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Review Article

On the k-Riemann-Liouville fractional integral and applications

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Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as differential calculus and goes back to times when G.W. Leibniz and I. Newton invented differential calculus. Fractional integrals and derivatives arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium. Very recently, Mubeen and Habibullah have introduced the k-Riemann-Liouville fractional integral defined by using the -Gamma function, which is a generalization of the classical Gamma function. In this paper, we presents a new fractional integration is called k-Riemann-Liouville fractional integral, which generalizes the k-Riemann-Liouville fractional integral. Then, we prove the commutativity and the semi-group properties of the k-Riemann-Liouville fractional integral and we give Chebyshev inequalities for k-Riemann-Liouville fractional integral. Later, using k-Riemann-Liouville fractional integral, we establish some new integral inequalities.

Keywords: Riemann-liouville fractional integral, convex function, hermite-hadamard inequality and hölder's inequality.

Mathematics Subject Classification: 26A33;26A51;26D15.

INTRODUCTİON

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation, and include nth derivatives and n-fold integrals (*n* denotes an integer number) as particular cases. Because of this, it would be ideal to have such physical and geometric interpretations of fractional-order operators, which will provide also a link to known classical interpretations of integer-order differentiation and integration.

Obviously, there is still a lack of geometric and physical interpretation of fractional integration and differentiation, which is comparable with the simple interpretations of their integer-order counterparts.

During the last two decades several authors have applied the fractional calculus in the field of sciences, engineering and mathematics (see,Atanackovic et. al (2009)-Atanackovic et. al (2010), Gorenflo and Mainardi (1997), Malinowska and Torres (2011)-Miller and Ross (1993),El-Nabulsi(2005)-Odzijewicz et. al (2012), Samko et. al (1993)). Mathematician Liouville, Riemann, and Caputo have done major work on fractional calculus, thus Fractional Calculus is a useful mathematical tool for applied sciences. Podlubny suggested a solution of more than 300 years old problem of geometric and physical interpretation of fractional integration and differentiation in 2002, for left-sided and right-sided of Riemann-Liouville fractional integrals (see, Anastassiou (2009), Belarbi and Dahmani (2009)-Dahmani et. al (2010), Katugopola

(2011), Latif and Hussain (2012), Sarikaya and Ogunmez (2012)-Sarikaya and Yaldiz (2013)).

The first is the Riemann-Liouville fractional integral of $\alpha \geq 0$ for a continuous function f on $[a,b]$ which is defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \geq 0, \ a < x \leq b.$$

motivated by the Cauchy integral formula

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{n-1} f(t)dt,$$

well-defined for $n \in \mathbb{N}^*$. The second is the Hadamard fractional integral introduced by Hadamard (1892), and given by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad \alpha > 0, \ x > a,$$

This is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} ... \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t} \right)^{n-1} f(t) \frac{dt}{t}$$

for $n \in \mathbb{N}^*$.

Recently, Diaz and Pariguan (2007) have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol that is generalization of the classical gamma and beta functions and the classical Pochhammer symbol.

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \ k > 0$$

Where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt.$$

Clearly, $\Gamma(x) = \lim_{k \to 1} \Gamma_k(x), \ \Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k})$ and $\Gamma_k(x+k) = x\Gamma_k(x)$. Furthermore, k -beta function defined as follows

$$B_k(x,y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt$$

So that $B_k(x,y) = \frac{1}{k} B(\frac{x}{k}, \frac{y}{k})$ and $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$. Later, under these definitions, Mubeen and Habibullah (2012) have introduced the k-fractional intregral of the Riemann-Liouville type as follows:

$${}_kJ^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad \alpha > 0, \ x > 0, \ k > 0.$$

Note that when $k \to 1$, then it reduces to the classical Riemann-liouville fractional integrals.

k -Riemann-Liouville fractional integral

Here we want to present the fractional integration which generalizes all of the above Rimann-Liouville fractional integrals as follows (see Romero (2013)): Let α be a real non negative number. Let f be piece wise continuous on $I' = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then for $t > 0$, we consider k -Riemann-Liouville fractional integral of f of order α

$$ {}_kJ_a^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a, \ k > 0. $$

Theorem 1. *Let* $f \in L_1[a,b], \ a > 0.$ *Then,* ${}_kJ_a^{\alpha} f(x)$ *exists almost everywhere on* $[a,b]$ *and* ${}_kJ_a^{\alpha} f(x) \in L_1[a,b]$.

Proof Define $P : \Delta := [a,b]\times[a,b] \to \mathrm{R}$ by $P(x,t) = \left[(x-t)^{\frac{\alpha}{k}-1}\right]_+$, that is,

$$ P(x,t) = \begin{cases} (x-t)^{\frac{\alpha}{k}-1} & , a \le t \le x \le b \\ 0 & , a \le x \le t \le b. \end{cases} $$

Thus, since P is measurable on $\Delta,$ we obtain

$$ \int_a^b P(x,t)dt = \int_a^x P(x,t)dt + \int_x^b P(x,t)dt = \int_a^x (x-t)^{\frac{\alpha}{k}-1} dt = \frac{k}{\alpha}(x-a)^{\frac{\alpha}{k}}. $$

By using the repeated integral, we obtain

$$ \int_a^b\left(\int_a^b P(x,t)\left|f(x)\right|dt\right)dx = \int_a^b \left|f(x)\right|\left(\int_a^b P(x,t)dt\right)dx $$

$$ = \frac{k}{\alpha}\int_a^b (x-a)^{\frac{\alpha}{k}}\left|f(x)\right|dx $$

$$ \le \frac{k}{\alpha}(b-a)^{\frac{\alpha}{k}}\int_a^b \left|f(x)\right|dx $$

$$ = \frac{k}{\alpha}(b-a)^{\frac{\alpha}{k}}\left\|f(x)\right\|_{L_1[a,b]} < \infty. $$

Therefore, the function $Q : \Delta \to \mathrm{R}$ such that $Q(x,t) := P(x,t) f(x)$ is integrable over Δ by Tonelli's theorem. Hence, by Fubini's theorem $\int_a^b P(x,t) f(x)dx$ is an integrable function on $[a,b]$, as a function of $t \in [a,b]$. That is, ${}_kJ_a^{\alpha} f(x)$ is integrable on $[a,b]$.

Theorem 2. *Let* $\alpha \ge 1, \ k > 0$ *and* $f \in L_1[a,b]$. *Then,* ${}_kJ_a^{\alpha} f \in C[a,b]$.

Proof For $\frac{\alpha}{k} = 1$ is trivial, thus we assume $\frac{\alpha}{k} \ne 1$. Let $x, y \in [a,b], \ x \le y$ and $x \to y$. Then we write

$$\left|{}_kJ_a^\alpha f(x)-{}_kJ_a^\alpha f(y)\right|$$

$$=\frac{1}{k\Gamma_k(\alpha)}\left|\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt-\int_a^y (y-t)^{\frac{\alpha}{k}-1} f(t)dt\right|$$

$$=\frac{1}{k\Gamma_k(\alpha)}\left|\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt-\int_x^y (y-t)^{\frac{\alpha}{k}-1} f(t)dt-\int_a^x (y-t)^{\frac{\alpha}{k}-1} f(t)dt\right|$$

$$\leq\frac{1}{k\Gamma_k(\alpha)}\left\{\int_a^x \left|(x-t)^{\frac{\alpha}{k}-1}-(y-t)^{\frac{\alpha}{k}-1}\right|\left|f(t)\right|dt+\int_x^y (y-t)^{\frac{\alpha}{k}-1}\left|f(t)\right|dt\right\}$$

$$\leq\frac{1}{k\Gamma_k(\alpha)}\left\{\int_a^x \left|(x-t)^{\frac{\alpha}{k}-1}-(y-t)^{\frac{\alpha}{k}-1}\right|\left|f(t)\right|dt+(y-x)^{\frac{\alpha}{k}-1}\left\|f(t)\right\|_{L_1[a,b]}\right\}$$

Since weget $(x-t)^{\frac{\alpha}{k}-1}\to(y-t)^{\frac{\alpha}{k}-1}$ as $x\to y,$ then

$$\left|(x-t)^{\frac{\alpha}{k}-1}-(y-t)^{\frac{\alpha}{k}-1}\right|\to 0$$

and also we have

$$\left|(x-t)^{\frac{\alpha}{k}-1}-(y-t)^{\frac{\alpha}{k}-1}\right|\leq 2(b-a)^{\frac{\alpha}{k}-1}.$$

Therefore, by dominated convergence theorem we obtain $\left|{}_kJ_a^\alpha f(x)-{}_kJ_a^\alpha f(y)\right|\to 0$ as $x\to y,$ that is, ${}_kJ_a^\alpha f\in C[a,b].$

Now, we give semi group properties of the k -Riemann-Liouville fractional integral:

Theorem 3. *Let f be continuous on I and let $\alpha,\beta>0,\ a>0.$ Then for all x ,*

$${}_kJ_a^\alpha\left[{}_kJ_a^\beta f(x)\right]={}_kJ_a^{\alpha+\beta} f(x)={}_kJ_a^\beta\left[{}_kJ_a^\alpha f(x)\right],\ k>0.$$

Proof By definition of the k -fractional integral and by using Dirichlet's formula, we have

$${}_kJ_a^\alpha\left[{}_kJ_a^\beta f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}\ {}_kJ_a^\beta f(t)dt$$

$$=\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}\left[\frac{1}{k\Gamma_k(\beta)}\int_a^t (t-\tau)^{\frac{\beta}{k}-1} f(\tau)d\tau\right]dt$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x f(\tau)\left[\int_\tau^x (x-t)^{\frac{\alpha}{k}-1}(t-\tau)^{\frac{\beta}{k}-1}dt\right]d\tau. \qquad (1)$$

The inner integral is evaluated by the change of variable $y=(t-\tau)/(x-\tau);$

$$\int_\tau^x (x-t)^{\frac{\alpha}{k}-1}(t-\tau)^{\frac{\beta}{k}-1}dt=(x-\tau)^{\frac{\alpha+\beta}{k}-1}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy$$

$$=(x-\tau)^{\frac{\alpha+\beta}{k}-1}kB_k(\alpha,\beta). \qquad (2)$$

According to the k -beta function and by (1) and (2), we obtain

$$ {}_kJ_a^{\alpha}\left[{}_kJ_a^{\beta} f(x)\right] = \frac{1}{k\Gamma_k(\alpha+\beta)} \int_a^x \left(x-\tau\right)^{\frac{\alpha+\beta}{k}-1} f(\tau)d\tau $$

$$ = {}_kJ_a^{\alpha+\beta} f(x). $$

This completes the proof of the Theorem 3.

Theorem 4. *Let* $\alpha, \beta > 0, \; a > 0.$ *Then there holds the formula,*

$$ {}_kJ_a^{\alpha}\left(\left(x-a\right)^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}\left(x-a\right)^{\frac{\alpha+\beta}{k}-1}, \quad k>0 \qquad (3) $$

where Γ_k *denotes the* k *-gamma function.*

Proof By definition of the k -fractional integral and by the change of variable $y = \left(x-t\right)/\left(x-a\right)$, we get

$$ {}_kJ_a^{\alpha}\left(\left(x-a\right)^{\frac{\beta}{k}-1}\right) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x \left(x-t\right)^{\frac{\alpha}{k}-1}\left(t-a\right)^{\frac{\beta}{k}-1} dt $$

$$ = \frac{\left(x-a\right)^{\frac{\alpha+\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy $$

$$ = \frac{\left(x-a\right)^{\frac{\alpha+\beta}{k}-1}}{\Gamma_k(\alpha)} B_k(\alpha,\beta), $$

which this completes the proof of the Theorem 4.

Remark 1. *For* $k=1$ *in (3), we arrive the formula*

$$ J_a^{\alpha}\left(\left(x-a\right)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(x-a\right)^{\alpha+\beta-1}. \qquad (4) $$

Corollary 1. *Let* $\alpha, \beta > 0.$ *Then, there holds the formula*

$$ {}_kJ_a^{\alpha}(1) = \frac{1}{\Gamma_k(\alpha+k)}\left(x-a\right)^{\frac{\alpha}{k}-2}. \qquad (5) $$

Remark 2. *For* $k=1$ *in (5), we get the formula*

$$ J_a^{\alpha}(1) = \frac{1}{\Gamma(\alpha+1)}\left(x-a\right)^{\alpha-2}. $$

Some new k -Riemann-Liouville fractional integral inequalities

Chebyshev inequalities can be represented in k -fractional integral forms as follows:

Theorem 5. *Let* f, g *be two synchronous on* $[0,\infty),$ *then for all* $t > a \geq 0, \; \alpha > 0, \; \beta > 0,$ *the following inequalities for* k *-fractional integrals hold:*

$$ {}_kJ_a^{\alpha}fg(t) \geq \frac{1}{J_a^{\alpha}(1)}\ {}_kJ_a^{\alpha}f(t)\ {}_kJ_a^{\alpha}g(t) \qquad (6)$$

$$ {}_kJ_a^{\alpha}fg(t)\ {}_kJ_a^{\beta}(1) + {}_kJ_a^{\beta}fg(t)\ {}_kJ_a^{\alpha}(1) \geq {}_kJ_a^{\alpha}f(t)\ {}_kJ_a^{\beta}g(t) + {}_kJ_a^{\alpha}g(t)\ {}_kJ_a^{\beta}f(t). \qquad (7)$$

Proof Since the functions f and g are synchronous on $[0,\infty)$, then for all $x, y \geq 0,$ we have

$$\left(f(x) - f(y)\right)\left(g(x) - g(y)\right) \geq 0.$$

Therefore

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x) \qquad (8)$$

Multiplying both sides of (8) by $\frac{1}{k\Gamma_k(\alpha)}(t-x)^{\frac{\alpha}{k}-1}$, then integrating the resulting inequality wit hrespest to x over (a,t) , we obtain

$$\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-x)^{\frac{\alpha}{k}-1} f(x)g(x)dx + \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-x)^{\frac{\alpha}{k}-1} f(y)g(y)dx$$

$$\geq \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-x)^{\frac{\alpha}{k}-1} f(x)g(y)dx + \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-x)^{\frac{\alpha}{k}-1} f(y)g(x)dx$$

i.e.

$$ {}_kJ_a^{\alpha}fg(t) + f(y)g(y)\ {}_kJ_a^{\alpha}(1) \geq g(y)\ {}_kJ_a^{\alpha}f(t) + f(y)\ {}_kJ_a^{\alpha}g(t). \qquad (9)$$

Multiplying both sides of (9) by $\frac{1}{k\Gamma_k(\alpha)}(t-y)^{\frac{\alpha}{k}-1}$, then integrating the resulting inequality with respect to y over (a,t) , we obtain

$$ {}_kJ_a^{\alpha}fg(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\alpha}{k}-1}dy + {}_kJ_a^{\alpha}(1)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\alpha}{k}-1} f(y)g(y)dy$$

$$\geq {}_kJ_a^{\alpha}f(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\alpha}{k}-1}g(y)dy + {}_kJ_a^{\alpha}g(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\alpha}{k}-1} f(y)dy,$$

that is,

$$ {}_kJ_a^{\alpha}fg(t) \geq \frac{1}{J_a^{\alpha}(1)}\ {}_kJ_a^{\alpha}f(t)\ {}_kJ_a^{\alpha}g(t)$$

and the first inequality is proved.

Multiplying both sides of (9) by $\frac{1}{k\Gamma_k(\alpha)}(t-y)^{\frac{\beta}{k}-1}$, then integrating the resulting inequality with respect to y over (a,t), we obtain

$$ {}_kJ_a^{\alpha}fg(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\beta}{k}-1}dy + {}_kJ_a^{\alpha}(1)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\beta}{k}-1} f(y)g(y)dy$$

$$\geq {}_kJ_a^{\alpha}f(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\beta}{k}-1}g(y)dy + {}_kJ_a^{\alpha}g(t)\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-y)^{\frac{\beta}{k}-1} f(y)dy,$$

that is

$$ {}_kJ_a^{\alpha}fg(t)\ {}_kJ_a^{\beta}(1) + {}_kJ_a^{\beta}fg(t)\ {}_kJ_a^{\alpha}(1) \geq {}_kJ_a^{\alpha}f(t)\ {}_kJ_a^{\beta}g(t) + {}_kJ_a^{\alpha}g(t)\ {}_kJ_a^{\beta}f(t)$$

and the second inequality is proved. The proof is completed.

Theorem 6. *Let f,g be two synchronous on $[0,\infty)$, $h\geq 0,$ then for all $t>a\geq 0,\ \alpha>0,\ \beta>0,$ the following in equalities for k -fractional integrals hold:*

$$\frac{1}{\Gamma_k(\beta+k)}(t-a)^{\frac{\beta}{k}-2}\ {}_kJ_a^{\alpha}fgh(t)+\frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\ {}_kJ_a^{\beta}fgh(t)$$

$$\geq\ {}_kJ_a^{\alpha}fh(t)\ {}_kJ_a^{\beta}g(t)+\ {}_kJ_a^{\alpha}gh(t)\ {}_kJ_a^{\beta}f(t)-\ {}_kJ_a^{\alpha}h(t)\ {}_kJ_a^{\beta}fg(t)-\ {}_kJ_a^{\alpha}fg(t)\ {}_kJ_a^{\beta}h(t)$$

$$+\ {}_kJ_a^{\alpha}f(t)\ {}_kJ_a^{\beta}gh(t)+\ {}_kJ_a^{\alpha}g(t)\ {}_kJ_a^{\beta}fh(t).$$

Proof: Since the functions f and g are synchronous on $[0,\infty)$ and $h\geq 0,$ then for all $x,y\geq 0,$ we have

$$(f(x)-f(y))(g(x)-g(y))(h(x)+h(y))\geq 0.$$

By opening the above, we get

$$f(x)g(x)h(x)+f(y)g(y)h(y)$$

$$\geq f(x)g(y)h(x)+f(y)g(x)h(x)-f(y)g(y)h(x)$$

$$-f(x)g(x)h(y)+f(x)g(y)h(y)+f(y)g(x)h(y). \qquad (10)$$

Multiplying both sides of (10) by $\frac{1}{k\Gamma_k(\alpha)}(t-x)^{\frac{\alpha}{k}-1}$, then integrating the resulting inequality with respect to x over (a,t), we obtain

$$\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}f(x)g(x)h(x)dx+f(y)g(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}dx$$

$$\geq g(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}f(x)h(x)dx+f(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}g(x)h(x)dx$$

$$-f(y)g(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}h(x)dx-h(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}f(x)g(x)dx$$

$$+g(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}f(x)dx+f(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}g(x)dx.$$

i.e,

$${}_kJ_a^{\alpha}fgh(t)+f(y)g(y)h(y)\ {}_kJ_a^{\alpha}(1)$$

$$\geq g(y)\ {}_kJ_a^{\alpha}fh(t)+f(y)\ {}_kJ_a^{\alpha}gh(t)-f(y)g(y)\ {}_kJ_a^{\alpha}h(t)-h(y)\ {}_kJ_a^{\alpha}fg(t)$$

$$+g(y)h(y)\ {}_kJ_a^{\alpha}f(t)+f(y)h(y)\ {}_kJ_a^{\alpha}g(t). \qquad (11)$$

Multiplying both sides of (11) by $\frac{1}{k\Gamma_k(\beta)}(t-y)^{\frac{\beta}{k}-1}$, then integrating the resulting inequality with respect to y over (a,t) , we obtain

$$ {}_kJ_a^\alpha fgh(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}dy+{}_kJ_a^\alpha(1)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}f(y)g(y)h(y)dy $$

$$ \geq {}_kJ_a^\alpha fh(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}g(y)dy+{}_kJ_a^\alpha gh(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}f(y)dy $$

$$ -{}_kJ_a^\alpha h(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}f(y)g(y)dy-{}_kJ_a^\alpha fg(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}h(y)dy $$

$$ +{}_kJ_a^\alpha f(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}g(y)h(y)dy+{}_kJ_a^\alpha g(t)\frac{1}{k\Gamma_k(\beta)}\int_a^t(t-y)^{\frac{\beta}{k}-1}f(y)h(y)dy. $$

that is

$$ {}_kJ_a^\alpha fgh(t)\,{}_kJ_a^\beta(1)+{}_kJ_a^\alpha(1)\,{}_kJ_a^\beta fgh(t)\geq {}_kJ_a^\alpha fh(t)\,{}_kJ_a^\beta g(t)+{}_kJ_a^\alpha gh(t)\,{}_kJ_a^\beta f(t) $$

$$ -{}_kJ_a^\alpha h(t)\,{}_kJ_a^\beta fg(t)-{}_kJ_a^\alpha fg(t)\,{}_kJ_a^\beta h(t) $$

$$ +{}_kJ_a^\alpha f(t)\,{}_kJ_a^\beta gh(t)+{}_kJ_a^\alpha g(t)\,{}_kJ_a^\beta fh(t) $$

which this completes the proof.

Corollary 2. *Let f,g be two synchronous on $[0,\infty)$, $h\geq 0$, then for all $t>a\geq 0$, $\alpha>0$, the following inequalities for k-fractional integrals hold:*

$$ \frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\;{}_kJ_a^\alpha fgh(t) $$

$$ \geq {}_kJ_a^\alpha fh(t)\,{}_kJ_a^\alpha g(t)+{}_kJ_a^\alpha gh(t)\,{}_kJ_a^\alpha f(t)-{}_kJ_a^\alpha h(t)\,{}_kJ_a^\alpha fg(t). $$

Theorem 7. *Let f,g and h be three monotonic functions defined on $[0,\infty)$ satisfying the following*

$$ (f(x)-f(y))(g(x)-g(y))(h(x)-h(y))\geq 0 $$

For all $x,y\in[a,t]$, then for all $t>a\geq 0$, $\alpha>0$, $\beta>0$, the following inequalities for k-fractional integrals holds:

$$ \frac{1}{\Gamma_k(\beta+k)}(t-a)^{\frac{\beta}{k}-2}\;{}_kJ_a^\alpha fgh(t)-\frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\;{}_kJ_a^\beta fgh(t) $$

$$ \geq {}_kJ_a^\alpha fh(t)\,{}_kJ_a^\beta g(t)+{}_kJ_a^\alpha gh(t)\,{}_kJ_a^\beta f(t)-{}_kJ_a^\alpha h(t)\,{}_kJ_a^\beta fg(t)+{}_kJ_a^\alpha fg(t)\,{}_kJ_a^\beta h(t) $$

$$ -{}_kJ_a^\alpha f(t)\,{}_kJ_a^\beta gh(t)-{}_kJ_a^\alpha g(t)\,{}_kJ_a^\beta fh(t). $$

Proof: The proof is similar to that given in Theorem 6.

Theorem 8. *Let f and g be two functions on $[0,\infty)$, then for all $t>a\geq 0$, $\alpha>0$, $\beta>0$, the following inequalities for k-fractional integrals hold:*

$$\frac{1}{\Gamma_k(\beta+k)}(t-a)^{\frac{\beta}{k}-2}\ {}_kJ_a^{\alpha} f^2(t)$$

$$+\frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\ {}_kJ_a^{\beta} g^2(t)$$

$$\geq 2\ {}_kJ_a^{\alpha} f(t)\ {}_kJ_a^{\beta} g(t) \qquad (12)$$

and

$${}_kJ_a^{\alpha} f^2(t)\ {}_kJ_a^{\beta} g^2(t)+\ {}_kJ_a^{\beta} f^2(t)\ {}_kJ_a^{\alpha} g^2(t)\geq 2\ {}_kJ_a^{\alpha} fg(t)\ {}_kJ_a^{\beta} fg(t). \qquad (13)$$

Proof Since,

$$\left(f(x)-g(y)\right)^2\geq 0$$

then we have

$$f^2(x)+g^2(y)\geq 2f(x)g(y). \qquad (14)$$

Multiplying both sides of (14) by $\frac{1}{k\Gamma_k(\alpha)}(t-x)^{\frac{\alpha}{k}-1}$ and $\frac{1}{k\Gamma_k(\beta)}(t-y)^{\frac{\beta}{k}-1}$, then integrating the resulting inequality with respect to x and y over (a,t) respectively, we obtain (12).
On the other hand, since

$$\left(f(x)g(y)-f(y)g(x)\right)^2\geq 0$$

then under procedures similar to the above we obtain (13).

Corollary 3. *Let f and g be two functions on $[0,\infty)$, then for all $t>a\geq 0,\ \alpha>0,$ the following inequalities for k -fractional integrals hold:*

$$\frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\left[{}_kJ_a^{\alpha} f^2(t)+\ {}_kJ_a^{\beta} g^2(t)\right]$$

$$\geq 2\ {}_kJ_a^{\alpha} f(t)\ {}_kJ_a^{\alpha} g(t),$$

and

$${}_kJ_a^{\alpha} f^2(t)\ {}_kJ_a^{\alpha} g^2(t)\geq \left[{}_kJ_a^{\alpha} fg(t)\right]^2.$$

Theorem 9. *Let $f\ :\ \mathrm{R}\to\mathrm{R}$ and defined by*

$$\bar{f}(x)=\int_a^x f(t)dt,\quad x>a\geq 0,$$

Then for $\alpha\geq k>0$

$${}_kJ_a^{\alpha} f(x)=\frac{1}{k}\ {}_kJ_a^{\alpha-k}\bar{f}(x)$$

Proof By definition of the k -fractional integral and by using Dirichlet's formula, we have

$$
{}_kJ_a^{\alpha}\bar{f}(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}\int_a^t f(u)dudt
$$

$$
=\frac{1}{k\Gamma_k(\alpha)}\int_a^x f(u)\int_u^x(x-t)^{\frac{\alpha}{k}-1}dtdu
$$

$$
=\frac{1}{\Gamma_k(\alpha+k)}\int_a^x(x-u)^{\frac{\alpha}{k}} f(u)du
$$

$$
=k\ {}_kJ_a^{\alpha+k}f(x).
$$

This completes the proof of Theorem 9.

We give the generalized Cauchy-Bunyakovsky-Schwarz inequality as follows:

Lemma 1. *Let* $f,g,h\ :[a,b]\to[0,\infty)$ *be continuous functions* $0\le a<b$. *Then*

$$
\left(\int_a^b g^m(t)h^r(t)f(t)dt\right)\left(\int_a^b g^n(t)h^s(t)f(t)dt\right)\ge\left(\int_a^b g^{\frac{m+n}{2}}(t)h^{\frac{r+s}{2}}(t)f(t)dt\right)^2. \quad (15)
$$

where m, n, x, y arbitrary real numbers.

Proof It is obvious that

$$
\int_a^b\left[\sqrt{g^m(t)h^r(t)f(t)}\sqrt{\int_a^b g^n(t)h^s(t)f(t)dt}-\sqrt{g^n(t)h^s(t)f(t)}\sqrt{\int_a^b g^m(t)h^r(t)f(t)dt}\right]^2dt\ge 0.
$$

Then, it follows that

$$
\int_a^b\Bigg[g^m(t)h^r(t)f(t)\int_a^b g^n(t)h^s(t)f(t)dt+g^n(t)h^s(t)f(t)\int_a^b g^m(t)h^r(t)f(t)dt
$$

$$
-2g^{\frac{m+n}{2}}(t)h^{\frac{r+y}{s}}(t)f(t)\sqrt{\int_a^b g^m(t)h^r(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^s(t)f(t)dt}\Bigg]dt\ge 0
$$

and also

$$
2\left(\int_a^b g^m(t)h^r(t)f(t)dt\right)\left(\int_a^b g^n(t)h^s(t)f(t)dt\right)
$$

$$
\ge 2\left(\int_a^b g^{\frac{m+n}{2}}(t)h^{\frac{r+s}{2}}(t)f(t)dt\right)\sqrt{\int_a^b g^m(t)h^r(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^s(t)f(t)dt,}
$$

which this give the required inequality.

Theorem 10. *Let* $f\in L_1[a,b]$. *Then*

$$
\left({}_kJ_a^{m(\frac{\alpha}{k}-1)+1}f^r(x)\right)\left({}_kJ_a^{n(\frac{\alpha}{k}-1)+1}f^s(x)\right)\ge\left({}_kJ_a^{\frac{m+n}{2}(\frac{\alpha}{k}-1)+1}f^{\frac{r+s}{2}}(x)\right)^2 \quad (16)
$$

For $k,m,n,r,s>0$ *and* $\alpha>1$.

Proof By taking $g(t)=(x-t)^{\frac{\alpha}{k}-1},\ f(t)=\frac{1}{k\Gamma_k(\alpha)}$ and $h(t)=f(t)$ in (15), we obtain

$$\left(\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{m(\frac{\alpha}{k}-1)} f^r(t)dt\right)\left(\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{n(\frac{\alpha}{k}-1)} f^s(t)dt\right)$$

$$\geq \left(\frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{m+n}{2}(\frac{\alpha}{k}-1)} f^{\frac{r+s}{2}}(t)dt\right)^2$$

which can be written as (16).

Remark 3. *For $k=1$ in (16), we get the following inequality*

$$\left(J_a^{m(\alpha-1)+1} f^r(x)\right)\left({}_k J_a^{n(\alpha-1)+1} f^s(x)\right) \geq \left({}_k J_a^{\frac{m+n}{2}(\alpha-1)+1} f^{\frac{r+s}{2}}(x)\right)^2.$$

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