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Nonlinear Boundary Value Problems for Hadamard Fractional Differential Inclusions with Integral Boundary Conditions

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Abstract

The authors establish sufficient conditions for the existence of solutions for nonlinear fractional differential inclusions involving the Hadamard type derivative with order $r \in (2, 3]$. Both cases of convex and nonconvex valued right hand side are considered.

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1 Introduction

In this paper we are concerned with the existence of solutions for the following nonlinear fractional differential inclusion with integral boundary value conditions

$$^{H}D^{r}y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, T], \quad 2 < r \leq 3, \tag{1.1}$$

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$$y(1) = y''(1) = 0, \tag{1.2}$$

$$y(T) = \int_1^T g(s, y(s))ds, \tag{1.3}$$

where ${}^H D^r$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given function.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. There has been a significant development in the theory of fractional differential equations in recent years; see the monographs of Hilfer [21], Kilbas et al. [23], Podlubny [27], and Momani et al. [26].

However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [3, 28]. The fractional derivative that Hadamard [18] introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. Detailed descriptions of the Hadamard fractional derivative and integral can be found in [6–8].

In this paper, we shall present two existence results for the problem (1.1)–(1.3), when in one case, the right hand side is convex valued, and in the other case, nonconvex valued. The first result relies on the nonlinear alternative of Leray–Schauder type, while the other is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler. These results extend to the multivalued case some previous results in the literature, and constitute a contribution in this emerging field.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \leq t \leq T\}$$

and we let $L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt.$$

The space $AC^1(J, \mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$, which are absolutely continuous, whose first derivative, y', is absolutely continuous.

Let $(X, \|.\|)$ be a Banach space. Let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup\limits_{x\in B}\{\sup\{|y| : y \in G(x)\}\}$).

G is called *upper semi-continuous* (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denote by $\mathrm{Fix}\, G$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be *measurable* if for every $y \in \mathbb{R}$, the function,

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\},$$

is measurable.

Definition 2.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{R}$,

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in AC^1(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([1, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e } t \in [1, T]\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |.|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}.$$

Definition 2.2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for each } x, y \in X,$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The following fixed point result for contraction multivalued maps is due to Covitz and Nadler [11].

Lemma 2.3. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

For more details on multivalued maps see the books of Aubin and Cellina [4], Aubin and Frankowska [5] and Castaing and Valadier [10].

Definition 2.4 (See [23])**.** The Hadamard fractional integral of order r for a function $h : [1, \infty) \to \mathbb{R}$ is defined as

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds, \ r > 0,$$

provided the integral exists.

Definition 2.5 (See [23])**.** For a function h given on the interval $[1, \infty)$, the r Hadamard fractional-order derivative of h, is defined by

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n-r)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \frac{h(s)}{s} ds, \ n-1 < r < n,$$

where $n = [r] + 1$ and $[r]$ denotes the integer part of r and $\log(.) = \log_e(.)$.

For convenience, we first recall the statement of the nonlinear alternative of Leray–Schauder.

Theorem 2.6 (Nonlinear alternative of Leray–Schauder type, see [16])**.** *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : U \to C$ a continuous and compact operator. Then either*
(a) T has fixed points, or
(b) There exist $u \in \partial U$ and $\lambda \in [0, 1]$ with $u = \lambda T(u)$.

3 Main Results

Let us start by defining what we mean by a solution of the problem (1.1)–(1.3).

Definition 3.1. A function $y \in AC^1([1, T], \mathbb{R})$ is said to be a solution of (1.1)–(1.3) if there exist a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that ${}^H D^r y(t) = v(t)$ on J, and the conditions (1.2) and (1.3) are satisfied.

Lemma 3.2. *Let $h, \ \rho : [1, \infty) \to \mathbb{R}$ be continuous functions . A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T \rho(s) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s) \frac{ds}{s}\right] \end{aligned} \qquad (3.1)$$

if and only if y is a solution of the nonlinear fractional problem

$$^{H}D^{r}y(t) = h(t), \text{ for a.e. } t \in J = [1,T], \quad 2 < r \leq 3, \tag{3.2}$$

$$y(1) = y''(1) = 0, \tag{3.3}$$

$$y(T) = \int_1^T \rho(s)ds. \tag{3.4}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2), we have

$$y(t) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} + c_3 + {}^{H}I^{r}h(t). \tag{3.5}$$

First of all, from $y(1) = 0$, we have $c_3 = 0$.

Now by differentiating y, we have

$$\begin{aligned} y'(t) &= c_1(r-1)\frac{(\log t)^{r-2}}{t} \\ &+ c_2(r-2)\frac{(\log t)^{r-3}}{t} + \frac{(r-1)}{t\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-2}h(s)\frac{ds}{s}. \end{aligned} \tag{3.6}$$

Differentiating y for the second time, we find

$$\begin{aligned} y''(t) &= c_1(r-1)(r-2)\frac{(\log t)^{r-3}}{t^2} - c_1(r-1)\frac{(\log t)^{r-2}}{t^2} \\ &+ c_2(r-2)(r-3)\frac{(\log t)^{r-4}}{t^2} - c_2(r-2)\frac{(\log t)^{r-3}}{t^2} \\ &+ \frac{(r-1)(r-2)}{t^2\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-3}h(s)\frac{ds}{s} \\ &- \frac{(r-1)}{t^2\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-2}h(s)\frac{ds}{s}. \end{aligned} \tag{3.7}$$

Using the conditions (3.3) and (3.4), we find

$$c_2 = 0$$

and

$$c_1 = \frac{\int_1^T \rho(s)ds - {}^{H}I^{r}h(T)}{(\log T)^{r-1}}.$$

Hence we get equation (3.1).

Conversely, it is clear that if y satisfies equation (3.1), then equations (3.2), (3.3) and (3.4) hold. $\square$

Theorem 3.3. *Assume the following hypotheses hold:*

(H_1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,p}(\mathbb{R})$ *is a Carathéodory multi-valued map.*

(H_2) *There exist* $p \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(|u|) \text{ for } t \in J \text{ and each } u \in \mathbb{R}.$$

(H_3) *There exists* $l \in L^1(J, \mathbb{R}^+)$, *with* $I^r l < \infty$, *such that*

$$H_d(F(t,u), F(t,\bar{u})) \le l(t)|u - \bar{u}| \quad \text{for every } u, \bar{u} \in \mathbb{R},$$

and

$$d(0, F(t,0)) \le l(t), \ \ a.e.\ t \in J.$$

(H_4) *There exists* $k > 0$ *such that*

$$\|g(t, y(t))\| \le k, \text{ for each, } (t,y) \in J \times \mathbb{R}$$

(H_5) *There exists a number* $M > 0$ *such that*

$$\frac{M}{2\frac{(\log T)^r \psi(M)}{\Gamma(r+1)}\|p\|_{L^1} + (T-1)k} > 1. \tag{3.8}$$

Then the problem (1.1)–(1.3) *has at least one solution on* J.

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the multivalued operator,

$$N(y) = \left\{ h \in C(J,\mathbb{R}) : \begin{array}{rcl} y(t) & = & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ & + & \dfrac{(\log t)^{r-1}}{(\log T)^{r-1}}[\displaystyle\int_1^T g(s,y(s))ds \qquad v \in S_{F,y} \\ & & \dfrac{1}{\Gamma(r)}\displaystyle\int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}], \end{array} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)–(1.3). We shall show that N satisfies the assumptions of nonlinear alternative of Leray–Schauder fixed point theorem. The proof will be given in several steps.

Step 1

We show that $N(y)$ is convex for each $y \in C(J,\mathbb{R})$. Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$, we have, for $i = 1, 2$,

$$\begin{aligned} h_i(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_i(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_i(s)\frac{ds}{s}\right]. \end{aligned}$$

Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} (dh_1 + (1-d)h_2)(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s} \\ &+ \frac{2(\log t)^{r-1}}{(\log T)^{r-1}}[\int_1^T g(s,y(s))ds \\ &- \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s}]. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2

We show that N maps bounded sets into bounded sets in $C(J,\mathbb{R})$. Let $B_{\mu_*} = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \le \mu_*\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_{\mu_*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right]. \end{aligned}$$

By (H_2), we have, for each $t \in J$

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T |g(s,y(s))|ds + \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\right] \\ &\le \frac{(\log T)^r}{\Gamma(r+1)}\int_1^t p(s)\psi(|y(s)|) + (T-1)kds \\ &+ \frac{(\log T)^r}{\Gamma(r+1)}\int_1^T p(s)\psi(|y(s)|)ds \\ &\le 2\frac{(\log T)^r\psi(\mu^*)}{\Gamma(r+1)}\int_1^T p(s)ds + (T-1)k. \end{aligned}$$

Thus
$$\|h\|_\infty \leq 2\frac{(\log T)^r\psi(\mu^*)}{\Gamma(r+1)}\|p\|_{L^1} + (T-1)k := \ell.$$

Step 3

We show that N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let B_{μ^*} be bounded set of $C(J,\mathbb{R})$ as in Step 2. Let $y \in B_{\mu^*}$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(t_2) - h(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{v(s)}{s}ds\right. \\
&+ \left.\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right| \\
&\leq \frac{p(s)\psi(|y(s)|)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s} \\
&+ \frac{p(s)\psi(|y(s)|)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzelà–Ascoli theorem, we can conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Step 4

We show that N has a closed graph. Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$, such that, for each $t \in J$

$$\begin{aligned}
h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right].
\end{aligned}$$

We must show that there exists $v_* \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned}
h_*(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_*(s)\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_*(s)\frac{ds}{s}].
\end{aligned}$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \ \text{ as } \ m \to \infty,$$

and

$$v_*(t) \in F(t, y_*(t)), \ \text{ a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \le |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then

$$|v_{n_m}(t) - v_*(t)| \le d(v_{n_m}(t), F(t, y_*(t)).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_*, and it follows that

$$|v_{n_m}(t) - v_*(t)| \le H_d(F(t, y_n(t)), F(t, y_*(t))) \le l(t)\|y_n - y_*\|_\infty.$$

Then

$$\begin{aligned}
|h_n(t) - h_*(t)| &\le \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left(\frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \right) \\
&\le \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\|h_n(t) - h_*(t)\|_\infty &\le \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \to 0,
\end{aligned}$$

as $m \to \infty$.

Step 5

We discuss a priori bounds on solutions. Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$. Then there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned}
h(t) &= \frac{\lambda}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\
&+ \lambda \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right].
\end{aligned}$$

This implies by (H_2) that, for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &\le \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s} \\
&+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T \|g(s,y(s))\|ds + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v(s)|\frac{ds}{s}\right] \\
&\le \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s)\psi(|y(s)|)ds + \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s)\psi(|y(s)|)ds \\
&\le 2\frac{(\log T)^r \psi(\mu^*)}{\Gamma(r+1)} \int_1^T p(s)ds + (T-1)k.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{2\frac{(\log T)^r \psi(\|y\|_\infty)}{\Gamma(r+1)} \|p\|_{L^1} + (T-1)k} < 1.$$

Then by condition (3.8), there exists $M > 0$ such that $\|y\|_\infty \neq M$. Let $U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < M\}$. The operator $N : \bar{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1]$. As a consequence of the nonlinear alternative of Leray–Schauder, we deduce that N has a fixed point $y \in \bar{U}$ which is a solution of the problem (1.1)–(1.3). This completes the proof. □

We present now a result for the problem (1.1)–(1.3) with a nonconvex valued right hand side. Our considerations are based on the fixed point result in Lemma 2.3.

Theorem 3.4. *Assume* (H_3) *and the following hypothesis holds:*

(H_6) $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$.

If

$$2\frac{(\log T)^r}{\Gamma(r+1)} \|l\|_{L^1} < 1 \tag{3.9}$$

then the problem (1.1)–(1.3) *has at least one solution on* J.

Proof. We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in two steps.

Step 1

We show $N(y) \in \mathcal{P}_{cl}(C(J,\mathbb{R}))$ for each $y \in C(J,\mathbb{R})$. Indeed, let $(y_n)_{n\geq 0} \subset N(y)$ be such that $y_n \to \bar{y}$ in $C(J,\mathbb{R})$. Then, $\bar{y} \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned} y_n(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right] \end{aligned}$$

Using the fact that F has compact values and from (H$_3$), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem implies that v_n converges strongly to v and hence $v \in S_{F,y}$. Then for each $t \in J$,

$$\begin{aligned} y_n(t) &\to \bar{y}(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right]. \end{aligned}$$

So, $\bar{y} \in N(y)$.

Step 2

We show there exists $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) \leq \gamma\|y-\bar{y}\|_\infty$ for each $y, \bar{y} \in C(J,\mathbb{R})$. Let $y, \bar{y} \in C(J,\mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in F(t,y(t))$ such that for each $t \in J$

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v_1(s)\frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[\int_1^T g(s,y(s))ds - \frac{1}{\Gamma(r)} int_1^T \left(\log\frac{T}{s}\right)^{r-1} v_1(s)\frac{ds}{s}\right]. \end{aligned}$$

From (H$_3$) it follows that

$$H_d(F(t,y(t)), F(t,\bar{y})(t)) \leq l(t)|y(t)-\bar{y}(t)|.$$

Hence, there exists $w \in F(t,\bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t)-\bar{y}(t)|,\ t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t)-\bar{y}(t)|\}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J$

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Let us define for each $v_2 \in J$,

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_2(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\int_1^T g(s, y(s)) ds - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_2(s) \frac{ds}{s} \right]. \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \right] \\ &\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \\ &+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[\frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \right] \\ &\leq \left[2 \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T l(s) ds \right] \|y - \bar{y}\|_\infty. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_\infty \leq \left[2 \frac{(\log T)^r}{\Gamma(r+1)} \|l\|_{L^1} \right] \|y - \bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N(y), N(\bar{y})) \leq \left[2 \frac{(\log T)^r}{\Gamma(r+1)} \|l\|_{L^1} \right] \|y - \bar{y}\|_\infty.$$

So by (3.9), N is a contraction and thus, by Lemma 2.3, N has a fixed point y which is solution to (1.1)–(1.3). The proof is complete. □

4 An Example

We apply Theorem 3.3, to the the following fractional differential inclusion,

$$^H D^r y(t) = h(t), \ \text{ for a.e. } t \in J = [1, T], \quad 2 < r \leq 3, \tag{4.1}$$

$$y(1) = y''(1) = 0, \tag{4.2}$$

$$y(T) = \int_1^T g(s, y(s))ds, \tag{4.3}$$

where ${}^H D^r$ is the Hadamard fractional derivative, $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given function. Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \le v \le f_2(t, y)$$

where $f_1, f_2 : [1, T] \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [1, T]$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [1, T]$, $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C([1, T], \mathbb{R}^+)$ and $\psi : [0, \infty) \mapsto (0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, y)|, |f_2(t, y)| \le p(t)\psi(|y|), \quad t \in [1, T], \text{ and all } y \in \mathbb{R}.$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous. Assume there exists $k > 0$ such that

$$\|g(t, y(t))\| \le k, \text{ for each, } (t, y) \in J \times \mathbb{R}.$$

Finally we assume that there exists a number $M > 0$ such that

$$\frac{M}{2\frac{(\log T)^r \psi(M)}{\Gamma(r+1)}\|p\|_{L^1} + (T-1)k} > 1. \tag{4.4}$$

Since all the conditions of Theorem 3.3 are satisfied, problem (4.1)–(4.3) has at least one solution y on $[1, T]$.

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