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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS WITH FRACTIONAL ORDER

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Abstract

In this paper, we shall establish sufficient conditions for the existence of solutions for a boundary value problem for fractional differential inclusions. Both cases of convex valued and nonconvex valued right hand sides are considered.

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1. Introduction

This paper deals with the existence of solutions for boundary value problems (BVP for short), for fractional order differential inclusions

$$^cD^\alpha y(t) \in F(t,y), \quad \text{a.e.} \quad t \in J := [0,T], \quad 0 < \alpha < 1, \tag{1}$$

$$ay(0) + by(T) = c, \tag{2}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), a, b, c are real constants with $a + b \neq 0$. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [11, 19, 20, 23, 28, 29, 31, 33]). There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al.* [26], Miller and Ross [30], Podlubny [33], Samko *et al.* [36] and the papers of Delbosco and Rodino [10], Diethelm *et al.* [11, 12, 13], El-Sayed [15, 16, 17], Kilbas and Marzan [25], Mainardi [28], Podlubny *et al.* [35], Yu and Gao [38] and the references therein. Very recently, some basic theory for initial value problems for fractional differential equations involving the Riemann-Liouville differential operator of order $\alpha \in (0,1]$ has been discussed by Lakshmikantham and Devi [27]. In [4, 6] the authors studied the existence and uniqueness of solutions of classes of functional differential equations with infinite delay and fractional order, and in [3] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. El-Sayed and Ibrahim [18] initiated the study of fractional multivalued differential inclusions. In the case where $\alpha \in (1,2]$, existence results for a fractional boundary value problem and the relaxation theorem were given by Ouahab [32].

Engineering problems require definitions of fractional derivatives allowing the use of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc. The same requirements apply to boundary conditions. The Caputo fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [22, 34].

In this paper, we shall present two existence results for the problem (1)–(2), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type, while the other is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler. These results extend to the multivalued case some previous results in the literature, and constitute a contribution of this emerging field. In particular, our results extend to the multivalued case those considered recently by Benchohra *et al.* in [5].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

Let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

$AC(J,\mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$, which are absolutely continuous. Let $(X, \|\cdot\|)$ be a Banach space. Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B}G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [9] and Hu and Papageorgiou [21].

Definition 2.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \longmapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y(t)) \ a.e.\ t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

When the right hand side is nonconvex valued, the following fixed point theorem will be used.

Lemma 2.3 [8]**.** *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition 2.4 ([26, 33])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.5 ([26, 33])**.** For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, $\alpha \in (0, 1)$, is defined by

$$\begin{aligned}(D^{\alpha}_{a+}h)(t) &= \frac{d^{\alpha}h(t)}{dt^{\alpha}} \\ &= \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_a^t (t-s)^{-\alpha}h(s)ds \\ &= \frac{d}{dt}\,I_a^{1-\alpha}h(t).\end{aligned}$$

Definition 2.6 ([26])**.** For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h of order $\alpha \in (0,1)$, is defined by

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}h'(s)ds.$$

3. Main results

In this section, we are concerned with the existence of solutions to the problem (1)–(2) when the right hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

Definition 3.1. A function $y \in AC(J,\mathbb{R})$ is said to be a solution of (1)–(2), if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,y(t))$, for a.e. $t \in J$, such that

$${}^cD^{\alpha}y(t) = v(t), \quad \text{a.e} \;\; t \in J,\; 0<\alpha<1,$$

and the function y satisfies condition (2).

For the existence of solutions to the problem (1)–(2), we need the following auxiliary lemma:

Lemma 3.2 [25]**.** *Let $0<\alpha<1$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \tag{3}$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$(4)\qquad {}^cD^\alpha y(t) = h(t), \quad t \in J,$$

$$(5)\qquad y(0) = y_0.$$

As a consequence of Lemma 3.2 we have the following result which is useful in what follows.

Lemma 3.3. *Let $0 < \alpha < 1$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$(6)\qquad \begin{aligned} y(t) \;=\; & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c\right] \end{aligned}$$

if and only if y is a solution of the fractional BVP

$$(7)\qquad {}^cD^\alpha y(t) = h(t), \quad t \in J,$$

$$(8)\qquad ay(0) + by(T) = c.$$

Proof. Assume y satisfies (7), then Lemma 3.2 implies that

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds.$$

From (8), a simple calculation gives

$$c_0 = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds\right].$$

Hence we get equation (6). Inversely, it is clear that if y satisfies equation (6), then equations (7)–(8) hold. ■

Theorem 3.4 *Assume the following hypotheses hold:*

(H1) $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map;*

(H2) *there exist* $p \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R};$$

(H3) *there exists* $l \in L^1(J, \mathbb{R}^+)$, *with* $I^\alpha l < \infty$ *such that*

$$H_d(F(t,u), F(t,\overline{u})) \leq l(t)|u - \overline{u}| \quad \text{for every } u, \overline{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \leq l(t), \ \text{a.e. } t \in J;$$

(H4) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\psi(M)\|I^\alpha p\|_\infty + \dfrac{|b|\psi(M)(I^\alpha p)(T)}{|a+b|} + \dfrac{|c|}{|a+b|}} > 1. \tag{9}$$

Then the BVP (1)–(2) *has at least one solution on* J.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the multivalued operator

$$N(y) =$$

$$= \left\{ h \in C(J, \mathbb{R}) : \begin{array}{l} h(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} v(s) ds \\ \quad - \dfrac{1}{a+b} \left[\dfrac{b}{\Gamma(\alpha)} \displaystyle\int_0^T (T-s)^{\alpha-1} v(s) ds - c \right], \ v \in S_{F,y} \end{array} \right\}.$$

■

Remark 3.5. Clearly, from Lemma 3.3, the fixed points of N are solutions to (1)–(2).

We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type ([14]). The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in C(J, \mathbb{R})$.

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_i(s)ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} v_i(s)ds - c\right], \quad i=1,2. \end{aligned}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} & (dh_1 + (1-d)h_2)(t) = \\ & = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds \\ & -\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds - c\right]. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2. N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Let $B_{\eta^*} = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \le \eta^*\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_{\eta^*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that for each

$$\begin{aligned} h(t) \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} v(s)ds - c\right], \ t \in J. \end{aligned}$$

By (H2) we have for each $t \in J$,

$$|h(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds$$

$$
\begin{aligned}
&+ \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|v(s)| + \frac{|c|}{|a+b|}\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\psi(|y(s)|)ds\\
&+ \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}p(s)\psi(|y(s)|)ds + \frac{|c|}{|a+b|}\\
&\leq \psi(\eta^*)I^\alpha(p)(t) + \frac{|b|\psi(\eta^*)I^\alpha(p)(T)}{|a+b|} + \frac{|c|}{|a+b|}\,.
\end{aligned}
$$

Thus

$$
\|h\|_\infty \leq \psi(\eta^*)\|I^\alpha(p)\|_\infty + \frac{|b|\psi(\eta^*)I^\alpha(p)(T)}{|a+b|} + \frac{|c|}{|a+b|} := \ell\,.
$$

Step 3. N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in Step 2, let $y \in B_{\eta^*}$ and $h \in N(y)$, then

$$
\begin{aligned}
|h(t_2)-h(t_1)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]v(s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}v(s)ds\Big|\\
&\leq \frac{\|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]ds\\
&\quad + \frac{|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&\leq \frac{\|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha+1)}\,[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha] + \frac{\|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\\
&\leq \frac{\|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha+1)}\,(t_2-t_1)^\alpha + \frac{\|p\|_\infty\psi(\eta^*)}{\Gamma(\alpha+1)}\,(t_1^\alpha - t_2^\alpha)\,.
\end{aligned}
$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : C(J, \mathbb{R}) \longrightarrow \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

Step 4. N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s) ds - c \right]. \end{aligned}$$

We have to show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_*(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_*(s) ds - c \right]. \end{aligned}$$

Since $F(t, \cdot)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exists $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1), \text{ a.e. } t \in J.$$

Since $F(\cdot, \cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \text{ as } m \to \infty$$

and

$$v_*(t) \in F(t, y_*(t)), \text{ a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))).$$

By a similar relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned}|h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_{n_m}(s) - v_*(s)|ds\\ &+ \frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|v_{n_m}(s) - v_*(s)|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)ds\|y_{n_m} - y_*\|_\infty\\ &+ \frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}l(s)ds\|y_{n_m} - y_*\|_\infty.\end{aligned}$$

Hence

$$\begin{aligned}\|h_{n_m} - h_*\|_\infty \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}l(s)ds\|y_{n_m} - y_*\|_\infty\\ &+\frac{|b|}{|a+b|}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}l(s)ds\|y_{n_m} - y_*\|_\infty \to 0 \text{ as } m \to \infty.\end{aligned}$$

Step 5. A priori bounds on solutions.

Let y be a possible solution to the problem (1)–(2). Then, there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned}y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds\\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds - c\right].\end{aligned}$$

This implies by (H2) that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| \le& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\
&+ \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}|v(s)| + \frac{|c|}{|a+b|} \\
\le& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\psi(|y(s)|)ds \\
&+ \frac{|b|}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}p(s)\psi(|y(s)|)ds + \frac{|c|}{|a+b|} \\
\le& \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds \\
&+ \frac{|b|\psi(\|y\|_\infty)}{\Gamma(\alpha)|a+b|}\int_0^T (T-s)^{\alpha-1}p(s)ds + \frac{|c|}{|a+b|} \\
\le& \psi(\|y\|_\infty)(I^\alpha p)(t) + \frac{|b|\psi(\|y\|_\infty)(I^\alpha p)(T)}{|a+b|} + \frac{|c|}{|a+b|}.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)\|I^\alpha p\|_\infty + \dfrac{|b|\psi(\|y\|_\infty)(I^\alpha p)(T)}{|a+b|} + \dfrac{|c|}{|a+b|}} \le 1.$$

Then by condition (9), there exists M such that $\|y\|_\infty \ne M$.

Let

$$U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M\}.$$

The operator $N : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [14], we deduce that N has a fixed point y in $\overline{U}$ which is a solution to the problem (1)–(2). This completes the proof. ■

We present now a result for the problem (1)–(2) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [8].

Theorem 3.6. *Assume* (H3) *and the following hypothesis holds:*

(H5) $F : J \times \mathbb{R} \longrightarrow P_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : J \to P_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$*;*

If

$$\|I^\alpha l\|_\infty + \frac{|b|(I^\alpha l)(T)}{|a+b|} < 1, \tag{10}$$

then the BVP (1)–(2) *has at least one solution on* J*.*

Remark 3.7. For each $y \in C(J, \mathbb{R})$, the set $S_{F,y}$ is nonempty since by (H5), F has a measurable selection (see [7], Theorem III.6).

***Proof of Theorem* 3.6.** We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1. $N(y) \in P_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$.

Indeed, let $(y_n)_{n \geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C(J, \mathbb{R})$. Then, $\tilde{y} \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s) ds - c \right]. \end{aligned}$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J, \mathbb{R})$ (the space endowed with the weak toplogy). An application of Mazur's theorem ([37]) implies that v_n converges strongly to v and hence $v \in S_{F,y}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s) ds - c \right]. \end{aligned}$$

So, $\tilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_\infty \text{ for each } y, \overline{y} \in C(J, \mathbb{R}).$$

Let $y, \overline{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1(t) \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_1(s) ds - c \right]. \end{aligned}$$

From (H3) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \leq l(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t, \overline{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \overline{y}(t)|, \ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [7]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \overline{y}(t)|.$$

Let us define for each $t \in J$

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_2(s) ds \\ &- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_2(s) ds - c \right]. \end{aligned}$$

Then for $t \in J$

$$\begin{aligned}|h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\ &+ \frac{|b|}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s) |y(s) - \overline{y}(s)| ds \\ &+ \frac{|b|}{\Gamma(\alpha)|a+b|} \int_0^T (T-s)^{\alpha-1} l(s) |y(s) - \overline{y}(s)| ds.\end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \left[\|I^\alpha l\|_\infty + \frac{|b|(I^\alpha l)(T)}{|a+b|} \right] \|y - \overline{y}\|_\infty.$$

From an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \left[\|I^\alpha l\|_\infty + \frac{|b|(I^\alpha l)(T)}{|a+b|} \right] \|y - \overline{y}\|_\infty.$$

So by (10), N is a contraction and thus, by Lemma 2.3, N has a fixed point y which is solution to (1)–(2). The proof is complete.

Remark 3.8. Our results for the BVP (1)–(2) are applied to initial value problems ($a = 1,\ b = 0$), terminal value problems ($a = 0,\ b = 1$) and anti-periodic solutions ($a = 1,\ b = 1,\ c = 0$).

4. AN EXAMPLE

We apply the main result of the paper (Theorem 3.4) to the following fractional differential inclusion

$$(11) \qquad {}^cD^\alpha y(t) \in F(t, y), \text{ a.e. } t \in J = [0, T], \quad 0 < \alpha \leq 1,$$

$$(12) \qquad y(0) = y_0.$$

Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

where $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$. We assume that for each $t \in J$, $f_1(t, \cdot)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C(J, \mathbb{R}^+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, y)|,\ |f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in J, \text{ and all } y \in \mathbb{R}.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [9]). Since all the conditions of Theorem 3.4 are satisfied, problem (11)–(12) has at least one solution y on J.

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