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# New Inequalities in Fractional Integrals

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**Abstract:** In this paper, we use the Riemann-Liouville fractional integral to present recent results on fractional integral inequalities. By considering the extended Chebyshev functional in the case of synchronous functions, we establish two main results. The first one deals with some inequalities using one fractional parameter. The second result concerns others inequalities using two fractional parameters.

**Keywords:** Chebyshev inequality; Fractional integral inequalities; Riemann-Liouville fractional integral

## 1 Introduction

Let us consider the functional

$$\begin{aligned} T(f,g,p,q) := \int_a^b q(x)dx \int_a^b p(x)f(x)g(x)\,dx + \int_a^b p(x)dx \int_a^b q(x)f(x)g(x)\,dx \\ - \left(\int_a^b q(x)f(x)\,dx\right)\left(\int_a^b p(x)g(x)\,dx\right) - \left(\int_a^b p(x)f(x)\,dx\right)\left(\int_a^b q(x)g(x)\,dx\right), \end{aligned} \tag{1}$$

where $f$ and $g$ are two integrable functions on $[a,b]$ and $p,q$ are positive integrable functions on $[a,b]$. If $f$ and $g$ are synchronous on $[a,b]$ ( i.e. $(f(x)-f(y))(g(x)-g(y)) \geq 0$, for any $x,y \in [a,b]$ ), then $T(f,g,p,q) \geq 0$ ( see [6, 8] ). The sign of this inequality is reversed if $f$ and $g$ are asynchronous on $[a,b]$ ( i.e. $(f(x)-f(y))(g(x)-g(y)) \leq 0$, for any $x,y \in [a,b]$ ). For $p(x)=q(x), x \in [a,b]$, we get the Chebyshev inequality [3]. In [9], Ostrowski established the following generalization of the Chebyshev inequality:
If $f$ and $g$ are two differentiable functions, synchronous on $[a,b]$, $p$ is a positive integrable function on $[a,b]$ and $|f'(x)| \geq m$, $|g'(x)| \geq r$, for $x \in [a,b]$, then

$$T(f,g,p) := T(f,g,p,p) \geq mrT(x-a,x-a;p) \geq 0. \tag{2}$$

If $f$ and $g$ are asynchronous on $[a,b]$, then

$$T(f,g,p) \leq mrT(x-a,b-x;p) \leq 0.$$

If $f$ and $g$ are two differentiable functions on $[a,b]$, $p$ is a positive integrable function on $[a,b]$ and $|f'(x)| \leq M$, $|g'(x)| \leq R$, for $x \in [a,b]$, then

$$|T(f,g,p)| \leq MRT(x-a,x-a;p) \leq 0. \tag{3}$$

Many researchers have given considerable attention to the functional $T(f,g,p)$ and a number of extensions, generalizations and variants have appeared in the literature, see [1, 2, 4, 7] and the references given therein.

The main purpose of this paper is to use the Riemann-Liouville fractional integral to establish some new fractional integral inequalities using the extended Chebyshev functional (1).

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# 2 Description of the Fractional Calculus

In the following, we will give the necessary notation and basic definitions. More details, one can consult [5,10].

**Definition 1** *A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty))$.*

**Definition 2** *A function $f(t), t > 0$ is said to be in the space $C_\mu^n, n \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.*

**Definition 3** *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, \ t > 0, J^0 f(t) = f(t), \tag{4}$$

*where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.*

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t); \ \alpha \geq 0, \ \beta \geq 0, \tag{5}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{6}$$

For the expression (4), when $f(t) = t^\mu$ we get another expression that will be used later:

$$J^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}; \ \alpha > 0, \ \ \mu > -1, \ t > 0. \tag{7}$$

**Remark 1** *In what follows we shall consider the real valued functions defined on the space $C_\mu, (\mu \geq -1)$.*

# 3 Main Results

**Theorem 2** *Let $f$ and $g$ be two synchronous functions on $[0,\infty)$ and let $r, p, q : [0,\infty) \to [0,\infty)$. Then for all $t > 0, \alpha > 0$, we have:*

$$\begin{aligned} &2J^\alpha r(t)\Big[J^\alpha p(t) J^\alpha(qfg)(t) + J^\alpha q(t) J^\alpha(pfg)(t)\Big] + 2J^\alpha p(t) J^\alpha(q)(t)) J^\alpha(rfg)(t) \geq \\ &J^\alpha r(t)\Big[J^\alpha(pf)(t) J^\alpha(qg)(t) + J^\alpha(qf)(t) J^\alpha(pg)(t)\Big] + J^\alpha p(t)\Big[J^\alpha(rf)(t) J^\alpha(qg)(t) + \\ &J^\alpha(qf)(t) J^\alpha(rg)(t)\Big] + J^\alpha q(t)\Big[J^\alpha(rf)(t) J^\alpha(pg)(t) + J^\alpha(pf)(t) J^\alpha(rg)(t)\Big]. \end{aligned} \tag{8}$$

**Lemma 3** *Let $f$ and $g$ be two synchronous functions on $[0,\infty)$ and let $v, w : [0,\infty) \to [0,\infty)$. Then for all $t > 0, \alpha > 0$, we have:*

$$J^\alpha v(t) J^\alpha(wfg)(t) + J^\alpha w(t) J^\alpha(vfg)(t) \geq J^\alpha(vf)(t) J^\alpha(wg)(t) + J^\alpha(wf)(t) J^\alpha(vg)(t). \tag{9}$$

**Proof.** Since the functions $f$ and $g$ are synchronous on $[0,\infty)$, then for all $\tau \geq 0, \rho \geq 0$, we have

$$\Big(f(\tau) - f(\rho)\Big)\Big(g(\tau) - g(\rho)\Big) \geq 0. \tag{10}$$

Therefore

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{11}$$

Multiplying both sides of (11) by $\frac{(t-\tau)^{\alpha-1} v(\tau)}{\Gamma(\alpha)}$, $\tau \in (0,t)$, we get

$$\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}v(\tau)f(\tau)g(\tau)+\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}v(\tau)f(\rho)g(\rho)\geq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}v(\tau)f(\tau)g(\rho)+\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}v(\tau)f(\rho)g(\tau). \tag{12}$$

Integrating (12) over $(0,t)$, we obtain:

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)f(\tau)g(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)f(\rho)g(\rho)d\tau\geq \frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)f(\tau)g(\rho)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)f(\rho)g(\tau)d\tau. \tag{13}$$

Consequently

$$J^\alpha(vfg)(t)+f(\rho)\,g(\rho)\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)d\tau\geq \frac{g(\rho)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)f(\tau)\,d\tau+\frac{f(\rho)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}v(\tau)g(\tau)\,d\tau. \tag{14}$$

So we have

$$J^\alpha(vfg)(t)+f(\rho)\,g(\rho)\,J^\alpha(v)(t)\geq g(\rho)\,J^\alpha(vf)(t)+f(\rho)\,J^\alpha(vg)(t). \tag{15}$$

Now multiplying both sides of (15) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho),\ \rho\in(0,t)$, we obtain:

$$\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho)J^\alpha(vfg)(t)+\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho)f(\rho)\,g(\rho)\,J^\alpha(v)(t)\geq \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho)g(\rho)\,J^\alpha(vf)(t)+\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho)f(\rho)\,J^\alpha(vg)(t). \tag{16}$$

Integrating (16) over $(0,t)$, we get:

$$J^\alpha(vfg)(t)\int_0^t\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}w(\rho)d\rho+\frac{J^\alpha(v)(t)}{\Gamma(\alpha)}\int_0^t w(\rho)f(\rho)g(\rho)(t-\rho)^{\alpha-1}d\rho\geq \frac{J^\alpha(vf)(t)}{\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}w(\rho)g(\rho)d\rho+\frac{J^\alpha(vg)(t)}{\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}w(\rho)f(\rho)d\rho. \tag{17}$$

Therefore

$$J^\alpha(w)(t)J^\alpha(vfg)(t)+J^\alpha(v)(t)J^\alpha(wfg)(t)\geq J^\alpha(vf)(t)J^\alpha(wg)(t)+J^\alpha(wf)(t)J^\alpha(vg)(t), \tag{18}$$

and this ends the proof of Lemma 3. ■
Proof of Theorem 2:
**Proof.** Putting $v=p,w=q$ and using Lemma 3, we can write:

$$J^\alpha(p)(t)J^\alpha(qfg)(t)+J^\alpha(q)(t)J^\alpha(pfg)(t)\geq J^\alpha(pf)(t)J^\alpha(qg)(t)+J^\alpha(qf)(t)J^\alpha(pg)(t). \tag{19}$$

Multiplying both sides of (19) by $J^\alpha(r)(t)$, we obtain:

$$J^\alpha(r)(t)\Big[J^\alpha(p)(t)J^\alpha(qfg)(t)+J^\alpha(q)(t)J^\alpha(pfg)(t)\Big]\geq J^\alpha(r)(t)\Big[J^\alpha(pf)(t)J^\alpha(qg)(t)+J^\alpha(qf)(t)J^\alpha(pg)(t)\Big]. \tag{20}$$

Putting $v=r,w=q$ and using again Lemma 3, we get:

$$J^\alpha(r)(t)J^\alpha(qfg)(t)+J^\alpha(q)(t)J^\alpha(rfg)(t)\geq J^\alpha(rf)(t)J^\alpha(qg)(t)+J^\alpha(qf)(t)J^\alpha(rg)(t). \tag{21}$$

Multiplying both sides of (21) by $J^\alpha(p)(t)$, we get:

$$J^\alpha(p)(t)\Big[J^\alpha(r)(t)J^\alpha(qfg)(t)+J^\alpha(q)(t)J^\alpha(rfg)(t)\Big]\geq J^\alpha(p)(t)\Big[J^\alpha(rf)(t)J^\alpha(qg)(t)+J^\alpha(qf)(t)J^\alpha(rg)(t)\Big]. \tag{22}$$

With the same arguments as before, we can obtain:

$$J^{\alpha}(q)(t)\Big[J^{\alpha}(r)(t)J^{\alpha}(pfg)(t)+J^{\alpha}(p)(t)J^{\alpha}(rfg)(t)\Big]\geq \\ J^{\alpha}(q)(t)\Big[J^{\alpha}(rf)(t)J^{\alpha}(pg)(t)+J^{\alpha}(pf)(t)J^{\alpha}(rg)(t)\Big]. \tag{23}$$

The required inequality (8) follows on adding the inequalities (20,22,23). ■

Our second result is:

**Theorem 4** *Let $f$ and $g$ be two synchronous functions on $[0,\infty)$ and let $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, $\beta>0$, we have:*

$$J^{\alpha}r(t)\Big[J^{\alpha}q(t)J^{\beta}(pfg)(t)+2J^{\alpha}p(t)J^{\beta}(qfg)(t)+J^{\beta}q(t)J^{\alpha}(pfg)(t)\Big]+ \\ \Big[J^{\alpha}p(t)J^{\beta}(q)(t))+J^{\beta}p(t)J^{\alpha}(q)(t)\Big]J^{\alpha}(rfg)(t)\geq \\ J^{\alpha}r(t)\Big[J^{\alpha}(pf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(pg)(t)\Big]+ \\ J^{\alpha}p(t)\Big[J^{\alpha}(rf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(rg)(t)\Big]+ \\ J^{\alpha}q(t)\Big[J^{\alpha}(rf)(t)J^{\beta}(pg)(t)+J^{\beta}(pf)(t)J^{\alpha}(rg)(t)\Big]. \tag{24}$$

**Remark 5** *Applying Lemma 3 for $\alpha=\beta$, we obtain Theorem 2 and for $\alpha=\beta=1, p(x)=q(x)=r(x)=1$, for any $x\in[0,\infty[$, we obtain the Chebyshev inequality on $[0,t]$, (see [3]).*

To prove Theorem 4, we need the following lemma:

**Lemma 6** *Let $f$ and $g$ be two synchronous functions on $[0,\infty]$ and let $v,w:[0,\infty]\to[0,\infty]$. Then for all $t>0,\alpha>0$, we have:*

$$J^{\alpha}v(t)J^{\beta}(wfg)(t)+J^{\beta}w(t)J^{\alpha}(vfg)(t)\geq J^{\alpha}(vf)(t)J^{\beta}(wg)(t)+J^{\beta}(wf)(t)J^{\alpha}(vg)(t) \tag{25}$$

Proof of Lemma 6:

**Proof.** Multiplying both sides of (15) by $\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho),\ \rho\in(0,t)$, we obtain:

$$\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho)J^{\alpha}(vfg)(t)+J^{\alpha}(v)(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho)f(\rho)g(\rho)\geq \\ \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho)g(\rho)J^{\alpha}(vf)(t)+\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho)f(\rho)J^{\alpha}(vg)(t). \tag{26}$$

Integrating (26) over $(0,t)$, we obtain

$$J^{\alpha}(vfg)(t)\int_0^t\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}w(\rho)d\rho+\frac{J^{\alpha}v(t)}{\Gamma(\beta)}\int_0^t w(\rho)f(\rho)g(\rho)(t-\rho)^{\beta-1}d\rho\geq \\ \frac{J^{\alpha}(vf)(t)}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}w(\rho)g(\rho)d\rho+\frac{J^{\alpha}(vg)(t)}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}w(\rho)f(\rho)d\rho. \tag{27}$$

Lemma 6 is thus proved. ■

Proof of Theorem 4:

**Proof.** Using Lemma 6 with $v=p,w=q$, we can write:

$$J^{\alpha}(p)(t)J^{\beta}(qfg)(t)+J^{\beta}(q)(t)J^{\alpha}(pfg)(t)\geq J^{\alpha}(pf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(pg)(t). \tag{28}$$

Multiplying both sides of (28) by $J^{\alpha}(r)(t)$, we obtain:

$$J^{\alpha}(r)(t)\Big[J^{\alpha}(p)(t)J^{\beta}(qfg)(t)+J^{\beta}(q)(t)J^{\alpha}(pfg)(t)\Big]\geq \\ J^{\alpha}(r)(t)\Big[J^{\alpha}(pf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(pg)(t)\Big]. \tag{29}$$

Using Lemma 6 with $v=r,w=q$ and then multiplying both sides of (29)by $J^{\alpha}(p)(t)$, we obtain:

$$J^{\alpha}(p)(t)\Big[J^{\alpha}(r)(t)J^{\beta}(qfg)(t)+J^{\beta}(q)(t)J^{\alpha}(rfg)(t)\Big]\geq \\ J^{\alpha}(p)(t)\Big[J^{\alpha}(rf)(t)J^{\beta}(qg)(t)+J^{\beta}(qf)(t)J^{\alpha}(rg)(t)\Big]. \tag{30}$$

With the same arguments, we can get:

$$J^{\alpha}(q)(t)\Big[J^{\alpha}(r)(t)J^{\beta}(pfg)(t)+J^{\beta}(p)(t)J^{\alpha}(rfg)(t)\Big]\geq \\ J^{\alpha}(q)(t)\Big[J^{\alpha}(rf)(t)J^{\beta}(pg)(t)+J^{\beta}(pf)(t)J^{\alpha}(rg)(t)\Big]. \tag{31}$$

The inequality (24) follows on adding the inequalities (29,30,31). ■

**Remark 7** *The inequalities (8) and (24) are reversed in the following cases:*
$a$. *The functions $f$ and $g$ asynchronous on* $[0,\infty)$.
$b$. *The functions $r, p, q$ are negative on* $[0,\infty)$.
$c$. *Two of the functions $r, p, q$ are positive and the third one is negative on* $[0,\infty)$.

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