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Nonexistence of subnormal solutions for linear complex differential equations

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Dedicated to my Parents

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Contents

Introduction

Nevanlinna Theory is a powerful tool from complex analysis, invented by Rolf Nevanlinna[1] in 1929, it's used to study the growth and behaviour of meromorphic functions on the complex plane. The Nevanlinna characteristic function $T(r, f)$ is a measure of a function's growth, and its associated counting function estimates how often certain values are taken. Nevanlinna theory has many applications in complex analysis and in theory of functions, in particular, it plays an important role in theory of complex differential equations. Using this tool, as well as other forms of modern complex analysis, we investigate several problems relating to complex differential equations, such as growth, oscillation, fix points, etc, of solutions of complex differential equations. The first one who made systematic studies in the applications of Nevanlinna theory into complex differential equations was H. Wittich[2] beginning from 1942.

In this thesis, we study the growth of solutions for complex differential equations with entire coefficients of the form

$$f^{(k)} + \left[P_{k-1}\left(e^{\alpha_{k-1}z}\right) + Q_{k-1}\left(e^{-\alpha_{k-1}z}\right)\right] f^{(k-1)} + \cdots + \left[P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right] f = 0$$

where $k \geq 2$ is an integer and $P_j(z)$, $Q_j(z)$ $(j = 0, \ldots, k-1)$ are polynomials in z; α_j $(j = 0, \ldots, k-1)$ are complex constants. It's well known that every solution of the equation above is entire function. We will see that under some hypotheses, all solutions of this equation are of infinite order of growth, for that, we use another conceptions to estimate the growth of solutions of infinite order, such as the hyper-order σ_2, e-type order σ_e. We call the solution with e-type order equals zero by subnormal solution.

This work is divided into three chapters.

In chapter one, we prepare for the next two chapters by giving the mathematical

[1] Rolf Herman Nevanlinna (October 22, 1895 – May 28, 1980) was one of the most famous Finnish mathematicians.

[2] Hans Wittich (May 4, 1911 – August 1, 1984) was a German mathematician.

background. We give the fundamental results and standard notations of Nevanlinna theory of meromorphic functions, and give some lemmas that we need in the next two chapters.

In chapter two, we study the existence of non-trivial subnormal solutions for second-order linear differential equations,

$$f'' + \left[P_1\left(e^z\right) + P_2\left(e^{-z}\right)\right] f' + \left[Q_1\left(e^z\right) + Q_2\left(e^{-z}\right)\right] f = 0,$$

where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z, with $\deg P_1 = \deg Q_1$and $\deg P_2 = \deg Q_2$, and

$$f'' + \left[P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right] f' + \left[Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right] f = 0,$$

where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z. α, β are complex constants. We show that under certain conditions these differential equations do not have subnormal solutions, also that the hyper-order of every solution equals one. This chapter is based on the work of Li and Yang [22].

In chapter three, we investigate the existence of subnormal solutions for a class of higher order complex differential equations. We generalize the results of chapter two, because it's natural to ask if the results in the case of second-order rest true for higher order. Throughout this chapter, we prove this possibility of the generalization.

Chapter 1

Nevanlinna theory

In this chapter we give the fundamental results and standard notations of Nevanlinna theory of meromorphic functions, and give some lemmas that we need in the next two chapters. For more details, see [10, 14, 17, 28].

Throughout this thesis, by meromorphic functions we always mean functions which are meromorphic in the complex plane (except at places explicitly stated).

1.1 Poisson-Jensen and Jensen formulae

Theorem 1.1.1 (Poisson-Jensen formula, [10, 14]) *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) denote its zeros (resp. poles), each taken into account according to its multiplicity. If $z = re^{i\theta}$ and $0 \leq r < R < \infty$, then*

$$\begin{aligned} \log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\varphi})\right| \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2} d\varphi \\ &\quad + \sum_{|a_j|<R} \log\left|\frac{R(z-a_j)}{R^2 - \overline{a_j}z}\right| - \sum_{|b_k|<R} \log\left|\frac{R(z-b_k)}{R^2 - \overline{b_k}z}\right|. \end{aligned} \tag{1.1}$$

Theorem 1.1.2 (Jensen formula, [14, 17]) *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) denote its zeros (resp. poles), each taken into account according to its multiplicity. Then, we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) - \sum_{|a_i|<r} \log\left(\frac{r}{|a_i|}\right). \tag{1.2}$$

Proof. We prove the formula (1.2) when f has no zeros or poles on $|z| = r$. Denote

$$g(z) := f(z) \prod_{|a_j|<r} \left(\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right) \prod_{|b_k|<r} \left(\frac{r^2 - \overline{b_k}z}{r(z-b_k)}\right)^{-1}, \tag{1.3}$$

then $g \neq 0, \infty$ in $|z| < r$ and $\log|g(z)|$ is a harmonic function. by the mean property of classical harmonic functions, we have

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g(re^{i\varphi})\right| d\varphi. \tag{1.4}$$

Since

$$|g(0)| = |f(0)| \prod_{|a_j|<r} \left(\frac{r}{|a_j|}\right) \prod_{|b_k|<r} \left(\frac{r}{|b_k|}\right)^{-1}, \tag{1.5}$$

we get that

$$\log|g(0)| = \log|f(0)| + \sum_{|a_j|<r} \log\left(\frac{r}{|a_j|}\right) - \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right). \tag{1.6}$$

For $z = re^{i\varphi}$, we have

$$\left|\frac{r^2 - \overline{a_j}z}{r(z-a_j)}\right| = \left|\frac{r^2 - \overline{b_k}z}{r(z-b_k)}\right| = 1$$

for all a_j, b_k. Then

$$\log\left|g(re^{i\varphi})\right| = \log\left|f(re^{i\varphi})\right|. \tag{1.7}$$

Substituting (1.6) and (1.7) in (1.4), we obtain

$$\log|f(0)| + \sum_{|a_j|<r} \log\left(\frac{r}{|a_j|}\right) - \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi,$$

hence, the formula of Jensen. ■

1.2 Nevanlinna characteristic function

Definition 1.2.1 (Unintegrated counting function, [17])

$a \in \mathbb{C}$ is given. Let f be a meromorphic function such that $f \not\equiv a$. Then $n(r, a, f)$ denotes the number of roots of the equation $f(z) - a = 0$ in the disk $|z| \leq r$, each root according to its multiplicity. And $n(r, \infty, f)$ denotes the number of poles of f in the disk $|z| \leq r$, each pole according to its multiplicity.

Definition 1.2.2 (Counting function, [17])

Let f be a meromorphic function. For $a \in \mathbb{C}$, we define

$$N(r, a, f) = N\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f)\log r, \quad f \not\equiv a,$$

and

$$N(r, \infty, f) = N(r, f) := \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f)\log r.$$

Lemma 1.2.1 **([17])** *Let f be a meromorphic function with a-points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $|z| \leq r$ such that $0 < |\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_n| \leq r$, each counted according to its multiplicity. Then*

$$\int_0^r \frac{n(t,a,f)}{t}dt = \int_0^r \frac{n(t,a,f)-n(0,a,f)}{t}dt = \sum_{0<|\alpha_j|\leq r} \log \frac{r}{|\alpha_j|}. \tag{1.8}$$

Proof. Denoting $|\alpha_j| = r_j$ for $j = 1, \ldots, n$, we obtain

$$\begin{aligned}
\sum_{0<|\alpha_j|\leq r} \log \frac{r}{|\alpha_j|} &= \sum_{j=1}^{n} \log \frac{r}{r_j} = n \log r - \sum_{j=1}^{n} \log r_j \\
&= \sum_{j=1}^{n} j \left(\log r_{j+1} - \log r_j\right) + n \left(\log r - \log r_n\right) \\
&= \sum_{j=1}^{n} \int_{r_j}^{r_{j+1}} \frac{j}{t} dt + \int_{r_n}^{r} \frac{n}{t} dt \\
&= \int_0^r \frac{n(t,a,f)}{t} dt.
\end{aligned}$$

■

Example 1.2.1

Let $f_1(z) = e^z$ and $f_2(z) = e^{az}$ $(a \in \mathbb{C})$. We know that $f_1(z)$ and $f_2(z)$ are entire functions, then they don't have poles. Hence,

$$n(r, \infty, f_1) = n(r, \infty, f_2) = 0,$$

and

$$N(r, f_1) = N(r, f_2) = 0.$$

Proposition 1.2.1 **([17])** *Let f be a meromorphic function with the Laurent expansion*

$$f(z) = \sum_{j=m}^{+\infty} c_j z^j, \ \ c_m \neq 0, \, m \in \mathbb{Z}.$$

Then

$$\log |c_m| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\varphi}) \right| d\varphi + N(r, f) - N\left(r, \frac{1}{f}\right).$$

Proof. Consider the function

$$h(z) = f(z)z^{-m} \ , \ \ z \in \mathbb{C}$$

It's clear that $m = n(0,0,f) - n(0,\infty,f)$ and $h(0) \neq 0,\infty$. The functions h and f have the same poles and zeros in $0 < |z| \leq r$. By the Jensen formula (1.2) and Lemma 1.2.1, we obtain

$$\begin{aligned}
\log|c_m| &= \log|h(0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|h(re^{i\varphi})\right| d\varphi + \sum_{|b_k|<r} \log\left(\frac{r}{|b_k|}\right) - \sum_{|a_i|<r} \log\left(\frac{r}{|a_i|}\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})r^{-m}\right| d\varphi \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + -m\log r \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + -\left[n(0,0,f) - n(0,\infty,f)\right]\log r \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi})\right| d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right).
\end{aligned}$$

■

Definition 1.2.3 ([10, 14, 17])

For any real number $x \geq 0$, we define

$$\log^+ x := \max(0; \log x).$$

Lemma 1.2.2 ([17]) *We have the following properties :*

1. $\log x \leq \log^+ x \quad (x \geq 0)$.

2. $\log^+ x \leq \log^+ y \quad (0 \leq x \leq y)$.

3. $\log x = \log^+ x - \log^+ \frac{1}{x} \quad (x > 0)$.

4. $|\log x| = \log^+ x + \log^+ \frac{1}{x} \quad (x > 0)$.

5. $\log^+\left(\prod_{j=1}^{n} x_j\right) \leq \sum_{j=1}^{n} \log^+ x_j \quad (x_j \geq 0, j = 1,\ldots,n)$.

6. $\log^+\left(\sum_{j=1}^{n} x_j\right) \leq \sum_{j=1}^{n} \log^+ x_j + \log n \quad (x_j \geq 0, j = 1, \ldots, n).$

Lemma 1.2.3 ([10, 14]) *For all $a \in \mathbb{C}$, we have*

$$\log^+|a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta. \tag{1.9}$$

Proof. Denote $f(z) = a - z$, and suppose that $|a| < 1$. By using Jensen formula (1.2) with $r = 1$, we obtain

$$\log|a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(e^{i\theta})\right| d\theta - \log\frac{1}{|a|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta + \log|a|$$

hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta = 0 = \log^+|a|.$$

If $|a| \geq 1$, then f has no zeros in the disc $|z| < 1$. Therefore,

$$\log^+|a| = \log|a| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|a - e^{i\theta}\right| d\theta.$$

■

Definition 1.2.4 (Proximity function, [17])

Let f be a meromorphic function. For $a \in \mathbb{C}$, we define the proximity function of f by

$$m(r, a, f) = m\left(r, \frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\varphi}) - a|} d\varphi, \quad f \not\equiv a,$$

and

$$m(r, \infty, f) = m(r, f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\varphi})\right| d\varphi.$$

Example 1.2.2

For the function $f_1(z) = e^z$, we have

$$\begin{aligned} m(r, f_1) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f_1(re^{i\varphi})\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|e^{re^{i\varphi}}\right| d\varphi \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\varphi d\varphi \\ &= \frac{r}{\pi}. \end{aligned}$$

And for the function $f_2(z) = e^{az}$, $a = |a|e^{i\theta} \in \mathbb{C}$, we have

$$\begin{aligned} m(r; f_2) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f_2(re^{i\varphi})\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r|a|e^{i(\varphi+\theta)}}\right| d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{r|a|e^{i\psi}}\right| d\psi \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r|a|\cos(\psi)d\psi \\ &= \frac{r|a|}{\pi}. \end{aligned}$$

Definition 1.2.5 (Characteristic function, [17])

For a meromorphic function f, we define its characteristic function as

$$T(r, f) := m(r, f) + N(r, f).$$

Example 1.2.3

We have

$$\begin{aligned} T(r, e^z) &= m(r, e^z) + N(r, e^z) = \frac{r}{\pi} + 0 = \frac{r}{\pi}, \\ T(r, e^{az}) &= m(r, e^{az}) + N(r, e^{az}) = |a|\frac{r}{\pi} + 0 = |a|\frac{r}{\pi}. \end{aligned}$$

Theorem 1.2.2 (Cartan, [14]) *Suppose that f is meromorphic in $|z| < R$. Then*

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta + \log^+ |f(0)|, \quad (0 < r < R). \tag{1.10}$$

Proof. By applying the Jensen formula (1.2) for the function $f(z) - e^{i\theta}$, we obtain

$$\log\left|f(0) - e^{i\theta}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\varphi + N(r, f) - N\left(r, e^{i\theta}, f\right). \tag{1.11}$$

We integrate both sides of (1.11) with respect to θ, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta}\right| d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\varphi\right] d\theta \\ &\quad + N(r, f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\varphi}) - e^{i\theta}\right| d\theta\right] d\varphi \\ &\quad + N(r, f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, e^{i\theta}, f\right) d\theta. \end{aligned}$$

Using (1.9), we deduce

$$\begin{aligned}\log^+|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta \\ &= m(r,f) + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta \\ &= T(r;f) - \frac{1}{2\pi}\int_0^{2\pi}N\left(r,e^{i\theta},f\right)d\theta\end{aligned}$$

hence, the formula (1.10). ■

1.3 The first main theorem

Theorem 1.3.1 (First main theorem of Nevanlinna, [17]) *Let f be a meromorphic function with the Laurent expansion*

$$f(z) - a = \sum_{j=m}^{+\infty} c_j z^j \ , \ \ c_m \neq 0, \ \ m \in \mathbb{Z}, \ \ a \in \mathbb{C}.$$

Then, we have

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) - \log|c_m| + \varphi(r,a) \tag{1.12}$$

where

$$|\varphi(r,a)| \leq \log^+|a| + \log 2.$$

Proof. Assume first $a = 0$. By Proposition 1.2.1 and Lemma 1.2.2(3), we obtain

$$\begin{aligned}\ln|c_m| &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\varphi})\right|d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\varphi})|}d\varphi + N(r,f) - N\left(r,\frac{1}{f}\right) \\ &= m(r,f) - m\left(r,\frac{1}{f}\right) + N(r,f) - N\left(r,\frac{1}{f}\right) \\ &= T(r,f) - T\left(r,\frac{1}{f}\right)\end{aligned}$$

hence

$$T\left(r,\frac{1}{f}\right) = T(r,f) - \ln|c_m| \tag{1.13}$$

with $\varphi(r,0) \equiv 0$.

Suppose now, that $a \neq 0$. We define $h(z) = f(z) - a$, then

$$\begin{aligned}N\left(r,\frac{1}{h}\right) &= N\left(r,\frac{1}{f-a}\right), \\ m\left(r,\frac{1}{h}\right) &= m\left(r,\frac{1}{f-a}\right), \\ N(r,h) &= N(r,f).\end{aligned}$$

Moreover,

$$\begin{aligned}\log^+|h| &= \log^+|f-a| \le \log^+|f| + \log^+|a| + \log 2,\\ \log^+|f| &= \log^+|f-a+a| = \log^+|h+a| \le \log^+|h| + \log^+|a| + \log 2.\end{aligned}$$

Integrating these inequalities we see that

$$\begin{aligned}m(r,h) &\le m(r,f) + \log^+|a| + \log 2,\\ m(r,f) &\le m(r,h) + \log^+|a| + \log 2.\end{aligned}$$

We put

$$\varphi(r,a) := m(r,h) - m(r,f)$$

satisfies $|\varphi(r,a)| \le \ln^+|a| + \ln 2$.

By applying the formula (1.13) for h, we obtain

$$\begin{aligned}T\left(r, \frac{1}{f-a}\right) &= T\left(r, \frac{1}{h}\right) = T(r,h) - \ln|c_m|\\ &= m(r,h) + N(r,h) - \ln|c_m|\\ &= \varphi(r,a) + m(r,f) + N(r,f) - \ln|c_m|\end{aligned}$$

hence, the result. ∎

Theorem 1.3.2 (Nevanlinna, [17]) *Let f be a meromorphic function not being identically equal to a constant. Then, for all $a \in \mathbb{C}$, we have*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1) \quad \text{as} \quad r \to +\infty.$$

Proposition 1.3.3 ([17]) *Let $f, f_1, \dots, f_n$ $(n \ge 1)$ be meromorphic functions and a, b, c and d be complex constants such that $ad - bc \ne 0$. Then*

1. $T\left(r, \prod_{k=1}^{n} f_k\right) \le \sum_{k=1}^{n} T(r, f_k)$
2. $T\left(r, \sum_{k=1}^{n} f_k\right) \le \sum_{k=1}^{n} T(r, f_k) + \log n$
3. $T(r, f^m) = mT(r,f)$, $\forall\, m \in \mathbb{N}$.
4. $T\left(r, \frac{af+b}{cf+d}\right) = T(r,f) + O(1)$ as $r \to +\infty$, $f \not\equiv -\frac{d}{c}$.

Theorem 1.3.4 ([17]) *A meromorphic function f is rational if and only if $T(r,f) = O(\log r)$.*

Lemma 1.3.1 *Let* $P(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, $(n \in \mathbb{N}^*)$ *be polynomial with constant coefficients and* f *meromorphic function. Then, for the composed function* $P \circ f$, *we have*

$$T(r, P(f)) = nT(r, f) + O(1).$$

The Lemma 1.3.1 is a particular case of Theorem due to G. Valiron and A. Mohon'ko, see [17, Page 29].

Example 1.3.1

Using Proposition 1.3.3 (4) , we obtain

$$\begin{aligned} T(r, ae^{nz} + b) &= T(r, e^{nz}) + O(1) \\ &= n\frac{r}{\pi} + O(1), \ \forall n \in \mathbb{N}^* \end{aligned}$$

and,

$$\begin{aligned} T(r, \cot z) &= T\left(r, \frac{\cos z}{\sin z}\right) \\ &= T\left(r, -i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}\right) \\ &= T\left(r, i\frac{e^{2iz} + 1}{e^{2iz} - 1}\right) \\ &= T(r, e^{2iz}) + O(1) \\ &= 2\frac{r}{\pi} + O(1). \end{aligned}$$

Let $P(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, $(n \in \mathbb{N}^*)$ be polynomial, by Lemma 1.3.1, we have

$$\begin{aligned} T\left(r, P(e^{az})\right) &= nT(r, e^{az}) + O(1) \\ &= n|a|\frac{r}{\pi} + O(1). \end{aligned}$$

1.4 Growth of meromorphic functions

Definition 1.4.1 (Order of growth, [14, 17])

Let f *be a meromorphic function. The order of growth of* f *is defined by*

$$\sigma(f) := \overline{\lim_{r \to +\infty}} \frac{\log T\left(r, f\right)}{\log r}.$$

Example 1.4.1

We have

$$\sigma(e^{az}) = \overline{\lim_{r \to +\infty}} \frac{\log\left(|a|\frac{r}{\pi}\right)}{\log r} = 1, \ \forall a \in \mathbb{C}.$$

By Example 1.3.1,

$$\sigma\left(P(e^{az})\right) = \overline{\lim_{r\to+\infty}} \frac{\log\left(n|a|\frac{r}{\pi} + O(1)\right)}{\log r} = 1, \ \forall a \in \mathbb{C}.$$

Theorem 1.4.1 **([10, 21])** *Let f, g be nonconstant meromorphic functions. Then*

1. $\sigma(f+g) \leq \max\{\sigma(f); \sigma(g)\}$.
2. $\sigma(fg) \leq \max\{\sigma(f); \sigma(g)\}$.
3. *If $\sigma(g) < \sigma(f)$ then $\sigma(f+g) = \sigma(fg) = \sigma(f)$.*

Theorem 1.4.2 **([14, 17, 21])** *Let f be an entire function and assume that $0 < r < R < +\infty$ and that the maximum modulus $M(r,f) = \max_{|z|=r} |f(z)| \geq 1$. Then*

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

Proof. Since f is entire, we have

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\varphi})\right| d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ M(r,f) d\varphi = \log M(r,f).$$

To prove the second inequality, take $z_0 = re^{i\theta}$ such that $M(r,f) = |f(z_0)|$. Furthermore, if $|z| < R$, then

$$\left|\frac{R(z-a_j)}{R^2 - \overline{a_j}z}\right| < 1.$$

Therefore, by applying the Poisson-Jensen formula (1.1), we obtain

$$\begin{aligned}
\log M(r,f) &= \log|f(z_0)| \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\varphi})\right| \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2} d\varphi + \sum_{|a_j|<R} \log\left|\frac{R(z-a_j)}{R^2 - \overline{a_j}z}\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\varphi})\right| \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\varphi) + r^2} d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\varphi})\right| \frac{R^2 - r^2}{(R-r)^2 + 2rR(1-\cos(\theta-\varphi))} d\varphi \\
&\leq \frac{R^2 - r^2}{(R-r)^2} \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(Re^{i\varphi})\right| d\varphi \\
&= \frac{R+r}{R-r} m(r,f) = \frac{R+r}{R-r} T(r,f).
\end{aligned}$$

■

Corollary 1.4.1 **([17, 21])** *Let f be an entire function. Then*

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.4.2 (Hyper-order[17, 21, 28])

Let f be a meromorphic function. The hyper-order of f is defined by

$$\sigma_2(f) := \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r},$$

and if f is entire, then

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log\log M(r,f)}{\log r}.$$

Remark 1.4.1 If $\sigma(f) < +\infty$, then $\sigma_2(f) = 0$.

Example 1.4.2

Let $h(z) = e^{e^z}$, we have $T(r,h) \sim \frac{e^r}{(2\pi^3 r)^{\frac{1}{2}}}$, as $r \to +\infty$. Then

$$\begin{aligned} \sigma(h) &= +\infty, \\ \sigma_2(h) &= 1. \end{aligned}$$

Y.M. Chiang and S.A. Gao [8] defined the e-type order of meromorphic function as follows :

Definition 1.4.3 ([8])

Let f be a meromorphic function. The e-type order of f is defined by

$$\sigma_e(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{r}.$$

Remark 1.4.2 **(i)** If $0 < \sigma_e(f) < +\infty$, then $\sigma_2(f) = 1$.

(ii) If $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$.

(iii) If $\sigma_2(f) = +\infty$, then $\sigma_e(f) = +\infty$.

Example 1.4.3

For all polynomial P, we have $\sigma_e(P) = 0$, and $\sigma_e(P(e^z)) = 0$.

Remark 1.4.3 By Remark 1.4.1 and Remark 1.4.2, we deduce that, every function f of finite order satisfies $\sigma_e(f) = 0$.

1.5 Estimates of the logarithmic derivative

The logarithmic derivative has an important role (as we will see) in study of the complex differential equations. For that, many researchers are interested in problem of finding best estimation of the logarithmic derivative, see [12, 14, 17]. For more details about new estimations for the growth of the logarithmic derivative,we refer to [19, 20].

Definition 1.5.1 ([15])

Let $E \subset [0, +\infty[$ be such a set. The linear and the logarithmic measures of E are defined to be

$$m(E) = \int_E dt \quad \text{and} \quad lm(E) = \int_{E\cap]1,+\infty[} \frac{dt}{t}$$

respectively. These may be finite or infinite.

Theorem 1.5.1 (Nevanlinna's main estimate of logarithmic derivative, [17]) *Let f be a transcendental meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = S\left(r, f\right),$$

where $S\left(r, f\right) = O\left(\log T\left(r, f\right) + \log r\right)$ outside of a possible exceptional set $E \subset [0, +\infty)$ with finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

Corollary 1.5.1 ([17]) *Let f be a transcendental meromorphic function and $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S\left(r, f\right),$$

where $S\left(r, f\right) = O\left(\log T\left(r, f\right) + \log r\right)$ outside of a possible exceptional set $E \subset [0, +\infty)$ with finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Theorem 1.5.2 ([12]) *Let f be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $i, j (0 \leq i < j)$, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{j-i}.$$

Lemma 1.5.1 ([12]) *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$. Let $H = \{(k_1, j_1), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$, for $i = 1, \ldots, q$. And let $\varepsilon > 0$ be a given constant. Then there exists a set $E \in [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi)\backslash E$, then there is a* *constant* $R_0 = R_0(\psi) > 1$ *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| = r \geq R_0$ *and for all* $(k; j) \in H$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(\sigma-1+\varepsilon)(k-j)}.$$

1.6 Wiman-Valiron theory

In this section we just give a short review of basic notions and most important results. See [17, 15]. Throughout this section, we assume that f is entire function with Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $(a_n)_{n\in\mathbb{N}}$ is sequence of complex numbers.

Definition 1.6.1 ([17, 15])

For a given $r > 0$*, we define the maximum term of* f *by*

$$\mu_f(r) = \max_{n\geq 0} |a_n| r^n,$$

and we define the central index $\nu_f(r)$ *as the greatest exponent* m *such that* $|a_m| r^m = \mu_f(r)$ *i.e.*

$$\left|a_{\nu_f(r)}\right| r^{\nu_f(r)} = \max_{n\geq 0} |a_n| r^n.$$

Theorem 1.6.1 ([17]) *Let* f *be an entire function of order* $\sigma(f) = \sigma$*. Then*

$$\sigma = \overline{\lim_{r\to+\infty}} \frac{\log \nu_f(r)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log \mu_f(r)}{\log r}.$$

Theorem 1.6.2 ([7]) *Let* f *be an entire function of infinite order with the hyper-order* $\sigma_2(f) < \infty$ *, and let* $\nu_f(r)$ *be the central index of* f*. Then*

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \nu_f(r)}{\log r}.$$

Theorem 1.6.3 (Wiman-Valiron, [15, 24]) *Let* f *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r; f)$*. Then for all* $|z|$ *outside a set* E *of* r *of finite logarithmic measure, we have*

$$f^{(k)}(z) = \left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1))\, f(z) \ , \quad (k \text{ is an integer}, \ r \notin E)$$

where $\nu_f(r)$ *is the central index of* f*.*

Lemma 1.6.1 ([5]) *Let f be an entire function of infinite order with $\sigma_2(f) = \alpha$ $(0 \le \alpha < \infty)$ and a set $E \subset [1, +\infty)$ have finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, $r_k \to \infty$, and such that*

1. *if $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then for any given ε_1 $(0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu_f(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\},$$

2. *if $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M > 0$, we have as r_k sufficiently large*

$$r_k^M < \nu_f(r_k) < \exp\{r_k^{\varepsilon_2}\}.$$

1.7 Some results about complex differential equations

In the theory of complex differential equations, the growth of solutions is a very important property. It is well known that all solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{1.14}$$

are entire functions, provided the coefficients $A_0(z) \not\equiv 0, A_1(z), \ldots, A_{k-1}(z)$ are entire, see [16, 17]. A classical result, due to Wittich [26], tells that all solutions of (1.14) are of finite order of growth if and only if all coefficients $A_j(z), j = 0, \ldots, k-1$, are polynomials, and if some of the coefficients $A_j(z), j = 0, \ldots, k-1$ are replaced by transcendental entire functions, then equation (1.14) has at least one solution of infinite order, see [9, 17].

There are other conceptions can be applied to get a more precise estimate of the growth of meromorphic function with infinite order, such as hyper-order and e-type order. In this work, we are interested in the case when the solutions of (1.14) are of infinite order and the e-type order vanishes.

Definition 1.7.1 ([11, 25])
If $f \not\equiv 0$ is a solution of the equation (1.14), and satisfies $\sigma_e(f) = 0$, then we say that f is a nontrivial subnormal solution of (1.14). For convenience, we also say that $f \equiv 0$ is a subnormal solution of (1.14).

Remark 1.7.1 If the equation (1.14) is non-homogeneous,then we define the subnormal solution as in the previous definition.

Now, we consider the differential equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.15}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. We have the following results.

Theorem 1.7.1 **([13])** *If $f \not\equiv 0$ is a solution of the equation (1.15) where $\sigma(f) < \infty$, then as $r \to +\infty$,*

$$T(r, B) \leq T(r, A) + O(\log r).$$

Proof. Suppose that $f \not\equiv 0$ is a solution of (1.15) where $\sigma(f) < \infty$. From (1.15) we have

$$B(z) = -\frac{f''}{f} - A(z)\frac{f'}{f}.$$

From Nevanlinna's main estimate of logarithmic derivative, we obtain that $m(r, B) \leq m(r, A) + O(\log r)$, as $r \to \infty$, and because $A(z)$ and $B(z)$ are entire functions, then $T(r, B) \leq T(r, A) + O(\log r)$, as $r \to \infty$. ■

Corollary 1.7.1 **([13])** *Let $A(z)$ and $B(z)$ be entire functions where either (i) $\sigma(A) < \sigma(B)$, or (ii) A is a polynomial and B is transcendental. Then every solution $f \not\equiv 0$ of (1.15) has infinite order.*

Proof. (i) Suppose that $\sigma(A) < \sigma(B)$ and $\sigma(f) < \infty$, from Theorem 1.7.1, we have $T(r, B) \leq T(r, A) + O(\log r)$, therefore $\sigma(A) \geq \sigma(B)$ which is a contradiction.

(ii) Suppose that A is a polynomial, B is transcendental and $\sigma(f) < \infty$, from Theorem 1.7.1, we have $T(r, B) \leq T(r, A) + O(\log r)$. As A is a polynomial, then $T(r, A) = O(\log r)$, hence $T(r, B) = O(\log r)$, that means B is a polynomial, which is again a contradiction.
■

Theorem 1.7.2 **([1])** *Let $A(z)$ and $B(z)$ be entire functions of finite order. If f is a solution of the equation (1.15), then $\sigma_2(f) \leq \max\{\sigma(A); \sigma(B)\}$.*

Proof. Set $\sigma = \max\{\sigma(A); \sigma(B)\}$. Then for any given $\varepsilon > 0$, when r is sufficiently large, we have

$$|A| \leq \exp\{r^{\sigma+\varepsilon}\}; \ |B| \leq \exp\{r^{\sigma+\varepsilon}\}. \tag{1.16}$$

From the Wiman-Valiron theory, there is a set $F \subset (1, +\infty)$ having logarithmic measure $lm(F) < \infty$, we can choose z satisfying $|z| = r \notin [0, 1] \cup F$ and $|f(z)| = M(r, f)$, then we get

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \quad j = 1, 2, \tag{1.17}$$

where $\nu_f(r)$ is the central index of $f(z)$. From (1.15) we have

$$\frac{f''}{f} = -A(z)\frac{f'}{f} + B(z). \tag{1.18}$$

Substituting (1.16) and (1.17) into (1.18), we obtain

$$\left(\frac{\nu_f(r)}{r}\right)^2 |1+o(1)| \leq \exp\{r^{\sigma+\varepsilon}\}\left(\frac{\nu_f(r)}{r}\right)|1+o(1)| + \exp\{r^{\sigma+\varepsilon}\}$$

where $|z| = r \notin [0,1] \cup F$ and $|f(z)| = M(r,f)$. Therefore,

$$\overline{\lim_{r\to+\infty}} \frac{\log\log \nu_f(r)}{\log r} \leq \sigma + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by Theorem 1.6.2 we have $\sigma_2(f) \leq \sigma$. ■

The Theorem 1.7.2 can be generalized to k^{th} order differential equation, as follows :

Theorem 1.7.3 ([4]) *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0,$$

then $\sigma_2(f) \leq \max\{\sigma(A_j) : j = 0, \ldots, k-1\}$.

1.8 Some lemmas and remarks

Lemma 1.8.1 ([17]) *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial with $a_n \neq 0$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r = |z| > r_0$ we have the inequalities*

$$(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n.$$

Proof. The assertion immediately follows from

$$\begin{aligned} \frac{|P(z)|}{|a_n|\,r^n} &= \left|\frac{P(z)}{a_n z^n}\right| \\ &= \left|\frac{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0}{a_n z^n}\right| \\ &= \left|1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right| \longrightarrow 1, \end{aligned}$$

as $r \longrightarrow +\infty$. ■

Remark 1.8.1 Let $P(z), Q(z)$ be polynomials in z with $\deg P = m, \deg Q = n$,

$$\begin{aligned} P(z) &= a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0, \\ Q(z) &= b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0, \end{aligned}$$

where a_u, b_v $(u = 0, \ldots, m; v = 0, \ldots, n)$ are complex constants, $a_m b_n \neq 0$. For $z = re^{i\theta}$ and $a_m = a+ib$, we denote $\delta(P, \theta) := a\cos(n\theta) - b\sin(n\theta)$. By Lemma 1.8.1, we can obtain that

$$\left|P\left(e^{\alpha z}\right) + Q\left(e^{-\alpha z}\right)\right| = \begin{cases} |a_m|\, e^{m\delta(\alpha z,\theta)r}(1+o(1)), & (\delta(\alpha z,\theta) > 0; r \to +\infty) \\ |b_n|\, e^{-n\delta(\alpha z,\theta)r}(1+o(1)), & (\delta(\alpha z,\theta) < 0; r \to +\infty) \end{cases} \tag{1.19}$$

$\forall \alpha \in \mathbb{C}$.

By the estimations (1.19), we obtain

$$\sigma\left(P\left(e^{\alpha z}\right) + Q\left(e^{-\alpha z}\right)\right) = 1.$$

Lemma 1.8.2 **([13, 17])** *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ holds for all $r > r_0$.*

Proof. Since E is of finite logarithmic measure, clearly E can contain at most finitely many disjoint intervals of the form $[r, \alpha r]$ for $r \geq 1$, it follows that there exists an $r_0 > 0$, such that for any $r > r_0$, the interval $[r, \alpha r]$ must contain a point t where $t \notin E \cup [0, 1]$. Then $g(r) \leq g(t) \leq h(t) \leq h(\alpha r)$. ■

Lemma 1.8.3 **([2])** *Let f be an entire function with $\sigma(f) = \sigma < +\infty$. Suppose there exists a set $E \cup [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi)\backslash E$, we have*

$$\left|f(re^{i\theta_0})\right| \leq Mr^k,$$

where $M = M(\theta_0) > 0$ is a constant and $k > 0$ is a constant independent of θ_0, then f is a polynomial with $\deg f \leq k$.

Lemma 1.8.4 **([13, 18])** *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to +\infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}|z_n|^{k-j}(1+o(1)), \quad (j = 0, \ldots, k-1).$$

Chapter 2

Nonexistence of subnormal solutions for second order complex differential equations

In this chapter, we study the existence of non-trivial subnormal solutions for second-order linear differential equations. We show that under certain conditions some differential equations do not have subnormal solutions, also that the hyper-order of every solution equals one. This chapter is based on the work of Li and Yang [22].

2.1 Introduction

The investigation of the existence of subnormal solutions of complex differential equations has been started by H. Wittich [25].

Wittich has given the general forms of all subnormal solutions of the equation

$$f'' + P\left(e^{z}\right) f' + Q\left(e^{z}\right) f = 0, \tag{2.1}$$

where $P\left(w\right)$ and $Q\left(w\right)$ are nonconstants polynomials in $w = e^{z}$ $(z \in \mathbb{C})$, and he proved the following theorem

Theorem 2.1.1 **([25])** *If $f \not\equiv 0$ is a subnormal solution of (2.1), then f must have the form*

$$f(z) = e^{cz}(a_0 + a_1 e^{z} + \cdots + a_m e^{mz}),$$

where $m \geq 0$ is an integer and c, $a_0, a_1, \ldots, a_m$ are constants with $a_0 a_m \neq 0$.

Based on the comparison of degrees of P and Q, Gundersen and Steinbart [11] refined Theorem 2.1.1 and obtained the exact forms of subnormal solutions of (2.1) as follows.

Theorem 2.1.2 ([11]) *Under the assumption of Theorem 2.1.1, the following statements hold.*

1. *If* $\deg P > \deg Q$ *and* $Q \not\equiv 0$, *then, any subnormal solution* $f \not\equiv 0$ *of (2.1) must have the form*
$$f(z) = a_0 + a_1 e^{-z} + \cdots + a_m e^{-mz},$$
where $m \geq 1$ *is an integer and* $a_0, a_1, \ldots, a_m$ *are constants with* $a_0 a_m \neq 0$.

2. *If* $Q \equiv 0$ *and* $\deg P \geq 1$, *then, any subnormal solution of (2.1) must be a constant.*

3. *If* $\deg P < \deg Q$, *then, the only subnormal solution of (2.1) is* $f \equiv 0$.

In 2007, Chen and Shon [3] studied the existence of subnormal solutions of the general equation
$$f'' + \left[P_1\left(e^z\right) + P_2\left(e^{-z}\right)\right] f' + \left[Q_1\left(e^z\right) + Q_2\left(e^{-z}\right)\right] f = 0, \tag{2.2}$$
where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z, and obtained the following results.

Theorem 2.1.3 ([3]) *Let* $P_j(z), Q_j(z)$ $(j = 1, 2)$ *be polynomials in* z. *If*
$$\deg Q_1 > \deg P_1 \text{ or } \deg Q_2 > \deg P_2,$$
then the equation (2.2) *has no nontrivial subnormal solution, and every solution of* (2.2) *satisfies* $\sigma_2(f) = 1$.

Theorem 2.1.4 ([3]) *Let* $P_j(z), Q_j(z)$ $(j = 1, 2)$ *be polynomials in* z. *If*
$$\deg Q_1 < \deg P_1 \text{ and } \deg Q_2 < \deg P_2$$
and $Q_1 + Q_2 \not\equiv 0$, *then the equation* (2.2) *has no nontrivial subnormal solution, and every solution of* (2.2) *satisfies* $\sigma_2(f) = 1$.

In 2013, Xiao [27] considered the problem about what the conditions that will guarantee the equation
$$f'' + P\left(e^{\alpha z}\right) f' + Q\left(e^{\beta z}\right) f = 0, \tag{2.3}$$
where $P(w)$ and $Q(w)$ are nonconstants polynomials in $w = e^z$ $(z \in \mathbb{C})$, α, β are complex constants, does not have a nontrivial subnormal solution.

2.2 Main results

Li and Yang [22] raised the following question,

Question. What can be said when $\deg P_1 = \deg Q_1$ and $\deg P_2 = \deg Q_2$ for (2.2) ?

To answer on this question, they proved the following theorem.

Theorem 2.2.1 ([22]) *Let*

$$\begin{aligned} P_1(z) &= a_n z^n + \cdots + a_1 z + a_0, \\ Q_1(z) &= b_n z^n + \cdots + b_1 z + b_0, \\ P_2(z) &= c_m z^m + \cdots + c_1 z + c_0, \\ Q_2(z) &= d_m z^m + \cdots + d_1 z + d_0, \end{aligned}$$

where a_i, b_i $(i = 0, \ldots, n)$, c_j, d_j $(j = 0, \ldots, m)$ *are constants,* $a_n b_n c_m d_m \neq 0$. *Suppose that* $a_n d_m = b_n c_m$ *and any one of the following three hypothesis holds :*

1. *There exists* i *satisfying* $(-\frac{b_n}{a_n})a_i + b_i \neq 0$, $0 < i < n$.
2. *There exists* j *satisfying* $(-\frac{b_n}{a_n})c_j + d_j \neq 0$, $0 < j < m$.
3. $(-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})(a_0 + c_0) + b_0 + d_0 \neq 0$.

Then (2.2) *has no nontrivial subnormal solution, and every nontrivial solution* f *satisfies* $\sigma_2(f) = 1$.

Example 2.2.1 ([22])

We remark that the equation

$$f'' + (e^{2z} + e^{-z} + 1)f' + (2e^{2z} + 2e^{-z} - 2)f = 0$$

has a subnormal solution $f_0 = e^{-2z}$. Here $n = 2$, $m = 1$, $a_2 = 1$, $b_2 = 2$, $a_1 = b_1 = 0$, $c_1 = 1$, $d_1 = 2$, $a_0 + c_0 = 1$, $b_0 + d_0 = -2$, $(-\frac{b_2}{a_2}) \cdot a_1 + b_1 = 0$, and $(-\frac{b_2}{a_2})^2 + (-\frac{b_2}{a_2})(a_0 + c_0) + b_0 + d_0 = 0$. This shows that the restrictions (i)–(iii) in Theorem 2.2.1 are sharp.

In the same article [22], Li and Yang have investigated the existence of subnormal solutions of the general form

$$f'' + \left[P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right] f' + \left[Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right] f = 0, \tag{2.4}$$

where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z. α, β are complex constants, and they proved the following results,

Theorem 2.2.2 ([22]) *Let*

$$\begin{aligned} P_1(z) &= a_{1m_1}z^{m_1}+\cdots+a_{11}z+a_{10},\\ P_2(z) &= a_{2m_2}z^{m_2}+\cdots+a_{21}z+a_{20},\\ Q_1(z) &= b_{1n_1}z^{n_1}+\cdots+b_{11}z+b_{10},\\ Q_2(z) &= b_{2n_2}z^{n_2}+\cdots+b_{21}z+b_{20}, \end{aligned}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k=1,2)$ *are integers,* $a_{1i_1}(i_1=0,\ldots,m_1)$, $a_{2i_2}(i_2=0,\ldots,m_2)$, $b_{1j_1}(j_1=0,\ldots,n_1)$, $b_{2j_2}(j_2=0,\ldots,n_2)$, α *and* β *are complex constants,* $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2}\neq 0$, $\alpha\beta\neq 0$. *Suppose* $m_1\alpha=c_1n_1\beta$ $(0<c_1<1)$ *or* $m_2\alpha=c_2n_2\beta$ $(0<c_2<1)$. *Then* (2.4) *has no nontrivial subnormal solution, and every nontrivial solution* f *satisfies* $\sigma_2(f)=1$.

Theorem 2.2.3 ([22]) *Let*

$$\begin{aligned} P_1(z) &= a_{1m_1}z^{m_1}+\cdots+a_{11}z+a_{10},\\ P_2(z) &= a_{2m_2}z^{m_2}+\cdots+a_{21}z+a_{20},\\ Q_1(z) &= b_{1n_1}z^{n_1}+\cdots+b_{11}z+b_{10},\\ Q_2(z) &= b_{2n_2}z^{n_2}+\cdots+b_{21}z+b_{20}, \end{aligned}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k=1,2)$ *are integers,* $a_{1i_1}(i_1=0,\ldots,m_1)$, $a_{2i_2}(i_2=0,\ldots,m_2)$, $b_{1j_1}(j_1=0,\ldots,n_1)$, $b_{2j_2}(j_2=0,\ldots,n_2)$, α *and* β *are complex constants,* $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2}\neq 0$, $\alpha\beta\neq 0$. *Suppose* $m_1\alpha=c_1n_1\beta$ $(c_1>1)$ *and* $m_2\alpha=c_2n_2\beta$ $(c_2>1)$. *Then* (2.4) *has no nontrivial subnormal solution, and every nontrivial solution* f *satisfies* $\sigma_2(f)=1$.

Example 2.2.2 ([22])

Note that the subnormal solution $f_0=e^{-z}+1$ satisfies the equation

$$f''-(e^{3z}+e^{2z}+e^{-z})f'-(e^{2z}+e^{-z})f=0.$$

Here $\alpha=\frac{1}{2}$, $\beta=1/3$, $m_1=6$, $m_2=2$, $n_1=6$, $n_2=3$, $m_1\alpha=\frac{3}{2}n_1\beta$ and $m_2\alpha=n_2\beta$. This shows that the restrictions that $m_1\alpha=c_1n_1\beta$ $(c_1>1)$ and $m_2\alpha=c_2n_2\beta\,(c_2>1)$ can not be omitted.

2.3 Proofs of the main results

2.3.1 Proof of Theorem 2.2.1

Suppose that $f(z)$ is a non-trivial subnormal solution of (2.2). Let

$$h(z)=e^{(b_n/a_n)z}f(z),$$

then $h(z)$ is a non-trivial subnormal solution of

$$h'' + \left[2(-\frac{b_n}{a_n}) + P_1(e^z) + P_2(e^{-z})\right] h'$$
$$+ \left[(-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})\big(P_1(e^z) + P_2(e^{-z})\big) + Q_1(e^z) + Q_2(e^{-z})\right] h = 0.$$

Since any one of the following three hypotheses holds:

(i) there exists i satisfying $(-\frac{b_n}{a_n})a_i + b_i \neq 0$, $0 < i < n$;

(ii) there exists j satisfying $(-\frac{b_n}{a_n})c_j + d_j \neq 0$, $0 < j < m$;

(iii)
$$(-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})(a_0 + c_0) + b_0 + d_0 \neq 0,$$

we obtain

$$(-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})(P_1(e^z) + P_2(e^{-z})) + Q_1(e^z) + Q_2(e^{-z}) \not\equiv 0. \tag{2.5}$$

From $a_n d_m = c_m b_n$, we obtain

$$\deg P_2(z) > m - 1 \geq \deg[(-\frac{b_n}{a_n})P_2(z) + Q_2(z)]. \tag{2.6}$$

Combining (2.5) and (2.6) with

$$\deg P_1(z) > n - 1 \geq \deg[(-\frac{b_n}{a_n})P_1(z) + Q_1(z)], \tag{2.7}$$

we obtain the conclusion by using Theorem 2.1.4.

2.3.2 Proof of Theorem 2.2.2

Suppose $f(\not\equiv 0)$ is a solution of (2.4), then f is an entire function. Next we will prove that f is transcendental. Since $Q_1(e^{\beta z}) + Q_2(e^{-\beta z}) \not\equiv 0$, we see that any nonzero constant can not be a solution of the (2.4). Now suppose that $f_0 = b_n z^n + \cdots + b_1 z + b_0$, $(n \geq 1, b_n, \ldots, b_0$ are constants, $b_n \neq 0)$ is a polynomial solution of (2.4).

(1) $m_1\alpha = c_1 n_1 \beta\ (0 < c_1 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z, \theta) = |\beta| \cos(\arg\beta + \theta) > 0$, then $\delta(\alpha z, \theta) = \frac{n_1 c_1}{m_1}\delta(\beta z, \theta) > 0$. From (2.4) and Lemma 1.8.1, that for a sufficiently large

r and $\varepsilon > 0$, we have

$$\begin{aligned} |b_n| r^n |b_{1n_1}| e^{n_1 \delta(\beta z,\theta) r}(1-o(1)) &\le \left|Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right| |f_0| \\ &\le |f_0''| + \left|P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right| |f_0'| \\ &\le |a_{1m_1}| e^{m_1 \delta(\alpha z,\theta) r} n(n-1) |b_n| r^{n-1}(1+o(1)) \\ &\le M_1 e^{m_1 \frac{n_1 c_1}{m_1} \delta(\beta z,\theta) r} r^{n-1}(1+o(1)) \\ &\le M_1 e^{n_1 c_1 \delta(\beta z,\theta) r} r^{n-1}(1+o(1)), \end{aligned} \tag{2.8}$$

where $M_1 > 0$ is some constant. Since $0 < c_1 < 1$, we see that (2.8) is a contradiction.

(2) $m_2 \alpha = c_2 n_2 \beta \, (0 < c_2 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z, \theta) = |\beta| \cos(\arg \beta + \theta) < 0$, then $\delta(\alpha z, \theta) = \frac{n_2 c_2}{m_2} \delta(\beta z, \theta) < 0$. From (2.4) and Lemma 1.8.1, that for a sufficiently large r and for $\varepsilon > 0$, we have

$$\begin{aligned} |b_n| r^n |b_{2n_2}| e^{-n_2 \delta(\beta z,\theta) r}(1-o(1)) &\le \left|Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right| |f_0| \\ &\le |f_0''| + \left|P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right| |f_0'| \\ &\le |a_{2m_2}| e^{-m_2 \delta(\alpha z,\theta) r} n(n-1) |b_n| r^{n-1}(1+o(1)) \\ &\le M_2 e^{-m_2 \frac{n_2 c_2}{m_2} \delta(\beta z,\theta) r} r^{n-1}(1+o(1)) \\ &\le M_2 e^{-n_2 c_2 \delta(\beta z,\theta) r} r^{n-1}(1+o(1)), \end{aligned} \tag{2.9}$$

where $M_2 > 0$ is some constant. Since $0 < c_2 < 1$, we see that (2.9) is also a contradiction. Thus we obtain that f is transcendental.

By Theorem 1.7.2 and $\max\{\sigma(P_1(e^{\alpha z})), \sigma(P_2(e^{-\alpha z})), \sigma(Q_1(e^{\beta z})), \sigma(Q_2(e^{-\beta z}))\} = 1$, we see that $\sigma_2(f) \le 1$. By Theorem 1.5.2, we can see that there exists a subset $E \subset (1, \infty)$ having a logarithmic measure $mlE < \infty$ and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B[T(2r, f)]^{j+1}, \quad j = 1, 2. \tag{2.10}$$

(1) Suppose $m_1 \alpha = c_1 n_1 \beta \, (0 < c_1 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z, \theta) > 0$, then $\delta(\alpha z, \theta) = \frac{n_1 c_1}{m_1} \delta(\beta z, \theta) > 0$. From (2.4), (2.10), that for a sufficiently large r and $r \notin$

$[0,1] \cup E$, we have

$$\begin{aligned} |b_{1n_1}| e^{n_1\delta(\beta z,\theta)r}(1-o(1)) &\leq \left|Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right| \\ &\leq \left|\frac{f''(z)}{f(z)}\right| + \left|P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right| \left|\frac{f'(z)}{f(z)}\right| \\ &\leq B\left[T(2r,f)\right]^3 + |a_{1m_1}| e^{m_1\delta(\alpha z,\theta)r} B\left[T(2r,f)\right]^2(1+o(1)) \\ &\leq C\left[T(2r,f)\right]^3 e^{m_1\frac{n_1c_1}{m_1}\delta(\beta z,\theta)r}(1+o(1)) \\ &\leq C\left[T(2r,f)\right]^3 e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)), \end{aligned} \tag{2.11}$$

where $C > 0$ is some constant. Since $0 < c_1 < 1$, by Lemma 1.8.2, (2.11), we obtain $\sigma_2(f) \geq 1$. So $\sigma_2(f) = 1$.

Next we prove that any $f(\not\equiv 0)$ is not subnormal. If f is subnormal, then for any $\varepsilon > 0$,

$$T(r,f) \leq e^{\varepsilon r}. \tag{2.12}$$

When taking $z = re^{i\theta}$, such that $\delta(\beta z,\theta) > 0$, by (2.11) and (2.12), we deduce that

$$\begin{aligned} |b_{1n_1}|e^{n_1\delta(\beta z,\theta)r}(1-o(1)) &\leq C[T(2r,f)]^3 e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)) \\ &\leq Ce^{6\varepsilon r} \cdot e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)). \end{aligned} \tag{2.13}$$

We see that (2.13) is a contradiction when $0 < \varepsilon < \frac{1}{6}n_1\delta(\beta z,\theta)(1-c_1)$. Hence (2.4) has no non-trivial subnormal solution and every solution f satisfies $\sigma_2(f) = 1$.

(2) Suppose $m_2\alpha = c_2n_2\beta\,(0 < c_2 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z,\theta) < 0$, then $\delta(\alpha z,\theta) = \frac{n_2c_2}{m_2}\delta(\beta z,\theta) < 0$. Using the similar method as in the proof of (1), we obtain the conclusion.

2.3.3 Proof of Theorem 2.2.3

Suppose that $f(\not\equiv 0)$ is a solution of (2.4), then f is an entire function. Next we will prove that f is transcendental. Since $Q_1(e^{\beta z}) + Q_2(e^{-\beta z}) \not\equiv 0$, we see that any nonzero constant can not be a solution of the equation (2.4). Now suppose that $f_0 = b_nz^n + \cdots + b_1z + b_0$, $(n \geq 1, b_n, \ldots, b_0$ are constants, $b_n \neq 0)$ is a polynomial solution of (2.4). Take $z = re^{i\theta}$, such that $\delta(\alpha z,\theta) = |\alpha|\cos(\arg\alpha + \theta) > 0$, then $\delta(\beta z,\theta) = \frac{m_1}{c_1n_1}\delta(\alpha z,\theta) > 0$.

From (2.4) and Lemma 1.8.1, that for a sufficiently large r and for $\varepsilon > 0$, we have

$$\begin{aligned} |b_n|\, nr^{n-1}\, |a_{1m_1}|\, e^{m_1\delta(\alpha z,\theta)r}(1-o(1)) &\leq \left|P_1\left(e^{\alpha z}\right)+P_2\left(e^{-\alpha z}\right)\right|\left|f_0'\right| \\ &\leq \left|f_0''\right| + \left|Q_1\left(e^{\beta z}\right)+Q_2\left(e^{-\beta z}\right)\right|\left|f_0\right| \\ &\leq |b_{1n_1}|\, e^{n_1\delta(\beta z,\theta)r} n(n-1)\,|b_n|\, r^n(1+o(1)) \\ &\leq Me^{n_1\frac{m_1}{c_1 n_1}\delta(\alpha z,\theta)r} r^n(1+o(1)) \\ &\leq Me^{\frac{m_1}{c_1}\delta(\alpha z,\theta)r} r^n(1+o(1)) \qquad (2.14) \end{aligned}$$

where $M>0$ is some constant. Since $c_1>1$, we see that (2.14) is a contradiction. Thus we obtain that f is transcendental.

First step. We prove that $\sigma(f)=\infty$. We assume that $\sigma(f)=\sigma<\infty$. By Lemma 1.5.1, we know that for any given $\varepsilon>0$, there exists a set $E\subset[0,2\pi)$ which has linear measure zero, such that if $\psi\in[0,2\pi)\setminus E$, then there is a constant $R_0=R_0(\psi)>1$, such that for all z satisfying $\arg z=\psi$ and $|z|=r\geq R_0$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq r^{\sigma-1+\varepsilon}. \qquad (2.15)$$

Let $H=\{\theta\in[0,2\pi):\delta(\alpha z,\theta)=0\}$. Then H is a finite set. Now we take a ray $\arg z=\theta\in[0,2\pi)\setminus(E\cup H)$, then $\delta(\alpha z,\theta)>0$ or $\delta(\alpha z,\theta)<0$. We divide the proof into the following two cases.

Case 1. If $\delta(\alpha z,\theta)>0$, then $\delta(\beta z,\theta)=\frac{m_1}{c_1n_1}\delta(\alpha z,\theta)>0$, $\delta(-\alpha z,\theta)<0$ and $\delta(-\beta z,\theta)<0$. We assert that $|f'(re^{i\theta})|$ is bounded on the ray $\arg z=\theta$. If $|f'(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 1.8.4, there exists a sequence of points $z_t=r_te^{i\theta}(t=1,2,\dots)$ such that as $r_t\to\infty$, $f'(z_t)\to\infty$ and

$$\left|\frac{f(z_t)}{f'(z_t)}\right| \leq r_t(1+o(1)). \qquad (2.16)$$

By (2.4), we obtain that

$$-\left[P_1(e^{\alpha z_t})+P_2(e^{-\alpha z_t})\right]=\frac{f''(z_t)}{f'(z_t)}+\left[Q_1(e^{\beta z_t})+Q_2(e^{-\beta z_t})\right]\frac{f(z_t)}{f'(z_t)}. \qquad (2.17)$$

From $\delta(\alpha z,\theta)>0$, we have

$$|P_1(e^{\alpha z_t})+P_2(e^{-\alpha z_t})| \geq |a_{1m_1}|e^{m_1\delta(\alpha z_t,\theta)r_t}(1-o(1)), \qquad (2.18)$$

$$|Q_1(e^{\beta z_t})+Q_2(e^{-\beta z_t})| \leq Me^{n_1\delta(\beta z_t,\theta)r_t}(1+o(1)). \qquad (2.19)$$

Substituting (2.15), (2.16), (2.18) and (2.19) in (2.17), we obtain

$$\begin{aligned} |a_{1m_1}|\, e^{m_1\delta(\alpha z_t,\theta)r_t}(1-o(1)) &\leq r_t^{\sigma-1+\varepsilon}+Me^{n_1\delta(\beta z_t,\theta)r_t}r_t(1+o(1)) \\ &\leq Mr_t^{\sigma+\varepsilon}e^{\frac{m_1}{c_1}\delta(\alpha z_t,\theta)r_t}(1+o(1)). \qquad (2.20) \end{aligned}$$

Since $c_1 > 1$, $\delta(\alpha z_t, \theta) > 0$, when $r_t \to \infty$, (2.20) is a contradiction. Hence $|f'(re^{i\theta})| \leq C$. So

$$|f(re^{i\theta})| \leq Cr. \tag{2.21}$$

Case 2. If $\delta(\alpha z, \theta) < 0$, then $\delta(\beta z, \theta) = \frac{m_2}{c_2 n_2}\delta(\alpha z, \theta) < 0$, $\delta(-\alpha z, \theta) > 0$ and $\delta(-\beta z, \theta) > 0$. Using the similar method as above, we can obtain that

$$|f(re^{i\theta})| \leq Cr. \tag{2.22}$$

Since the linear measure of $E \cup H$ is zero, by (2.21), (2.22) and Lemma 1.8.3, we know that $f(z)$ is a polynomial, which contradicts the assumption that $f(z)$ is transcendental. Therefore $\sigma(f) = \infty$.

Second step. We prove that (2.4) has no non-trivial subnormal solution. Now suppose that (2.4) has a non-trivial subnormal solution f_0. By the conclusion in the first step, $\sigma(f_0) = \infty$. By Theorem 1.7.2, we see that $\sigma_2(f_0) \leq 1$. Set $\sigma_2(f_0) = \omega \leq 1$. By Theorem 1.5.2, we see that there exists a subset $E_1 \subset (1, \infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f_0^{(j)}(z)}{f_0(z)}\right| \leq B[T(2r, f_0)]^3, \quad (j = 1, 2). \tag{2.23}$$

From the Wiman-Valiron theory, there is a set $E_2 \subset (1, \infty)$ having finite logarithmic measure, so we can choose z satisfying $|z| = r \notin E_2$ and $|f_0(z)| = M(r, f_0)$. Thus, we have

$$\frac{f_0^{(j)}(z)}{f_0(z)} = \left(\frac{\nu_{f_0}(r)}{z}\right)^j (1 + o(1)), \quad j = 1, 2, \tag{2.24}$$

where $\nu_{f_0}(r)$ is the central index of $f_0(z)$.

By Lemma 1.6.1, we see that there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f_0(z_n)| = M(r_n, f_0)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin [0, 1] \cup E_1 \cup E_2$, $r_n \to \infty$, and if $\omega > 0$, we see that for any given ε_1 $(0 < \varepsilon_1 < \omega)$, and for sufficiently large r_n,

$$\exp\{r_n^{\omega - \varepsilon_1}\} < \nu_{f_0}(r_n) < \exp\{r_n^{\omega + \varepsilon_1}\}, \tag{2.25}$$

and if $\omega = 0$, then by $\sigma(f_0) = \infty$ and Lemma 1.6.1, we see that for any given ε_2 $(0 < \varepsilon_2 < 1/2)$, and for any sufficiently large M, as r_n is sufficiently large,

$$r_n^M < \nu_{f_0}(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{2.26}$$

From (2.25) and (2.26), we obtain that

$$\nu_{f_0}(r_n) > r_n, \quad r_n \to \infty. \tag{2.27}$$

For θ_0, let $\delta = \delta(\alpha z, \theta_0) = |\alpha| \cos(\arg \alpha + \theta_0)$, then $\delta < 0$, or $\delta > 0$, or $\delta = 0$. We divide this proof into three cases.

Case 1. $\delta > 0$. By $\theta_n \to \theta_0$, we see that there is a constant $N(> 0)$ such that, as $n > N$, $\delta(\alpha z_n, \theta_n) > 0$. Since f_0 is a subnormal solution, for any given ε $(0 < \varepsilon < \frac{1}{12}(1 - \frac{1}{c_1})\delta(\alpha z_n, \theta_n))$, we have

$$[T(2r_n, f_0)]^3 \le e^{6\varepsilon r_n} \le e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n, \theta_n) r_n}. \tag{2.28}$$

By (2.23), (2.24), (2.28), we have

$$\begin{aligned} \left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^j (1 + o(1)) &= \left|\frac{f_0^{(j)}(z_n)}{f_0(z_n)}\right| \\ &\le B\left[T(2r_n, f)\right]^3 \\ &\le Be^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n, \theta_n) r_n}, \ j = 1, 2. \end{aligned} \tag{2.29}$$

Since $\delta(\alpha z_n, \theta_n) > 0$, from (2.4), (2.24), we obtain that

$$\begin{aligned} \frac{\nu_{f_0}(r_n)}{r_n} |a_{1m_1}| e^{m_1 \delta(\alpha z_n, \theta_n) r_n} (1 - o(1)) &\le \left|\frac{f_0'(z_n)}{f_0(z_n)} (P_1(e^{\alpha z_n}) + P_2(e^{-\alpha z_n}))\right| \\ &= \left|\frac{f_0''(z_n)}{f_0(z_n)} + \left[Q_1(e^{\beta z_n}) + Q_2(e^{-\beta z_n})\right]\right| \\ &\le \left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^2 (1 + o(1)) + |b_{1n_1}| e^{n_1 \delta(\beta z_n, \theta_n) r_n} (1 + o(1)) \\ &\le M_1 \left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^2 e^{\frac{m_1}{c_1}\delta(\alpha z_n, \theta_n) r_n} (1 + o(1)). \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we can obtain

$$|a_{1m_1}| e^{m_1(1-\frac{1}{c_1})\delta(\alpha z_n, \theta_n) r_n} (1 - o(1)) \le M_1 B e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n, \theta_n) r_n} (1 + o(1)). \tag{2.31}$$

Since $c_1 > 1$ and $m_1 \ge 1$, we see that (2.31) is a contradiction.

Case 2. $\delta < 0$. By $\theta_n \to \theta_0$, we see that there is a constant $N(> 0)$ such that, as $n > N$, $\delta(\alpha z_n, \theta_n) < 0$. Since f_0 is a subnormal solution, for any given ε $(0 < \varepsilon < -\frac{1}{12}(1 - \frac{1}{c_2})\delta(\alpha z_n, \theta_n))$, we have

$$[T(2r_n, f_0)]^3 \le e^{6\varepsilon r_n} \le e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n, \theta_n) r_n}. \tag{2.32}$$

By (2.23), (2.24), (2.32) we have

$$\begin{aligned}\left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^j(1+o(1)) &= \left|\frac{f_0^{(j)}(z_n)}{f_0(z_n)}\right| \\ &\leq B[T(2r_n,f_0)]^3 \\ &\leq Be^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}, \quad j=1,2. \end{aligned} \tag{2.33}$$

By (2.24) and (2.4), we obtain

$$\begin{aligned}\frac{\nu_{f_0}(r_n)}{r_n}|a_{2m_2}|e^{-m_2\delta(\alpha z_n,\theta_n)r_n}(1-o(1)) &\leq \left|\frac{f_0'(z_n)}{f_0(z_n)}\left(P_1(e^{\alpha z_n})+P_2(e^{-\alpha z_n})\right)\right| \\ &= \left|\frac{f_0''(z_n)}{f_0(z_n)}+\left[Q_1(e^{\beta z_n})+Q_2(e^{-\beta z_n})\right]\right| \\ &\leq \left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^2(1+o(1))+|b_{2n_2}|e^{-n_2\delta(\beta z_n,\theta_n)r_n}(1+o(1)) \\ &\leq M_2\left(\frac{\nu_{f_0}(r_n)}{r_n}\right)^2 e^{-\frac{m_2}{c_2}\delta(\alpha z_n,\theta_n)r_n}(1+o(1)). \end{aligned} \tag{2.34}$$

From (2.33) and (2.34), we can deduce that

$$|a_{2m_2}|e^{-m_2(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}(1-o(1)) \leq M_2Be^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}(1+o(1)). \tag{2.35}$$

Since $c_2>1$ and $m_2\geq 1$, we see that (2.35) is a contradiction.

Case 3. $\delta=0$. Then $\theta_0\in H=\{\theta|\theta\in[0,2\pi),\delta(\alpha z,\theta)=0\}$. Since $\theta_n\to\theta_0$, for any given $\varepsilon>0$, we see that there is an integer $N(>0)$, as $n>N$, $\theta_n\in[\theta_0-\varepsilon,\theta_0+\varepsilon]$ and $z_n=r_ne^{i\theta_n}\in\overline{\Omega}=\{z:\theta_0-\varepsilon\leq\arg z\leq\theta_0+\varepsilon\}$. By Theorem 1.5.2, there exists a subset $E_3\subset(1,\infty)$ having finite logarithmic measure and a constant $B>0$, such that for all z satisfying $|z|=r\notin[0,1]\cup E_3$, we have

$$\left|\frac{f_0''(z)}{f_0'(z)}\right|\leq B[T(2r,f_0')]^2. \tag{2.36}$$

Now we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z=\theta\in\overline{\Omega}\setminus\{\theta_0\}$. Denote $\Omega_1=[\theta_0-\varepsilon,\theta_0)$, $\Omega_2=(\theta_0,\theta_0+\varepsilon]$. We can easily see that when $\theta_1\in\Omega_1,\theta_2\in\Omega_2$, then $\delta(\alpha z,\theta_1)\cdot\delta(\alpha z,\theta_2)<0$. Without loss of generality, we suppose that $\delta(\alpha z,\theta)>0\,(\theta\in\Omega_1)$ and $\delta(\alpha z,\theta)<0\,(\theta\in\Omega_2)$.

Since when $\theta\in\Omega_1$, $\delta(\alpha z,\theta)>0$. Recall f_0 is subnormal, then for any given ε $(0<\varepsilon<\frac{1}{8}(1-\frac{1}{c_1})\delta(\alpha z,\theta))$,

$$[T(2r,f_0')]^2\leq e^{4\varepsilon r}\leq e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z,\theta)r}. \tag{2.37}$$

We assert that $|f_0'(re^{i\theta})|$ is bounded on the ray $\arg z=\theta$. If $|f_0'(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 1.8.4, there exists a sequence $\{y_j=R_je^{i\theta}\}$ such that

$R_j \to \infty$, $f_0'(y_j) \to \infty$ and

$$\left|\frac{f_0(y_j)}{f_0'(y_j)}\right| \le R_j(1+o(1)). \tag{2.38}$$

By (2.36) and (2.37), we see that for sufficiently large j,

$$\left|\frac{f_0''(y_j)}{f_0'(y_j)}\right| \le B[T(2R_j, f_0')]^2 \le Be^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha y_j,\theta)R_j}. \tag{2.39}$$

By (2.4), we deduce that

$$\begin{aligned} |a_{1m_1}|e^{m_1\delta(\alpha y_j,\theta)R_j}(1-o(1)) &\le \left|P_1(e^{\alpha y_j}) + P_2(e^{-\alpha y_j})\right| \\ &\le \left|\frac{f_0''(y_j)}{f_0'(y_j)}\right| + \left|Q_1(e^{\beta y_j}) + Q_2(e^{-\beta y_j})\right|\left|\frac{f_0(y_j)}{f_0'(y_j)}\right| \\ &\le C_1 e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha y_j,\theta)R_j} e^{n_1\delta(\beta y_j,\theta)R_j} R_j(1+o(1)) \\ &\le C_1 e^{[\frac{1}{2}(1-\frac{1}{c_1})+\frac{m_1}{c_1}]\delta(\alpha y_j,\theta)R_j} R_j(1+o(1)). \end{aligned} \tag{2.40}$$

Since $\delta(\alpha y_j, \theta) > 0$, $c_1 > 1$, we know that when $R_j \to \infty$, (2.40) is a contradiction. Hence

$$|f_0(re^{i\theta})| \le Cr, \tag{2.41}$$

on the ray $\arg z = \theta \in \Omega_1$.

When $\theta \in \Omega_2$, $\delta(\alpha z, \theta) < 0$. Recall f_0 is subnormal, then for any given ε $(0 < \varepsilon < -\frac{1}{8}(1-\frac{1}{c_2})\delta(\alpha z,\theta))$,

$$[T(2r, f_0')]^2 \le e^{4\varepsilon r} \le e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z,\theta)r}. \tag{2.42}$$

We assert that $|f_0'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f_0'(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, using the similar proof as above, we can obtain that

$$|a_{2m_2}|e^{-m_2(1-\frac{1}{c_2})\delta(\alpha y_j,\theta)R_j}(1-o(1)) \le C_2 e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha y_j,\theta)R_j} R_j(1+o(1)) \tag{2.43}$$

Since $\delta(\alpha y_j, \theta) < 0$ and $c_2 > 1$, we know that when $R_j \to \infty$, (2.43) is a contradiction. Hence

$$|f_0(re^{i\theta})| \le Cr, \tag{2.44}$$

on the ray $\arg z = \theta \in \Omega_2$. By (2.41) and (2.44), we see that $|f_0(re^{i\theta})|$ satisfies

$$|f_0(re^{i\theta})| \le Cr, \tag{2.45}$$

on the ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$. However, since f_0 is transcendental and $\{z_n = r_n e^{i\theta_n}\}$ satisfies $|f_0(z_n)| = M(r_n, f_0)$, we see that for any large $N(> 2)$, as n is sufficiently large,

$$|f_0(z_n)| = |f_0(r_n e^{i\theta_n})| \ge r_n^N. \tag{2.46}$$

Since $z_n \in \overline{\Omega}$, by (2.45) and (2.46), we see that for sufficiently,large n,

$$\theta_n = \theta_0.$$

Thus for sufficiently large n, $\delta(\alpha z_n, \theta_n) = 0$ and

$$|P_1(e^{\alpha z_n}) + P_2(e^{-\alpha z_n})| \leq C, \quad |Q_1(e^{\beta z_n}) + Q_2(e^{-\beta z_n})| \leq C. \tag{2.47}$$

By (2.4) and (2.24), we obtain that

$$\begin{aligned} & -\left(\frac{\nu_{f_0}(r_n)}{z_n}\right)^2 (1+o(1)) \\ & = \left(P_1(e^{\alpha z_n}) + P_2(e^{-\alpha z_n})\right)\left(\frac{\nu_{f_0}(r_n)}{z_n}\right)(1+o(1)) + [Q_1(e^{\beta z_n}) + Q_2(e^{-\beta z_n})]. \end{aligned} \tag{2.48}$$

By (2.47), (2.48) and (2.27) we obtain that

$$\nu_{f_0}(r_n) \leq 2Cr_n, \tag{2.49}$$

by (2.25) (or (2.26)), we see that (2.49) is a contradiction. Hence (2.4) has no non-trivial subnormal solution.

Third step. We prove that all solutions of (2.4) satisfies $\sigma_2(f) = 1$. If there is a solution f_1 satisfying $\sigma_2(f_1) < 1$, then $\sigma_e(f_1) = 0$, that is to say f_1 is subnormal, but this contradicts the conclusion in Step 2. Hence $\sigma_2(f) = 1$. This completes the proof of Theorem 2.2.3.

Chapter 3

Nonexistence of subnormal solutions for a class of higher order complex differential equations

In this chapter, we investigate the existence of subnormal solutions for a class of higher order complex differential equations. We generalize the result of N. Li and L. Z. Yang [22], L. P. Xiao [27] and also result of Z. X. Chen and K. H. Shon [3].

3.1 Introduction

In [11], Gundersen and Steinbart considered the higher order non-homogeneous linear differential equation

$$f^{(k)} + P_{k-1}(e^z) f^{(k-1)} + \cdots + P_0(e^z) f = Q_1(e^z) + Q_2\left(e^{-z}\right), \tag{3.1}$$

where $Q_1(z), Q_2(z), P_0(z), \ldots, P_{k-1}(z)$ are polynomials in z. They obtained the following Theorem.

Theorem 3.1.1 ([11]) *Suppose in equation (3.1) we have $k \geq 2$ and*

$$degP_0 > degP_j \tag{3.2}$$

for all $1 \leq j \leq k-1$. Then any subnormal solution f of (3.1) must have the form

$$f(z) = S_1(e^z) + S_2(e^{-z}),$$

where $S_1(z)$ and $S_2(z)$ are polynomials in z.

From the proof of Theorem 3.1.1, we see that the condition (3.2) in Theorem 3.1.1 guarantees that the corresponding homogeneous differential equation

$$f^{(k)} + P_{k-1}\left(e^{z}\right) f^{(k-1)} + \cdots + P_{0}\left(e^{z}\right) f = 0 \tag{3.3}$$

of (3.1) has no nontrivial subnormal solutions.

In [6], Chen and Shon investigated the existence and estimated the number of nontrivial subnormal solutions of the equation (3.3).

Chen and Shon in [4] and Liu and Yang in [23] improved the Theorems 2.1.3, 2.1.4 to higher periodic differential equation

$$f^{(k)} + \left[P_{k-1}\left(e^{z}\right) + Q_{k-1}\left(e^{-z}\right)\right] f^{(k-1)} + \cdots + \left[P_{0}\left(e^{z}\right) + Q_{0}\left(e^{-z}\right)\right] f = 0 \tag{3.4}$$

and they proved the following results.

Theorem 3.1.2 ([23, 4]) *Let $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ be polynomials in z with* $\deg P_j = m_j$, $\deg Q_j = n_j$*. If P_0 satisfies*

$$m_0 > \max\{m_j : 1 \leq j \leq k-1\} = m$$

or Q_0 satisfies

$$n_0 > \max\{n_j : 1 \leq j \leq k-1\} = n,$$

then (3.4) has no nontrivial subnormal solution, and every solution of (3.4) is of hyper-order $\sigma_2(f) = 1$.

Theorem 3.1.3 ([4]) *Let $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ be polynomials in z with* $\deg P_j = m_j$, $\deg Q_j = n_j$*, and $P_0 + Q_0 \not\equiv 0$. If there exists m_s, n_d $(s, d \in \{0, \ldots, k-1\})$ satisfying both the inequalities*

$$\begin{aligned} m_s &> \max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\} = m, \\ n_d &> \max\{n_j : j = 0, \ldots, d-1, d+1, \ldots, k-1\} = n, \end{aligned}$$

then (3.4) has no nontrivial subnormal solution, and every solution of (3.4) is of hyper-order $\sigma_2(f) = 1$.

3.2 Main results

Firstly, we want to answer to the question,

Question. Can Theorems 2.2.2, 2.2.3 be generalized to higher order differential equation ? And we will prove the following results.

Let

$$f^{(k)}+\left[P_{k-1}\left(e^{\alpha_{k-1}z}\right)+Q_{k-1}\left(e^{-\alpha_{k-1}z}\right)\right]f^{(k-1)}+\cdots+\left[P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\right]f=0, \quad (3.5)$$

where

$$\begin{aligned} P_j(z) &= a_{jm_j}z^{m_j}+a_{j(m_j-1)}z^{m_j-1}+\cdots+a_{j0},\ j=0,\ldots,k-1,\\ Q_j(z) &= b_{jn_j}z^{n_j}+b_{j(n_j-1)}z^{n_j-1}+\cdots+b_{j0},\ j=0,\ldots,k-1 \end{aligned}$$

and $m_j \geq 1, n_j \geq 1$ $(j=0,\ldots,k-1; k\geq 2)$ are integers, $a_{ju}\neq 0, b_{jv}\neq 0$ and $\alpha_j \neq 0$ $(j=0,\ldots,k-1;\ u=0,\ldots,m_j;\ v=0,\ldots,n_j)$ are complex constants.

Theorem 3.2.1 **([29])** *Suppose that*

$$c_j m_0 \alpha_0 = m_j \alpha_j,\ 0<c_j<1, \forall j=1,\ldots,k-1$$

or

$$d_j n_0 \alpha_0 = n_j \alpha_j,\ 0<d_j<1, \forall j=1,\ldots,k-1,$$

then the equation (3.5) has no nontrivial subnormal solution, and every solution of (3.5) satisfies $\sigma_2(f)=1$.

Theorem 3.2.2 **([29])** *If* $P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\not\equiv 0$, *and if there exists* $s,t\in\{0,\ldots,k-1\}$ *such that*

$$\begin{cases} m_s\alpha_s = c_j m_j \alpha_j,\ c_j>1,\ j=0,\ldots,s-1,s+1,\ldots,k-1,\\ n_t\alpha_t = d_j n_j \alpha_j,\ d_j>1,\ j=0,\ldots,t-1,t+1,\ldots,k-1, \end{cases}$$

then the equation (3.5) has no nontrivial subnormal solution, and every solution of (3.5) satisfies $\sigma_2(f)=1$.

As a generalization to higher order equation of Theorem 1.5 and Theorem 1.6 in [27], we have the following results.

Theorem 3.2.3 **([29])** *Let*

$$P_j\left(e^{\alpha_j z}\right)=a_{jm_j}e^{m_j\alpha_j z}+a_{j(m_j-1)}e^{(m_j-1)\alpha_j z}+\cdots+a_{j0},\ j=0,\ldots,k-1,$$

where $m_j \geq 1$ $(j = 0, \ldots, k-1; k \geq 2)$ *are integers,* $a_{ju} \neq 0$ *and* $\alpha_j \neq 0$ $(j = 0, \ldots, k-1;$ $u = 0, \ldots, m_j)$ *are complex constants. Suppose that* $c_j m_0 \alpha_0 = m_j \alpha_j$, $0 < c_j < 1$, $\forall j = 1, \ldots, k-1$. *Then the equation*

$$f^{(k)} + P_{k-1}\left(e^{\alpha_{k-1} z}\right) f^{(k-1)} + \cdots + P_0\left(e^{\alpha_0 z}\right) f = 0 \quad (3.6)$$

has no nontrivial subnormal solution, and every solution satisfies $\sigma_2(f) = 1$.

Theorem 3.2.4 ([29]) *Let*

$$P_j^*\left(e^{\alpha_j z}\right) = a_{jm_j} e^{m_j \alpha_j z} + a_{j(m_j-1)} e^{(m_j-1)\alpha_j z} + \cdots + a_{j1} e^{\alpha_j z}, \quad j = 0, \ldots, k-1,$$

where $m_j \geq 1$ $(j = 1, \ldots, k-1; k \geq 2)$ *are integers,* $a_{ju} \neq 0$ *and* $\alpha_j \neq 0$ $(j = 0, \ldots, k-1;$ $u = 0, \ldots, m_j)$ *are complex constants. Suppose that* $P_0\left(e^{\alpha_0 z}\right) \not\equiv 0$ *and there exists* $s \in \{1, \ldots, k-1\}$ *such that* $c_j m_s \alpha_s = m_j \alpha_j$, $0 < c_j < 1, \forall j = 0, \ldots, s-1, s+1, \ldots, k-1$. *Then the equation*

$$f^{(k)} + P_{k-1}^*\left(e^{\alpha_{k-1} z}\right) f^{(k-1)} + \cdots + P_0^*\left(e^{\alpha_0 z}\right) f = 0 \quad (3.7)$$

has no nontrivial subnormal solution, and every solution satisfies $\sigma_2(f) = 1$.

In [23], Liu-Yang gave an example that shows that in Theorem 3.1.2, if there exists $\deg P_i = \deg P_j$ and $\deg Q_i = \deg Q_j$ $(i \neq j)$, then the equation (3.4) may have a nontrivial subnormal solution.

Example ([23, page 610]). A subnormal solution $f = e^{-z}$ satisfies the following equation

$$f^{(n)} + f^{(n-1)} + \cdots + f'' + \left(e^{2z} + e^{-2z}\right) f' + \left(e^{2z} + e^{-2z}\right) f = 0,$$

where n is an odd number.

Question. What can we say when $\deg P_0 = \deg P_1$ and $\deg Q_0 = \deg Q_1$ in the equation (3.4)? We have the following result.

Theorem 3.2.5 ([29]) *Let* $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ *be polynomials in* z *with* $\deg P_0 = \deg P_1 = m, \deg Q_0 = \deg Q_1 = n$, $\deg P_j = m_j$, $\deg Q_j = n_j$ $(j = 2, \ldots, k-1)$, *let*

$$\begin{aligned} P_1(z) &= a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \\ P_0(z) &= b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0, \\ Q_1(z) &= c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0, \\ Q_0(z) &= d_n z^n + d_{n-1} z^{n-1} + \cdots + d_0, \end{aligned}$$

where a_u, b_u, c_v, d_v *(*$u = 0, \ldots, m; v = 0, \ldots, n$*) are complex constants,* $a_m b_m c_n d_n \neq 0$. *If* $a_m d_n = b_m c_n$, $m > \max\{m_j : j = 2, \ldots, k-1\}$, $n > \max\{n_j : j = 2, \ldots, k-1\}$ *and* $e^{-(b_m/a_m)z}$ *is not a solution of* (3.4)*, then the equation* (3.4) *has no nontrivial subnormal solution, and every solution* f *of* (3.4) *satisfies* $\sigma_2(f) = 1$.

Example. This example shows that the Theorem 3.2.5, is not a particular case (and is different) of Theorems 3.1.2, 3.1.3. Consider the differential equation

$$f''' + \left(e^z + e^{-z}\right) f'' + \left(e^{3z} - e^{-2z}\right) f' + \left(-2e^{3z} + 2e^{-2z}\right) f = 0.$$

By Theorems 3.1.2, 3.1.3, we can't say anything about the existence or nonexistence of nontrivial subnormal solutions, because neither hypotheses of Theorem 3.1.2 nor of Theorem 3.1.3 are satisfied. But, we can see that all hypotheses of Theorem 3.2.5 are satisfied, then, we guarantee that the above equation has no nontrivial subnormal solution. In fact, we have $k = 3$, $P_2(e^z) = e^z, Q_2(e^z) = e^{-z}, P_1(e^z) = e^{3z}, Q_1(e^z) = -e^{-2z}, P_0(e^z) = -2e^{3z}$ and $Q_0(e^z) = 2e^{-2z}$. $m = 3$, $n = 2$, $m > 1 = \deg P_2$, $n > 1 = \deg Q_2$. $a_m = 1, b_m = -2, c_n = -1$ and $d_n = 2$, and we have $a_m d_n = b_m c_n$. It's clear that $e^{-(b_m/a_m)z} = e^{2z}$ is not a solution of the equation above.

Remark 1. In Theorem 3.2.5, if the equation (3.4) accepts $e^{-(b_m/a_m)z}$ as a solution, then (3.4) has a subnormal solution. But, if $e^{-(b_m/a_m)z}$ doesn't satisfy (3.4), is there another subnormal solution may satisfy (3.4)? The conditions of Theorem 3.2.5 guarantee that, if (3.4) doesn't accept $e^{-(b_m/a_m)z}$ as a subnormal solution, then (3.4) doesn't accept any other subnormal solution.

Remark 2. In Theorem 3.2.5, we can replace the condition " $e^{-(b_m/a_m)z}$ is not a solution of (3.4) " by many partial conditions. For example

1. $P_j(0) + Q_j(0) = 0$, $(j = 0, \ldots, k-1)$.
2. $P_j(0) + Q_j(0) = 1$, $(j = 0, \ldots, k-1)$ and $a_m \neq b_m$.
3. $P_j(0) + Q_j(0) = 1$, $(j = 0, \ldots, k-1)$, $a_m = b_m$ and k is even number.
4. $P_j(0) + Q_j(0) = 0, P_l(0) + Q_l(0) = 1$ $(j = 0, \ldots, s;\ l = s+1, \ldots, k-1)$, $a_m = b_m$ and s, k are both even or both odd. And so on.

Remark 3. In Theorem 2.2.1, the hypotheses (1-3) can be replaced by the condition "$e^{-(b_n/a_n)z}$ is not a solution of (2.2)".

3.3 Proofs of main results

3.3.1 Proof of Theorem 3.2.1

(1) Suppose that f is a nontrivial solution of (3.5), then f is an entire function. Since $P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\not\equiv 0$, then every nonzero constant is not a solution of (3.5). Now, suppose that $f_0=a_n z^n+\cdots+a_0$ ($n\geq 1$; $a_0,\ldots,a_n$ are constants, $a_n\neq 0$) is a polynomial solution of (3.5). If $c_j m_0\alpha_0=m_j\alpha_j$, $(0<c_j<1, \forall j=1,\ldots,k-1)$, then we choose $z=re^{i\theta}$, such that $\delta(\alpha_0 z,\theta)=|\alpha_0|\cos(\arg\alpha_0+\theta)>0$, then $\delta(\alpha_j z,\theta)=\dfrac{c_j}{m_j}m_0\delta(\alpha_0 z,\theta)>0$, $(\forall j=1,\ldots,k-1)$. By Lemma 1.8.1 and (3.5) for a sufficiently large r, we have

$$\begin{aligned}|a_n|\,|a_{0m_0}|\,e^{m_0\delta(\alpha_0 z,\theta)r}r^n(1+o(1)) &= \left|P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\right||f_0| \\ &\leq \left|f_0^{(k)}\right|+\sum_{j=1}^{k-1}\left|P_j\left(e^{\alpha_j z}\right)+Q_j\left(e^{-\alpha_j z}\right)\right|\left|f_0^{(j)}\right| \\ &\leq Me^{cm_0\delta(\alpha_0 z,\theta)r}r^n(1+o(1)),\end{aligned}$$

where $0<c=\max\{c_j: j=1,\ldots,k-1\}<1$. This is a contradiction. Then (3.5) has no nonzero polynomial solution. If $d_j n_0\alpha_0=n_j\alpha_j$, $(0<d_j<1,\forall j=1,\ldots,k-1)$, then we choose $z=re^{i\theta}$, such that $\delta(\alpha_0 z,\theta)=|\alpha_0|\cos(\arg\alpha_0+\theta)<0$, then $\delta(\alpha_j z,\theta)=\dfrac{d_j}{n_j}n_0\delta(\alpha_0 z,\theta)<0$, $(\forall j=1,\ldots,k-1)$. Using the similar method as in the case $\delta(\alpha_0 z,\theta)>0$, we obtain

$$|a_n|\,|b_{0n_0}|\,e^{-n_0\delta(\alpha_0 z,\theta)r}r^n(1+o(1))\leq Me^{-dn_0\delta(\alpha_0 z,\theta)r}r^n(1+o(1)),$$

where $0<d=\max\{d_j: j=1,\ldots,k-1\}<1$. This is a contradiction. So, (3.5) has no nonzero polynomial solution.

(2) By Theorem 1.5.2 we can see that there exists a set $E\subset(1,\infty)$ with finite logarithmic measure and there is a constant $B>0$ such that for all z satisfying $|z|=r\notin E\cup[0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq B\left[T(2r,f)\right]^{k+1},\ j=1,\ldots,k. \tag{3.8}$$

Suppose that $f\not\equiv 0$ is a subnormal solution, then $\sigma_e(f)=0$. Hence, for all $\varepsilon>0$, and for sufficiently large r, we have

$$T(r,f)<e^{\varepsilon r}. \tag{3.9}$$

Substituting (3.9) into (3.8) with sufficiently large $|z|=r\notin E\cup[0,1]$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq Be^{2\varepsilon(j+1)r}\leq Be^{2\varepsilon(k+1)r},\ j=1,\ldots,k. \tag{3.10}$$

(i) Suppose that $c_j m_0 \alpha_0 = m_j \alpha_j$, $(0 < c_j < 1, \forall j = 1, \ldots, k-1)$. Take $z = re^{i\theta}$ such that $r \notin E \cup [0,1]$ and $\delta(\alpha_0 z, \theta) = |\alpha_0| \cos(\arg \alpha_0 + \theta) > 0$, then $\delta(\alpha_j z, \theta) = \dfrac{c_j}{m_j} m_0 \delta(\alpha_0 z, \theta) > 0, (\forall j = 1, \ldots, k-1)$. Therefore

$$\left|P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right| = \left|a_{0m_0}\right| e^{m_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)), \tag{3.11}$$

$$\begin{aligned} \left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right| &= \left|a_{jm_j}\right| e^{m_j \delta(\alpha_j z, \theta) r}(1 + o(1)) \\ &= \left|a_{jm_j}\right| e^{c_j m_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)) \\ &\leq D e^{c m_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)), \end{aligned} \tag{3.12}$$

where $D = \max\limits_{1 \leq j \leq k-1}\{\left|a_{jm_j}\right|\}$ and $0 < c = \max\limits_{1 \leq j \leq k-1}\{|c_j|\} < 1$. Substituting (3.10), (3.11) and (3.12) into (3.5), we obtain

$$\begin{aligned} \left|a_{0m_0}\right| e^{m_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)) &= \left|P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right| \\ &\leq \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right|\left|\frac{f^{(j)}}{f}\right| \\ &\leq B e^{2\varepsilon(k+1) r} + (k-1) D B e^{c m_0 \delta(\alpha_0 z, \theta) r} e^{2\varepsilon(k+1) r}(1 + o(1)). \end{aligned}$$

Hence,

$$\left|a_{0m_0}\right| e^{m_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)) \leq M e^{[c m_0 \delta(\alpha_0 z, \theta) + 2\varepsilon(k+1)] r}(1 + o(1)) \tag{3.13}$$

for some constant $M > 0$. Since $0 < c < 1$, we can see that (3.13) is a contradiction when

$$0 < \varepsilon < \frac{1-c}{2(k+1)} m_0 \delta(\alpha_0 z, \theta).$$

Hence, the equation (3.5) has no nontrivial subnormal solution.

(ii) Suppose that $d_j n_0 \alpha_0 = n_j \alpha_j$, $(0 < d_j < 1, \forall j = 1, \ldots, k-1)$. We choose $z = re^{i\theta}$, such that $r \notin E \cup [0,1]$ and $\delta(\alpha_0 z, \theta) = |\alpha_0| \cos(\arg \alpha_0 + \theta) < 0$, then $\delta(\alpha_j z, \theta) = \dfrac{d_j}{n_j} n_0 \delta(\alpha_0 z, \theta) < 0$, $(\forall j = 1, \ldots, k-1)$. Using the similar method as in the proof of (i) above, we obtain

$$\left|b_{0n_0}\right| e^{-n_0 \delta(\alpha_0 z, \theta) r}(1 + o(1)) \leq M e^{[-d n_0 \delta(\alpha_0 z, \theta) + 2\varepsilon(k+1)] r}(1 + o(1)), \tag{3.14}$$

where $0 < d = \max\limits_{1 \leq j \leq k-1}\{|d_j|\} < 1$, and for some constant $M > 0$. We see that (3.14) is a contradiction when

$$0 < \varepsilon < -\frac{1-d}{2(k+1)} n_0 \delta(\alpha_0 z, \theta).$$

Hence, (3.5) has no nontrivial subnormal solution.

(3) By Theorem 1.7.3, every solution f of (3.5) satisfies $\sigma_2(f) \leq 1$. Suppose that $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$, i.e., f is subnormal solution and this contradicts the conclusion above. So $\sigma_2(f) = 1$.

3.3.2 Proof of Theorem 3.2.2

Suppose that $f \not\equiv 0$ is a solution of equation (3.5), then f is an entire function. Since $P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right) \not\equiv 0$, then f cannot be nonzero constant.

(1) We will prove that f is a transcendental function. We assume that f is polynomial solution to (3.5), and we set

$$f(z) = a_n z^n + \cdots + a_0,$$

where $n \geq 1$, $a_0, \ldots, a_n$ are constants with $a_n \neq 0$. Suppose that $s \leq t$. If $n \geq s$, then $f^{(s)} \not\equiv 0$, and we have

$$\left|f^{(s)}(z)\right| = A_n^s \left|a_n\right| r^{n-s}(1+o(1)) , \quad |z| = r \to \infty.$$

Take $z = re^{i\theta}$ such that $\delta(\alpha_s z, \theta) > 0$, therefore

$$\left|P_s(e^{\alpha_s z}) + Q_s(e^{-\alpha_s z})\right| = \left|a_{sm_s}\right| e^{m_s \delta(\alpha_s z, \theta) r}(1+o(1))$$

and we obtain

$$\left|P_s(e^{\alpha_s z}) + Q_s(e^{-\alpha_s z})\right| \left|f^{(s)}(z)\right| = A_n^s \left|a_n\right| \left|a_{sm_s}\right| e^{m_s \delta(\alpha_s z, \theta) r} r^{n-s}(1+o(1)). \tag{3.15}$$

From $m_s \alpha_s = c_j m_j \alpha_j$, we obtain $m_j \delta(\alpha_j z, \theta) = \frac{m_s}{c_j} \delta(\alpha_s z, \theta) > 0$, $(c_j > 1, j = 0, \ldots, s-1, s+1, \ldots, k-1)$. Hence,

$$\begin{aligned} \left|P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha_j z})\right| &= \left|a_{jm_j}\right| e^{m_j \delta(\alpha_j z, \theta) r}(1+o(1)) \\ &= \left|a_{jm_j}\right| e^{\frac{m_s}{c_j} \delta(\alpha_s z, \theta) r}(1+o(1)) \end{aligned}$$

and we have

$$\left|f^{(j)}(z)\right| = \begin{cases} A_n^j \left|a_n\right| r^{n-j}(1+o(1)), & \text{if } j \leq n \\ 0, & \text{if } j > n. \end{cases}$$

Then, we obtain for $j = 0, \ldots, s-1, s+1, \ldots, k-1$

$$\begin{aligned} \left|P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha_j z})\right| \left|f^{(j)}(z)\right| &= A_n^j \left|a_n\right| \left|a_{jm_j}\right| e^{\frac{m_s}{c_j} \delta(\alpha_s z, \theta) r} r^{n-j}(1+o(1)) \\ &\leq De^{Cm_s \delta(\alpha_s z, \theta) r} r^n (1+o(1)), \end{aligned} \tag{3.16}$$

where $D = \max\limits_j \{A_n^j \left|a_n\right| \left|a_{jm_j}\right|\}$ and $0 < C = \max\limits_j \{\frac{1}{c_j}\} < 1$. Substituting (3.15) and (3.16) into (3.5), we obtain

$$\begin{aligned} A_n^s \left|a_n\right| \left|a_{sm_s}\right| e^{m_s \delta(\alpha_s z, \theta) r} r^{n-s}(1+o(1)) &= \left|P_s(e^{\alpha_s z}) + Q_s(e^{-\alpha_s z})\right| \left|f^{(s)}(z)\right| \\ &\leq \left|f^{(k)}(z)\right| + \sum_{j=0, j\neq s}^{k-1} \left|P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha_j z})\right| \left|f^{(j)}(z)\right| \\ &\leq Me^{Cm_s \delta(\alpha_s z, \theta) r} r^n (1+o(1)), \end{aligned} \tag{3.17}$$

where M some constant. Since, $0 < C < 1$ and $\delta(\alpha_s z, \theta) > 0$, we can see that (3.17) is a contradiction. If $n < s$, then $f^{(s)} \equiv 0$. Since $P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right) \not\equiv 0$, we can rewrite (3.5) as

$$f(z) = -\sum_{j=1}^{n} \frac{P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha_j z})}{P_0(e^{\alpha_0 z}) + Q_0(e^{-\alpha_0 z})} f^{(j)}(z) \tag{3.18}$$

and we obtain a contradiction since the left side of equation (3.18) is a polynomial function but the right side is a transcendental meromorphic function. Hence, every solution of (3.5) is transcendental function.

(2) Now, we will prove that every solution f of (3.5) satisfies $\sigma(f) = +\infty$. We assume that $\sigma(f) = \sigma < +\infty$. By Lemma 1.5.1, we know that for any given $\varepsilon > 0$ there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, and for each $\psi \in [0, 2\pi)\backslash E$, there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$, and $|z| = r \geq R_0$, we have for $l \leq k-1$

$$\left|\frac{f^{(j)}(z)}{f^{(l)}(z)}\right| \leq |z|^{(\sigma-1+\varepsilon)(j-l)}; \quad j = l+1, \ldots, k. \tag{3.19}$$

Let $H = \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$, H is a finite set. By the hypotheses of Theorem 3.2.2, we have $H = \{\theta \in [0, 2\pi) : \delta(\alpha_j z, \theta) = 0\}$ $(j = 0, \ldots, k-1)$. We take $z = re^{i\theta}$, such that $\theta \in [0, 2\pi)\backslash E \cup H$. Then, $\delta(\alpha_s z, \theta) > 0$ or $\delta(\alpha_s z, \theta) < 0$. If $\delta(\alpha_s z, \theta) > 0$, then $\delta(\alpha_j z, \theta) > 0$ for all $j = 0, \ldots, s-1, s+1, \ldots, k-1$. We assert that $\left|f^{(s)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(s)}(z)\right|$ is unbounded, then by Lemma 1.8.4, there exists an infinite sequence of points $z_u = r_u e^{i\theta}$ $(u = 1, 2, \ldots)$ where $r_u \to +\infty$ such that $f^{(s)}(z_u) \to \infty$ and

$$\left|\frac{f^{(j)}(z_u)}{f^{(s)}(z_u)}\right| \leq \frac{1}{(s-j)!}|z_u|^{s-j}(1+o(1)), \quad (j = 0, \ldots, s-1). \tag{3.20}$$

By (3.5) we obtain

$$\begin{aligned}
|a_{sm_s}| e^{m_s \delta(\alpha_s z_u, \theta) r_u}(1+o(1)) &= \left|P_s(e^{\alpha_s z_u}) + Q_s(e^{-\alpha_s z_u})\right| \\
&\leq \left|\frac{f^{(k)}(z_u)}{f^{(s)}(z_u)}\right| + \sum_{j=0, j\neq s}^{k-1} \left|P_j(e^{\alpha_j z_u}) + Q_j(e^{-\alpha_j z_u})\right| \left|\frac{f^{(j)}(z_u)}{f^{(s)}(z_u)}\right| \\
&\leq r_u^{(\sigma-1+\varepsilon)(k-s)} + \sum_{j>s} |a_{jm_j}| e^{m_j \delta(\alpha_j z_u, \theta) r_u} r_u^{(\sigma-1+\varepsilon)(j-s)} \\
&\quad + \sum_{j<s} \frac{1}{(s-j)!} |a_{jm_j}| e^{m_j \delta(\alpha_j z_u, \theta) r_u} r_u^{s-j}(1+o(1)) \\
&\leq M e^{C m_s \delta(\alpha_s z_u, \theta) r_u} r_u^{(\sigma-1+\varepsilon)(k-s)+s}(1+o(1)),
\end{aligned} \tag{3.21}$$

for some $M > 0$. Since $0 < C = \max_j\{\frac{1}{c_j}\} < 1$, and $\delta(\alpha_s z_u, \theta) > 0$, then (3.21) is a contradiction when $r_u \to +\infty$. Hence, $\left|f^{(s)}(z)\right|$ is bounded on the ray $\arg z = \theta$. Therefore,

$$\left|f(re^{i\theta})\right| \le C_1 r^s. \tag{3.22}$$

If $\delta(\alpha_s z, \theta) < 0$, then $\delta(\alpha_j z, \theta) < 0$ for all $j = 0, \dots, s-1, s+1, \dots, k-1$, in particular $\delta(\alpha_t z, \theta) < 0$, i.e., $-n_t\delta(\alpha_t z, \theta) > 0$. We assert that $\left|f^{(t)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(t)}(z)\right|$ is unbounded then by Lemma 1.8.4, there exists an infinite sequence of points $z_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(t)}(z_u) \to \infty$ and

$$\left|\frac{f^{(j)}(z_u)}{f^{(t)}(z_u)}\right| \le \frac{1}{(t-j)!}|z_u|^{t-j}(1+o(1)),\ (j = 0, \dots, t-1).$$

We obtain

$$|b_{tm_t}|\, e^{-n_t\delta(\alpha_t z_u, \theta) r_u}(1+o(1)) \le M e^{-Dn_t\delta(\alpha_t z_u,\theta) r_u} r_u^{(\sigma-1+\varepsilon)(k-t)+t}(1+o(1)) \tag{3.23}$$

for some constant $M > 0$. Since $0 < D = \max\limits_j\{\frac{1}{d_j}\} < 1$ and $-n_t\delta(\alpha_t z, \theta) > 0$, we see that (3.23) is a contradiction when $r_u \to +\infty$. Thus

$$\left|f(re^{i\theta})\right| \le C_2 r^t. \tag{3.24}$$

Since the linear measure of $E \cup H$ is zero, by (3.22), (3.24) and Lemma 1.8.3, we conclude that f is polynomial, which contradicts the fact that f is transcendental. Therefore $\sigma(f) = +\infty$.

(3) Finally, we will prove that (3.5) has no non trivial subnormal solution. Suppose that (3.5) has a subnormal solution f. So, $\sigma(f) = \infty$, and by Theorem 1.7.3, we see that $\sigma_2(f) \le 1$. Set $\sigma_2(f) = \mu \le 1$. By Theorem 1.5.2, there exists a set $E_1 \subset (1, \infty)$ having a finite logarithmic measure, and there is a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B\left[T(2r, f)\right]^{k+1},\quad j = 1, \dots, k. \tag{3.25}$$

From Wiman-Valiron theory, there is a set $E_2 \subset (1, \infty)$ having finite logarithmic measure, so we can choose z satisfying $|z| = r \notin E_2$ and $|f(z)| = M(r, f)$. Thus, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1+o(1)),\quad j = 1, \dots, k. \tag{3.26}$$

By Lemma 1.6.1, we can see that there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim\limits_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin [0, 1] \cup E_1 \cup E_2$, $r_n \to \infty$, and such that

1. if $\mu > 0$, then for any given ε_1 $(0 < \varepsilon_1 < \mu)$,

$$\exp\{r_n^{\mu-\varepsilon_1}\} < \nu_f(r_n) < \exp\{r_n^{\mu+\varepsilon_1}\} \tag{3.27}$$

2. if $\mu = 0$, and since $\sigma(f) = \infty$, then for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M > 0$, we have as r_n sufficiently large

$$r_n^M < \nu_f(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{3.28}$$

From (3.27) and (3.28), we obtain that

$$\nu_f(r_n) > r_n, \quad r_n \to \infty. \tag{3.29}$$

Since θ_0 may belong to $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) > 0\}$, or $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) < 0\}$, or $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$, we divide the proof into three cases.

Case 1. $\theta_0 \in \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) > 0\}$. By, $\theta_n \to \theta_0$, there exists $N > 0$ such that, as $n > N$, we have $\delta(\alpha_s z_n, \theta_n) > 0$. Since f is subnormal, then for any given $\varepsilon > 0$, we have

$$T(r, f) \le e^{\varepsilon r}. \tag{3.30}$$

By (3.25), (3.26) and (3.30), we obtain

$$\left(\frac{\nu_f(r_n)}{r_n}\right)^j (1 + o(1)) = \left|\frac{f^{(j)}(z_n)}{f(z_n)}\right| \le B\left[T(2r_n, f)\right]^{k+1} \le e^{2(k+1)\varepsilon r}, \quad j = 1, \dots, k. \tag{3.31}$$

Because $\delta(\alpha_s z_n, \theta_n) > 0$, then $\delta(\alpha_j z_n, \theta_n) > 0$ $(j = 0, \dots, s-1, s+1, \dots, k-1)$, and we have

$$\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right| = |a_{sm_s}|\, e^{m_s \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)) \tag{3.32}$$

and

$$\begin{aligned} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| &= \left|a_{jm_j}\right| e^{m_j \delta(\alpha_j z_n, \theta_n) r_n}(1 + o(1)) \\ &= \left|a_{jm_j}\right| e^{\frac{m_s}{c_j} \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)) \\ &\le M e^{C m_s \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)), \quad j \ne s, \end{aligned} \tag{3.33}$$

where $M = \max\limits_j\{\left|a_{jm_j}\right|\}$ and $0 < C = \max\limits_j\{\frac{1}{c_j}\} < 1$. By (3.29), (3.31), (3.32), (3.33) and (3.5), we obtain

$$|a_{sm_s}|\, e^{m_s \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)) \le kMB e^{C m_s \delta(\alpha_s z_n, \theta_n) r_n} e^{2(k+1)\varepsilon r_n}(1 + o(1)). \tag{3.34}$$

Since $0 < C < 1$ and $\delta(\alpha_s z_n, \theta_n) > 0$ we can see that (3.34) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < \frac{1-C}{2(k+1)} m_s \delta(\alpha_s z_n, \theta_n).$$

Case 2. $\theta_0 \in \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) < 0\}$. By, $\theta_n \to \theta_0$, there exists $N > 0$ such that, as $n > N$, we have $\delta(\alpha_s z_n, \theta_n) < 0$, then $\delta(\alpha_j z_n, \theta_n) > 0$ $(j = 0, \dots, s-1, s+1, \dots, k-1)$. In particular $\delta(\alpha_t z_n, \theta_n) < 0$, i.e., $-n_t \delta(\alpha_t z_n, \theta_n) > 0$. We have

$$\left|P_t(e^{\alpha_t z_n}) + Q_t(e^{-\alpha_t z_n})\right| = |b_{tn_t}|\, e^{-n_t \delta(\alpha_t z_n, \theta_n) r_n} (1 + o(1)) \tag{3.35}$$

and

$$\begin{aligned} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| &= \left|b_{jn_j}\right| e^{n_j \delta(\alpha_j z_n, \theta_n) r_n} (1 + o(1)) \\ &= \left|b_{jn_j}\right| e^{\frac{n_t}{d_j} \delta(\alpha_t z_n, \theta_n) r_n} (1 + o(1)) \\ &\leq M e^{D n_t \delta(\alpha_t z_n, \theta_n) r_n} (1 + o(1)), \quad j \neq t, \end{aligned} \tag{3.36}$$

where $M = \max_j \{\left|b_{jn_j}\right|\}$ and $0 < D = \max_j \{\frac{1}{d_j}\} < 1$. By (3.29), (3.31), (3.35), (3.36) and (3.5), we obtain

$$|b_{tn_t}|\, e^{-n_t \delta(\alpha_t z_n, \theta_n) r_n} (1 + o(1)) \leq kMB e^{-D n_t \delta(\alpha_t z_n, \theta_n) r_n} e^{2(k+1)\varepsilon r_n} (1 + o(1)). \tag{3.37}$$

Since $0 < D < 1$ and $-n_t \delta(\alpha_t z_n, \theta_n) > 0$ we can see that (3.37) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < -\frac{1-D}{2(k+1)} n_t \delta(\alpha_t z_n, \theta_n).$$

Case 3. $\theta_0 \in H = \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$. By, $\theta_n \to \theta_0$, for any given $\gamma > 0$, there exists $N > 0$ such that, as $n > N$, we have $\theta_n \in [\theta_0 - \gamma, \theta_0 + \gamma]$ and $z_n = r_n e^{i\theta_n} \in S(\theta_0) = \{z : \theta_0 - \gamma \leq \arg z \leq \theta_0 + \gamma\}$. By Theorem 1.5.2, there exists a set $E_3 \subset (1, \infty)$ having finite logarithmic measure, and there a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, we have for $l \leq k - 1$

$$\left|\frac{f^{(j)}(z)}{f^{(l)}(z)}\right| \leq B \left[T(2r, f)\right]^{k+1}, \quad j = l+1, \dots, k. \tag{3.38}$$

Now, we consider the growth of $f(re^{i\theta})$ on the ray $\arg z = \theta \in [\theta_0 - \gamma, \theta_0) \cup (\theta_0, \theta_0 + \gamma]$. Denote $S_1(\theta_0) = [\theta_0 - \gamma, \theta_0)$ and $S_2(\theta_0) = (\theta_0, \theta_0 + \gamma]$. We can easily see that when $\theta_1 \in S_1(\theta_0)$ and $\theta_2 \in S_2(\theta_0)$ then $\delta(\alpha_s z, \theta_1)\delta(\alpha_s z, \theta_2) < 0$. Without loss of the generality, we suppose that $\delta(\alpha_s z, \theta) > 0$ for $\theta \in S_1(\theta_0)$ and $\delta(\alpha_s z, \theta) < 0$ for $\theta \in S_2(\theta_0)$. For

$\theta \in S_1(\theta_0)$, we have $\delta(\alpha_s z, \theta) > 0$. Since, f is subnormal, then for any given $\varepsilon > 0$, we have

$$T\left(r, f\right) \leq e^{\varepsilon r}. \tag{3.39}$$

We assert that $\left|f^{(s)}(re^{i\theta})\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(s)}(z)\right|$ is unbounded then by Lemma 1.8.4, there exists an infinite sequence of points $w_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(s)}(w_u) \to \infty$ and

$$\left|\frac{f^{(j)}(w_u)}{f^{(s)}(w_u)}\right| \leq \frac{1}{(s-j)!} r_u^{s-j}(1+o(1)) \leq r_u^s(1+o(1)), \quad j = 0, \dots, s-1. \tag{3.40}$$

By (3.38) and (3.39), we obtain

$$\left|\frac{f^{(j)}(w_u)}{f^{(s)}(w_u)}\right| \leq B\left[T\left(2r_u, f\right)\right]^{k+1} \leq e^{2(k+1)\varepsilon r_u}, \ \ j = s+1, \dots, k. \tag{3.41}$$

By (3.5), (3.32), (3.33), (3.40) and (3.41), we deduce

$$\left|a_{sm_s}\right| e^{m_s \delta(\alpha_s z_n, \theta) r_u}(1+o(1)) \leq kMBe^{Cm_s\delta(\alpha_s w_u, \theta) r_u} e^{2(k+1)\varepsilon r_u} r_u^s (1+o(1)). \tag{3.42}$$

Since $0 < C < 1$ and $\delta(\alpha_s w_u, \theta) > 0$ we can see that (3.42) is a contradiction when $r_u \to \infty$ and

$$0 < \varepsilon < \frac{1-C}{2(k+1)} m_s \delta(\alpha_s w_u, \theta).$$

Hence,

$$\left|f(re^{i\theta})\right| \leq M_1 r^s \tag{3.43}$$

on the ray $\arg z = \theta \in [\theta_0 - \gamma, \theta_0)$. For $\theta \in S_2(\theta_0)$, we have $\delta(\alpha_s z, \theta) < 0$, $\delta(\alpha_t z, \theta) < 0$ and we assert that $\left|f^{(t)}(re^{i\theta})\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(t)}(z)\right|$ is unbounded then by Lemma 1.8.4, there exists an infinite sequence of points $w_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(t)}(w_u) \to \infty$ and

$$\left|\frac{f^{(j)}(w_u)}{f^{(t)}(w_u)}\right| \leq \frac{1}{(t-j)!} r_u^{t-j}(1+o(1)) \leq r_u^t(1+o(1)), \quad j = 0, \dots, t-1. \tag{3.44}$$

By (3.38) and (3.39), we obtain

$$\left|\frac{f^{(j)}(w_u)}{f^{(t)}(w_u)}\right| \leq B\left[T\left(2r_u, f\right)\right]^{k+1} \leq e^{2(k+1)\varepsilon r_u}, \ j = t+1, \dots, k. \tag{3.45}$$

By (3.5), (3.35), (3.36), (3.44) and (3.45), we deduce

$$\left|b_{tn_t}\right| e^{-n_t \delta(\alpha_t w_u, \theta) r_u}(1+o(1)) \leq kMBe^{-Dn_t\delta(\alpha_t w_u, \theta) r_u} e^{2(k+1)\varepsilon r_n} r_u^t (1+o(1)). \tag{3.46}$$

Since $0 < D < 1$ and $-n_t\delta(\alpha_t z_n, \theta_n) > 0$ we can see that (3.46) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < -\frac{1-D}{2(k+1)} n_t \delta(\alpha_t z_n, \theta_n).$$

Hence,

$$\left| f(re^{i\theta}) \right| \leq M_2 r^t \tag{3.47}$$

on the ray $\arg z = \theta \in (\theta_0, \theta_0 + \gamma]$. By (3.43) and (3.47), we have

$$\left| f(re^{i\theta}) \right| \leq M r^k \tag{3.48}$$

on the ray $\arg z = \theta \neq \theta_0$, $z \in S(\theta_0)$. Since f has infinite order and $\{z_n = r_n e^{i\theta_n} \in S(\theta_0)\}$ satisfies $|f(z_n)| = M(r_n, f)$, we see that for any large $N > 0$, and as n sufficiently large, we have

$$\left| f(r_n e^{i\theta_n}) \right| \geq \exp\{r_n^N\}. \tag{3.49}$$

Then, from (3.48) and (3.49), we get $Mr_n^k \geq \exp\{r_n^N\}$ that is a contradiction. Hence, (3.5) has no nontrivial subnormal solution.

(4) By Theorem 1.7.3, every solution f of (3.5) satisfies $\sigma_2(f) \leq 1$. Suppose that $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$, i.e., f is subnormal solution and this contradicts the conclusion above. So $\sigma_2(f) = 1$.

3.3.3 Proof of Theorem 3.2.3

We consider $Q_j(z) \equiv 0$ $(j = 1, \dots, k-1)$ in (3.5). By similar method of proof to Theorem 3.2.1, we conclude the result.

3.3.4 Proof of Theorem 3.2.4

We consider $Q_j(z) \equiv 0$ $(j = 1, \dots, k-1)$ in (3.5). We use the same method as in the proof of Theorem 3.2.2. Just in the case when $\delta(\alpha_s z, \theta) < 0$, we use the fact that $\left| f^{(k)}(z) \right|$ is bounded on the ray $\arg z = \theta$. If $\left| f^{(k)}(z) \right|$ is unbounded then by Lemma 1.8.4, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \dots)$ where $r_n \to +\infty$ such that $f^{(k)}(z_n) \to \infty$ and

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \leq r_n^k (1 + o(1)), \ (j = 0, \dots, k-1). \tag{3.50}$$

By the definition of $P_j^*(e^{\alpha_j z})$, and because $\delta(\alpha_s z, \theta) < 0$, i.e., $\delta(\alpha_j z, \theta) < 0, \forall j$, by $m_s\alpha_s = c_j m_j \alpha_j$. Then, we can write

$$\left|P_j^*(e^{\alpha_j z_n})\right| = |a_{j1}|\, e^{\delta(\alpha_j z_n, \theta) r_n}(1 + o(1)). \tag{3.51}$$

By (3.7), (3.50) and (3.51), we have

$$\begin{aligned} 1 &\leq \sum_{j=0}^{k-1} \left|P_j^*(e^{\alpha_j z_n})\right| \left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \\ &\leq \sum_{j=0}^{k-1} |a_{j1}|\, e^{\delta(\alpha_j z_n, \theta) r_n} r_n^k (1 + o(1)). \end{aligned} \tag{3.52}$$

Since $\delta(\alpha_j z, \theta) < 0, \forall j$, then (3.52) is a contradiction as $r_n \to \infty$. Thus, $\left|f^{(k)}(z)\right| \leq M$, so $|f(z)| \leq M r^k$.

3.3.5 Proof of Theorem 3.2.5

Suppose that f is a nontrivial subnormal solution of (3.4). Let

$$h(z) = f(z) e^{(b_m/a_m) z}.$$

Then h is a nontrivial subnormal solution of the equation

$$h^{(k)} + \sum_{j=0}^{k-1} \left[R_j\left(e^z\right) + S_j\left(e^{-z}\right)\right] h^{(j)} = 0, \tag{3.53}$$

where

$$R_j\left(e^z\right) + S_j\left(e^{-z}\right) = C_k^j \left(-\frac{b_m}{a_m}\right)^{k-j} + \sum_{l=j}^{k-1} C_l^j \left(-\frac{b_m}{a_m}\right)^{l-j} \left[P_l\left(e^z\right) + Q_l\left(e^{-z}\right)\right].$$

Because $m > \max\{m_j : j = 2, \ldots, k-1\}$ and $n > \max\{n_j : j = 2, \ldots, k-1\}$, we have

$$\begin{aligned} \deg R_1 &= \deg P_1 = m, \\ \deg S_1 &= \deg Q_1 = n. \end{aligned}$$

From $a_m d_n = b_m c_n$, we see in the formula

$$\begin{aligned} R_0\left(e^z\right) + S_0\left(e^{-z}\right) &= \left(-\frac{b_m}{a_m}\right)^k + \sum_{l=2}^{k-1} \left(-\frac{b_m}{a_m}\right)^l \left[P_l\left(e^z\right) + Q_l\left(e^{-z}\right)\right] \\ &\quad + \left(-\frac{b_m}{a_m}\right) \left[P_1\left(e^z\right) + Q_1\left(e^{-z}\right)\right] + \left[P_0\left(e^z\right) + Q_0\left(e^{-z}\right)\right] \end{aligned}$$

that

$$\begin{aligned} \deg R_0 &< m, \\ \deg S_0 &< n. \end{aligned}$$

Then, we have

$$\begin{aligned} \deg R_1 &= m > \deg R_j : j = 0, 2, \ldots, k-1, \\ \deg S_1 &= n > \deg S_j : j = 0, 2, \ldots, k-1 \end{aligned}$$

and since $e^{-(b_m/a_m)z}$ is not a solution of (3.4), then

$$R_0\left(e^z\right) + S_0\left(e^{-z}\right) = \left(-\frac{b_m}{a_m}\right)^k + \sum_{l=0}^{k-1}\left(-\frac{b_m}{a_m}\right)^l \left[P_l\left(e^z\right) + Q_l\left(e^{-z}\right)\right] \not\equiv 0.$$

By applying Theorem 3.1.3 on the equation (3.53), we obtain the conclusion.

Conclusion and perspective

Throughout this work, we have been talking about the possibility of generalization of some results related to second-order complex differential equations to higher-order complex differential equations in analogous manner or different manner, and extension for other results. For example, we generalized the results of Li and Yang : Theorem 2.2.2 and Theorem 2.2.3 to Theorem 3.2.1 and Theorem 3.2.2, and these last two theorems are extensions for results of Liu and Yang, Chen and Shon : Theorem 3.1.2 and Theorem 3.1.3.

Theorem 2.2.3 is generalized to Theorem 3.2.5 and we considered at that case, the equation

$$f^{(k)}+\left[P_{k-1}\left(e^{z}\right)+Q_{k-1}\left(e^{-z}\right)\right]f^{(k-1)}+\cdots+\left[P_{1}\left(e^{z}\right)+Q_{1}\left(e^{-z}\right)\right]f'+\left[P_{0}\left(e^{z}\right)+Q_{0}\left(e^{-z}\right)\right]f=0$$

with $\deg P_1 = \deg P_0$ and $\deg Q_1 = \deg Q_0$.
From that, we hope to solve the next problem :

What can be said about the subnormal solutions of the equation above if we suppose that

$$\deg P_j = \deg P_0 \text{ and } \deg Q_j = \deg Q_0 \ ; \ \forall j = 0, \ldots, k-1 \ ?$$

or, What are the hypotheses that guarantee that the equation above doesn't have subnormal solution ?

We hope also, study the existence or nonexistence of subnormal solutions of the equation of the general form

$$f'' + P(e^{A})f' + Q(e^{B})f = 0$$

where $P(z), Q(z)$ are polynomials in z, with $\deg P = \deg Q$ and , $A(z)$ and $B(z)$ are polynomials in z with $\deg A = \deg B$ or $A(z)$ and $B(z)$ are transcendental entire functions with $\sigma_p(A) = \sigma_p(B)$; here, σ_p denote the p-iterated order. See [23].

Another problem is about the existence or nonexistence of subnomral solutions of the equation

$$f'' + \left[P_1(e^{A}) + Q_1(e^{-A})\right] f' + \left[P_1(e^{B}) + Q_2(e^{-B})\right] f = 0$$

where $P(z), Q(z), A(z)$ and $B(z)$ are polynomials in z.

Other questions are raised about differential polynomials generated by the nontrivial solutions and especially nontrivial subnormal solutions of all forms of differential equations mentioned in this thesis. The problems related to the differential polynomials are about estimate the growth of order and if possible estimate the e-type order, oscillation theory, etc.

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