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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS INVOLVING RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE ON THE HALF-LINE

Ravi P. Agarwal[1], Mouffak Benchohra[2],
Samira Hamani[2] and Sandra Pinelas[3]

[1]Department of Mathematical Sciences
Florida Institute of Technology, Melboune, Florida, 32901-6975, USA

[2]Laboratoire de Mathématiques
Université de Sidi Bel-Abbès, B.P. 89, 22000, Sidi Bel-Abbès, Algérie

[3]Department of Mathematics
Azores University, R. Mãe de Deus, 9500-321 Ponta Delgada, Portugal
Corresponding author email: agarwal@fit.edu

Abstract. In this paper, we establish sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential equations involving the Riemann-Liouville fractional derivative on infinite intervals. This result is based on the nonlinear alternative of Leray-Schauder type combined with the diagonalization method.

Keywords. Boundary value problem; Differential equation; Riemann-Liouville fractional derivative; Fractional integral; Existence; Fixed point; Infinite intervals; Diagonalization process.

AMS (MOS) subject classification: 26A33; 34A60; 34B15.

1 Introduction

This paper deals with the existence of solutions for the boundary value problems (BVP for short) for fractional order differential equations of the form

$$D^{\alpha}y(t) = f(t, y(t)), \text{ for each } t \in J = [0, \infty), \quad 1 < \alpha \leq 2, \tag{1}$$

$$y(0) = 0, \; y \text{ bounded on } [0, \infty), \tag{2}$$

where D^{α} is the Riemann-Liouville fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function.

Differential equations of fractional order have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [10, 11, 12, 15, 19, 20] and the references therein). There has been a significant development in fractional differential and partial differential equations in recent years; we refer to the monographs by Kilbas *et al* [17], Lakshmikantham *et al.* [18], Podlubny [21], Samko *et al* [23] and the papers by Agarwal *et al* [1], Delbosco and Rodino [9], Diethelm *et al* [10], Kilbas and Marzan [16], Mainardi [19], Zhang [24] and the references therein. In [7, 8] the authors studied the existence and uniqueness of solutions of classes of initial value problems for functional differential equations with infinite delay and fractional order, and in [6] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [14, 22].

In this paper, we present existence results for the BVP (1)-(2). We use the nonlinear alternative of Leray-Schauder type [13] combined with the diagonalization process used widely for integer order differential equations; see for instance [2, 3, 4].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

Definition 2.1 *The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,$$

where Γ is the Gamma function. When $a=0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$, $\varphi_\alpha(t)=0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the Delta function.

Definition 2.2 *For a function h defined on the interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of h is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds.$$

Here and hereafter $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

More details on fractional derivatives and integrals and their properties can be found in [17, 21].

3 Existence of solutions

Let us start by defining what we mean by a solution of BVP (1)–(2).

Definition 3.1 *A function $y \in C(J, \mathbb{R})$ is said to be a solution of BVP (1)–(2), if y satisfies the equation*

$$D^\alpha y(t) = f(t, y(t)), \text{ for each } t \in J, \ 1 < \alpha \leq 2,$$

and the condition (2).

For the existence of solutions for BVP (1)–(2), we need the following auxiliary lemma:

Lemma 3.2 *[9] Let $\alpha > 0$. If we assume $h \in C((0,T), \mathbb{R}) \bigcap L((0,T), \mathbb{R})$, then the fractional differential equation*

$$D^\alpha h(t) = 0$$

has solutions

$$h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{n-1}, \ \text{ for } c_i \in \mathbb{R}, \ i = 1, 2, ..., n.$$

Lemma 3.3 *[9] Assume $h \in C((0,T), \mathbb{R}) \bigcap L((0,T), \mathbb{R})$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^\alpha D^\alpha h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{n-1}$$

for some constants $c_i, \ i = 1, 2, ..., n$.

As a consequence of Lemmas 3.2 and 3.3, we have the following result which provides the integral formulation for BVP (1)-(2).

Lemma 3.4 *Let $1 < \alpha \leq 2$ and let $\sigma : [0, T] \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \int_0^T G(t,s)\sigma(s)ds, \tag{3}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}(T-s)^{\alpha-2}}{T^{\alpha-2}\Gamma(\alpha)}, & 0 \leq s \leq t \\ -\dfrac{t^{\alpha-1}(T-s)^{\alpha-2}}{T^{\alpha-2}\Gamma(\alpha)}, & t \leq s < T \end{cases} \tag{4}$$

if and only if y is a solution of the fractional BVP

$$D^{\alpha}y(t) = \sigma(t), \quad t \in [0,T], \tag{5}$$

$$y(0) = 0, \quad y'(T) = 0. \tag{6}$$

Proof: Assume that y satisfies (5); then Lemma 3.3 implies that

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds.$$

From (6), a simple calculation yields $c_2 = 0$ and

$$c_1 = \frac{-1}{(\alpha-1)T^{\alpha-2}\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\sigma(s)ds,$$

whence, equation (3). Conversely, it is clear that if y satisfies equation (3), then equations (5)-(6) hold.

Remark 3.5 *For each $T > 0$, the function $t \mapsto \int_0^T |G(t,s)|ds$ is continuous on $[0,T]$, and hence is bounded.*

In this section we assume that there exists $T_n \in J$, $n \in \mathbb{N}$, with

$$0 < T_1 < T_2 < \ldots < T_n < \ldots \text{ with } T_n \to \infty \text{ as } n \to \infty.$$

In the sequel we set $J_n := [0, T_n]$.

Theorem 3.6 *Assume the following hypotheses hold:*

($\mathcal{H}_1$) *$f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous,*

($\mathcal{H}_2$) *there exist $p \in C(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that*

$$|f(t,u)| \le p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R};$$

($\mathcal{H}_3$)

$$\sup_{c\in(0,\infty)} \frac{c}{p_n^* \psi(c)\tilde{G}_n} > 1, \tag{7}$$

where

$$\tilde{G}_n = \sup\left\{ \int_0^{T_n} |G_n(t,s)|ds, \ t \in J_n \right\},$$

$$p_n^* = \sup\{p(s), \ s \in J_n\},$$

and

$$G_n(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t^{\alpha-1}(T_n-s)^{\alpha-2}}{T_n^{\alpha-2}\Gamma(\alpha)}, & 0 \le s \le t \\ -\dfrac{t^{\alpha-1}(T_n-s)^{\alpha-2}}{T_n^{\alpha-2}\Gamma(\alpha)}, & t \le s < T_n. \end{cases} \tag{8}$$

Then BVP (1)-(2) has at least one solution on J.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$D^\alpha y(t) = f(t, y(t)), \ \ t \in J_n, \quad 1 < \alpha \leq 2, \tag{9}$$

$$y(0) = 0, \ \ y'(T_n) = 0. \tag{10}$$

We begin by showing that (9)-(10) has a solution $y_n \in C(J_n, \mathbb{R})$ with

$$|y_n(t)| \leq M \text{ for each } t \in J_n,$$

for some constant $M > 0$. Here $C(J_n, \mathbb{R})$ is the Banach space of all continuous functions from J_n into $\mathbb{R}$ with the norm

$$\|y\|_n = \sup\{|y(t)| : \quad t \in J_n\}.$$

Consider the operator $N : C(J_n, \mathbb{R}) \longrightarrow C(J_n, \mathbb{R})$ defined by

$$(Ny)(t) = \int_0^{T_n} G_n(t, s) f(s, y(s)) ds,$$

where the Green's function $G_n(t, s)$ is given by (8). Clearly, from Lemma 3.4, the fixed points of N are solutions to (9)–(10). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder [13]. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_q\}$ be a sequence such that $y_q \to y$ in $C(J_n, \mathbb{R})$. Then for each $t \in J_n$.

$$|(Ny_q)(t) - (Ny)(t)| \leq \int_0^{T_n} |G_n(t, s) f(s, y_q(s)) - f(s, y(s))| ds.$$

Let $\rho > 0$ be such that

$$\|y_q\|_n \leq \rho, \ \ \|y\|_n \leq \rho.$$

By $(\mathcal{H}_2)$ we have

$$|G(\cdot, s)\|f(s, y_q(s)) - f(s, y(s))\| \leq 2\psi(\rho)|G(\cdot, s)|p(s) \in L^1(J, \mathbb{R}_+).$$

Since f is continuous, the Lebesgue dominated convergence theorem implies that

$$\|Ny_q - Ny\|_n \to 0 \quad \text{as} \quad q \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in $C(J_n, \mathbb{R})$.*

Let $B_{\eta^*} = \{y \in C(J_n, \mathbb{R}) : \|y\|_n \leq \eta^*\}$ be a bounded set in $C(J_n, \mathbb{R})$ and $y \in B_{\eta^*}$. Then for each $t \in J_n$, we have by $(\mathcal{H}_2)$

$$\begin{aligned} |(Ny)(t)| &\leq \int_0^{T_n} |G_n(t,s)||f(s,y(s))|ds \\ &\leq \psi(\|y\|_n)p_n^* \int_0^{T_n} |G_n(t,s)|ds. \end{aligned}$$

Thus

$$\|Ny\|_n \leq \psi(\eta^*)p_n^* \tilde{G}_n := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C(J_n, \mathbb{R})$.*

Let $\tau_1, \tau_2 \in J_n$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of $C(J_n, \mathbb{R})$ as in Step 2, $y \in B_{\eta^*}$ then

$$\begin{aligned} |(Ny)(\tau_2) - (Ny)(\tau_1)| &\leq \int_0^{T_n} |G(\tau_2, s) - G(\tau_1, s)|f(s, y(s))|ds \\ &\leq p_n^* \psi(\eta^*) \int_0^{T_n} |G(\tau_2, s) - G(\tau_1, s)|ds. \end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem, we conclude that N is completely continuous.

Step 4: *A priori bounds on solutions.*

Choose $M > 0$ with

$$\frac{M}{p_n^* \psi(M) \tilde{G}_n} > 1. \tag{11}$$

This constant exists by (7).

Let y be such that $y = \lambda(Ny)$ for $\lambda \in [0, 1]$. Then, for each $t \in J_n$,

$$\begin{aligned} |y(t)| &\leq \int_0^{T_n} G_n(t,s)p(s)\psi(|y(s)|)ds \\ &\leq p_n^* \psi(\|y\|_n) \int_0^{T_n} |G_n(t,s)|ds \\ &\leq p_n^* \psi(\|y\|_n) \tilde{G}_n. \end{aligned}$$

Thus

$$\frac{\|y\|_n}{p_n^* \tilde{G}_n \psi(\|y\|_n)} \leq 1. \tag{12}$$

Conditions (11) and (12) imply that $\|y\|_n \neq M$. Let

$$U = \{y \in C(J_n, \mathbb{R}) : \|y\|_n < M\}.$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda(Ny)$ for some $\lambda \in (0,1)$. Moreover, the operator $N : \overline{U} \to C(J_n, \mathbb{R})$ is completely continuous. Therefore, we deduce that N has a fixed point y_n in $\overline{U}$, a solution of BVP (9)–(10) with

$$|y_n(t)| \leq M \text{ for each } t \in J_n.$$

Step 5: *Diagonalization process*

We will use diagonalization process. For $k \in \mathbb{N}$, let

$$u_k(t) = \begin{cases} y_k(t), & t \in [0, T_k], \\ y_k(T_k) & t \in [T_k, \infty). \end{cases} \tag{13}$$

Let $S = \{u_k\}_{k=1}^{\infty}$. Notice that

$$|u_k(t)| \leq M \text{ for } \ t \in [0, T_1],\ k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t \in [0, T_1]$ we have

$$u_k(t) = \int_0^{T_1} G_1(t,s) f(s, u_k(s)) ds.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [0, T_1]$ we have

$$u_k(t) - u_k(x) = \int_0^{T_1} [G_1(t,s) - G_1(x,s)] f(s, u_k(s)) ds$$

and by $(\mathcal{H}_2)$, we have

$$|u_k(t) - u_k(x)| \leq p_1^* \psi(M) \int_0^{T_1} |G_1(t,s) - G_1(x,s)| ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_1^* of $\mathbb{N}$ and a function $z_1 \in C([0,T_1], \mathbb{R})$ with $u_k \to z_1$ in $C([0,T_1], \mathbb{R})$ as $k \to \infty$ through N_1^*. Let $N_1 = N_1^* \backslash \{1\}$. Notice that

$$|u_k(t)| \leq M \text{ for } t \in [0, T_2],\ k \in \mathbb{N}_2.$$

Also for $k \in \mathbb{N}_1$ and $t, x \in [0, T_2]$ we have

$$|u_k(t) - u_k(x)| \leq p_2^* \psi(M) \int_0^{T_2} |G_2(t,s) - G_2(x,s)| ds.$$

The Arzelà-Ascoli Theorem guarantees that there is a subsequence N_2^* of N_1 and a function $z_2 \in C([0,T_2], \mathbb{R})$ with $u_k \to z_2$ in $C([0,T_2], \mathbb{R})$ as $k \to \infty$ through N_2^*. Note that $z_1 = z_2$ on $[0, T_1]$ since $N_2^* \subseteq N_1$. Let $N_2 = N_2^* \backslash \{2\}$. Proceed inductively to obtain for $m \in \{3, 4, ...\}$ a subsequence N_m^* of N_{m-1} and a function $z_m \in C([0,T_m], \mathbb{R})$ with $u_k \to z_m$ in $C([0,T_m], \mathbb{R})$ as $k \to \infty$

through N_m^*. Let $N_m = N_m^* \backslash \{m\}$.
Define a function y as follows. Fix $t \in (0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq T_m$. Then define $y(t) = z_m(t)$. Then $y \in C([0, \infty), \mathbb{R})$, $y(0) = 0$ and $|y(t)| \leq M$ for $t \in [0, \infty)$. Again fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $s \leq T_m$. Then for $n \in N_m$ we have

$$u_n(t) = \int_0^{T_m} G_m(t, s) f(s, u_n(s)) ds,$$

Let $n \to \infty$ through N_m to obtain

$$z_m(t) = \int_0^{T_m} G_m(x, s) f(s, z_m(s)) ds,$$

i.e

$$y(t) = \int_0^{T_m} G_m(t, s) f(s, y(s)) ds.$$

We can use this method for each $x \in [0, T_m]$, and for each $m \in \mathbb{N}$. Thus

$$D^\alpha y(t) = f(t, y(t)), \text{ for } t \in [0, T_m]$$

for each $m \in \mathbb{N}$ and $\alpha \in (1, 2]$. This completes the proof of the theorem.

4 An example

Consider the boundary value problem

$$D^\alpha y(t) = \frac{1}{e^t + 1} |y(t)|^\delta, \text{ for } t \in J = [0, \infty), \quad 1 < \alpha \leq 2, \tag{14}$$

$$y(0) = 0, \ y \text{ is bounded on } [0, \infty), \tag{15}$$

where D^α is the Riemann-Liouville fractional derivative, and $\delta \in (0, 1)$. Set

$$f(t, u) = \frac{1}{e^t + 1} u^\delta, \text{ for each } (t, u) \in J \times [0, \infty).$$

It is clear that conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied with

$$p(t) = \frac{1}{e^t + 1}, \text{ for each } t \in J,$$

and

$$\psi(u) = u^\delta, \text{ for each } u \in [0, \infty).$$

From (8) we have for $s \leq t$

$$\int_0^t G_n(t, s) ds = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{\alpha-1}(T_n - t)^{\alpha-1} - t^{\alpha-1} T_n^{\alpha-1}}{(\alpha - 1) T_n^{\alpha-2} \Gamma(\alpha)}$$

and for $t \leq s$

$$\int_t^{T_n} G_n(t,s)ds = \frac{-t^{\alpha-1}(T_n - t)^{\alpha-1}}{(\alpha-1)T_n^{\alpha-2}\Gamma(\alpha)}.$$

Also

$$\sup_{c\in(0,\infty)} \frac{c}{p_n^*\psi(c)\tilde{G}_n} = \sup_{c\in(0,\infty)} \frac{c}{\psi(c)} = \sup_{c\in(0,\infty)} \frac{c}{c^\delta} = \infty,$$

hence $(\mathcal{H}_3)$ is satisfied. Then by Theorem 3.6, BVP (14)-(15) has a bounded solution on $[0,\infty)$.

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