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Measure of Noncompactness and Caputo-Hadamard Fractional Differantial Equations in Banach Spaces

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Abstract. In this paper, we discuss the existence of solutions for a boundary value problem of Caputo-Hadamard fractional differential equation . This result is based on Mönch's fixed point theorem and the technique of measures of noncompactness.
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1. Introduction

Fractional differential equations have attracted lots of attention in mathematics and physics because of their applications in memory and hereditary of various material. For details on fractional models and qualitative analysis, see monographs [1]-[6] and the papers [9], [12], [14] and [18].
Fractional calculus has also emerged as a powerful modeling tool for many real world problems. For examples and recent development of the topic, see the monographs of Hilfer [23], Kilbas *et al.* [26], Podlubny [28], and the papers by Agarwal *et al.* [3], Benchohra *et al.* [14, 13], Benhamida *et al.* [16] and Kilbas *et al.* [25].
The fractional derivative that Hadamard introduced in 1892, as that in 2012 F. Jarad, D. Baleanu and T. Abdeljawad [24], is considered a new type of fractional derivative which is called the Caputo Hadamard derivative that is a new approach obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo Hadamard derivative of a constant is zero. The most important advantage of Caputo Hadamard is that it brought a new definition through which the integer order initial conditions can be defined for fractional.A detailed description of the Hadamard fractional derivative and integral can be found in [18]-[20] and [15], [17].

In this paper we consider the existence of solutions for the following fractional differential equation

$$ {}^{c}_{H}D^{r}y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [1, T], \quad 0 < r \leq 1, \tag{1}$$

$$ ay(1) + by(T) = c, \tag{2}$$

where ${}^{c}_{H}D^{r}$ is the Caputo-Hadamard fractional derivatives, $f : [1, T] \times E \to E$ is a given function, $a, b, c \in \mathbb{R}$ such that $a + b \neq 0$.

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We will use the technique of measures of noncompactness which is often used in several branches of nonlinear analysis. Especially, that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in Akhmerov et al. [6], Alvàrez [7], Banaŝ et al. [8]-[12], Guo et al. [21], Mönch [27] and Szufla[29].

The principal goal here is to prove the existence of solutions for the above problem using Mönchs fixed point theorem and its related Kuratowski measure of noncompactness.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C([1,T],E)$ be the Banach space of all continuous functions from $[1,T]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \leq t \leq T\}.$$

Let $L^1([1,T],E)$ be the Banach space of measurable functions $y : [1,T] \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt.$$

Let the space

$$AC^n_\delta([a,b],E) = \{h : [a,b] \to E : \delta^{n-1}h(t) \in AC([a,b],E)\},$$

were $\delta = t\dfrac{d}{dt}$ is the Hadamard derivative and $AC([a,b],E)$ is the space of absolutely continuous functions on $[a,b]$.

Definition 2.1. *([26]). The Hadamard fractional integral of order r for a function $h : [1,+\infty) \to \mathbb{R}$ is defined as*

$$_HI^r h(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds,\ r > 0,$$

where Γ is the Gamma function.

Definition 2.2. *([26]). For a function h given on the interval $[1,+\infty)$, The Hadamard derivative of order r, is defined by*

$$\begin{aligned}(_HD^r h)(t) &= \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds, n-1<r<n\\ &= \delta^n(I^{n-r}h)(t),\end{aligned}$$

where $n = [r]+1$ and $[r]$ denotes the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.3. *([26])Let $r \geq 0$ $n = [r]+1$, $[r]$ denotes the integer part of r and $h \in AC^n_\delta[1,+\infty)$, . The Hadamard-Caputo fractional derivative of order r defined by*

$$\begin{aligned}(^C_HD^r h)(t) &= \frac{1}{\Gamma(n-r)}\int_1^t\left(\log\frac{t}{s}\right)^{n-r-1}\delta^n h(s)\frac{ds}{s}, n-1<r<n.\\ &= I^{n-r}(\delta^n h)(t).\end{aligned}$$

Lemma 2.4. *([26]). Let $h \in AC^n_\delta[1,+\infty)$,and $r > 0$ then*

$$I^r({}^C_H D^r h)(t) = h(t) - \sum_{i=0}^{n-1} \frac{\delta^i y(1)}{i!}(\log t)^i.$$

We recall the definition of the Kuratowski measure of noncompactness, and then summarize the main properties of this measure.

Definition 2.5. *([6, 9]) Let E be a Banach space and let Ω_E be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 \, : B \subset \bigcup_{j=1}^{m} B_j \text{ and } diam(B_j) \leq \epsilon\}\,; here\, B \in \Omega_E.$$

Properties:

(1) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(2) $\alpha(B) = \alpha(\overline{B})$.

(3) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(4) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(5) $\alpha(cB) = |c|\alpha(B), c \in \mathbb{R}$.

(6) $\alpha(conB) = \alpha(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively. The details of α and its properties can be found in [6, 9].

Definition 2.6. *A map $f : J \times E \to E$ is said to be Carathéodory if*

(1) $t \to f(t,u)$ is measurable for each $u \in E$.

(2) $u \to f(t,u)$ is continuous for almost all $t \in J$.

Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t) \, : v \in V, \}\, t \in J,$$
$$V(J) = \{v(t) \, : v \in V, t \in J\}.$$

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.7. *([27],[4]) Let D be a bounded, closed and convex subset of a Banach space E such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{co}N(V) \text{ or } V = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0, \tag{3}$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.8. *([21]) If $V \subset C(J,E)$ is a bounded and equicontinuous set, then*

(i) The function $t \to \alpha(v(t))$ is continuous on J

(ii)

$$\alpha(\{\int_J y(t)dt \, : y \in V\}) \leq \int_J \alpha(V(t))dt. \tag{4}$$

3. Main Results

Let us start by defining what we meant by a solution of the problem (1)–(2).

Definition 3.1. *A function $y \in AC^1_\delta([1,T],E)$ is said to be a solution of (1)–(2) if y satisfies the equation ${}^c_H D^r y(t) = f(t,y(t))$ on J, and the conditions (2).*

For the existence of solutions for the problem (1)–(2),we need the following auxiliary lemma.

Lemma 3.2. *Let $h : [1,+\infty) \to E$ be a continuous function. A function y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} h(s) \frac{d}{ds} - \frac{b}{\Gamma(r)(a+b)} \int_1^T (\log \frac{T}{s})^{r-1} h(s) \frac{d}{ds} + \frac{c}{(a+b)} \tag{5}$$

if and only if y is a solution of the fractional BVP

$$D^r y(t) = h(t) \qquad 0 < \alpha \le 1 \tag{6}$$

$$ay(1) + by(T) = c. \tag{7}$$

Proof. Assume y satisfies (6),then Lemma 2.4 implies that

$$y(t) = I^r f(t) + c_1 \tag{8}$$

The condition implies that

$$ay(1) + by(T) = bI^r f(T) + (a+b)c_1 = c$$

so

$$c_1 = \frac{c - bI^r f(T)}{(a+b)}$$

Finally ,we obtain the solution (5)

$$y(t) = I^r f(t) - \frac{b}{(a+b)} I^r f(T) + \frac{c}{(a+b)}.$$

Conversely, it is clear that if y satisfies equation (5), then equations (6)-(7) hold.

In the following,we prove existence results, for the boundary value problem (1)-(2) by using a Mönch fixed point theorem.

Theorem 3.3. *Assume the following hypotheses hold:*

(H1) The function $f : J \times E \longrightarrow E$ satisfies the Carathéodory conditions.

(H2) There exists $p \in L^1(J, \mathbb{R}_+)$, such that

$$\|f(t,y)\| \le p(t)\|y\| \text{ for a.e. } t \in J \text{ and each } y \in E.$$

(H3) For almost each $t \in J$ and each bounded set $B \subset E$, we have

$$\alpha(f(t,B)) \le p(t)\alpha(B),$$

where α is the Kuratowski measure of noncompacteness and $J_{t,k} = [t-k, t]$.

Then the BVP (1)–(2) has at least one solution in $C(J, B)$, provided that

$$(1 + \frac{|b|}{|a+b|})({}_H I^r p)(T) < 1. \tag{9}$$

Proof. Transform the problem (1)–(2) into a fixed point problem .Consider the operator

$$\begin{aligned}(Ny)(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} f(s, y(s))\frac{ds}{s} \\ &- \frac{b}{(a+b)\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} f(s, y(s))\frac{ds}{s} + \frac{c}{a+b}.\end{aligned} \tag{10}$$

Remark 3.4. *Clearly, from Lemma 3.2, the fixed points of N are solutions to (1)–(2).*

Let

$$R \geq \frac{\frac{|c|}{|a+b|}}{1 - (1 + \frac{|b|}{|a+b|})({}_H I^r p)(T)},$$

and consider the set

$$D_R = \{y \in C(J, E) \; : \|y\|_\infty \leq R\}.$$

Clearly, the subset D_R is closed, bounded and convex. We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, D_R)$. Then, for each $t \in J$,

$$\begin{aligned}|(Ny_n)(t) - (Ny)(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |f(s, y_n(s)) - f(s, y(s))|\frac{ds}{s} \\ &+ \frac{|b|}{|a+b|\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |f(s, y_n(s)) - f(s, y(s))|\frac{ds}{s}\end{aligned}$$

Let

$$\|y_n\|_\infty \leq R \text{ and } \|y\|_\infty \leq R.$$

By (H2) we have

$$\|f(s, y_n(s)) - f(s, y(s))\| \leq 2Rp(s) := \sigma(s); \;\; \sigma \in L^1(J, \mathbb{R}_+).$$

Since f satisfies the Carathéodory condition, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: *N maps D_R into itself.* For each $y \in D_R$, by (H2) and (9), we have for each $t \in J$,

$$\begin{aligned}|N(y)(t)| &\leq \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |f(s, y(s))|\frac{ds}{s} \\ &+ \frac{|b|}{|a+b|\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-1} |f(s, y(s))|\frac{ds}{s}. + \frac{|c|}{|a+b|} \\ &\leq (1 + \frac{|b|}{|a+b|})R({}_H I^r p)(T) + \frac{|c|}{|a+b|} \\ &\leq R.\end{aligned}$$

Step 3: *$N(D_R)$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_R) \subset C(J,E)$ is bounded.
For the equicontinuity of $N(D_R)$, let $t_1,\ t_2 \in J,\ t_1 <\ t_2$, and $y \in D_R$. We have

$$\begin{aligned}
|(Ny)(t_2)-(Ny)(t_1)| &\leq \frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_1}{s}\right)^{r-1}\right]|f(s,y(s))|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}|f(s,y(s))|\frac{ds}{s}\\
&\leq \frac{R}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_1}{s}\right)^{r-1}\right]p(s)\frac{ds}{s}\\
&+ \frac{R}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}p(s)\frac{ds}{s}\\
&\leq \frac{R\|p\|_{l^\infty}}{\Gamma(r+1)}\left[(\log t_2)^r-(\log t_1)^r\right].
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V \subset \overline{co}(N(V)\cup\{0\})$.
Since V is bounded and equicontinuous, and therefore the function $\vartheta \to \vartheta(t)=\alpha(V(t))$ is continuous on J.By (H2)-(H3), Lemma 2.8 and the properties of the measure ÃÂ±, we have, for each $t \in J$

$$\begin{aligned}
\vartheta(t) &\leq \alpha(N(V)(t)\cup\{0\})\\
&\leq \alpha(N(V)(t))\\
&\leq \frac{1}{\Gamma(r)}\int_1^{t}\left(\log\frac{t}{s}\right)^{r-1}p(s)\alpha(V(s))\frac{ds}{s}\\
&+ \frac{|b|}{|a+b|\Gamma(r)}\int_1^{T}\left(\log\frac{T}{s}\right)^{r-1}p(s)\alpha(V(s))\frac{ds}{s}\\
&\leq (1+\frac{|b|}{|a+b|})\int_1^{T}\frac{1}{\Gamma(r)}\left(\log\frac{T}{s}\right)^{r-1}p(s)\alpha(V(s))\frac{ds}{s}.
\end{aligned}$$

This means that

$$\|\vartheta\|_\infty\left(1-(1+\frac{|b|}{|a+b|})({}_HI^rp)(T)\right)\leq 0.$$

By (9), it follows that $\|\vartheta\|_\infty=0$, that is, $\vartheta(t)=0$ for each $t\in J$, and so $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 2.7, we conclude that N has a fixed point which is a solution of the problem (1)–(2).

4. Example

Let

$$E=l^1=\{(y_1,y_2,...,y_n,..),\sum_1^\infty|y_n|<\infty\}$$

be our Banach space with norm

$$\|y\|_E=\sum_1^\infty|y_n|$$

we apply the main result of the paper to the following system of fractional differential equations

$$ {}^c_HD^ry(t)=\frac{(\log t)^3}{48}|y_n(t)|,\ \text{for a.e. } y\in E,\ t\in J=[1,e],\quad 0<r\leq 1, \tag{11}$$

$$y(1, \frac{2}{4}, \frac{2}{8}, ...) + y(e, \frac{2e}{4}, \frac{2e}{8}, ...) = (1, \frac{2}{4}, \frac{2}{8}, ...), \tag{12}$$

here $T = e, \quad a = b = c = 1$, Where Where

$$f_n(t, y) = \frac{(\log t)^3}{48} |y_n(t)|, \ \ (t, y) \in J \times E,$$

and

$$y = (y_1, y_2, ..., y_n, ...)$$

Set

$$f = (f_1, f_2, ..., f_n, ...)$$

Clearly, condition (H1) holds, and conditions (H2)hold with

$$p(t) = \frac{(\log t)^3}{48}.$$

and

$$(I^r p)(e) = \frac{\Gamma(4)}{48\Gamma(r+4)} = \frac{1}{2\Gamma(r+4)}$$

so, we shall check that condition (9) is satisfied. Indeed,

$$(1 + \frac{|b|}{|a+b|})({}_H I^r p)(T) = \frac{3}{4\Gamma(r+4)} < 1 \text{ , if } \Gamma(r+4) > 3/4, \tag{13}$$

which is satisfied for some $r \in (0, 1]$. Then by theorem (3.3), the problem (11)-(12) has has at least one solution on $[1, e]$ for values of r satisfying (13).

References

[1] Agarwal, R. P., Zhou, Y. and He, Y. (2010). Existence of fractional neutral functional differential equations. Computers & Mathematics with Applications, 59(3): 1095-1100.

[2] Agarwal, R. P., Zhou, Y., Wang, J. and Luo, X. (2011). Fractional functional differential equations with causal operators in Banach spaces. Mathematical and Computer Modelling, 54(5-6): 1440-1452.

[3] Agarwal, R. P., Benchohra, M. and Hamani, S. (2010). A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Applicandae Mathematicae, 109(3) :973-1033.

[4] Meehan, M. and O'Regan, D. (2001). Fixed Point Theory and Applications. Cambridge Tracts in Mathematics. Cambridge University Press.

[5] Ahmed, B. and Ntouyas, S. K. (2014). Initial value problems for hybrid Hadamard fractional equations. EJDE, 2014: 1-8.

[6] Akhmerov, R. R., Kamenski M. I., Patapov, A. S., Rodkina, A. E. and Sadovski, B. N. (1992). Measure of noncompactness and condensing operators: Translated from the 1986 Russian original by A. Iacop. Operator theory: Advances and Applications, 55, Birhhauser Verlag, Bassel.

[7] Alvárez, J. C. (1985). Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces. Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid, 79: 53-66.

[8] Banaś, J. (1997). Applications of measures of weak noncompactness and some classes of operators in the theory of functional equations in the Lebesgue space. Nonlinear Analysis: Theory, Methods & Applications, 30(6): 3283-3293.

[9] Bana‘s, J. and Goebel, K. (1980). Measures of Noncompactness in Banach Spaces,In lecture Notes in Pure and Applied Mathematics, Vol 60, Marcel Dekker, New York.

[10] Bana‘s, J. and Olszowy, L. (2001). Measures of noncompactness related to monotonicity, Comment. Math. 41: 13-23.

[11] Bana‘s, J. and Rzepka, B. (2003). An application of a measure of noncompactness in the study of asymptotique stability, Appl. Math. Lett. 16: 1-6.

[12] Bana's, J. and Sadarangani, K. (2008). On some measures of noncompactness in the space of continous functions, Nonlinear Anal. 60(2): 377-383.

[13] Benchohra, M. and Hamani, S. (2009). Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal. 71: 2391-2396.

[14] Benchohra, M., Hamani, S. and Ntouyas, S. K. (2008). Boundary value problems for differential equations with fractional order, Surv. Math. Appl. 3: 1-12.

[15] Benhamida, W., Hamani, S. and Henderson, J. (2016). A Boundary Value Problem for Fractional Differential Equations with Hadamard Derivative and Nonlocal Conditions, PanAmerican Math. 26: 1-11

[16] Benhamida, W., Graef, J. R. and Hamani, S. (2018). Boundary value problems for fractional differential equations with integral and anti-periodic conditions in a Banach space, Prog. Frac. Differ. Appl. 4(2): 65-70.

[17] Benhamida, W., Graef, J. R. and Hamani, S. (2018). Boundary value problems for Hadamard fractional differential equations with nonlocal multi-point boundary conditions, Frac-Diff Calculus, 8(1): 165-176.

[18] Butzer, P. L., Kilbas A. A. and Trujillo, J. J. (2002). Composition of Hadamard-type fractional integration operators and the semigroup property, J. Math. Anal. Appl. 269: 387-400.

[19] Butzer, P. L., Kilbas A. A. and Trujillo, J. J. (2002). Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, J. Math. Anal. Appl. 269: 1-27.

[20] Butzer, P. L., Kilbas A. A. and Trujillo, J. J. (2002). Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, J. Math. Anal. Appl. 270: 1-15.

[21] Guo, D., Lakshmikantham, V. and Liu, X. (1996). Nonlinear integral equations in abstract spaces, Mathematics and its Applications, 373, Kluwer Academic Publishers, Dordrecht, The Netherlands.

[22] Hadamard, H. (1892). Essai sur l'etude des fonctions donnees par leur development de Taylor, J. Mat. Pure Appl. Ser. 8: 101-186.

[23] Hilfer, R. (2000). Applications of Fractional Calculus in Physics, World Scientific, Singapore.

[24] Jarad, F., Abdeljawad, T. and Baleanu, D. (2012). Caputo-type modification of the Hadamard fractional derivatives. Advances in Difference Equations, 2012(1): 142.

[25] Kilbas, A. A. and Marzan, S. A. (2005). Nonlineardifferential equations with the Caputo fractional derivative in the space of continuously differentiable functions, Differential Equations 41: 84-89.

[26] Kilbas, A. A., Srivastava H. M. and Trujillo, J. J. (2006). Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam.

[27] Mönch, H. (1980). Boundary value problem for nonlinear ordinary differential equations of second order in Banach spaces, Nonlinear Anal. 4: 985-999.

[28] Podlubny, I . (1999). Fractional Differential Equation, Academic Press, San Diego.

[29] Szufla, S. (1986). On the application of measure of noncompactness to existence theorems, Rendiconti del Seminaro Mathematico Della Universita di Padova, 75: 1-14.