + BKC)^T\left(XE + E_0Y^T\right)\right)\right\} \prec 0. \tag{3.52}$$

[1] A complex square matrix U is unitary if its conjugate transpose U^* is also its inverse, that is, if $U^*U = UU^* = I$

Inequality (3.52) is equivalent to

$$W + \mathrm{Sym}\left\{\Theta \otimes \left((KC)^T G\right)\right\} \prec 0 \tag{3.53}$$

with

$$W = sym\left\{\Theta \otimes \left(A^T \left(XE + E_0 Y^T\right)\right)\right\}, G = B^T \left(XE + E_0 Y^T\right). \tag{3.54}$$

Inequality (3.53) is equivalent to

$$W + \mathrm{Sym}\left\{\left(I_2 \otimes (KC)^T\right)(\Theta \otimes G)\right\} \prec 0.$$

Then a sufficiently small scalar $\gamma > 0$ can always be found such that

$$W + \mathrm{Sym}\left\{F^T (\Theta \otimes G)\right\} + \gamma F^T F \prec 0, \tag{3.55}$$

with

$$F = I_2 \otimes (KC). \tag{3.56}$$

Rewriting the matrix $\Theta \otimes G$ by

$$\Theta \otimes G = Q + \widetilde{Q}, \tag{3.57}$$

where

$$Q = \begin{bmatrix} G\sin\theta & 0_{m\times n} \\ 0_{m\times n} & G\sin\theta \end{bmatrix}, \widetilde{Q} = \begin{bmatrix} 0_{m\times n} & G\cos\theta \\ -G\cos\theta & 0_{m\times n} \end{bmatrix}, \tag{3.58}$$

the inequality (3.55) can be rewritten as follows:

$$W + \gamma\left(F^T + \frac{1}{\gamma}Q^T\right)\left(F + \frac{1}{\gamma}Q\right) - \frac{1}{\gamma}Q^T Q + F^T\widetilde{Q} + \widetilde{Q}^T F \prec 0. \tag{3.59}$$

Choosing

$$KC = -\frac{\sin\theta}{\gamma}G, \tag{3.60}$$

we get $\left(F^T + \frac{1}{\gamma}Q^T\right)\left(F + \frac{1}{\gamma}Q\right) = 0_{2n}$ and $F^T\widetilde{Q} + \widetilde{Q}^T F = 0_{2n}$. Then, (3.52) leads to inequality (3.50).

Since $\mathrm{rank}\, C = p$, from Lemma 3.5.1, we get $C = U[S\ 0]V^T$. Then from (3.60), we obtain (3.51).

Sufficiency - Suppose that (3.50) and (3.51) hold for some matrices $X \succ 0$, Y and some real $\gamma > 0$. Indeed, for $KC = -\frac{\sin\theta}{\gamma}G$, (3.50) is equivalent to

$$W + \gamma\left(F^T + \frac{1}{\gamma}Q^T\right)\left(F + \frac{1}{\gamma}Q\right) - \frac{1}{\gamma}Q^T Q \prec 0, \tag{3.61}$$

or equivalently

$$W + F^T Q + Q^T F + \gamma F^T F \prec 0$$

Knowing that $\gamma F^T F \succeq 0$, we obtain

$$W + F^T Q + Q^T F \prec 0$$

From (3.57), with $F^T \widetilde{Q} + \widetilde{Q}^T F = 0_{2n}$, we get

$$W + F^T (\Theta \otimes G) + (\Theta \otimes G)^T F \prec 0$$

which is equivalent to

$$\mathrm{Sym}\left\{\Theta \otimes \left((A + BKC)^T \left(XE + E_0 Y^T\right)\right)\right\} \prec 0$$

Using Theorem (3.4.1), we conclude that system (3.49) is admissible. This completes the proof of Theorem (3.5.1). □

Remark 3.5.1. *Note that, in most practical applications, the state variables are not available. Then, an observer-based controller or output feedback controller are often considered. Here, the very challenging problem of static output feedback control is studied. Indeed, a necessary and sufficient condition to design a such controller is proposed to ensure the admissibility of the closed-loop system.*

Remark 3.5.2. *If all state variables are measurable which means that the output matrix C is equal to I_n, the problem treated becomes a stabilization by state feedback control of the SFOS* (3.6)*, since the control* (3.48) *takes the form $u(t) = Kx(t)$. Therefore, the gain matrix K is given by*

$$K = -\frac{\sin\theta}{\gamma} B^T \left(XE + E_0 Y^T\right)$$

3.5.2 Numerical Example

In the following, two numerical examples are presented to demonstrate the effectiveness of the theoretical results.

Example 3.5.1. *Consider the linear singular fractional-order system* (3.6) *with the following data:*

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & -1 \\ 4 & 1 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \; \alpha = 1.2$$

It can be seen that the system (3.6) *is regular and impulse free, since* $\det(s^\alpha E - A) = 5s^{2\alpha} + 26s^\alpha - 3$*; but it is not asymptotically stable because it has two finite modes* $-5.3129, 0.1129$*, the first one lives in the stable region whereas it is not the same case for the second one. Indeed, the time response of the SFOS* (3.6) *with* $u(t) = 0$ *and initial condition* $x(0) = [1, 0, -0.1401]^T$ *is shown in figure* (3.4) *which confirms that the system* (3.6) *is not asymptotically stable and its states are not convergent.*

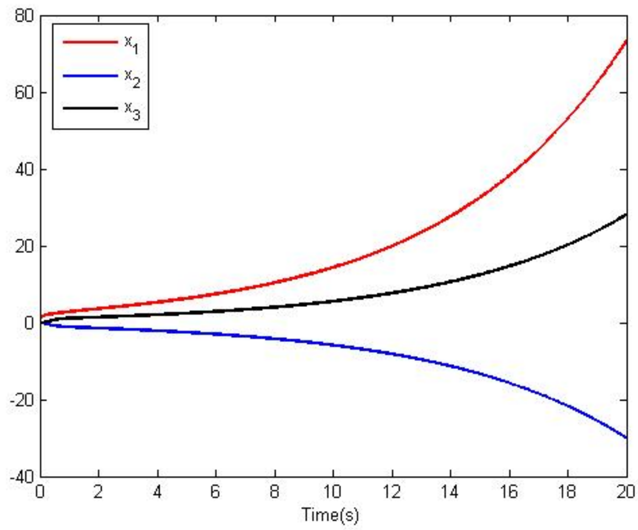

Figure 3.4: Time response of the system in example 3.4.1

The objective is to design an output feedback control law $u(t) = Ky(t)$ *that stabilizes the closed-loop system* (3.49)*. To this end, the inequality* (3.50) *in Theorem* (3.5.1) *is solved by using Matlab control toolbox. A feasible solution solution of* (3.50) *is given by*

$$X = \begin{bmatrix} 0.0181 & -0.0312 & 0.0000 \\ -0.0312 & 0.0553 & 0.0000 \\ 0.0000 & 0.0000 & 32.7563 \end{bmatrix}, \; Y = \begin{bmatrix} -0.0185 \\ -0.0102 \\ 0.0220 \end{bmatrix},$$

and the controller gain K is

$$K = -5.0375.$$

We can now easily verify that the closed-loop system (3.49)*, ie.* $(E, A + BKC, \alpha)$*, is admissible. This result is also confirmed by simulation in figure* (3.5) *and the corresponding control input is given in figure* (3.6)*, which shows that the pair* $(E, A + BKC)$ *is asymptotically stable and its states converge to zero.*

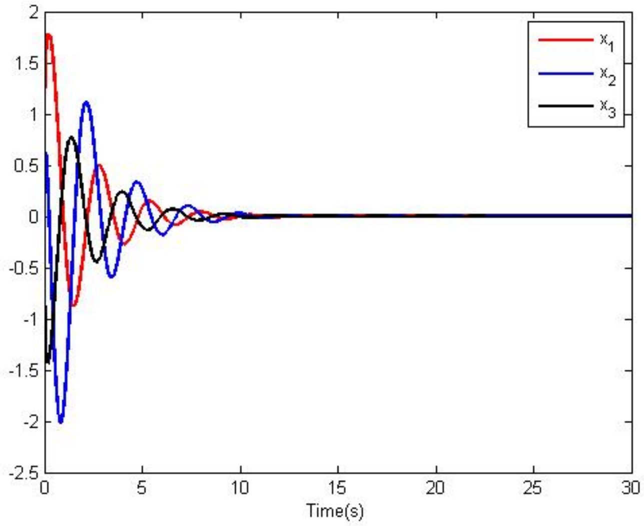

Figure 3.5: Time response of the selected system in Example 3.4.1 with $u(t) = Ky(t)$.

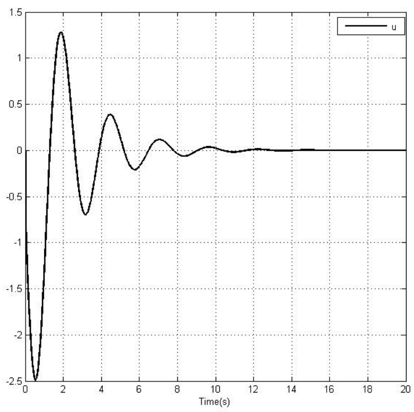

Figure 3.6: Time response of the control input $u(t)$.

3.5.3 Observer-based control for singular fractional-order systems

In the following, we deals with the design of observer-based controller for closed-loop fractional-order linear system (3.6), with $0 < \alpha \leq 1$, to be admissible. Indeed, we consider a Luenberger-type fractional-order linear observer of the form:

$$\begin{cases} ED^{\alpha}\hat{x}(t) & = & A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right) \\ \hat{y}(t) & = & C\hat{x}(t) \end{cases} \tag{3.62}$$

coupled with the control law

$$u(t) = K\hat{x}(t) \tag{3.63}$$

where K and L are the parameter gains to determine. The closed-loop system is given by

$$\breve{E}D^{\alpha}v(t) = A_{cl}v(t) \tag{3.64}$$

with $e(t) = x(t) - \hat{x}(t)$ and

$$\breve{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, v(t) = \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix}, A_{cl} = \begin{bmatrix} A + BK & LC \\ 0 & A - LC \end{bmatrix}$$

Conditions of the admissibility for the system (3.64), or the pair $\left(\breve{E}, A_{cl}\right)$, are given by the following results.

Lemma 3.5.2. *The system* (3.64) *is admissible if and only if the pairs* $(E, A + BK)$ *and* $(E, A - LC)$ *are admissible.*

Proof. We know that the system (3.64) is admissible if and only if the pair $\left(\breve{E}, A_{cl}\right)$ is (i) regular, (ii) impulse free and (iii) stable.
Condition (i) means that $\det(s^{\alpha}\breve{E} - A_{cl}) \neq 0,$ for some $s \in \mathbb{C}$. Since

$$\det(s^{\alpha}\breve{E} - A_{cl}) = \det(s^{\alpha}E - (A + BK). \det(s^{\alpha}E - (A - LC)) \tag{3.65}$$

we conclude that the condition (i) is equivalent to that the pairs $(E,\ A + BK)$ and $(E, A - LC)$ are simultaneously regular.
Condition (ii) means that for $\lambda \in \mathbb{C}$, $\deg\left(\det(\lambda\breve{E} - A_{cl})\right) = rank\breve{E} = 2r$. Note also that

$$\deg\left(\det(\lambda\breve{E} - A_{cl})\right) = \deg\left(\det(\lambda E - (A + BK)\right) + \deg\left(\det(\lambda E - (A - LC)\right)$$

Since $\deg\left(\det(\lambda E - (A + BK)\right) \leq r$ and $\deg\left(\det(\lambda E - (A - LC)\right) \leq r$, we get

$$\deg\left(\det(\lambda E - (A + BK))\right) = \deg\left(\det(\lambda E - (A - LC)\right) = r$$

it follows that the pairs $(E, A + BK)$ and $(E, A - LC)$ are impulse free.
Finally from (3.65), we deduce that stability of the pair $\left(\breve{E}, A_{cl}\right)$ (condition (iii)) is equivalent to the stability of $(E, A + BK)$ and $(E, A - LC)$.
Consequently, the admissibility of the pair $\left(\breve{E}, A_{cl}\right)$ is equivalent to the admissibility of $(E, A + BK)$ and $(E, A - LC)$ separately. □

Based on the results of Lemma (3.5.2) and Corollary (3.4.1), the determination of gains K and L, such that the system (3.64) is admissible, are derived through the following theorem.

Theorem 3.5.2. *The closed-loop system* (3.64) *is admissible if and only if there exist matrices* $X_1 = X_1^* \in \mathbb{C}^{n\times n} \succ 0$, $X_2 = X_2^* \in \mathbb{C}^{n\times n} \succ 0$, $Y_1 \in \mathbb{R}^{n\times(n-r)}$, $Y_2 \in \mathbb{R}^{n\times(n-r)}$, *scalars* $\gamma_1 > 0$ *and* $\gamma_2 > 0$ *satisfying*

$$\mathrm{Sym}\left\{A^T\left((zX_1 + \overline{z}\overline{X}_1)E + E_0Y_1^T\right)\right\} - \frac{1}{\gamma_1}Q_1^TQ_1 \prec 0 \tag{3.66}$$

$$\mathrm{Sym}\left\{A\left((zX_2 + \overline{z}\overline{X}_2)E^T + \widetilde{E}_0Y_2^T\right)\right\} - \frac{1}{\gamma_2}Q_2^TQ_2 \prec 0 \tag{3.67}$$

with

$$Q_1 = B^T\left((zX_1 + \overline{z}\overline{X}_1)E + E_0Y_1^T\right) \tag{3.68}$$

$$Q_2 = -C\left((zX_2 + \overline{z}\overline{X}_2)E^T + \widetilde{E}_0Y_1^T\right) \tag{3.69}$$

where $E_0 \in \mathbb{R}^{n\times(n-r)}$ *and* $\widetilde{E}_0 \in \mathbb{R}^{n\times(n-r)}$ *are full column rank satisfying* $E^TE_0 = 0,\ E\widetilde{E}_0 = 0$ *and* $z = e^{j(1-\alpha)\frac{\pi}{2}}$,
The gain matrices are given by

$$K = -\frac{1}{\gamma_1}B^T\left((zX_1 + \overline{z}\overline{X}_1)E + E_0Y_1^T\right) \tag{3.70}$$

and

$$L = \frac{1}{\gamma_2}\left((zX_2 + \overline{z}\overline{X}_2)E^T + \widetilde{E}_0Y_2^T\right)^T C^T \tag{3.71}$$

Proof. Necessity- Assume that the system (3.64) is admissible, then by Lemma (3.5.2) the pairs $(E, A + BK)$ and $(E, A - LC)$ are admissible . By Corollary (3.4.1), the pair $(E, A + BK)$ is admissible if and only if there exist matrix $X_1 = X_1^* \in \mathbb{C}^{n\times n} \succ 0$, matrix $Y_1 \in \mathbb{R}^{n\times(n-r)}$ satisfying

$$\mathrm{Sym}\left\{(A + BK)^T\left((zX_1 + \overline{z}\overline{X}_1)E + E_0Y_1^T\right)\right\} \prec 0 \tag{3.72}$$

which is equivalent to

$$W + K^T Q_1 + Q_1^T K \prec 0 \tag{3.73}$$

with

$$W = \mathrm{Sym}\left\{A^T\left((zX_1 + \overline{z}\overline{X}_1)E + E_0 Y_1^T\right)\right\}$$

and Q_1 is given by (3.68). Then a sufficiently small scalar $\gamma_1 > 0$ can always be found such that

$$W + K^T Q_1 + Q_1^T K + \gamma_1 K^T K \prec 0 \tag{3.74}$$

or equivalently

$$W + \gamma_1\left(K + \frac{1}{\gamma_1}Q_1\right)^T\left(K + \frac{1}{\gamma_1}Q_1\right) - \frac{1}{\gamma_1}Q_1^T Q_1 \prec 0 \tag{3.75}$$

Then with the controller gain K given by (3.70), we get (3.66). By following the same approach, we get (3.67) with gain (3.71).
Sufficiency- Now suppose that condition (3.66) holds with the controller gain (3.70) for some matrices $X_1 = X_1^* \in \mathbb{C}^{n\times n} \succ 0$, $Y_1 \in \mathbb{R}^{n\times(n-r)}$ and a scalar $\gamma_1 > 0$.
Indeed, from (3.66) and (3.70) we get (3.74). Since $\gamma_1 K^T K \succeq 0$, we obtain (3.72). Using Theorem (3.4.1), we deduce that the pair $(E, A + BK)$ is admissible.

By the same reasoning, the proof of the admissibility of the pair $(E, A - LC)$ is derived. Which completes the proof of Theorem (3.5.2). □

Example 3.5.2. *Consider the singular continuous fractional-order system described in (3.64) with the following data:*

$$E = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 1 & 3 \\ 0 & -1 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ -0.5 & 1.5 & 2.7 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 4 \\ -1 & 0 \\ -3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \alpha = 0.5 \tag{3.76}$$

The finite modes of the pair (E, A) are $\left\{\frac{5}{23}, 1\right\}$, then the unforced singular fractional order system (E, A) is unstable, which means that it is not admissible as can be seen in Figure 3.7.

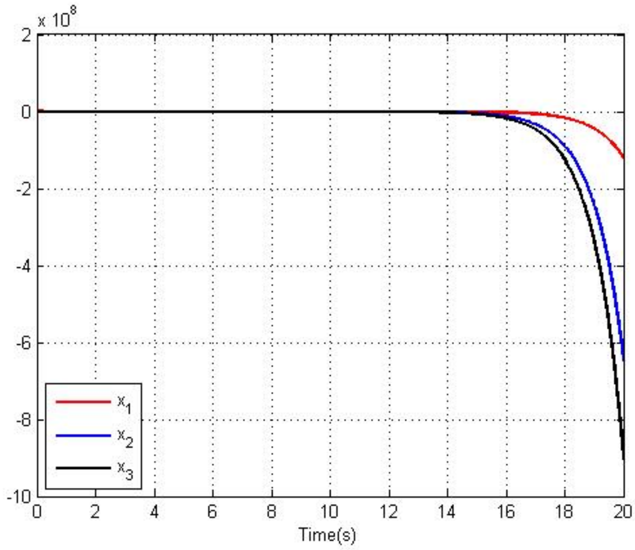

Figure 3.7: State responses of the selected system in Example (3.5.2)

In the following, our objective is to design an observer-based controller such that the closed loop system defined in (3.64), with the control law (3.62)-(3.63), is admissible.
Solving the design conditions (3.66) and (3.67) of Theorem 3.5.2 with $\gamma_1 = 0.9$ *and* $\gamma_2 = 0.5$*, we get the following results:*

$$X_1 = \begin{bmatrix} 1.5178 & -0.0445 & 1.2720 \\ -0.0445 & 4.0088 & 0.0445 \\ 1.2720 & 0.0445 & 1.5178 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} -4.3503 \\ -0.3467 \\ -11.5210 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 1.6308 & 0.2260 & 0.0215 \\ 0.2260 & 1.8035 & 0.4877 \\ 0.0215 & 0.4877 & 0.1797 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.3797 \\ 0.3401 \\ -0.0354 \end{bmatrix}.$$

The gain matrices can be designed as
$$K = \begin{bmatrix} -1.3897 & 1.8558 & -0.3648 \\ -1.8918 & 12.1136 & -2.8097 \\ 1.4474 & -1.4541 & -0.2180 \end{bmatrix}, L = \begin{bmatrix} 6.2675 & -1.5834 \\ 1.6911 & 6.3791 \\ -0.0399 & 0.1303 \end{bmatrix}.$$

We can easily verify that $(E, A + BK)$ *and* $(E, A - LC)$ *are admissible.*
Indeed, in one hand we have

$$\det(\lambda E - A - BK) = -24.6521\lambda^2 - 83.2307\lambda - 37.2117$$

and

$$\det(\lambda E - A + LC) = 206.1997\lambda^2 + 1.454210^3\lambda + 2.0663 \times 10^3$$

which means that the systems $(E, A + BK)$ *and* $(E, A - LC)$ *are regular and impulse-free.*
In the other hand, we have

$$\text{spec}(E, A + BK) = \{-2.8458, -0.5304\}$$

and

$$\text{spec}(E, A - LC) = \{-5.0797, -1.9728\}$$

meaning that the eigenvalues of the pairs $(E, A + BK)$ *and* $(E, A - LC)$ *lie in the stable region. We conclude that the closed loop system of the singular fractional-order system (3.76) is now admissible. The histories of the system states and the observation errors are given in Figures 3.8 and 3.9 respectively.*

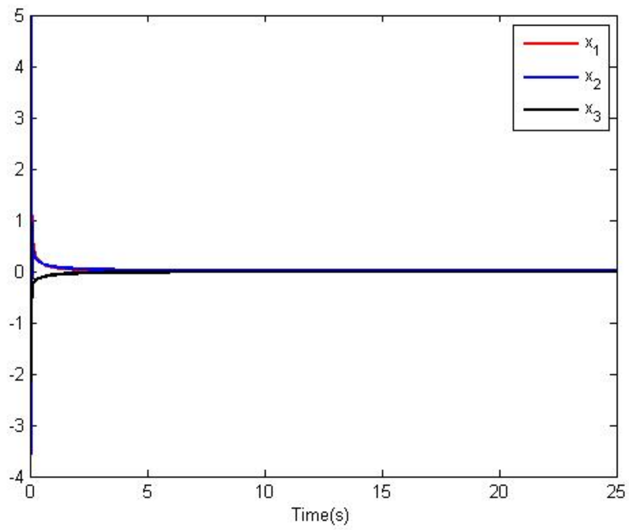

Figure 3.8: State responses of the colsed-loop system in Example (3.5.2)

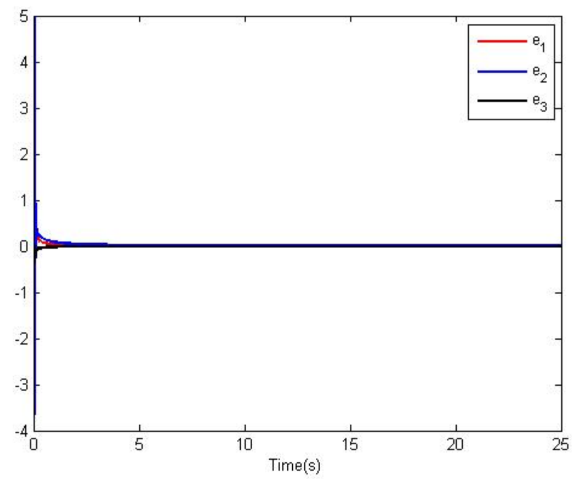

Figure 3.9: The observation errors in Example (3.5.2)

3.6 Conclusion

This chapter deals with the study of the singular fractional order linear continuous time systems. As a first step, we give explicit formulas for the solution of singular fractional continuous time state space models, based on the Caputo definition of fractional derivatives, this definition involves only a finite number of initial conditions and compatibility requirements. It is also useful for several practical applications. As we all know that for singular systems, we need to consider not only stability but also the regularity and the non-impulsiveness. Specifically, regularity guarantees the existence and the uniqueness of a solution to a given singular system, while non-impulsiveness ensures no infinite dynamical modes in such system. Analysis and synthesis for singular fractional order systems were investigated in some papers. For example in [97], singular fractional order systems are considered with differentiation order between 1 and 2 and the obtained results in terms of $\mathcal{LMI}$s , under the assumption that the system is regular and impulse free, are only sufficient conditions to get asymptotic stabilization. These results are derived using the decomposition of the original system with Weierstrass canonical form. For the same class of systems with alpha between 0 and 2, results derived for the stability and stabilization problem are also just sufficient conditions in [118]. In [94], the robust stabilization of uncertain descriptor systems with the fractional order derivative belonging (0,2) was treated using the concept of the normalization to check sufficient conditions. Improvements in our work compared to that shown previously are such that our result ensures the three criterion to get admissibility and stabilization of singular fractional order systems. Necessary and sufficient conditions are derived in terms of $\mathcal{LMI}$s where the matrices of the original system are involved. Using the obtained result, a static output feedback controller and an output feedback based on observer-based controller are then designed, for the cases $(1,2)$ and $(0,1)$ respectively, to ensure the admissibility of the closed-loop system.

Concluding Remarks

The methodologies developed in this thesis are essentially theoretical. They are dedicated to the analysis and synthesis of control laws for linear systems described by fractional-order singular models. Their institutions appeal exclusively to the second lyapunov method and to the $\mathcal{LMI}$ formalism. This work belongs to one of the axes of the command theory of linear complex systems, the complexity being in the non-integer order of the derivation of the differential equations describing a class of singular models. . The results reported in this dissertation can be viewed as extensions of some existing results in the literature of linear singular systems to their homologous of fractional-order. The study we have conducted is organized in two parts: The first part deals with the analysis of the admissibility of fractional-order singular linear systems, the second part relates to the stabilization. The main achievements are summarized and future research topics are discussed in this concluding chapter.

Basic concepts for linear time-invariant descriptor systems are recalled in Chapter 1 as preliminaries. Fundamental and important results, such as regularity, admissibility, equivalent realizations, system decomposition and temporal response are reviewed. The defnitions of controllability and observability are also presented. Conditions for stabilizability and detectability are recalled in the end of this chapter.

Chapter 2 has been devoted to dynamic systems described by differential equations of a non-integer order and to the main results of the literature on these systems. In first time, We present the theory of non-integer derivation: different types of non-integral derivation (Grünwald-Leitnikov, Riemann-Liouville and Caputo), Laplace transformation, gamma function, Mittag-Leffler functions, . . . We also justify the choice of the derivation in the sense of Caputo for further developments presented in this manuscript. Analysis of the properties of non-integer models then revealed that only non-integer commensurable models can be described by a pseudo state representation. This representation is similar in writing to that of the integer order models. In second time, the characterization of stability of fractional-order systems is highlighted. In particular, we can see that the asymptotic stability of

a commensurable non-integer model can be evaluated by the position in the complex plane of the eigenvalues of its state matrix. Emphasis is then placed on the command of this class of systems. Our contribution in this chapter appears at the minimum energy control problem where a control law is defined such that the performance index of the system is minimized.
Chapter 3 is devoted to our contributions on linear singular fractional-order systems [79, 80]. As it well known that for singular systems, we need to consider not only stability but also the regularity and the non-impulsiveness. Specifically, regularity guarantees the existence and the uniqueness of a solution to a given singular system, while non-impulsiveness ensures no infinite dynamical modes in such system. This chapter serves to present, in a first step, necessary and sufficient conditions of the admissibility for linear singular fractional-order systems in both cases of the fractional-order α satisfying $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$. These conditions are derived in terms of strict $\mathcal{LMI}$s. It should be noted that these results are obtained under no assumption, without decomposition of the matrices of the original systems and neither from the standardization. Then, the second step is devoted to the control problem. Note that in most practical applications, usually not all the state variables are accessible for the feedback and all the designer knows are the output and input of the plant. In this case, the observer-based control or output feedback control is often needed. For the case $1 \leq \alpha < 2$, a static output feedback controller is designed to ensure the admissibility of the closed-loop system, and for the case $0 < \alpha \leq 1$, an output feedback based on observer-based controller is considered for the closed-loop system to be admissible.

Perspectives

Despite these developments, some areas deserve further reflection. However, extensions can be made to our work. The perspectives remain numerous and must be oriented towards the diminution of the conservatism of the conditions that can occur at the following levels.

1. Our results can be extended to uncertain non-linear singular fractional-order systems.
2. Further works will be focused on robust admissibility of uncertain singular fractional order systems.

appendix A

Linear Algebra Recall

A.1 Positive Definite Matrices

A.1.1 Definitions

The study of matrices is quite old. Leibnitz is one of the founders of the analysis which developed the theory of determinants in 1693 to facilitate the resolution of differential equations. The matrices are now used for multiple applications and serve in particular to represent the coefficients of linear equations.The matrices are now used for multiple applications and serve in particular to represent the coefficients of linear equations [2, 43, 53].

Definition A.1.1. *Let A a square matrix.*

- *A matrix $A \in \mathbb{R}^{n \times n}$ is called symmetric if and only if*

$$A^T = A \tag{A.1}$$

- *A matrix $A \in \mathbb{C}^{n \times n}$ is called hermitian if and only if*

$$A^* = A \tag{A.2}$$

Definition A.1.2. *1. If A is a symmetric matrix.*

- *A is called positive definite if and only if*

$$\forall v \in \mathbb{R}^n (v \neq 0), v^T A v > 0 \tag{A.3}$$

- *A is called positive semidefinite if and only if*

$$\forall v \in \mathbb{R}^n, v^T A v \geq 0 \tag{A.4}$$

2. *If A is a hermitian matrix.*

- *A is called positive definite if and only if*

$$\forall v \in \mathbb{C}^n (v \neq 0), v^T A v > 0 \tag{A.5}$$

- *A is called positive semidefinite if and only if*

$$\forall v \in \mathbb{C}^n, v^* A v \geq 0 \tag{A.6}$$

Here is a simple but fundamental fact.

Theorem A.1.1. *A Hermitian matrix is positive definite if and only if all its eigenvalues are positive.*

Proof. (Necessity) Suppose that $v \in \mathbb{C}^n$ is an eigenvector of the Hermitian matrix $A \in \mathbb{C}^{n \times n}$ corresponding to the eigenvalue λ. Then

$$v^* A v = \lambda v^* v \tag{A.7}$$

Since the eigenvector v is nonzero, it follows that $v^* v = \|v_2\| > 0$.
If A is positive definite, then

$$\lambda = \frac{v^* A v}{v^* v} > 0 \tag{A.8}$$

Hence, all eigenvalues of a hermitian positive definite matrix must be positive.
(Sufficiency) suppose $A \in \mathbb{C}^{n \times n}$ is a hermitian matrix whose eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are all positive. Then let $u_1, \cdots, u_n$ denote an orthonormal basis of eigenvectors, so that any $v \in \mathbb{C}^n$ can be written as

$$v = \sum_{i=1}^{n} \alpha_i u_i \tag{A.9}$$

then

$$Av = \sum_{i=1}^{n} \alpha_i \lambda_i u_i$$

It follows that

$$v^* A v = (\sum_{j=1}^{n} \bar{\alpha}_j u_j^*)(\sum_{i=1}^{n} \alpha_i \lambda_i u_i)$$

which is the same as

$$v^*Av = \sum_{j=1}^{n}\sum_{i=1}^{n}\bar{\alpha}_j\alpha_i\lambda_i u_j^* u_i$$

Since $u_j^* u_i = 0, \forall i \neq j$ and $u_j^* u_i = 1, i = j$, we obtain

$$v^*Av = \sum_{j=1}^{n}\lambda_j \|\alpha_j\|^2 \geq \lambda_1 \sum_{j=1}^{n} \|\alpha_j\|^2$$

If $v \neq 0$, then $0 \neq \|v\|_2^2 = \sum_{i=1}^{n} \|\alpha_i\|^2$, and since all the eigenvalues are positive, we must have

$$v^*Av > 0$$

which means that the matrix A is positive definite. □

A.1.2 Properties

- If A is a hermitian matrix then

  $$A \succ 0 \;\Leftrightarrow\; A^* \succ 0 \tag{A.10}$$

  Indeed,
  As $A = A^*$ then $\mathrm{spec}(A) = \mathrm{spec}(A^{*)}$ and since $A \succ 0$, all eigenvalues of A are positive (see theorem (A.1.1)) which implies that all eigenvalues of A^* are also positive.
  To prove the other implication, it is sufficient to use the same reasoning as in the necessity.

- If A is a hermitian matrix then

  $$A \succ 0 \;\Leftrightarrow\; \bar{A} \succ 0 \tag{A.11}$$

  Proof. (Necessity) Assume that $A \succ 0$, then $A = A^*$. This implies

  $$\overline{A} = \overline{A^*} = \overline{A}^*$$

  which means that $\bar{A}$ is hermitian.
  In the other hand, for all $v \in \mathbb{C}^n, v \neq 0$, we have

  $$v^*\bar{A}v = \overline{\overline{v^*A\overline{v}}} = \overline{\overline{v}^*A\overline{v}} > 0$$

  since $A \succ 0$. We conclude that $\bar{A} \succ 0$.
  (Sufficiency) From the necessity, if $\overline{A} \succ 0$ then $\overline{\overline{A}} = A \succ 0$. □

A.2 Proof of Lemma (2.4.3)

(Necessity). A hermitian matrix $M \in \mathbb{C}^{n\times n}$ is said to be positive definite if and only if for every non-zero column vector $v \in \mathbb{C}^n$

$$v^* A v > 0 \tag{A.12}$$

Without loss of generality, we can define

$$v = x + jy$$

such that $x, y \in \mathbb{R}^n$ and $\|x\|_2 + \|y\|_2 \neq 0$.
Therefore from $M = A + jB \succ 0$, we can get

$$\begin{aligned} v^* M v &= (x - jy)^{\mathrm{T}}(A + jB)(x + jy) \\ &= (x^{\mathrm{T}}Ax - x^{T}By + y^{\mathrm{T}}Ay + y^{\mathrm{T}}Bx) \\ &+ j(x^{\mathrm{T}}Ay + x^{\mathrm{T}}Bx - y^{\mathrm{T}}Ax + y^{\mathrm{T}}By) \succ 0 \end{aligned}$$

Then, we can deduce that

$$x^{\mathrm{T}}Ax - x^{T}By + y^{\mathrm{T}}Ay + y^{\mathrm{T}}Bx > 0 \tag{a}$$

$$x^{\mathrm{T}}Ay + x^{\mathrm{T}}Bx - y^{\mathrm{T}}Ax + y^{\mathrm{T}}By = 0 \tag{b}$$

The inequality (a) is the same as

$$\begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > 0 \tag{A.13}$$

and note that the inequality (A.13) is established for all nonzero real vector

$$z = \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

then we obtain

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \succ 0$$

which is exactly (2.82).
To obtain (2.81), it suffises to see that (a) can be rewritten as

$$\begin{bmatrix} y^{\mathrm{T}} & x^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} > 0$$

(Sufficiency) Assume that the inequality (2.82) is verified. Then we have

$$\begin{cases} A^{\mathrm{T}} & = & A \\ B^{\mathrm{T}} & = & -B \end{cases} \tag{A.14}$$

This leads to

$$M^{*} = (A + jB)^{*} = A^{\mathrm{T}} - jB^{\mathrm{T}} = M$$

So, we conclude that the matrix M is hermitian.

For any nonzero complex vector $v = x + jy$ with $x, y \in \mathbb{R}^{n}$, we have

$$v^{*}Mv = (x^{\mathrm{T}}Ax - x^{T}By + y^{\mathrm{T}}Ay + y^{\mathrm{T}}Bx) + j(x^{\mathrm{T}}Ay + x^{\mathrm{T}}Bx - y^{\mathrm{T}}Ax + y^{\mathrm{T}}By)$$

According to (A.14), we have

$$(x^{\mathrm{T}}Ay)^{\mathrm{T}} = y^{\mathrm{T}}Ax$$

and

$$\begin{cases} (x^{\mathrm{T}}Bx)^{\mathrm{T}} & = -x^{\mathrm{T}}Bx \\ (y^{\mathrm{T}}By)^{\mathrm{T}} & = -y^{\mathrm{T}}By \end{cases} \Rightarrow \begin{cases} x^{\mathrm{T}}Bx & = 0 \\ y^{\mathrm{T}}By & = 0 \end{cases}$$

then we get that

$$x^{\mathrm{T}}Ay + x^{\mathrm{T}}Bx - y^{\mathrm{T}}Ax + y^{\mathrm{T}}By = 0$$

Therefore

$$\begin{aligned} v^{*}Mv &= (x^{\mathrm{T}}Ax - x^{T}By + y^{\mathrm{T}}Ay + y^{\mathrm{T}}Bx) \\ &= \begin{bmatrix} x^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} > 0 \end{aligned}$$

Thus one complete the proof.

appendix B

$\mathcal{LMI}$ Regions

B.1 A Brief History of LMIs in Control Theory

The history of $\mathcal{LMI}$s in the analysis of dynamical systems goes back more than 100 years, in about 1890 by the early work of Lyapunov [77], which remains to this day one of the most powerful methodologies for stability analysis. The story begins when Lyapunov published his seminal work introducing what we now call Lyapunov theory or the Lyapunov second method that can be summarized by the following statement.

Theorem B.1.1. *(Second Lyapunov Method). System $\dot{x}(t) = f(t, x(t)/$ is stable (globally asymptotically stable around the origin) if there exists a real-valued function $V(x, t)$ such that:*
$V(0, t) = 0, \;\; \forall t \geq 0$
$V(x, t) > 0, \;\; \forall x \neq 0, \;\; \forall t \geq 0$
$\lim_{\|x\| \to \infty} V(x, t) = \infty$
$\dot{V}(x, t) < 0, \;\; \forall x \neq 0, \;\; \forall t \geq 0$
$V(x, t)$ is called the Lyapunov function.

In the case of a linear system of the form

$$\dot{x}(t) = Ax(t) \tag{B.1}$$

with arbitrary matrix A, the existence of a quadratic Lyapunov function

$$V(x, t) = x^{\mathrm{T}} P x, \;\; P \succ 0$$

is a necessary and sufficient condition for asymptotic stability. Finding a positive definite matrix P ensuring that Lyapunov function $V(x,t)$ decreases along the trajectories of the system can be achieved via the following theorem.

Theorem B.1.2. *Integer order system* (B.1) *is asymptotically stable if and only if there exists a positive definite matrix P such that:*

$$A^T P + PA \prec 0 \tag{B.2}$$

Note that the theorem (B.1.2) is satisfied if and only if the eigenvalues of A lie in the open left half plane. This Lyapunov characterization of stability has been extended to a variety of regions by Gutman [47]. More details can be seen in the next few sections.

B.2 Linear Matrix Inequalities

Definition B.2.1. *A strict linear matrix inequality $\mathcal{LMI}$ has the form*

$$M(x) = M_0 + \sum_{k=1}^{n} x_k M_k \succ 0 \tag{B.3}$$

where $x \in \mathbb{R}^n$ is the variable and the symmetric matrices $M_k = M_k^T \in \mathbb{R}^{n\times n},\ \ k = 0, 1, \cdots, n$ are given.

Remark B.2.1.

- *The term linear matrix inequality is used in the literature on systems and control, but the terminology is not consistent with the expression $F(x) \succ 0$ since F is not a linear function. The term affine matrix inequality may better correspond to the formulation.*

- *A nonstrict $\mathcal{LMI}$ has the form*

$$M(x) = M_0 + \sum_{k=1}^{n} x_k M_k \succeq 0$$

- *The linear matrix inequalities $F(x) \prec 0$ and$F(x) \prec G(x)$ where F, G are affine functions, are special cases of* (B.3) *since they can be reformulated in the form $\mathcal{LMI}$*

$$-F(x) \succ 0 \tag{B.4}$$

$$G(x) - F(x) \succ 0 \tag{B.5}$$

- *The objective is to find $x \in \mathbb{R}^n$ satisfying the inequality* (B.3). *This choice of x is called "**feasibility problem**".*
  The $\mathcal{LMI}$ (B.3)*(2.1) is a convex constraint on x, i.e., the set $\{x/F(x) \succ 0\}$ is convex.*

- *Multiple $\mathcal{LMI}$s*

$$M^{(1)}(x) > 0, \cdots, M^{(p)}(x) > 0 \tag{B.6}$$

  can be expressed as the single $\mathcal{LMI}$

$$\mathrm{diag}(M^{(1)}(x), \cdots, M^{(p)}(x)) > 0 \tag{B.7}$$

- *Nonlinear (convex) inequalities are converted to $\mathcal{LMI}$ form using Schur complements. The basic idea is as follows: the $\mathcal{LMI}$*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} \succ 0 \tag{B.8}$$

  where $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$ and $S(x)$ depend affinely on x, is equivalent to

$$\begin{cases} R(x) & \succ 0 \\ Q(x) - S(x)R^{-1}(x)S^T(x) & \succ 0 \end{cases} \tag{B.9}$$

  or equivalently

$$\begin{cases} Q(x) & \succ 0 \\ R(x) - S^T(x)Q^{-1}(x)S(x) & \succ 0 \end{cases} \tag{B.10}$$

  Indeed, the proof is easily done by multiplying (B.8) *to the right by:*

$$\begin{bmatrix} I & 0 \\ -R^{-1}(x)S^T(x) & I \end{bmatrix} \tag{B.11}$$

  And to the left by the transpose of this last matrix. These two matrices being defined, then an equivalent condition is obtained:

$$\begin{bmatrix} Q(x) - S(x)R^{-1}(x)S^T(x) & 0 \\ 0 & R \end{bmatrix} \tag{B.12}$$

B.3 Definition of $\mathcal{LMI}$ Region

The class of LMI regions defined below turns out to be suitable for LMI-based synthesis.

Definition B.3.1. *A subset $\mathcal{D}$ of the complex plane is called an $\mathcal{LMI}$ region if there exist a symmetric matrix $\alpha \in \mathbb{R}^{n\times n}$ and a matrix $\beta \in \mathbb{R}^{n\times n}$ such that*

$$\mathcal{D} = \{z \in \mathbb{C} / f_{\mathcal{D}}(z) = \alpha + z\beta + \bar{z}\beta^T \prec 0\} \tag{B.13}$$

It can be noted that the characteristic function $f_{\mathcal{D}}$ of variable z takes values in the space of $n \times n$ Hermitian matrices. As a result, $\mathcal{LMI}$ regions are convex. Moreover, $\mathcal{LMI}$ regions are symmetric with respect to the real axis since for any $z \in \mathcal{D}$, $\overline{f_{\mathcal{D}}(z)} = f_{\mathcal{D}}(\bar{z})$. This last property is often verified by the regions used for the study of the $\mathcal{D}$-stability[1] of a matrix A since the spectrum of a matrix is auto-conjugate.

Theorem B.3.1. *[29, 32] Let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{D}$ an $\mathcal{LMI}$ region defined by* (B.13). *The matrix A is $\mathcal{D}$-stable if and only if there exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that*

$$M_{\mathcal{D}}(A, X) = \alpha \otimes X + \beta \otimes (AX) + \beta^T \otimes (AX)^T \prec 0 \tag{B.14}$$

In practical applications, $\mathcal{LMI}$ regions are often specified as the intersection of elementary regions, such as conic sectors, disks, or vertical half-planes.
Given $\mathcal{LMI}$ regions $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_N$ and their associated characteristic functions $f_{\mathcal{D}_i}, i = 1, 2, ..., N$ such that

$$f_{\mathcal{D}_i}(z) = \alpha_i + z\beta_i + \bar{z}\beta_i^{\mathrm{T}}, \qquad i = 1, 2, ..., N$$

The intersection

$$\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2 \cap ... \cap \mathcal{D}_N \tag{B.15}$$

is also an $\mathcal{LMI}$ region with characteristic function

$$f_{\mathcal{D}}(z) = \operatorname*{diag}_{i=1}^{N} f_{\mathcal{D}_i}(z) \tag{B.16}$$

If $\mathcal{D}$-stability of a matrix A in region $\mathcal{D}$ defined in (B.15) is of interest, Theorem (B.3.1) should be applied to the overall characteristic function $f_{\mathcal{D}}(z)$ defined by (B.16). It has shown (see corollary 2.3 in [29] that the eigenvalues of the matrix A belong to the region $\mathcal{D}$ if and only if there exists a positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that for any $i = 1, 2, ..., N$

$$M_{\mathcal{D}_i}(A, X) = \alpha_i \otimes X + \beta_i \otimes (AX) + \beta_i^{\mathrm{T}} \otimes (AX)^{\mathrm{T}} \prec 0 \tag{B.17}$$

[1] A matrix A is called $\mathcal{D}$-stable if and only if all its eigenvalues lie in a subregion $\mathcal{D}$ of the complex plane

i.e.,

$$M_{\mathcal{D}}(A,X) = \alpha \otimes X + \beta \otimes (AX) + \beta^{\mathrm{T}} \otimes (AX)^{\mathrm{T}} = \operatorname*{diag}_{i=1}^{N} M_{\mathcal{D}_i}(A,X) \prec 0 \tag{B.18}$$

where

$$\alpha = \operatorname*{diag}_{i=1}^{N}(\alpha_i), \quad \beta = \operatorname*{diag}_{i=1}^{N}(\beta_i)$$

This last inequality clearly shows that the region $\mathcal{D}$ can itself be formulated as an $\mathcal{LMI}$ region. This shows that the $\mathcal{LMI}$ approach makes it possible to consider the same matrix X for all subregions of the intersection while preserving the necessity of the $\mathcal{D}$-stability condition.

B.4 Examples of $\mathcal{LMI}$ Regions

The inequality (B.14) can also be written in the following form

$$\begin{bmatrix} \alpha_{11}X + \beta_{11}AX + \beta_{11}(AX)^{\mathrm{T}} & \dots & \alpha_{1n}X + \beta_{1n}AX + \beta_{n1}(AX)^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ \alpha_{n1}X + \beta_{n1}AX + \beta_{1n}(AX)^{\mathrm{T}} & \dots & \alpha_{nn}X + \beta_{nn}AX + \beta_{nn}(AX)^{\mathrm{T}} \end{bmatrix} \prec 0 \tag{B.19}$$

where $\alpha = (\alpha_{ij})_{1\leq i,j\leq n}$ and $\beta = (\beta_{ij})_{1\leq i,j\leq n}$.
Below are a few examples of $\mathcal{LMI}$ regions:

- **Open left half-plane**

$$\mathrm{Re}(z) < 0 \Leftrightarrow z + \bar{z} < 0 \tag{B.20}$$

It is enough to take $\alpha = 0$ and $\beta = 1$. From the expression (B.20), we deduce the following $\mathcal{LMI}$

$$AX + (AX)^{\mathrm{T}} \prec 0, X \succ 0 \tag{B.21}$$

The $\mathcal{LMI}$ (B.21) is the characterization of the asymptotic stability of a dynamical system $\dot{x}(t) = Ax(t)$ introduced in terms of $\mathcal{LMI}$ by the Lyapunov theorem.

- **a-Stability**

$$\mathrm{Re}(z) < -a \Leftrightarrow z + \bar{z} + 2a < 0 \tag{B.22}$$

It is enough to take $\alpha = 2a$ and $\beta = 1$, the following $\mathcal{LMI}$ is obtained

$$2aX + AX + (AX)^{\mathrm{T}} \prec 0, \;\; X \succ 0 \tag{B.23}$$

- **Vertical strip**

$$a_1 < \mathrm{Re}(z) < a_2 \Leftrightarrow 2a_1 < z + \bar{z} < 2a_2 \tag{B.24}$$

Using the intersection property (B.16), we obtain

$$\begin{bmatrix} -2a_2 + z + \bar{z} & 0 \\ 0 & 2a_1 - z - \bar{z} \end{bmatrix} \prec 0 \tag{B.25}$$

which leads to the $\mathcal{LMI}$ (B.18)

$$\begin{bmatrix} -2a_2X + AX + (AX)^{\mathrm{T}} & 0 \\ 0 & -2a_1X - (AX + (AX)^{\mathrm{T}}) \end{bmatrix} \prec 0, \quad X \succ 0 \tag{B.26}$$

with

$$\alpha = \begin{bmatrix} -2a_2 & 0 \\ 0 & 2a_1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Horizontal strip**

$$|\mathrm{Im}(z)| < a \Leftrightarrow \|z - \bar{z}\| < 2a \tag{B.27}$$

According to the following complex numbers property. For any complex number z

$$\|z\|^2 = z\bar{z} \tag{B.28}$$

The inequality (B.27) is equivalent to

$$-4a^2 - (z - \bar{z})(z - \bar{z}) < 0 \tag{B.29}$$

The application of the Shur complement to (B.29) leads to

$$\begin{bmatrix} -2a & z - \bar{z} \\ -z + \bar{z} & -2a \end{bmatrix} \prec 0 \tag{B.30}$$

It is enough to take $\alpha = \begin{bmatrix} -2a & 0 \\ 0 & -2a \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the following $\mathcal{LMI}$ is then obtained

$$\begin{bmatrix} -2aX & AX - (AX)^{\mathrm{T}} \\ -AX + (AX)^{\mathrm{T}} & -2aX \end{bmatrix} \prec 0, \ X \succ 0 \tag{B.31}$$

- **The disk $D(q, r)$ with center $(-q, 0)$ and radius r**

The disk $D(q, r)$ is characterized by

$$\|z + q\| < r \Leftrightarrow \|z + q\|^2 < r^2 \tag{B.32}$$

According to the complex numbers property (B.28) the inequality (B.32) is the same as

$$-r^2 + (z + q)(\bar{z} + q) < 0 \tag{B.33}$$

Applying the Shur complement, (B.33) is equivalent to

$$\begin{bmatrix} -r & q + z \\ q + \bar{z} & -r \end{bmatrix} \prec 0 \tag{B.34}$$

which corresponds to

$$\alpha = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{B.35}$$

consequently the following $\mathcal{LMI}$ is obtained

$$\begin{bmatrix} -rX & qX + AX \\ qX + (AX)^{\mathrm{T}} & -rX \end{bmatrix} \prec 0, \quad X \succ 0 \tag{B.36}$$

- **Conic sector with apex at the origin and inner angle 2θ**

The conic sector with apex at the origin and inner angle $2\theta, 0 < \theta < \frac{\pi}{2}$ caught in the left half plane is resulting from the intersection of the half complex plane defined by

$$\operatorname{Re}(z)\sin(\theta) + \operatorname{Im}(z)\cos(\theta) < 0 \tag{B.37}$$

with the half complex plane defined by

$$\operatorname{Re}(z)\sin(\theta) - \operatorname{Im}(z)\cos(\theta) < 0 \tag{B.38}$$

Then, the conic sector is characterized by

$$\tan(\theta) > \frac{|\operatorname{Im}(z)|}{-\operatorname{Re}(z)} \tag{B.39}$$

which is the same as

$$\operatorname{Re}(z)\tan(\theta) < -|\operatorname{Im}(z)| \tag{B.40}$$

Since $\text{Re}(z) < 0$ and $\tan(\theta) > 0$ because $0 < \theta < \frac{\pi}{2}$, then the inequality (B.40) is equivalent to

$$(\text{Re}(z))^2 \tan^2(\theta) > |\text{Im}(z)|^2 \tag{B.41}$$

i.e.,

$$(\text{Re}(z))^2 \sin^2(\theta) - \cos^2(\theta) \, |\text{Im}(z)|^2 > 0$$

According to the property (B.28), the last inequality is equivalent to

$$(z + \bar{z})^2 \sin^2(\theta) + (z - \bar{z})(z - \bar{z}) \cos^2(\theta) > 0 \tag{B.42}$$

According to the Shur complement, (B.42) is equivalent to

$$\begin{bmatrix} \sin(\theta)(z + \bar{z}) & \cos(\theta)(z - \bar{z}) \\ -\cos(\theta)(z - \bar{z}) & \sin(\theta)(z + \bar{z}) \end{bmatrix} \prec 0 \tag{B.43}$$

It is enough to take

$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -cos(\theta) & \sin(\theta) \end{bmatrix}$$

and then, we obtain the $\mathcal{LMI}$

$$\begin{bmatrix} \sin(\theta)(AX + (AX)^{\mathsf{T}}) & \cos(\theta)(AX - (AX)^{\mathsf{T}}) \\ -\cos(\theta)(AX - (AX)^{\mathsf{T}}) & \sin(\theta)(AX + (AX)^{\mathsf{T}}) \end{bmatrix} \prec 0 \tag{B.44}$$

appendix C

$\mathcal{GLMI}$ Regions

The concept of Generalized LMI ($\mathcal{GLMI}$) regions is recalled here.

C.1 Definition of a $\mathcal{GLMI}$ region

Definition C.1.1. *[8, 32]A region $\mathcal{D}$ of the complex plane is a $\mathcal{GLMI}$ region of order l if there exist square complex matrices $\theta_k \in \mathbb{C}^{l\times l}$, $\psi_k \in \mathbb{C}^{l\times l}$, $H_k \in \mathbb{C}^{l\times l}$ and $J_k \in \mathbb{C}^{l\times l}$, $\forall k \in \{1, 2, ..., m\}$, such that*

$$\mathcal{D} = \{z \in \mathbb{C} : \exists[\omega_1 \, ... \, \omega_m]^T \in \mathbb{C}^m s.t. f_{\mathcal{D}}(z, \omega) \prec 0, g_{\mathcal{D}}(\omega) = 0_{l\times l}\} \tag{C.1}$$

where

$$f_{\mathcal{D}}(z, \omega) = \sum_{k=1}^{m} (\theta_k \omega_k + \theta_k^* \bar{\omega}_k + \psi_k z \omega_k + \psi_k^* \bar{\omega}_k \bar{z}) \tag{C.2}$$

and

$$g_{\mathcal{D}}(\omega) = \sum_{k=1}^{m} (H_k \omega_k + J_k \bar{\omega}_k) \tag{C.3}$$

C.2 Stability in a $\mathcal{GLMI}$ region

Definition C.2.1. *A matrix A is said to be $\mathcal{D}$-stable if and only if its eigenvalues are strictly located in region $\mathcal{D}$ of the complex plane.*

In the case where $\mathcal{D}$ is a $\mathcal{GLMI}$ region of the form (C.1), the following theorem presents a necessary and sufficient $\mathcal{LMI}$ condition for matrix A to be $\mathcal{D}$-stable. A proof can be found in [8, 32].

Theorem C.2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\mathcal{D}$ a $\mathcal{GLMI}$ region. A is $\mathcal{D}$-stable if and only if there exist m matrices $X_k \in \mathbb{C}^{n\times n}$, $\forall k \in \{1, 2, ..., m\}$, such that:*

$$\sum_{k=1}^{m}(\theta_k \otimes X_k + \theta_k^* \otimes X_k^* + \psi_k \otimes (AX_k) + \psi_k^* \otimes (AX)^*) \prec 0 \tag{C.4}$$

and

$$\sum_{k=1}^{m}(H_k \otimes X_k + J_k \otimes X_k^*) = 0_{nl\times nl} \tag{C.5}$$

C.3 $\mathcal{D}$-stability in the union of convex sub-regions

This paragraph is devoted to the means by which many simple polynomial regions can be formulated as $\mathcal{GLMI}$ regions. Indeed, these regions actually result from unions of convex polynomial sub-regions, not necessarily symmetrical with respect to the real axis. We describe here polynomial regions, often used in pole placement problems, and which can also be described by a $\mathcal{GLMI}$ formulation.

C.3.1 First-order $\mathcal{GLMI}$ regions

$\mathcal{GLMI}$ regions of first order are the half plane (see [8]). Indded, a half plane in the complex plane is defined by

$$\mathcal{D} = \{z = x + jy \in \mathbb{C}/d_0 + d_1x + d_2y < 0\} \tag{C.6}$$

where d_0, d_1, d_2 are real constants. knowing that for any complex number $z = x + jy$, we have

$$x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2j} \tag{C.7}$$

we can deduce that

$$\mathcal{D} = \left\{z = x + jy \in \mathbb{C}/d_0 + d_1\frac{z+\bar{z}}{2} + d_2\frac{z-\bar{z}}{2j} < 0\right\} \tag{C.8}$$

which is the same as

$$\mathcal{D} = \left\{z = x + jy \in \mathbb{C}/d_0 + \frac{d_1 - jd_2}{2}z + \frac{d_1 + jd_2}{2}\bar{z} < 0\right\} \tag{C.9}$$

This formulation is identifiable with the polynomial formulation

$$\mathcal{D} = \{z = x + jy \in \mathbb{C}/\alpha + \beta z + \beta^*\bar{z} < 0\} \tag{C.10}$$

with

$$\alpha = d_0,\ \ \beta = \frac{d_1 - jd_2}{2},\ \ \beta^* = \frac{d_1 + jd_2}{2} \tag{C.11}$$

this polynomial formulation is identifiable with a $\mathcal{GLMI}$ region of first order with $m = 1$. It is enough to see that we can take

$$\theta_1 = \alpha,\ \omega = 1,\ \psi_1 = \beta,\ H_1 = -J_1 = 1 \tag{C.12}$$

C.3.2 $\mathcal{GLMI}$ Formulation of the union of first order sub-regions

Let $\mathcal{D}$ be the region of the complex plane described by

$$\mathcal{D} = \bigcup_{k=1}^{m} \mathcal{D}_k \tag{C.13}$$

where $\mathcal{D}_k$ is a sub-region (We restrict ourselves to regions of the first order) described by (C.10)

$$\mathcal{D}_k = \left\{ z \in \mathbb{C} / \alpha_k + \beta_k z + \beta_k^* \bar{z} < 0 \right\},\ \ \forall k \in \{1, 2, \ldots, m\} \tag{C.14}$$

As presented in [8], a union of m $\mathcal{GLMI}$ regions of the form (C.14) is also a $\mathcal{GLMI}$ region of the form (C.1) with order

$$l = m + 1 \tag{C.15}$$

and for any $k \in \{1, \ldots, m\}$, we have

$$\theta_k = \frac{1}{2} \begin{bmatrix} \Theta_k & 0_{1\times m} \\ 0_{m\times 1} & -\varepsilon_k^m \end{bmatrix} \tag{C.16}$$

$$\psi_k = \begin{bmatrix} \Psi_k & 0_{1\times m} \\ 0_{m\times 1} & 0_{m\times m} \end{bmatrix} \tag{C.17}$$

$$H_k = -J_k = \varepsilon_{k+1}^{m+1} \tag{C.18}$$

$$\Theta_k = \begin{bmatrix} \alpha_k & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \tag{C.19}$$

$$\Psi_k = \begin{bmatrix} \beta_k & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \tag{C.20}$$

and

$$\varepsilon_p^q \in \mathbb{R}^{q\times q} \text{ and } \begin{cases} \varepsilon_p^q(\rho,\sigma)=1 & \text{if } \rho=\sigma=p \\ \varepsilon_p^q(\rho,\sigma)=0 & \text{else} \end{cases} \tag{C.21}$$

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المساهمة في تحليل ومراقبة الانظمة الخطية الفردية ذات رتبة غير طبيعية

المنهجيات التي وضعت في هذه الأطروحة هي النظرية أساسا. فهي مخصصة لتحليل وتجميع قوانين الرقابة على الأنظمة الخطية ذات رتبة غير طبيعية.النتائج المحصل عليها تعتمد هذا العمل ينتمي إلى واحدة من محاور نظرية قيادة الأنظمة المعقدة الخطية حيث التعقيد يجري في النظام غير .LM اساسا و بشكل حصري على طريقة ليابونوف الثانية والشكلية صحيح للاشتقاق فيي المعادلات التفاضلية التي تصف فئة من نماذج فريدة.
النتائج المعلنة في هذه الأطروحة يمكن أن ينظر إليه باعتبارها امتدادا لبعض النتائج الموجودة في أدبيات النظم الفريدة الخطية إلى مثيلها ذات رتبة اشتقاق غير طبيعية.
الدراسة التي أجريناها والتي نظمت في قسمين: يتناول الجزء الأول التحليل و الجزء الثاني يتعلق بالاستقرار.
تم مراجعة المفاهيم الأساسية للنظم الخطية الثابتة الوقت في الفصل 1 مثل الانتظام والمقبولية والاستجابة الزمنية.
وخصص الفصل 2 لوصف الاشتقاق والتكامل من الرتب غير طبيعية. في المرحلة الأولى، تم تقديم نظرية الاشتقاق غير صحيح: أنواع مختلفة من اشتقاق غير يتجزأ (جرونوالد- ليينتيكوب، ريمان - ليوفيل وكابوتو)، تحول لابلاس، وظيفة غاما، وظائف ميتاج‌ليفلر، نبرر أيضا اختيار الاشتقاق بمعنى كابوتوفي هذا العمل.
ويبدو لنا مساهمة في هذا الفصل في مشكلة الحد الأدنى للتحكم في الطاقة حيث يتم تعريف قانون مراقبة حيث أن مؤشر أداء النظام إلى الحد الأدنى.
تلبية بين0 و1 والحالة بين 1 و 2 و هذه النتائج هي على شكلLMI صارمة. في خطوة أولى، نقدم الشروط الضرورية والكافية لمقبولية هذا النوع من الأنظمة في كلتا الحالتين ل α
تم تصميم وحدة تحكم ردود الفعل الناتج ثابت 2> α≤1 وبالنسبة لحالة 1≥α>0 ، يعتبر ردود الفعل الناتج على أساس تحكم على أساس مراقب لنظام الحلقة المغلقة لتكون مقبول.
لضمان قبول نظام.

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الكلمات المفتاحية : حساب التفاضل والتكامل الكسري؛ أنظمة فريدة؛ نظام الأهلية عدم المساواة في المصفوفة الخطية(LMIS).

Contribution à l’analyse et le contrôle des systèmes linéaires singuliers d’ordre Fractionnaire

Les méthodologies développées dans cette thèse sont essentiellement théoriques. Ils sont dédiés à l'analyse et à la synthèse de lois de contrôle pour des systèmes linéaires décrits par des modèles d'ordre fractionnaires. Leurs institutions font appel exclusivement à la seconde méthode de Lyapunov et au formalisme LMI. Les résultats rapportés dans cette dissertation peuvent être considérés comme des extensions de certains résultats existants dans la littérature de systèmes linéaires singuliers à leur homologue de l'ordre fractionnaire. L'étude que nous avons menée est organisée en deux parties: La première partie traite de l'analyse de l’admissibilité des systèmes linéaires singuliers d’ordre fractionnaire, la deuxième partie se rapporte à la stabilisation. Notre contribution apparaît au niveau du problème minimal de contrôle d'énergie dans lequel une loi de contrôle est définie de telle sorte que l'indice de performance du système est minimisé. Des conditions nécessaires et suffisantes d'admissibilité pour les systèmes à ordre fractionnaire singulier linéaire dans les deux cas de l'ordre fractionnaire α satisfaisant à $0 < \alpha \leq 1$ et $1 \leq \alpha < 2$. Ces conditions sont dérivées en termes de LMIs strictes. Dans ce cas, la commande basée sur l'observateur ou la commande de retour de la sortie est souvent nécessaire. Pour le cas $1 \leq \alpha < 2$, un régulateur de retour de sortie statique est conçu pour assurer l'admissibilité du système en boucle fermée et pour le cas $0 < \alpha \leq 1$, un contrôle par retour de sortie basé sur observateur est considéré pour que le système en boucle fermée soit admissible.

Mots-Clés : Calcul fractionnaire, Systèmes singuliers, Admissibilité, Inégalités matricielles linéaires

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Contribution to Analysis and control of singular Linear Fractional-Order Systems

The methodologies developed in this thesis are essentially theoretical. They are dedicated to the analysis and synthesis of control laws for linear systems described by fractional-order singular models. Their institutions appeal exclusively to the second Lyapunov method and to the *LMI* formalism. The results reported in this dissertation can be viewed as extensions of some existing results in the literature of linear singular systems to their homologous of fractional-order.   The study we have conducted is organized in two parts: The first part deals with the analysis of the admissibility of fractional-order singular linear systems, the second part relates to the stabilization. Chapter 2 has been devoted to dynamic systems described by differential equations of a non-integer order and to the main results of the literature on these systems.   Our contribution appears at the minimum energy control problem where a control law is defined such that the performance index of the system is minimized. Chapter 3 is devoted to our contributions on linear singular fractional-order systems . For singular systems, we need to consider not only stability but also the regularity and the non-impulsiveness. Specifically, regularity guarantees the existence and the uniqueness of a solution to a given singular system, while non-impulsiveness ensures no infinite dynamical modes in such system. This chapter serves to present, in a first step, necessary and sufficient conditions of the admissibility for linear singular fractional-order systems in both cases of the fractional-order α satisfying $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$. These conditions are derived in terms of strict *LMI*s. For the case $1 \leq \alpha < 2$ a static output feedback controller is designed to ensure the admissibility of the closed-loop system, and for the case $0 < \alpha \leq 1$, an output feedback based on observer-based controller is considered for the closed-loop system to be admissible.

Key Words: Fractional calculus, Singular systems, Admissibility, Linear matrix equalities